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Regularization Methods to Approximate Solutions of Variational Inequalities

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Abstract

In this paper, we study the regularization methods to approximate the solutions of the variational inequalities with monotone hemi-continuous operator having perturbed operators arbitrary. Detail, we shall study regularization methods to approximate solutions of following variational inequalities: $\langle Ax - y_0, x - z\rangle \geq 0, \forall x \in D,$ and $\langle Az - y_0, x - z\rangle \geq f(z) - f(x), \forall x \in D,$ with operator A being monotone hemi-continuous form real Banach reflexive X into its dual space X^*, but instead of knowing the exact data (y_0, A), we only know its approximate data (y_δ, A_h) satisfying certain specified conditions and D is a nonempty convex closed subset of X; the real function f defined on X is assumed to be lower semi-continuous, convex and is not identical to infinity. At the same time, we will evaluate the convergence rate of the approximate solution. The regularization methods here are different from the previous ones.

Keywords

Ill-Posed Problem, Variational Inequality, Regularization Method, Monotone Operator, Hemi-Continuous Operator, Lower Semi-Continuous Function

1. Introduction

In this paper, we study the regularization methods to approximate the solutions of the following variational inequalities

$$\langle Ax - y_0, x - z\rangle \geq 0, \forall x \in D, \quad (1)$$

and

$$\langle Az - y_0, x - z\rangle \geq f(z) - f(x), \forall x \in D, \quad (2)$$

where A is a hemi-continuous monotone operator form real Banach reflexive X

into its dual space X^* and D is a nonempty convex closed subset of X and $\langle g,z\rangle$ is the value of the linear functional $g \in X^*$ at $z \in X$, *i.e.* $g(z)=\langle g,z\rangle$. We suppose that (1) or (2) has the exact solutions in D. For the variational inequality (2), the real function f defined on X is assumed to be lower semi-continuous, convex and is not identical to infinity.

These variational inequalities are used in the study of nonlinear problems in physics and engineering, especially in finding solutions to problems in non-linear partial differential equations [1].

The problem to approximate solutions of variational inequalities (1) or (2) has been posed for a long time, and has been studied by many famous mathematicians in the world, including mathematicians Al'ber Ya. I., Abramov A., Ramm A. G., Riyazantseva P., Browder F. E., Liskovets O. A., who have contributed many foundational works [2]-[19].

In the 1960s, Browder F. E. was the first to study the problem of finding approximate solutions of (1), (2) and to study the stability of the solutions of these problems by a method that he called the monotone operator method [9].

After Tikhonov A. N. introduced the variational method, which was later called the Tikhonov regularization method to solve the ill-posed problems in works [20] [21] [22] [23], Albert Ia. I. was the first to study the stability of the solution of variational inequality (1) according to the Tikhonov regularization method by adding the conjugate operator [4] [5] [6]. Actually, this idea of Browder F. E. has been used in works [9] but has not been prominent in the method.

Liskovevtz O. A. [15] [16] [17] [18], Nguyen Van Kinh [24] [25] [26] [27] [28], have achieved many results in this field, especially for variational inequality (1), where the noise operator of the exact operator is non-monotone. However, there are still many unresolved open issues. This paper will address some of these open problems. Detail, we shall study regularization methods to approximate solutions of variational inequalities (1) and (2) with operator A being monotone hemi-continuous form real Banach reflexive X into its dual space X^*, but instead of knowing the exact data (y_0, A), we only know its approximate data (y_δ, A_h) satisfying certain specified conditions and D is a nonempty convex closed subset of X.

The paper structure consists of 3 sections: Section 1 the introduction briefly summarizes the recent research results and comes up with problems that need to be studied; Section 2 presents regularization method for variational inequality (1) and Section 3 presents regularization for general variational inequality (2).

2. Regularization Method to Approximate Variational Inequality

First, we give the definition and some properties of topology on non-empty subsets of a topological space, some properties of the variational inequality with the hemi-continuous, monotone operator and dual mapping.

Suppose that X is a topology space. We denote $S(X)$ as the set of all non-empty subsets of X. We will define a topology on $S(X)$ as follows:

Definition 2.1 [29] Topology on the set $S(X)$ that has a basis topology, which is of the form $\{M \in S(X): M \subset U\}$, where U is an arbitrary open set of X. This topology is called B-topology on $S(X)$. If topology space X is topology defined by metric then B-topology is called β-topology.

The above topological definition can be found in [29]. The following are some characteristics of this topology.

Proposition 2.1 [29] If the sequence of the sets $\{M_\sigma, \sigma \in \Sigma\}$ converges to the set $M \subset X$, denoted by $M_\sigma \xrightarrow{B} M$ and if the sequence of the sets $\{m_\sigma, \sigma \in \Sigma\}$, with $\varnothing \neq m_\sigma \subset M_\sigma$ and $M \subset N \subset X$ then $m_\sigma \xrightarrow{B} N$.

Proposition 2.2 [29] If the sequence of the sets $\{M_\sigma, \sigma \in \Sigma\}$ B-converges to $M \subset X$, $M \neq \varnothing$, then arbitrary subsequence of this sequence also B-converges to the set M.

It is easy to see that the sequence of sets $\{M_\sigma, \sigma \in \Sigma\}$ B-converges to the set M, which means that for any open set U containing M, then for σ "large enough" then $M_\sigma \subset U$. If X is a metric space, then what has just been said is expressed by the following proposition:

Proposition 2.3 [29] Suppose that X is a metric space. Then we have

$$M_\sigma \xrightarrow{\beta} M \Leftrightarrow \beta(M_\sigma, M) \to 0,$$

where $\beta(M_\sigma, M) = \sup_{x \in M_\sigma} d(x, M) = \sup_{x \in M_\sigma} \inf_{y \in M} d(x, y)$, which is called the deviation of the set M_σ with respect to the set M.

Proposition 2.4 [29] Suppose that the sequence of sets $\{M_\sigma, \sigma \in \Sigma\}$ in Hausdorff topology space X, has the following property:

Taking arbitrarily $x_\sigma \in M_\sigma$, we get a sequence of points $\{x_\sigma, \sigma \in \Sigma\}$ in X. Any subsequence of this sequence has one of the following properties:

1) $\{x_\varphi\}' \cap M \neq \varnothing$, where $\{x_\varphi\}'$ is the set of limit points of the sequence $\{x_\varphi\}$.

2) $\varnothing \neq \{x_\varphi\}' \subset M$, where M is a some subset of X.

Then, $M_\sigma \xrightarrow{B} M$.

Conversely, if M is a compact set and $M_\sigma \xrightarrow{B} M$, then both properties (1) and (2) are true if we replace the set M by the set $\bar{M}$.

The operator $A: D(A) \subset X \to X^*$ is called monotone if it satisfies the following condition

$$\langle Ax - Ay, x - y \rangle \geq 0, \forall x, y \in D(A). \tag{3}$$

The operator A is called strictly monotone if (3) with an equals sign occurs if and only if $x = y$.

The operator A is called hemi-continuous at $x_0 \in D(A)$ if for any x such that:

$$x_0 + tx \in D(A), \forall t \in (0, \alpha], \alpha = \alpha(x) > 0 \text{ and } \forall \{t_n\}, t_n \in (0, \alpha], t_n \to 0$$
$$\Rightarrow A(x_0 + t_n x) \rightharpoonup A(x_0), \text{ as } n \to \infty.$$

A is called hemi-continuous on $D(A)$ if it is continuous at every point $x \in D(A)$.

An extension of Minty's Lemma is that the following variational inequalities are equivalent:

Lemma 2.1 [29] If $A: D(A) \subset X \to X^*$ is a monotone and hemi-continuous, then the following variational inequalities are equivalent, *i.e.*, they have the same of the solution sets:

$$\langle Ax - y, x - z \rangle \geq \varepsilon \|x - z\|, \forall x \in D, \tag{4}$$

$$\langle Az - y, x - z \rangle \geq \varepsilon \|x - z\|, \forall x \in D, \tag{5}$$

where $\varepsilon > 0$ and D is convex closed subset of $D(A)$.

Lemma 2.2 Let $g(t)$ be a non-negative real, continuous, and $B: D(B) \subset X \to X^*$ be a hemi-continuous at $z \in D(B)$, $D(B)$ be a convex subset of X and $\varepsilon > 0$. If

$$\langle Bx - y, x - z \rangle \geq \varepsilon g(\|x\|)\|x - z\|, \forall x \in D(B) \tag{6}$$

then

$$\langle Bz - y, x - z \rangle \geq \varepsilon g(\|z\|)\|x - z\|, \forall x \in D(B) \tag{7}$$

Proof. Suppose that $z \in D(B)$ satisfies (6). Let $\omega \in D(B)$, then $m = tz + (1-t)\omega \in D(B)$, with $0 < t < 1$, because $D(B)$ is convex and $z, \omega \in D(B)$.

By replacing x in the variational inequality (6) by m, we obtain

$$\langle Bm - y, m - z \rangle \geq \varepsilon g(\|m\|)\|m - z\|, \forall \omega \in D(B)$$

or

$$(1-t)\langle Bm - y, \omega - z \rangle \geq \varepsilon (1-t) g(\|m\|)\|m - z\|, \forall z, \omega \in D(B).$$

Therefore

$$\langle Bm - y, \omega - z \rangle \geq \varepsilon g(\|tz + (1-t)\omega\|)\|\omega - z\|, \forall z, \omega \in D(B) \tag{8}$$

Since $g(t)$ is a continuous function, then $g(\|tz + (1-t)\omega\|) \to g(\|z\|)$ as $t \to 1$ and since B is hemi-continuous at z, then the left hand side of (8) converges to $\langle Bz - y, \omega - z \rangle$ as $t \to 1$. Therefore, we have

$$\langle Bz - y, x - z \rangle \geq \varepsilon g(\|z\|)\|x - z\|, \forall x \in D(B).$$

Lemma is proved.

The mapping $U: X \to X^*$ is called *d*ual mapping of the norm space X if it satisfies the following conditions

$$\|Ux\| = \|x\|, \langle Ux, x \rangle = \|Ux\|\|x\| = \|x\|^2, \forall x \in X.$$

This concept was first studied by Browder F. E. [8] and Vaiberg M. M. [30].

The norm space X is called strictly convex if it has the following property

$$\|x + y\| = \|x\| + \|y\|, x, y \in X \Rightarrow \{x, y\} \text{ is linearly dependent.}$$

Next, we state some properties of duality mapping, which are used in the following sections.

