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On Signed Domination of Grid Graph

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Abstract

Let $G(V, E)$ be a finite connected simple graph with vertex set $V(G)$. A function $f: V(G) \to \{-1,1\}$ is a signed dominating function if for every vertex $v \in V(G)$, the sum of closed neighborhood weights of v is greater or equal to 1. The signed domination number $\gamma_s(G)$ of G is the minimum weight of a signed dominating function on G. In this paper, we calculate the signed domination numbers of the Cartesian product of two paths P_m and P_n for $m = 6$, 7 and arbitrary n.

Keywords

Grid Graph, Cartesian Product, Signed Dominating Function, Signed Domination Number

1. Introduction

Let G be a finite simple connected graph with vertex set $V(G)$. The neighborhood of v, denoted $N(v)$, is set $\{u: uv \in E(G)\}$ and the closed neighborhood of v, denoted $N[v]$, is set $N(v) \cup \{v\}$. The function f is a signed dominating function if for every vertex $v \in V$, the closed neighborhood of v contains more vertices with function value 1 than with -1. The weight of f is the sum of the values of f at every vertex of G. The signed domination number of G, $\gamma_s(G)$, is the minimum weight of a signed dominating function on G.

In [1] [2] [3] [4], Dunbar *et al.* introduced this concept, in [5] Haas and Wexler had found the signed domination number of $P_2 \times P_n$ and $P_2 \times C_n$. In [6] Hosseini gave a lower and upper bound for the signed domination number for any graph. In [7] Hassan, Al Hassan and Mostafa had found the signed domination number of $P_m \times P_n$ for $m = 3, 4, 5$ and arbitrary n.

We consider when we represent the $P_m \times P_n$ graph. The weight of the black circle is 1, and the white circles refer to the graph vertices which weight -1.

Let f be a signed dominating function of the $P_m \times P_n$ and $A = \{v \in V : f(v) = 1\}$,

$V = \{v \in V : f(v) = -1\}$, then $\gamma_s(P_m \times P_n) = m \cdot n - 2|B| = |A| - |B|$. Let K_j be the j^{th} column vertices, and also $A_j = \{v \in K_j : f(v) = 1\}$, $B_j = \{v \in K_j : f(v) = -1\}$.

2. Main Results

In this paper we will show tow theorem to find the signed domination number of Cartesian product of $P_m \times P_n$.

Theorem 2.1. For $n \geq 1$ then

$$\gamma_s(P_6 \times P_n) = \begin{cases} 2n; & \text{If } n \equiv 1(\text{mod}\,5), \\ 2n+2; & \text{If } n \equiv 2(\text{mod}\,5), \\ 2n+4; & \text{If } n \equiv 0,3,4(\text{mod}\,5). \end{cases}$$

Proof:

Let f be a signed dominating function of ($P_6 \times P_n$), then for any j were $2 \leq j \leq n-3$, then $\sum_{k=j-1}^{j+2} |B_K| \leq 8$. We discuss the following cases:

Case a. $|B_j| = 4$:

we notice that the first and last columns can't include four of the B set vertices, but in the case $2 \leq j \leq n-3$ and $|B_j| = 4$, then the vertices (1, j), (3, j), (4, j), (6, j) $\in B$, and all of the $j-1^{th}$, $j+1^{th}$ column's vertices don't contain any one of the B set vertices, so the (1, $j+2$), (6, $j+2$) vertices, then the $j+2^{th}$ column includes three of the B set vertices at most (Figure 1).

Case b. $|B_j| = 3$:

We discuss the following cases:

b-1. If (1, j), (3, j), (4, j) $\in B$ then both of the $j-1^{th}$, $j+1^{th}$ columns include at most one of the B set vertices, then the $j+2^{th}$ column includes at most three of the B set vertices.

b-2. If (1, j), (3, j), (5, j) $\in B$ then the $j-1^{th}$ and $j+1^{th}$ columns include at most two of the B set vertices, and the $j+1^{th}$ column includes three of the B set vertices.

b-3. If (1, j), (3, j), (6, j) $\in B$ then both of the $j-1^{th}$, $j+1^{th}$ columns include at most one of the B set vertices. And the $j+2^{th}$ column includes two of the B set vertices.

b-4. If (1, j), (4, j), (5, j) $\in B$ then only one of the $j-1^{th}$, $j+1^{th}$ columns include at most one of the B set vertices, so (1, $j+2$) $\in A$, then the $j+2^{th}$ column includes at most three of the B set vertices.

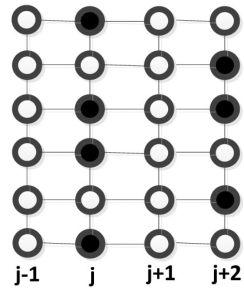

Figure 1. Case a.

b-5. If $(1, j), (4, j), (6, j) \in B$ then both of the $j-1^{\text{th}}$, $j+1^{\text{th}}$ columns include at most one of the B set vertices. Also $(1, j+2)$, $(4, j+2)$ and $(6, j+2) \in A$ then only two of the $j+2^{\text{th}}$ vertices belong to B set.

b-6. If $(2, j), (3, j), (6, j) \in B$ then only one of the $j-1^{\text{th}}$, $j+1^{\text{th}}$ column's vertices belong to the B set vertices, then the $j+2^{\text{th}}$ column include at most four of the B set vertices (Figure 2).