Proposition 2.5 [30] The dual mapping $U: X \to X^*$ is single if dual space X^* is strictly convex.

The mapping $U: X \to X^*$ is called coercive if

$$\langle Ux, x\rangle \geq c(\|x\|)\|x\|, \forall x \in X,$$

where $c(t)$ is a non-negative real function and $c(t) \to +\infty$ as $t \to +\infty$.

Proposition 2.6 [30] The dual mapping $U: X \to X^*$ is coercive, monotone. Moreover, if X is a strictly convex space then U is strictly monotone.

Proposition 2.7 [30] If X is a reflexive real space and its dual space is strictly convex then the dual mapping $U: X \to X^*$ is hemi-continuous.

From the definition of the dual mapping, it follows that

$$\langle Ux - Uy, x - y\rangle \geq (\|x\| - \|y\|)^2, \forall x, y \in X.$$

Now we begin to present the regularization method to approximate solution of the variational inequality (1).

In the section, we assume that is a real reflective Banach space X with the conjugate space X^* being strictly convex. Then, the dual mapping $U: X \to X^*$ is single, coercive, monotone and hemi-continuous (Proposition 2.5, Proposition 2.6, Proposition 2.7).

As we already know the problem of finding the solution of the variational inequalities (1) is generally an ill-posed problem. Therefore, it is necessary to study some variational methods solving such problems to ensure that the solutions are stable, *i.e.* the solutions depend continuously on small changes to the exact data. We will present the regularization method with small parameter to approximate the variational inequality (1).

Suppose with exact data (y_0, A), the solution set of the variational inequality (1) is $Z_0 \neq \varnothing$. Since A is a hemi-continuous monotone operator from real Banach reflexive X into its dual space X^* and D is a nonempty convex closed subset of X, then Z_0 is a convex, closed set [30]. Instead of knowing the exact data (y_0, A), we only know its approximate data (y_δ, A_h) satisfying the following conditions

$$\|y_\delta - y_0\| \leq \delta, \tag{9}$$

and

$$\|A_h x - Ax\| \leq hg(\|x\|), \forall x \in D(A_h), \tag{10}$$

where the operator $A_h: D(A_h) \subset X \to X^*$, $D(A_h) = D(A) = D \subset X$, A_h is an arbitrary operator, not necessarily monotone or hemi-continuous, and the real function $g(t)$ is non-negative, continuous and increasing no faster than some linear function, *i.e.*, there exist $M \geq 0, N \geq 0$ such that

$$g(t) \leq Mt + N, \forall t \geq 0. \tag{11}$$

In this work, we construct an approximate solution of (1), according to the regularization method, is the solutions of the variational inequality

$$\langle A_h x + \alpha Ux - y_\delta, x - z\rangle \geq -\varepsilon g(\|x\|)\|x - z\|, \forall x \in D, \tag{12}$$

where $\varepsilon \geq 0$ is a small parameter.

Remark 2.1

1) If the operator A_h is non-monotone, non-hemicontinuous then in [17], Liskovets O. A. studied the Tikhonov regularization solution of (1) which is the solution of (12) with $g(t) \equiv 1$, *i.e.*, the solution of the variational inequality:

$$\langle A_h x + \alpha Ux - y_\delta, x - z\rangle \geq -\varepsilon\|x - z\|, \forall x \in D. \tag{13}$$

2) If the operator A_h is non-monotone, non-hemicontinuous, then in [18], Liskovets O. A. studied the approximate solution of (1) which is the solution of the variational inequality:

$$\langle A_h z + \alpha Uz - y_\delta, x - z\rangle \geq -\varepsilon g(\|z\|)\|x - z\|, \forall x \in D. \tag{14}$$

Since the operator A_h is arbitrary, the two sets of solutions for (12) and (14) are different, so the author's and [18]'s methods are different. The following example demonstrates it:

Let $X = \mathbb{R}$ be real normal space, therefore, its conjugate space also be $\mathbb{R}$ and its dual mapping U be identity mapping. Let $D = [-1,1]$, $y_0 = 0$.

Suppose that the exact operator $A: D \to \mathbb{R}$, is defined by

$$Ax = \begin{cases} x+1, & -1 \leq x < 0 \\ 1, & 0 \leq x \leq 1 \end{cases}$$

Clearly the operator A is monotone and hemi-continuous on D.

The approximate operator A_h of the operator A is defined by

$$A_h x = \begin{cases} x+1, & -1 \leq x < 0 \\ 1+h, & x = 0 \\ 1, & 0 < x \leq 1 \end{cases}$$

Therefore, the operator A_h is non-monotone, non-hemicontinuous on D and

$$\|A_h x - Ax\| \leq h, \forall x \in D.$$

Let $g(t) \equiv 1, 0 < \varepsilon - h < 1, \varepsilon < 1$. Therefore, the variational inequality (12) has the form

$$\langle A_h x + \alpha x, x - 0\rangle = (A_h x + \alpha x)x \geq -\varepsilon|x - 0|, \forall x \in D = [-1,1].$$

Consequently, $z = 0$ is the solution of the variational inequality (12).

On the other hand, the following inequality is not true

$$(1+h)x \geq -\varepsilon|x|, \forall x \in [-1,1].$$

So $z = 0$ is not the solution of the variational inequality (14).

Therefore, the solution sets of (12) and (14) are different.

In the following, we study the existence of an approximate solution and the stability of the solution of the regualaiation method (12).

We denote the solution set of (12) as Z_{Δ}^{α}, where $\Delta=(h,\varepsilon,\delta)$. The following theorem asserts the existence of a solution of (12), that is, the existence of the Tikhonov regularization operator.

Theorem 2.1 If $\varepsilon \ge h$, then $Z_{\Delta}^{\alpha} \ne \varnothing$.

Proof. We consider the following variational inequality

$$\langle Ax+\alpha Ux-y_{\delta}, x-z\rangle \ge 0, \forall x\in D. \tag{15}$$

By assumption, the operators A and U are single, monotone, hemicontinuous, so $A+\alpha U$ is also single, monotone, hemicontinuous.

+ If D is non-bounded, then the operator $A+\alpha U$ is coercive on D. Indeed, we have

$$\begin{aligned}\frac{\langle Ax+\alpha Ux, x-x_0\rangle}{\|x\|} &= \frac{\langle Ax-Ax_0, x-x_0\rangle}{\|x\|}+\frac{\langle Ax_0, x-x_0\rangle}{\|x\|}+\alpha\frac{\langle Ux, x-x_0\rangle}{\|x\|}\\ &\ge -\frac{\|Ax_0\|\|x-x_0\|}{\|x\|}+\alpha\frac{\langle Ux,x\rangle}{\|x\|}-\alpha\frac{\langle Ux,x_0\rangle}{\|x\|}\\ &\ge -\frac{\|Ax_0\|\|x-x_0\|}{\|x\|}+\alpha\frac{\|x\|^2}{\|x\|}-\alpha\frac{\|x\|\|x_0\|}{\|x\|}\\ &= -\frac{\|Ax_0\|\|x-x_0\|}{\|x\|}+\alpha\|x\|-\alpha\|x_0\|.\end{aligned}$$

Therefore, if $x\in D, \|x\|\to+\infty$ then $\dfrac{\langle Ax+\alpha Ux, x-x_0\rangle}{\|x\|}\to+\infty$

So, the operator is coercive on D.

Since $A+\alpha U$ is monotone, hemi-continuous, coercive, according to [8] [31] the variational inequality (15) has a solution in D.

+ If D is bounded, according to [32], the variational inequality (15) has a solution in D.

We will show that the solutions of (15) are all solutions of (12). Indeed, taking $z\in D$ as an arbitrary solution of (15), we have

$$\begin{aligned}\langle Ax+\alpha Ux-y_{\delta}, x-z\rangle &= \langle A_hx-Ax+Ax+\alpha Ux-y_{\delta}, x-z\rangle\\ &= \langle A_hx-Ax, x-z\rangle+\langle Ax+\alpha Ux-y_{\delta}, x-z\rangle.\end{aligned}$$

Since $z\in D$ is the solution (15) then

$$\langle Ax+\alpha Ux-y_{\delta}, x-z\rangle \ge 0, \forall x\in D.$$

Consequently,

$$\begin{aligned}\langle Ax+\alpha Ux-y_{\delta}, x-z\rangle &\ge \langle A_hx-Ax, x-z\rangle\\ &\ge -\|A_hx-Ax\|\|x-z\|\\ &\ge -hg(\|x\|)\|x-z\|\\ &\ge -\varepsilon g(\|x\|)\|x-z\|, (\varepsilon>h).\end{aligned}$$

So, $z\in Z_{\Delta}^{\alpha}$.

Theorem is proved.

Theorem 2.2 If $\varepsilon \ge h$ and $M\dfrac{h+\varepsilon}{\alpha} < k_1 < 1$, $\dfrac{\delta}{\alpha} < k_2 < +\infty$, where k_1, k_2 are constants, Z_Δ^α is the set of the solution of (12) then the sequence of the sets Z_Δ^α is uniformly bounded in *X*.

Proof. By assumption $\varepsilon \ge h$ and by Theorem 2.1. then $Z_\Delta^\alpha \ne \varnothing$. Let $z \in Z_\Delta^\alpha$, we have

$$\langle A_h x + \alpha U z - y_\delta, x - z\rangle \ge -\varepsilon g(\|x\|)\|x - z\|, \forall x \in D$$

or

$$\langle A_h x - Ax, x - z\rangle + \langle Ax + \alpha Ux - y_\delta, x - z\rangle \ge -\varepsilon g(\|x\|)\|x - z\|, \forall x \in D$$

$$\Rightarrow \langle Ax + \alpha Ux - y_\delta, x - z\rangle \ge -\langle A_h x - Ax, x - z\rangle - \varepsilon g(\|x\|)\|x - z\|, \forall x \in D$$

Consequently

$$\langle Ax + \alpha Ux - y_\delta, x - z\rangle \ge -(h+\varepsilon) g(\|x\|)\|x - z\|, \forall x \in D. \tag{16}$$

Since the operator is monotone, hemi-continuous and Lemma 2.2, from (16) it follows that

$$\langle Az + \alpha Uz - y_\delta, x - z\rangle \ge -(h+\varepsilon) g(\|z\|)\|x - z\|, \forall x \in D$$

or

$$\langle Az - y_0, x - z\rangle + \langle y_0 - y_\delta, x - z\rangle + (h+\varepsilon) g(\|z\|)\|x - z\| \ge \alpha\langle Uz, z - x\rangle.$$

So

$$\begin{aligned}&\langle Az - y_0, x - z\rangle + \left[\delta + (h+\varepsilon) g(\|z\|)\right]\|x - z\| \\ &\ge \alpha\langle Uz, z - x\rangle = \alpha\|z\|\|z - x\|, \forall x \in D.\end{aligned} \tag{17}$$

In (17) we take $x = x_0 \in Z_0 \subset D$, with x_0 is the solution of (1), we have

$$\langle Az - y_0, x_0 - z\rangle \le 0, \forall z \in Z_\Delta^\alpha.$$

Consequently

$$\left[\delta + (h+\varepsilon) g(\|z\|)\right]\|x_0 - z\| \ge \alpha\|z\|^2 - \alpha\|x_0\|\|z\|, \forall z \in Z_\Delta^\alpha. \tag{18}$$

Since $g(t) \le Mt + N, \forall t \ge 0$, then from (18) it follows that

$$\begin{aligned}&\left[M\frac{h+\varepsilon}{\alpha} - 1\right]\|z\|^2 + \left[\frac{\delta}{\alpha} + M\frac{h+\varepsilon}{\alpha}\|x_0\| + N\frac{h+\varepsilon}{\alpha}\|x_0\|\right]\|z\| \\ &+ N\frac{h+\varepsilon}{\alpha}\|x_0\| \ge 0, \forall z \in Z_\Delta^\alpha.\end{aligned} \tag{19}$$

By assumption $M\dfrac{h+\varepsilon}{\alpha} - 1 < 0$, $\dfrac{\delta}{\alpha} < k_2 < +\infty$, and (19) is a quadratic inequality in terms of $\|z\|$, then the set of $\|z\|$ in (19) is bounded. Since k_1, k_2 are constants which do not depend on $h, \varepsilon, \delta, \alpha$, then the sequence of the sets Z_Δ^α is uniformly bounded, *i.e.*, there exists a constant *k* which do not depend on $h, \varepsilon, \delta, \alpha$ such that

$$\|z\| \le k, \forall z \in Z_\Delta^\alpha, \forall \Delta = (h, \varepsilon, \delta), \forall \alpha.$$

Theorem is proved.

Remark 2.2

1) If $\dfrac{h+\varepsilon+\delta}{\alpha}$ is uniformly bounded then the sequence of sets Z_Δ^α, with Z_Δ^α being the set of the solutions of (13), is uniformly bounded [18].

2) The conditions for the sequence of sets Z_Δ^α, with Z_Δ^α being the set of the solutions of (14), is uniformly bounded , which is the same as theorem above [17].

3) Instead of regularization for the solution of the variational inequality (1), we regulate for the solution of the equation $Ax = y_0$, where A is given exactly and y_δ is an approximation of y_0 satisfying (9), in [3] [4] the condition is uniformly bounded the sequence of solution sets of (13) with A_h replaced by A, is $\dfrac{\delta}{\alpha} < k$, with k constant.

The following theorems talk about the stability of the solution of the variational inequality (1) according to the Tikhonov regularization method.

Theorem 2.3 With the same assumptions as in Theorem 2.2, the sequence of the sets Z_Δ^α, B-weakly converges to exactly solution set Z_0 of the variational inequality (1) as $\alpha \to 0$.

Proof. Let arbitrary $z_\Delta^\alpha \in Z_\Delta^\alpha$, from Theorem 2.2 it follows that the sequence $\{z_\Delta^\alpha\}$ is uniformly bounded. Since X is a reflexive space and $\{z_\Delta^\alpha\} \subset X$, there exists a subsequence $\{z_\Delta^\beta\}$ of the sequence such that it weakly converges to $z_* \in D$ (since $z_\Delta^\beta \in D$ and D is weakly closed). We shall denote $z_\Delta^\beta \rightharpoonup z_* \in D$.

Since z_Δ^β is the solution of (12), we have

$$\left\langle A_h x + \alpha U x - y_\delta, x - z_\Delta^\beta \right\rangle \geq -\varepsilon g\left(\|x\|\right)\left\|x - z_\Delta^\beta\right\|, \forall x \in D. \tag{20}$$

The argument is the same as in the proof of Theorem 2.2, using Lemma 2.2, from (20) it follows that

$$\left\langle A z_\Delta^\beta + \beta U z_\Delta^\beta - y_\delta, x - z_\Delta^\beta \right\rangle \geq -(h+\varepsilon) g\left(\left\|z_\Delta^\beta\right\|\right)\left\|x - z_\Delta^\beta\right\|, \forall x \in D. \tag{21}$$

Since the operator $A + \beta U$ is monotone, from (21) it follows that

$$\left\langle A x + \beta U x - y_\delta, x - z_\Delta^\beta \right\rangle \geq -(h+\varepsilon) g\left(\left\|z_\Delta^\beta\right\|\right)\left\|x - z_\Delta^\beta\right\|, \forall x \in D. \tag{22}$$

Since the function $g(t)$ is continuous and the sequence $\{z_\Delta^\beta\}$ is bounded, the sequence $\left\{g\left\|z_\Delta^\beta\right\|\right\}$ is bounded. Therefore, in (22) taking $\beta \to 0$, therefore, $\delta \to 0$, we have

$$\left\langle Ax - y_0, x - z_* \right\rangle \geq 0, x \in D.$$

Consequently, $z_* \in Z_0$.

From this, according to Proposition 2.4, it follows that the sequence of the sets $\{Z_\Delta^\alpha\}$ B-weakly converges to exactly solution set Z_0 of the variational inequality (1) as $\alpha \to 0$.

Theorem is proved.

We denote

$$Z_* := \arg\min_{z \in Z_0}\{\|z\|\} = \left\{z : \|z\| = \min_{z \in Z_0}\|u\|\right\}.$$

Since X is a reflexive real Banach space and Z_0 is a closed convex subset in X, then $Z_* \neq \varnothing$.

Theorem 2.4 The sequence of the sets $\{Z_\Delta^\alpha\}$ B-weakly converges to Z_* as $\alpha \to 0$, $\frac{h+\varepsilon+\delta}{\alpha} \to 0$. Moreover, let arbitrary $z_\Delta^\alpha \in Z_\Delta^\alpha$ and $z_* \in Z_*$ then

$$\|z_\Delta^\alpha\| \to \|z_*\|, \text{ as } \alpha \to 0, \frac{h+\varepsilon+\delta}{\alpha} \to 0.$$

Proof. + We first prove $z_\Delta^\alpha \rightharpoonup z_*$ as $\alpha \to 0$, $\frac{h+\varepsilon+\delta}{\alpha} \to 0$.

Since $\alpha \to 0$, $\frac{h+\varepsilon+\delta}{\alpha} \to 0$, then we take $\alpha, \frac{h+\varepsilon+\delta}{\alpha}$ is small enough such that

$$\varepsilon \geq h, M\frac{h+\varepsilon}{\alpha} < k_1 < 1, \frac{\delta}{\alpha} < k_2 < +\infty \tag{23}$$

Therefore, let arbitrary $z_\Delta^\alpha \in Z_\Delta^\alpha$, then the sequence $\{z_\Delta^\alpha\}$ is uniformly bounded (Theorem 2.2). Since X is a reflexive space and $\{z_\Delta^\alpha\} \subset X$, there exists a subsequence $\{z_\Delta^\beta\}$ of the sequence such that $z_\Delta^\beta \rightharpoonup z_* \in D$ (since $z_\Delta^\beta \in D$ and D is weakly closed) as $\beta \to 0$, $\frac{h+\varepsilon+\delta}{\beta} \to 0$. The argument is the same as in the proof of Theorem 2.3, we have $z_* \in Z_*$.

Since z_Δ^β is a solution of (12), we have

$$\left\langle A_h x + \beta U x - y_\delta, x - z_\Delta^\beta \right\rangle \geq -\varepsilon g\left(\|x\|\right)\left\|x - z_\Delta^\beta\right\|, \forall x \in D.$$

From the above inequalities, it follows that

$$\left\langle Ax + \beta U x - y_\delta, x - z_\Delta^\beta \right\rangle \geq -\left(h+\varepsilon\right) g\left(\|x\|\right)\left\|x - z_\Delta^\beta\right\|, \forall x \in D.$$

From the above inequality, according to Lemma 2.2, it follows that

$$\left\langle Az_\Delta^\beta + \beta U z_\Delta^\beta - y_\delta, x - z_\Delta^\beta \right\rangle \geq -\left(h+\varepsilon\right) g\left(\left\|z_\Delta^\beta\right\|\right)\left\|x - z_\Delta^\beta\right\|, \forall x \in D. \tag{24}$$

Since the operator $A + \beta U$ is monotone, from (24) it follows that

$$\left\langle Ax + \beta U x - y_\delta, x - z_\Delta^\beta \right\rangle \geq -\left(h+\varepsilon\right) g\left(\left\|z_\Delta^\beta\right\|\right)\left\|x - z_\Delta^\beta\right\|, \forall x \in D.$$

or

$$\begin{aligned}&\left\langle Ax - y_0, x - z_\Delta^\beta \right\rangle + \left\langle y_0 - y_\delta, x - z_\Delta^\beta \right\rangle + \left(h+\varepsilon\right) g\left(\left\|z_\Delta^\beta\right\|\right)\left\|x - z_\Delta^\beta\right\| \\ &\geq \beta\left\langle Ux, z_\Delta^\beta \right\rangle, \forall x \in D.\end{aligned} \tag{25}$$

In (25), taking $x \in Z_0 \subset D$, we have

$$\left\langle Ax - y_0, x - z_\Delta^\beta \right\rangle \leq 0$$

Hence, from (25) it follows that

$$\delta\left\|x - z_\Delta^\beta\right\| + \left(h+\varepsilon\right) g\left(\left\|z_\Delta^\beta\right\|\right)\left\|x - z_\Delta^\beta\right\| \geq \beta\left\langle Ux, z_\Delta^\beta \right\rangle, \forall x \in Z_0. \tag{26}$$

Since sequence $\{z_\Delta^\beta\}$ is bounded and $g(t)$ is continuous, in (26) taking $\beta \to 0$, $\frac{h+\varepsilon+\delta}{\beta} \to 0$, we obtain

$$\langle Ux, z_* - x\rangle \le 0, x \in Z_0. \tag{27}$$

From (27), according to Lemma of Minty, it follows that

$$\langle Uz_*, z_* - x\rangle \le 0, x \in Z_0. \tag{28}$$

Due to the property of the dual mapping *U*, from (28) it follows that

$$\|z_*\|^2 \le \langle Uz_*, x\rangle \le \|z_*\|\|x\|, \forall x \in Z_0.$$

Hence

$$\|z_*\| \le \|x\|, \forall x \in Z_0.$$

Consequently,

$$z_* \in Z_*.$$

Therefore, the sequence of sets $\{Z_\Delta^\alpha\}$ B- weakly converges to Z_* as $\alpha \to 0$, $\frac{h+\varepsilon+\delta}{\alpha} \to 0$.

+ Next, we prove $\|z_\Delta^\alpha\| \to \|z_*\|$, as $\alpha \to 0$, $\frac{h+\varepsilon+\delta}{\alpha} \to 0$.

Since the operator *A* is monotone, from (24) it follows that

$$\langle Az_\Delta^\beta + \beta Uz_\Delta^\beta - y_\delta, x - z_\Delta^\beta\rangle \ge -(h+\varepsilon) g(\|z_\Delta^\beta\|)\|x - z_\Delta^\beta\|, \forall x \in D.$$

The argument is the same as in the above proof and from the above inequality, we have

$$\delta\|x - z_\Delta^\beta\| + (h+\varepsilon) g(\|z_\Delta^\beta\|)\|x - z_\Delta^\beta\| \ge \beta\langle Uz_\Delta^\beta, z_\Delta^\beta - x\rangle, \forall x \in Z_0,$$

or

$$\frac{\delta}{\beta}\|x - z_\Delta^\beta\| + \frac{h+\varepsilon}{\beta} g(\|z_\Delta^\beta\|)\|x - z_\Delta^\beta\| \ge \langle Uz_\Delta^\beta, z_\Delta^\beta - x\rangle, \forall x \in Z_0. \tag{29}$$

In (29), replacing *x* by z_*, we have

$$\langle Uz_\Delta^\beta, z_\Delta^\beta - z_*\rangle \le \frac{\delta}{\beta}\|z_* - z_\Delta^\beta\| + \frac{h+\varepsilon}{\beta} g(\|z_\Delta^\beta\|)\|z_* - z_\Delta^\beta\|. \tag{30}$$

According to the property of the dual mapping *U*, from (30) it follows that

$$(\|z_*\| - \|z_\Delta^\beta\|)^2 \le \langle Uz_* - Uz_\Delta^\beta, z_* - z_\Delta^\beta\rangle = \langle Uz_*, z_* - z_\Delta^\beta\rangle + \langle Uz_\Delta^\beta, z_* - z_\Delta^\beta\rangle.$$

$$(\|z_*\| - \|z_\Delta^\beta\|)^2 \le \langle Uz_*, z_* - z_\Delta^\beta\rangle + \frac{\delta}{\beta}\|z_* - z_\Delta^\beta\| + \frac{h+\varepsilon}{\beta} g(\|z_\Delta^\beta\|)\|z_* - z_\Delta^\beta\| \tag{31}$$

In (31), taking $\beta \to 0$, $\frac{h+\varepsilon+\delta}{\beta} \to 0$, we obtain $\|z_\Delta^\beta\| \to \|z_*\|$.

The above happens for every subsequence $\{\|z_\Delta^\beta\|\}$ of the sequence $\|z_\Delta^\alpha\|$, so the sequence itself also converges to $\|z_*\|$ as $\alpha \to 0$, $\frac{h+\varepsilon+\delta}{\alpha} \to 0$.

Theorem is proved.

Corollary 2.1 If *X* is a *E-space* [33], then $\{Z_\Delta^\alpha\}$ B-converges to Z_* as $\alpha \to 0$, $\frac{h+\varepsilon+\delta}{\alpha} \to 0$.

Proof. It follows from Theorem 2.4.

Corollary 2.2 Suppose that X is a E-space and the set Z_* has only one element (this happens, for example, A is a stricly monotone operator and X is a strictly convex space). Therefore, if z_Δ^α arbitrarily in Z_Δ^α, then $z_\Delta^\alpha \to z_*$, as $\alpha \to 0$, $\frac{h+\varepsilon+\delta}{\alpha} \to 0$.

Proof. It follows from Theorem 2.4.

If we don't know if (1) has a solution before, then the following theorem tells us that.

Theorem 2.5 The necessary and sufficient condition for the variational inequality (1) to have a solution is that the sequence of sets $\{Z_\Delta^\alpha\}$ B-weakly converges to some unique bounded subset of X as $\alpha \to 0$, $\frac{h+\varepsilon+\delta}{\alpha} \to 0$.

Proof. Suppose that the sequence of sets $\{Z_\Delta^\alpha\}$ B-weakly converges to bounded subset $Z \subset X$ as $\alpha \to 0$, $\frac{h+\varepsilon+\delta}{\alpha} \to 0$. Since X is a reflexive Banach space and by Proposition 2.1, without loss of generality, we can assume that Z is a weakly compact set.

Let arbitrary $z_\Delta^\alpha \in Z_\Delta^\alpha$, then there exists the subsequence $\{z_\Delta^\beta\}$ of the sequence such that $z_\Delta^\beta \rightharpoonup z_* \in D \cap Z$ as $\alpha \to 0$, $\frac{h+\varepsilon+\delta}{\alpha} \to 0$ (by Proposition 2.4). The argument is the same as in the proof of Theorem 2.3, it follows that satisfies

$$\langle Ax - y_0, x - z_* \rangle \geq 0, \forall x \in D.$$

Hence, z_* is a solution of (1).

Conversely, if $Z_0 \neq \varnothing$, then from Theorem 2.4, it follows that the sequence of sets $\{Z_\Delta^\alpha\}$ B- weakly converges to bounded subset as $\alpha \to 0, \frac{h+\varepsilon+\delta}{\alpha} \to 0$.

Theorem is proved.

We now present an example where the above regularization method can be applied. This example is slightly modified in [18] [30].

As usual, the notation $L_p[0,1]$, $1 < p \leq 2$, is the normed space of real functions p-integrable in the Lebesgue sense on the interval [0,1]. As we all know, $L_p[0,1]$ is a reflecxive Banach space, and its conjugate space is $L_q[0,1]$, with $\frac{1}{p} + \frac{1}{q} = 1$, $L_q[0,1]$ is strictly convex.

Let $k(s,t)$ be a non-negative real function, continuous on the variable s,t on the unit square $[0,1]\times[0,1]$ and $\rho(s,t)$ is also a definite and continuous real function on the unit square, without decreasing according to the variable t and satisfies the condition

$$|\rho(s,t)| \leq a(s) + b|t|^{p-1}, \forall t \in [0,1],$$

where $a(s)$ is a continuous, non-negative real function on $[0,1]$, and b is a positive constant.

The operator A is defined as:

$$Ax = \int_0^1 k(s,t)\rho(s,x(s))\mathrm{d}s. \tag{32}$$

According to [30], this operator acts from $L_p[0,1]$ to $L_q[0,1]$, with the domain

$$D(A) = \{x(s) \in L_p[0,1]: 0 \le x(s) \le 1, \text{ a.e. on } [0,1]\}$$

Operator A is monotone. Indeed, let $x(t), y(t) \in D(A)$, we have

$$\begin{aligned}
&\langle Ax - Ay, x - y\rangle \\
&= \int_0^1\left[\int_0^1 (k(s,t)\rho(s,x(s)) - k(s,t)\rho(s,y(s))\mathrm{d}s\right][x(t) - y(t)]\mathrm{d}t \\
&= \int_0^1\int_0^1 k(s,t)[\rho(s,x(s)) - \rho(s,y(s))][x(t) - y(t)]\mathrm{d}s\mathrm{d}t.
\end{aligned}$$

Since the function $(k(s,t)$ is not negative and $\rho(s,t)$ is not decrease respect to the variable t, the right hand side of the above inequality is non-negative. So,

$$\langle Ax - Ay, x - y\rangle \ge 0, \forall x, y \in D(A).$$

According to [30] the operator hemi-continuous.

The perturbed operator A_h of A is defined as follows

$$A_h x = \int_0^1 k_h(s,t)\rho_h(s,x(s))\mathrm{d}s, \tag{33}$$

where $k_h(s,t)$ is a measurable real function on the unit square as follows

$$|k_h(s,t) - k(s,t)| \le h, \forall (s,t) \in [0,1]\times[0,1], \tag{34}$$

and $\rho_h(s,t)$ is also a measurable real function on the unit square as follows

$$|\rho_h(s,t) - \rho(s,t)| \le h\left[c(s) + d|t|^{p-1}\right], \forall (s,t) \in [0,1]\times[0,1], \tag{35}$$

with $c(s)$ is a non negative real function in $L_q[0,1]$.

According to [30], from (34), (35) it follows that A_h acts from $L_p[0,1]$ to $L_q[0,1]$, with the domain $D(A)$.

Clearly, in general, A_h is not monotone and not continuous. Moreover, after calculating and estimating, we have

$$\|A_h x - Ax\| \le h[b + hd + k_0 d]\|x\|_{L_p[0,1]}^{p-1} + A_0 + (h + k_0)C_0, \tag{36}$$

where

$$A_0 = \left(\int_0^1 |a(s)|^q \mathrm{d}s\right)^{1/q}, C_0 = \left(\int_0^1 |c(s)|^q \mathrm{d}s\right)^{1/q}, k_0 = \max_{0\le s,t\le 1} k(s,t), 0 < h < 1.$$

Since $1 < p \le 2$, then $0 < p - 1 \le 1$.

By using the inequality

$$t^\alpha \le 1 + \alpha t, \forall t \in [0,1], 0 < \alpha < 1,$$

we have an estimate of the function g as follows:

$$g(t) \le Mt + N,$$

where $g(t)=(b+d+dk_0)t^{\alpha}+A_0+(1+k_0)C_0$, $M=b+d+dk_0$ and $N=b+d+dk_0+A_0+(1+k_0)C_0$.

We have g is continuous and

$$g(t)\le Mt+N, \forall t\ge 0.$$

Therefore, we can apply the above regularization method to find the approximate solution of the variational inequality (1) with the operator A is given by (32) and its perturbed operator A_h is given by (33).

3. Regularizer of General Variational Inequality

3.1. Regularization Method Solving the General Variational Inequality-Strong Stability

In this subsection, we consider X to be a real reflexive space and satisfies the following conditions:

1) There exists a mapping U from X to the conjugate space X^*, which maps the bounded subsets of X to the bounded subset of X^*;

2) For every fixed x_1 that belongs to X, then $\frac{\langle Ux, x-x_1\rangle}{\|x\|}\to+\infty$, as $\|x\|\to+\infty$ (this condition is often called U as satisfying the coercive condition);

3) For every bounded sequence $\{x_n\}\subset X$ and $x\in X$ such that:

$$\langle Ux_n-Ux, x_n-x\rangle\to 0, \text{ as } n\to\infty \Rightarrow x_n\to x, \text{ as } n\to\infty.$$

Remark 3.1 If X is a uniformly convex space and the conjugate space X^* is a strictly convex space, then the dual mapping $U: X\to X^*$ satisfies the above conditions [30].

We consider the following variational inequality

$$\langle Az-y_0, x-z\rangle\ge f(z)-f(x), \forall x\subset D, \tag{37}$$

where A is a monotone, hemi-continuous, acting from X into X^*, with domain $D(A)\subset X$, $f: X\to(-\infty,+\infty]$, $f\ne+\infty$, is lower semi-continuous and D is a convex closed subset of $D(A)$.

As we know the solution of (37) is $z\in D$ which satisfies (37). The class of problems of the form (37) is often called the general variational inequalities. This class of problems contains special cases: the first kind operator equations $Ax=y$, where A is a monotonous operator; variational inequality problems on convex sets with monotone operators on convex sets.

If $f\equiv 0$ then we have studied in the Section 2.

The existence of solutions of (37) has been studied by Browder F. E. and many other mathematicians in the world, some typical works can be mentioned as [10] [11] [34].

For the first time, the author studies the approximate solution of the variational inequality (37) by the Tikhonov regularization method [24] [25] [26] [27] [28], as follows:

Assume that for exact data (y_0, A) the exact solution set of (37) is $Z_0\ne\varnothing$.

In fact, instead of knowing this exact data, we only know its approximate data (y_δ, A_h) that satisfies the conditions

$$\|y_\delta - y_0\| \le \delta \tag{38}$$

and

$$\|A_h x - Ax\| \le h, \forall x \in D(A_h) = D(A) \tag{39}$$

where $A_h : D(A_h) \subset X \to X^*$, A_h is not necessarily a monotone operator or hemi-continuous.

The author has studied the approximate solution of (37) with noisy data (y_δ, A_h) according to Tikhonov regularization method with small parameter, which is the solutions of the following variational inequality:

$$\langle A_h z + \alpha U z - y_\delta, x - z\rangle \ge f(z) - f(x) - \varepsilon\|x - z\|, \forall x \in D. \tag{40}$$

The solutions of (40) are the elements $z \in D$ that satisfy (40). Let's also denote the solution set of (40) is Z_Δ^α, with $\Delta = (h, \varepsilon, \delta)$.

The following lemma is necessary for proving the existence of a solution of (5.2).

Lemma 3.1 [12] Suppose T is a operator with defined on the convex subset $D(T)$ of X into X^*, hemi-continuous on $D(T)$ and f is a below semi-continuous function from X into $(-\infty, +\infty]$ with f is not identical to infinity. If z is a solution of the variational inequality

$$\langle Tx - \omega, x - z\rangle \ge f(z) - f(x), \forall x \in D(T),$$

then it is also a solution of the variational inequality

$$\langle Tz - \omega, x - z\rangle \ge f(z) - f(x), \forall x \in D(T).$$

Theorem 3.1 If $\varepsilon \ge h$, then $Z_\Delta^\alpha \ne \varnothing$.

Proof. We consider the following variational inequality

$$\langle A_h z + \alpha U z - y_\delta, x - z\rangle \ge f(z) - f(x), \forall x \in D. \tag{41}$$

We will show that the variational inequality (41) has solutions and that these solutions are also solutions of the variational inequality (37).

From $f \ne +\infty$ and the property (2) of the operator U it follows that there exists some non-negative real $\gamma(t)$ with $\gamma(t) \to +\infty$, as $t \to +\infty$ such that

$$\langle Ux, x - x_1\rangle \ge \gamma(\|x\|)\|x\|, \forall x \in X. \tag{42}$$

Since f is a below semi-continuous, convex function, there exist a constant $k > 0$ such that

$$f(x) \ge -k\|x\| + f(0), \forall x \in X, \|x\| > 1. \tag{43}$$

Indeed, suppose that (43) is not true, therefore there exists some sequence $\{x_n\} \subset X$, with $\|x_n\| > 1, \forall n$ and $\|x_n\| \to +\infty$, as $n \to \infty$, such that

$$f(x_n) \le -n\|x_n\| + f(0), \forall n.$$

Consequently

$$\frac{1}{\|x_n\|} f(x_n) - \frac{1}{\|x_n\|} f(0) \le -n, \forall n$$

or

$$\frac{1}{\|x_n\|} f(x_n) + \left(1 - \frac{1}{\|x_n\|}\right) f(0) \le -n + f(0), \forall n. \quad (44)$$

From f is a convex function and (44), it follows that

$$f\left(\frac{x_n}{\|x_n\|}\right) \le -n + f(0). \quad (45)$$

The inequality (45) is contract with the lower semi-continuous property of f which is mapping any bounded subset of X to bounded set.

Using the monotony of the operator A, we have

$$\begin{aligned} \langle Ax, x - x_1 \rangle &= \langle Ax - Ax_1, x - x_1 \rangle + \langle Ax_1, x - x_1 \rangle \ge \langle Ax_1, x - x_1 \rangle \\ &\ge -\|Ax_1\| \|x - x_1\|, \forall x \in X \end{aligned} \quad (46)$$

From (41), (43) and (46), it follows that

$$\langle Ax + \alpha Ux, x - x_1 \rangle + f(x) \ge \left[\gamma\left(\|x\| - \|Ax_1\| - k\right)\right]\|x\| - \|Ax_1\|\|x_1\| + f(0). \quad (47)$$

From (47), it follows that

$$\frac{\langle Ax + \alpha Ux, x - x_1 \rangle}{\|x\|} \to +\infty, \text{ as } \|x\| \to \infty. \quad (48)$$

+ If D is not bounded, from the monotone, hemi-continuous of the operator $A + \alpha U$, (48), according to [31] it follows that the variational inequality (41) has the solutions belong to D.

+ If D is bounded, according to [32], it follows that the variational inequality (41) has the solutions belong to D.

Let z be arbitrary solution of (41), we have

$$\begin{aligned} \langle A_h z + \alpha Uz - y_\delta, x - z \rangle &= \langle A_h z - Az, x - z \rangle + \langle Az + \alpha Uz - y_\delta, x - z \rangle \\ &\ge -\|A_h z - Az\|\|x - z\| + f(z) - f(x) \\ &\ge -\varepsilon\|x - z\| + f(z) - f(x), \forall x \in D. \end{aligned} \quad (49)$$

So, z is a solution of the variational inequality (40), *i.e.*, $Z_\Delta^\alpha \ne \varnothing$.

In case D is an unbounded set, then the following theorem states that the sequence of sets Z_Δ^α is uniformly bounded.

Theorem 3.2 If $\varepsilon \ge h$ and $\frac{\varepsilon + h + \delta}{\alpha} < k$, k is a constant, then the sequence of the sets Z_Δ^α is uniformly bounded.

Proof. Since $\varepsilon \ge h$, then $Z_\Delta^\alpha \ne \varnothing$ (Theorem 3.1). Let $z_\Delta^\alpha \in Z_\Delta^\alpha$, then we have

$$\left\langle A_h z_\Delta^\alpha + \alpha U z_\Delta^\alpha - y_\delta, x - z_\Delta^\alpha \right\rangle \ge f\left(z_\Delta^\alpha\right) - f(x) - \varepsilon\left\|x - z_\Delta^\alpha\right\|, \forall x \in D.$$

Consequently,

$$\begin{aligned} & h\left\|x - z_\Delta^\alpha\right\| + \delta\left\|x - z_\Delta^\alpha\right\| + \left\langle A z_\Delta^\alpha - y_0, x - z_\Delta^\alpha \right\rangle + \alpha\left\langle U z_\Delta^\alpha, x - z_\Delta^\alpha \right\rangle \\ & \ge f\left(z_\Delta^\alpha\right) - f(x) - \varepsilon\left\|x - z_\Delta^\alpha\right\|, \forall x \in D. \end{aligned} \quad (50)$$

In inequality (50) we take $x \in Z_0$ then

$$\left\langle Az_\Delta^\alpha - y_0, x - z_\Delta^\alpha \right\rangle \le f\left(z_\Delta^\alpha\right) - f\left(x\right), \forall x \in Z_0.$$

Therefore, in inequality (50) we replace x by x_1, we get

$$\frac{\varepsilon + h + \delta}{\alpha}\left\|x_1 - z_\Delta^\alpha\right\| \ge \left\langle Uz_\Delta^\alpha, z_\Delta^\alpha - x_1 \right\rangle, \forall z_\Delta^\alpha \in Z_\Delta^\alpha.$$

Since $\frac{\varepsilon + h + \delta}{\alpha} < k$, from the above inequality implies

$$k\left\|z_\Delta^\alpha - x_1\right\| \ge \left\langle Uz_\Delta^\alpha, z_\Delta^\alpha - x_1 \right\rangle. \tag{51}$$

Since k is not dependent on h, ε, δ and since the mapping U turns bounded set into bounded set, the sequence $\left\{z_\Delta^\alpha\right\}$ is uniformly bounded. From this, deduce the sequence of sets Z_Δ^α is also uniformly bounded.

Theorems 3.3 and 3.4 below talk about the stability of the solution of inequality (41) by the Tikhonov regularization method.

Theorem 3.3 With the assumptions given in Theorem 3.2, then the sequence of sets Z_Δ^α B-weakly converges to the set Z_0, as $\alpha \to 0$.

Proof. Let $z_\Delta^\alpha \in Z_\Delta^\alpha$, then $\left\{z_\Delta^\alpha\right\}$ is uniformly bounded (Theorem 3.2). Since X is a reflexive Banach space, there exists a subsequence $\left\{z_\Delta^\beta\right\}$ of this sequence such that $z_\Delta^\beta \rightharpoonup z \in D$, as $\beta \to 0$. We prove that $z \in Z_o$.

The argument is the same as in the proof of Theorem 3.2, we get the inequality (50) and in this inequality we replace z_Δ^α by z_Δ^β, we get

$$\begin{aligned} &\left(h + \varepsilon + \delta\right)\left\|x - z_\Delta^\beta\right\| + \left\langle Ax - y_0, x - z_\Delta^\beta \right\rangle \\ &\ge f\left(z_\Delta^\beta\right) - f\left(x\right) + \beta\left\langle Ux, z_\Delta^\beta - x \right\rangle, \forall x \in D. \end{aligned} \tag{52}$$

Due to the lower semi-continuity of f and $\frac{h + \varepsilon + \delta}{\alpha} < k$, in (52) given $\beta \to 0$, we get

$$\left\langle Ax - y_0, x - z \right\rangle \ge f\left(z\right) - f\left(x\right), \forall x \in D.$$

Since A is hemi-continuous, according to Lemma 3.1, from the above variational inequality, it implies that

$$\left\langle Az - y_0, x - z \right\rangle \ge f\left(z\right) - f\left(x\right), \forall x \in D.$$

So, $z \in Z_0$, that means $\left\{Z_\Delta^\alpha\right\}$ B-weakly converges to Z_0, as $\alpha \to 0$.

We enter the following notation

$$Z_* = \left\{z_* \in Z_0 : \left\langle Uz_*, x - z_* \right\rangle \ge 0, \forall x \in Z_0\right\}.$$

Due to the coercive property of the mapping U on the closed convex set Z_0, the set $Z_0 \ne \varnothing$. The following theorem gives a stronger result than Theorem 3.3.

Theorem 3.4 If $\varepsilon \ge h$, then $\left\{Z_\Delta^\alpha\right\}$ B-converges to Z_*, as $\alpha \to 0$, $\frac{\varepsilon + h + \delta}{\alpha} \to 0$.

Proof. Since $\alpha \to 0$, $\frac{\varepsilon+h+\delta}{\alpha} \to 0$, then $\frac{\varepsilon+h+\delta}{\alpha}$ is small enough, then the sequence of sets Z_Δ^α is unformly bounded (Theorem 3.2). Let $z_\Delta^\alpha \in Z_\Delta^\alpha$, then $\left\{z_\Delta^\alpha\right\}$ is bounded. Since X is a reflexive Banach space, there exists a subsequence $\left\{z_\Delta^\beta\right\}$ of the sequence such that $z_\Delta^\beta \rightharpoonup z_* \in D$, as $\beta \to 0$, $\frac{\varepsilon+h+\delta}{\beta} \to 0$.

The same argument as in the proof of Theorem 3.3, we obtain $z_* \in Z_0$.

From (52) it follows that

$$\frac{\varepsilon+h+\delta}{\alpha}\left\|x-z_\Delta^\beta\right\| \ge \left\langle Uz_\Delta^\beta, z_\Delta^\beta - x\right\rangle \ge \left\langle Ux, z_\Delta^\beta - x\right\rangle, \forall x \in Z_0. \tag{53}$$

In (53) given $\beta \to 0$, $\frac{\varepsilon+h+\delta}{\beta} \to 0$, we get

$$\left\langle Ux, z_* - x\right\rangle \le 0, \forall x \in Z_0, \text{ or } \left\langle Ux, x - z_*\right\rangle \ge 0, \forall x \in Z_0.$$

According to Minty's Lemma [30], from the above variational inequality, we get

$$\left\langle Uz_*, x - z_*\right\rangle \ge 0, \forall x \in Z_0.$$

So, $z_* \in Z_*$.

In inequality (53) we replace x by z_*, we get

$$\frac{\varepsilon+h+\delta}{\beta}\left\|z_* - z_\Delta^\beta\right\| + \left\langle Uz_*, z_* - z_\Delta^\beta\right\rangle \ge \left\langle Uz_\Delta^\beta - Uz_*, z_\Delta^\beta - z_*\right\rangle. \tag{54}$$

We have

$$\left\langle Uz_\Delta^\beta - Uz_*, z_\Delta^\beta - z_*\right\rangle \to 0, \text{ as } \beta \to 0, \frac{\varepsilon+h+\delta}{\beta} \to 0 \tag{55}$$

Due to property (3) of the mapping U, from (55) it follows that

$$z_\Delta^\beta \to z_*, \text{ as } \beta \to 0, \frac{\varepsilon+h+\delta}{\beta} \to 0.$$

This means the sequence $\left\{Z_\Delta^\alpha\right\}$ B- converges strongly to Z_*, as $\alpha \to 0$, $\frac{\varepsilon+h+\delta}{\alpha} \to 0$.

Theorem is proved.

Corollary 3.1 With the same assumptions as in Theorem 3.4, the sequence of sets Z_Δ^α B-converges strongly to Z_0, as $\alpha \to 0$, $\frac{\varepsilon+h+\delta}{\alpha} \to 0$.

Proof. It is easy to deduce from Theorem 3.4 and Proposition 2.1

3.2. Regularization Method Solving the General Variational Inequality with Small Parameter-Weakly Stability

In this subsection, we consider X as a real reflexive Banach space with its conjugate space X^* being strictly convex. Therefore, dual mapping $U: X \to X^*$ is single, monotone and hemi-continuous [34].

For the first time, the author studies the approximate solution of the variational inequality (37) by Tikhonov regularization method with small parameter [24] [25] [26] [27] [28] as follows:

Assume that for exact data (y_0, A) the exact solution set of (37) is $Z_0 \neq \varnothing$. In fact, instead of knowing this exact data, we only know its approximate data (y_δ, A_h) that satisfies the conditions (38) and (39) and $A_h : D(A_h) \subset X \to X^*$, A_h is not necessarily a monotone operator or hemi-continuous.

The author has studied the approximate solution of (37) with noisy data (y_δ, A_h) according to Tikhonov regularization method Tikhonov with small parameter, which is the solutions of the following variational inequality:

$$\langle A_h x + \alpha Ux - y_\delta, x - z\rangle \geq f(z) - f(x) - \varepsilon g(\|x\|)\|x - z\|, \forall x \in D. \tag{56}$$

The solutions of (56) are the elements $z \in D$ that satisfy (56). Let's also denote the solution set of (56) is Z_Δ^α, with $\Delta = (h, \varepsilon, \delta)$.

Theorem 3.5. If $\varepsilon \geq h$ then the variational inequality (56) has a solution.

Proof. We consider the following variational inequality

$$\langle Az + \alpha Uz - y_\delta, x - z\rangle \geq f(z) - f(x), \forall x \in D. \tag{57}$$

The same argument as in the proof of Theorem 3.1, the variational inequality (57) has a solution in *D*. Let *z* be a solution of (57). Due to the monotony of $A + \alpha U$, from (57) it follows that

$$\langle Ax + \alpha Ux - y_\delta, x - z\rangle \geq f(z) - f(x), \forall x \in D. \tag{58}$$

We will prove that every solution of (58) is a solution of (57). Indeed, let *z* be a solution of (58), we have

$$\begin{aligned}\langle A_h x + \alpha Ux - y_\delta, x - z\rangle &= \langle A_h x - Ax, x - z\rangle + \langle Ax + \alpha Ux - y_\delta, x - z\rangle \\ &\geq -hg(\|x\|)\|x - z\| + f(z) - f(x) \\ &\geq -\varepsilon g(\|x\|)\|x - z\| + f(z) - f(x), \forall x \in D.\end{aligned}$$

So, $z \in Z_\Delta^\alpha$.

Theorem is proved.

In the case that the set D is not bounded, then the following theorem states that the uniformly boundedness of the sequence of sets Z_Δ^α.

Theorem 3.6 If $\varepsilon \geq h$, $\dfrac{M(h+\varepsilon)}{\alpha} < k_1 < 1$, $\dfrac{\delta}{\alpha} < k_2$ (k_1, k_2: constant), then the sequence of the sets Z_Δ^α of (55) is uniformly bounded in *X*.

Proof. Since $\varepsilon \geq h$, then $Z_\Delta^\alpha \neq \varnothing$ (Theorem 3.5). Let $z \in Z_\Delta^\alpha$, we have

$$\langle A_h x + \alpha Ux - y_\delta, x - z\rangle \geq f(z) - f(x) - \varepsilon g(\|x\|)\|x - z\|, \forall x \in D.$$

From the variational inequality above, it follows that

$$\begin{aligned}&\left[(\varepsilon + h) g(\|x\|) + \delta\right]\|x - z\| + \langle Ax - y_0, x - z\rangle + \alpha\langle Ux, z - x\rangle \\ &\geq f(z) - f(x), \forall x \in D.\end{aligned} \tag{59}$$

Due to Lemma 2.2, Theorem 3.1, from (59) it follows that

$$\begin{aligned}&\left[(\varepsilon + h) g(\|z\|) + \delta\right]\|x - z\| + \langle Az - y_0, x - z\rangle + \alpha\langle Uz, z - x\rangle \\ &\geq f(z) - f(x), \forall x \in D.\end{aligned} \tag{60}$$

Due to the monotony of A, from (60) it follows that

$$\begin{aligned}&\left[(\varepsilon+h)g(\|z\|)+\delta\right]\|x-z\|+\langle Ax-y_0,x-z\rangle+\alpha\langle Uz,z-x\rangle\\&\geq f(z)-f(x),\forall x\in D.\end{aligned}\tag{61}$$

From (58), (59) it follows that

$$\left[\frac{\varepsilon+h}{\alpha}g(\|z\|)+\frac{\delta}{\alpha}\right]\|x-z\|\geq\langle Uz,z-x\rangle,\forall z\in Z_0.\tag{62}$$

Due to $g(t)\leq Mt+N,\forall t\geq 0$, from (60) it follows that

$$\begin{aligned}&\left[\frac{M(\varepsilon+h)}{\alpha}-1\right]\|z\|^2+\left[\frac{\varepsilon+h}{\alpha}M\|x\|+N)+\frac{\delta}{\alpha}+\|x\|\right]\|z\|\\&+\left[\frac{M(\varepsilon+h)}{\alpha}+\frac{\delta}{\alpha}\right]\|x\|\geq 0,\forall x\in Z_0\end{aligned}\tag{63}$$

Since $\frac{M(\varepsilon+h)}{\alpha}<k_1<1$, $\frac{\delta}{\alpha}<k_2<+\infty$, from (63) it follows that the sequence $\{Z_\Delta^\alpha\}$ is uniformly bounded.

Theorem is proved.

The following theorem states B-weakly stability of the solution of the variational inequality (37) according to the Tikhonov regularization method with small parameters.

Theorem 3.7 With the same assumptions as in Theorem 3.6, then the sequence $\{Z_\Delta^\alpha\}$ B-weakly converges to Z_0, as $\alpha\to 0$.

Proof. Since $\varepsilon\geq h$, $\frac{M(h+\varepsilon)}{\alpha}<k_1<1$, $\frac{\delta}{\alpha}<k_2<\infty$, then the sequence $\{Z_\Delta^\alpha\}$ is uniformly bounded. Let $z_\Delta^\alpha\in Z_\Delta^\alpha$, then the sequence $\{z_\Delta^\alpha\}$ is bounded in the reflexive Banach space X. Therefore, there exists a subsequence $\{z_\Delta^\beta\}$ of this sequence such that $z_\Delta^\beta\rightharpoonup z\in D$ (since D is the weakly closed set). Since z_Δ^β is a solution of (56), we have

$$\langle A_hx+\alpha Ux-y_\delta,x-z_\Delta^\beta\rangle\geq f(z_\Delta^\beta)-f(x)-\varepsilon g(\|x\|)\|x-z_\Delta^\beta\|,\forall x\in D.$$

Consequently,

$$\begin{aligned}&\left[(\varepsilon+h)g(\|x\|)+\delta\right]\|x-z_\Delta^\beta\|+\langle Ax-y_0,x-z_\Delta^\beta\rangle+\beta\langle Ux,x-z_\Delta^\beta\rangle\\&\geq f(z_\Delta^\beta)-f(x),\forall x\in D.\end{aligned}$$

Due to Lemma 2.2 and Lemma 3.1, the monotony of A and U, from the above inequality, it follows that

$$\begin{aligned}&\left[(\varepsilon+h)g(\|z_\Delta^\beta\|)+\delta\right]\|x-z_\Delta^\beta\|+\langle Ax-y_0,x-z_\Delta^\beta\rangle+\beta\langle Ux,x-z_\Delta^\beta\rangle\\&\geq f(z_\Delta^\beta)-f(x),\forall x\in D.\end{aligned}\tag{64}$$

In (64), taking $\beta\to 0$, we obtain

$$\langle Ax-y_0,x-z\rangle\geq f(z)-f(x),\forall x\in D.$$

Due to Lemma 3.1, from the above inequality, it follows that

$$\left\langle Az - y_0, x - z \right\rangle \geq f(z) - f(x), \forall x \in D.$$

So, $z \in Z_0$. It means that the sequence $\{Z_\Delta^\alpha\}$ B-weakly converges to Z_0, as $\alpha \to 0$.

Theorem is proved.

We denote $Z_* = \arg\max_{z \in Z_0} \{\|z\|\}$, then $Z_* \neq \varnothing$ (since Z_0 is the convex closed subset in reflexive Banach space *X*.

The following theorem gives us a stronger result than the above theorem.

Theorem 3.8 If $\varepsilon \geq h$, then the sequence $\{Z_\Delta^\alpha\}$ B-strongly converges to Z_*, as $\alpha \to 0$, $\frac{\varepsilon + h + \delta}{\alpha} \to 0$. Moreover, if let $z_\Delta^\alpha \in Z_\Delta^\alpha$, $z_* \in Z_*$, then $\|z_\Delta^\alpha\| \to \|z_*\|$, as $\alpha \to 0$, $\frac{\varepsilon + h + \delta}{\alpha} \to 0$.

Proof. Since $\alpha \to 0$, $\frac{\varepsilon + h + \delta}{\alpha} \to 0$, then we consider $\alpha, \varepsilon, h, \delta$ such that

$$\frac{M(h + \varepsilon)}{\alpha} < k_1 < 1, \frac{\delta}{\alpha} < k_2 < \infty. \tag{65}$$

With Δ, α satisfying (65), then the sequence $\{Z_\Delta^\alpha\}$ is uniformly bounded (Theorem 3.6). Let $z_\Delta^\alpha \in Z_\Delta^\alpha$, then sequence $\{z_\Delta^\alpha\}$ is bounded in the reflexive Banach space *X*. Therefore, there exists a subsequence $\{z_\Delta^\beta\}$ of this sequence such that $z_\Delta^\beta \rightharpoonup z_* \in Z_0$ as $\beta \to 0$, $\frac{\varepsilon + h + \delta}{\beta} \to 0$. The same argument as in the proof of Theorem 3.7, we get the variational inequality same (64), that is

$$\begin{aligned} &\left[(\varepsilon + h) g\left(\|z_\Delta^\beta\|\right) + \delta\right] \|x - z_\Delta^\beta\| + \left\langle Ax - y_0, x - z_\Delta^\beta \right\rangle + \beta \left\langle Ux, x - z_\Delta^\beta \right\rangle \\ &\geq f\left(z_\Delta^\beta\right) - f(x), \forall x \in D. \end{aligned}$$

In this variational inequality, we take $x \in Z_0 \subset D$, then

$$\left\langle Ax - y_0, x - z_\Delta^\beta \right\rangle \leq f\left(z_\Delta^\beta\right) - f(x).$$

Consequently,

$$\left[(\varepsilon + h) g\left(\|z_\Delta^\beta\|\right) + \delta\right] \|x - z_\Delta^\beta\| + \beta \left\langle Ux, x - z_\Delta^\beta \right\rangle \geq 0, \forall x \in Z_0. \tag{66}$$

or

$$\left[\frac{\varepsilon + h}{\beta} g\left(\|z_\Delta^\beta\|\right) + \frac{\delta}{\beta}\right] \|x - z_\Delta^\beta\| + \left\langle Ux, x - z_\Delta^\beta \right\rangle \geq 0, \forall x \in Z_0.$$

Since the sequence $\{z_\Delta^\beta\}$ is bouned and the function $g(t)$ is continuous, then the sequence $\left\{g\left(\|z_\Delta^\beta\|\right)\right\}$ is bounded. Therefore, in (66) taking $\beta \to 0$, $\frac{\varepsilon + h + \delta}{\beta} \to 0$, we obtain

$$\left\langle Ux, x - z_* \right\rangle \geq 0, \forall x \in Z_0.$$

Due to Lemma of Minty, from the above variational inequality, it follows that

$$\left\langle Uz_*, x - z_* \right\rangle \geq 0, \forall x \in Z_0.$$

Due to the property of dual mapping U, from the above variational inequality, it follows that

$$\|x\| \ge \|z_*\|, \forall x \in Z_0.$$

So, $z_* \in Z_*$, that is, the sequence $\{Z_\Delta^\alpha\}$ B-weakly converges to Z_*, as $\alpha \to 0$, $\frac{\varepsilon+h+\delta}{\alpha} \to 0$.

On the other hand, we have

$$\begin{aligned}\left(\|z_*\|-\|z_\Delta^\beta\|\right)^2 &\le \left\langle Uz_* - Uz_\Delta^\beta, z_* - z_\Delta^\beta\right\rangle = \left\langle Uz_*, z_* - z_\Delta^\beta\right\rangle + \left\langle Uz_\Delta^\beta, z_\Delta^\beta - z_*\right\rangle \\ &\le \left\langle Uz_*, z_* - z_\Delta^\beta\right\rangle + \left[\frac{\varepsilon+h}{\beta} g\left(\|z_\Delta^\beta\|\right) + \frac{\delta}{\beta}\right]\|z_\Delta^\beta - z_*\|\end{aligned} \tag{67}$$

From (67) it follows that

$$\left\langle Uz_\Delta^\beta, z_\Delta^\beta - z_*\right\rangle \le \left[\frac{\varepsilon+h}{\beta} g\left(\|z_\Delta^\beta\|\right) + \frac{\delta}{\beta}\right]\|z_\Delta^\beta - z_*\|$$

In (67) taking $\beta \to 0$, $\frac{\varepsilon+h+\delta}{\beta} \to 0$, we obtain

$$\|z_\Delta^\beta\| \to \|z_*\|, \text{ as } \beta \to 0, \frac{\varepsilon+h+\delta}{\beta} \to 0.$$

This proves that the sequence $\left\{\|z_\Delta^\alpha\|\right\}$ has only one limit point, so the sequence itself converges to $\|z_*\|$, as $\alpha \to 0$, $\frac{\varepsilon+h+\delta}{\alpha} \to 0$.

Corollary 3.2 Necessary and sufficient condition for the sequence $\{Z_\Delta^\alpha\}$ B-weakly converges to some bounded subset of X as $\alpha \to 0$, $\frac{\varepsilon+h+\delta}{\alpha} \to 0$ the variational inequality (40) has a solution.

Proof. Similar to the proof of theorem 2.5.

Remark 3.2: If A_h is monotone operator and $f \equiv 0$, the above results are the same with results in [18].

We now study the approximate solution of the variational inequality (37) according to Tikhonov regularization method with small parameters, which is the solutions of the following variational inequality:

$$\left\langle A_h z + \alpha Uz - y_\delta, x - z\right\rangle \ge f(z) - f(x) - \varepsilon g\left(\|z\|\right)\|x-z\|, \forall x \in D, \tag{68}$$

where A_h, U, f, g, y_δ the same assumption of the first of this subset.

The solution of the variational inequality (68) is the elements $z \in D$ that satisfy (68). We still denote the solution set of (68) as the set Z_Δ^α.

Theorem 3.9 If $\varepsilon \ge h$ then the variational inequality (68) has a solution.

Proof. The same argument as in the proof of Theorem 3.5, we have the following inequality which has a solution:

$$\left\langle Az + \alpha Uz - y_\delta, x - z\right\rangle \ge f(z) - f(x), \forall x \in D. \tag{69}$$

It is easy to show that every solution of (69) is a solution of (68). So, $Z_\Delta^\alpha \ne \varnothing$.

Theorem 3.10 If $\varepsilon \ge h$, $\frac{M(h+\varepsilon)}{\alpha} < k_1 < 1$, $\frac{\delta}{\alpha} < k_2$ (k_1, k_2: constant), then the sequence of the sets Z_Δ^α of (56) is uniformly bounded in *X*.

Proof. Since $\varepsilon \ge h$, then $Z_\Delta^\alpha \ne \varnothing$ (Theorem 3.9). Let $z_\Delta^\alpha \in Z_\Delta^\alpha$, we have

$$\left\langle A_h z_\Delta^\alpha + \alpha U z_\Delta^\alpha - y_\delta, x - z_\Delta^\alpha \right\rangle \ge f\left(z_\Delta^\alpha\right) - f(x) - \varepsilon g\left(\left\|z_\Delta^\alpha\right\|\right)\left\|x - z_\Delta^\alpha\right\|, \forall x \in D.$$

Consequently,

$$\begin{aligned}&\left[(\varepsilon+h) g\left(\left\|z_\Delta^\alpha\right\|\right) + \delta\right]\left\|x - z_\Delta^\alpha\right\| + \left\langle A z_\Delta^\alpha - y_0, x - z_\Delta^\alpha \right\rangle \\ &\ge f\left(z_\Delta^\alpha\right) - f(x) - \alpha\left\langle U z_\Delta^\alpha, x - z_\Delta^\alpha \right\rangle, \forall x \in D.\end{aligned}$$

Since A is the monotone operator, from the above inequality it follows that

$$\begin{aligned}&\left[(\varepsilon+h) g\left(\left\|z_\Delta^\alpha\right\|\right) + \delta\right]\left\|x - z_\Delta^\alpha\right\| + \left\langle A x - y_0, x - z_\Delta^\alpha \right\rangle \\ &\ge f\left(z_\Delta^\alpha\right) - f(x) - \alpha\left\langle U z_\Delta^\alpha, x - z_\Delta^\alpha \right\rangle, \forall x \in D\end{aligned} \tag{70}$$

We have

$$\left\langle A x - y_0, x - z_\Delta^\alpha \right\rangle \le f\left(z_\Delta^\alpha\right) - f(x), \forall x \in Z_0.$$

Consequently,

$$\left[(\varepsilon+h) g\left(\left\|z_\Delta^\alpha\right\|\right) + \delta\right]\left\|x - z_\Delta^\alpha\right\| + \alpha\left\langle U z_\Delta^\alpha, x - z_\Delta^\alpha \right\rangle \ge 0, \forall x \in Z_0. \tag{71}$$

Since $g(t) \le Mt + N, \forall t \ge 0$, from (71) it follows that

$$\begin{aligned}&\left[\frac{M(\varepsilon+h)}{\alpha} - 1\right]\left\|z_\Delta^\alpha\right\| + \left[\frac{\varepsilon+h}{\alpha} M \|x\| + N) + \frac{\delta}{\alpha} + \|x\|\right]\left\|z_\Delta^\alpha\right\| \\ &+\left[\frac{M(\varepsilon+h)}{\alpha} + \frac{\delta}{\alpha}\right]\|x\| \ge 0, \forall x \in Z.\end{aligned} \tag{72}$$

Since $\frac{M(h+\varepsilon)}{\alpha} < k_1 < 1, \frac{\delta}{\alpha} < k_2 < \infty$, from (72) the sequence $\left\{Z_\Delta^\alpha\right\}$ is uniformly bounded.

Theorem is proved.

The following theorem talks about B-weakly stability of the solution of the variational inequality (37) by the Tikhonov regularization method with small parameter (68).

Theorem 3.11 With the assumption as in Theorem 3.10. Then, the sequence $\left\{Z_\Delta^\alpha\right\}$ B-weakly converges to Z_0, as $\alpha \to 0$.

Proof. From the assumption of Theorem 3.10, then $Z_\Delta^\alpha \ne \varnothing$. Let $z_\Delta^\alpha \in Z_\Delta^\alpha$, then the sequence $\left\{z_\Delta^\alpha\right\}$ is bounded in the reflexive Banach *X*, therefore, there exists a subsequence $\left\{z_\Delta^\beta\right\}$ of this sequence such that $z_\Delta^\beta \rightharpoonup z \in D$, as $\beta \to 0$. Since z_Δ^β is a solution of (65), we have

$$\left\langle A_h z_\Delta^\beta + \beta U z_\Delta^\beta - y_\delta, x - z_\Delta^\beta \right\rangle \ge f\left(z_\Delta^\beta\right) - f(x) - \varepsilon g\left(\left\|z_\Delta^\beta\right\|\right)\left\|x - z_\Delta^\beta\right\|, \forall x \in D.$$

Consequently,

$$\begin{aligned}&\left[(\varepsilon+h) g\left(\left\|z_\Delta^\beta\right\|\right) + \delta\right]\left\|x - z_\Delta^\beta\right\| + \left\langle A z_\Delta^\beta - y_0, x - z_\Delta^\beta \right\rangle + \beta\left\langle U z_\Delta^\beta \right\rangle \\ &\ge f\left(z_\Delta^\beta\right) - f(x), \forall x \in D.\end{aligned} \tag{73}$$

Since the operators A,U are monotone, from (73) it follows that

$$\begin{aligned}&\left[(\varepsilon+h)g\left(\left\|z_\Delta^\beta\right\|\right)+\delta\right]\left\|x-z_\Delta^\beta\right\|+\left\langle Ax-y_0,x-z_\Delta^\beta\right\rangle+\beta\left\langle Ux,x-z_\Delta^\beta\right\rangle\\&\geq f\left(z_\Delta^\beta\right)-f(x),\forall x\in D.\end{aligned}\tag{74}$$

Since the function $g(t)$ is continuous and lower semi continuous and the sequence $\left\{\left\|z_\Delta^\beta\right\|\right\}$ is bounded, then in (74) taking $\beta\to 0$, we obtain

$$\langle Ax-y_0,x-z\rangle\geq f(z)-f(x),\forall x\in D.$$

So, $z\in Z_0$, that is, the sequence $\left\{Z_\Delta^\alpha\right\}$ B-weakly converges to Z_0, as $\alpha\to 0$.

Theorem is proved.

Theorem 3.12 If $\varepsilon\geq h$, $\alpha\to 0$, $\dfrac{\varepsilon+h+\delta}{\alpha}\to 0$, then the sequence $\left\{Z_\Delta^\alpha\right\}$ B-weakly converges to Z_*. Moreover, let $z_\Delta^\alpha\in Z_\Delta^\alpha$ and $z_*\in Z_*$, then $\left\|z_\Delta^\alpha\right\|\to\left\|z_*\right\|$, as $\alpha\to 0$, $\dfrac{\varepsilon+h+\delta}{\alpha}\to 0$.

Proof. Since $\alpha\to 0$, $\dfrac{\varepsilon+h+\delta}{\alpha}\to 0$, we consider $\varepsilon,h,\delta,\alpha$ being small such that

$$\frac{M(\varepsilon+h)}{\alpha}<k_1<1,\frac{\delta}{\alpha}<k_2<+\infty.\tag{75}$$

With $\varepsilon,h,\delta,\alpha$ satisfying (75), then $Z_\Delta^\alpha\neq\varnothing$ and is uniformly bounded. Let $z_\Delta^\alpha\in Z_\Delta^\alpha$, then the sequence $\left\{z_\Delta^\alpha\right\}$ is bounded in X is a reflexive Banach space, there exists a subsequence $\left\{z_\Delta^\beta\right\}$ of this sequence such that $z_\Delta^\beta\rightharpoonup z_*\in Z_0$, as $\beta\to 0$, $\dfrac{\varepsilon+h+\delta}{\beta}\to 0$.

The same argument as in the proof of Theorem 3.10, we obtain

$$\left[\frac{\varepsilon+h}{\beta}g\left(\left\|z_\Delta^\beta\right\|\right)+\frac{\delta}{\beta}\right]\left\|x-z_\Delta^\beta\right\|+\left\langle Ux,x-z_\Delta^\beta\right\rangle\geq 0,\forall x\in Z_0.\tag{76}$$

In (76), taking $\beta\to 0$, $\dfrac{\varepsilon+h+\delta}{\beta}\to 0$, due to the continulty of the function $g(t)$ and the boundedness of the sequence $\left\{\left\|z_\Delta^\beta\right\|\right\}$, we obtain

$$\langle Ux,x-z_*\rangle\geq 0,\forall x\in Z_0.\tag{77}$$

Due to Lemma of Minty, from (74) it follows that

$$\langle Uz_*,x-z_*\rangle\geq 0,\forall x\in Z_0.$$

Consequently,

$$\|z_*\|\leq\|x\|,\forall x\in Z_0.$$

So, $z_*\in Z_*$.

On the other hand,

$$\begin{aligned}\left(\|z_*\|-\left\|z_\Delta^\beta\right\|\right)^2&\leq\left\langle Uz_*-Uz_\Delta^\beta,z_*-z_\Delta^\beta\right\rangle=\left\langle Uz_*,z_*-z_\Delta^\beta\right\rangle+\left\langle Uz_\Delta^\beta,z_\Delta^\beta-z_*\right\rangle\\&\leq\left\langle Uz_*,z_*-z_\Delta^\beta\right\rangle+\left[\frac{\varepsilon+h}{\beta}g\left(\left\|z_\Delta^\beta\right\|\right)+\frac{\delta}{\beta}\right]\left\|z_\Delta^\beta-z_*\right\|,\forall x\in Z_0.\end{aligned}\tag{78}$$

In (78) taking $\beta \to 0$, $\frac{\varepsilon+h+\delta}{\beta} \to 0$, we obtain

$$\left\|z_\Delta^\beta\right\| \to \left\|z_*\right\|, \text{ as } \beta \to 0, \frac{\varepsilon+h+\delta}{\beta} \to 0.$$

The above argument shows that the sequence $\left\{\left\|z_\Delta^\alpha\right\|\right\}$ has only one limit point $\left\|z_*\right\|$, so it is this sequence that converges to $\left\|z_*\right\|$, as $\alpha \to 0$, $\frac{\varepsilon+h+\delta}{\alpha} \to 0$.

Theorem is proved.

Corollary 3.2 Necessary and sufficient condition for the sequence $\left\{Z_\Delta^\alpha\right\}$ B-weakly converges to some bounded subset of X as $\alpha \to 0$, $\frac{\varepsilon+h+\delta}{\alpha} \to 0$ the variational inequality (37) has a solution.

Proof. Similar to the proof of Theorem 2.5.

Remark 3.3: We can use Tikhonov regularization methods in Subsections 2.2, 2.3 to find the approximate solution in the given example above.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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