Case c. $|B_j| = 2$:

We discuss the following cases:

c-1. If $(1, j), (3, j) \in B$ then all of the $j-1^{\text{th}}$, $j+1^{\text{th}}$, $j+2^{\text{th}}$ columns include at most two of the B set vertices (Figure 3).

c-2. If $(1, j), (4, j) \in B$ and the $j-1^{\text{th}}$ column include two of the B set vertices then the $j+1^{\text{th}}$ column include at most one of the B set vertices, so the $j+2^{\text{th}}$ column include at most three vertices (Figure 4).

c-3. If $(1, j), (5, j) \in B$ or $(1, j), (6, j) \in B$, then all of the $j-1^{\text{th}}$, $j+1^{\text{th}}$, $j+2^{\text{th}}$ columns include at most two of the B set vertices (Figure 5).

c-4. If $(2, j), (3, j) \in B$ then if the $j-1^{\text{th}}$ column includes two of the B set vertices, then the $j+1^{\text{th}}$ column includes at most one of the B set vertices, so the $j+2^{\text{th}}$ column includes at most three vertices (Figure 6).

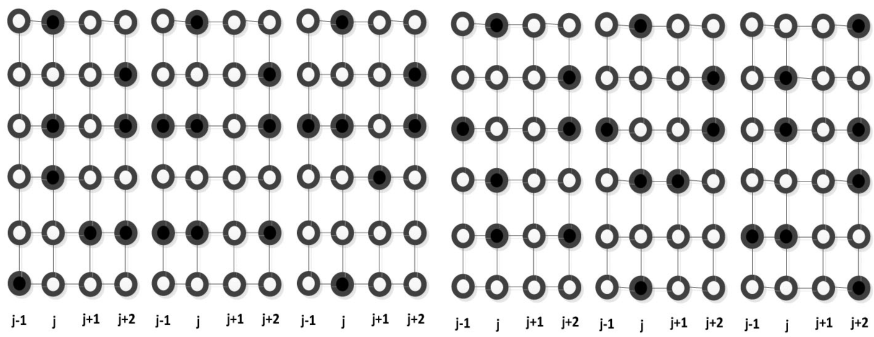

Figure 2. Case b.

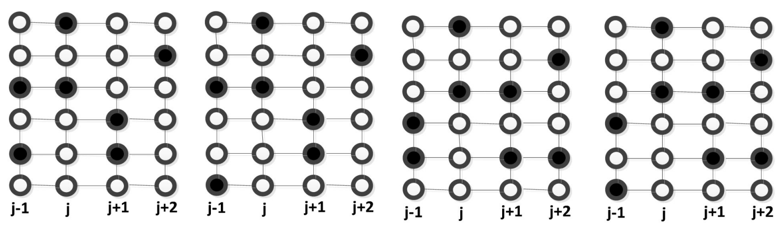

Figure 3. Case c-1.

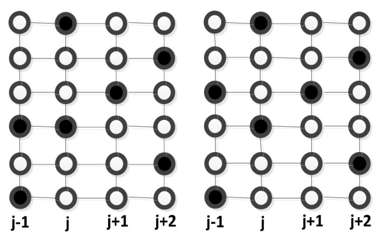

Figure 4. Case c-2.

c-5. If (2, *j*), (4, *j*) ∈ B then the $j-1^{\text{th}}$ column includes at most three of the B set vertices, it is (2, $j-1$), (4, $j-1$), (6, $j-1$) ∈ B, so the $j+1^{\text{th}}$ column includes one of the B set vertices, also the $j+2^{\text{th}}$ column includes three of the B set vertices and both of the $j-2^{\text{th}}$, $j+3^{\text{th}}$ columns don't include any one of the B set vertices, so the $j+4^{\text{th}}$ column includes four of the B set vertices and the $j-3^{\text{th}}$ column includes three of the B set vertices. then the eight columns include sixteen of the B set vertices. In other cases stay $\sum_{k=j-1}^{j+2}|B_K| \le 8$ (**Figure 7**).

c-6. If (2, *j*), (5, *j*) ∈ B then all of the $j-1^{\text{th}}$, $j+1^{\text{th}}$, $j+2^{\text{th}}$ columns include at most two of the B set vertices (**Figure 8**).

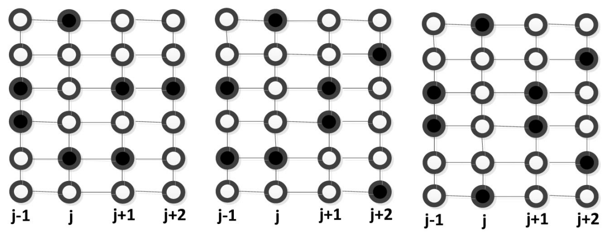

Figure 5. Case c-3.

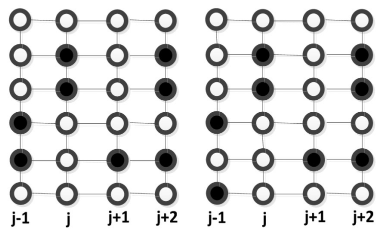

Figure 6. Case c-4.

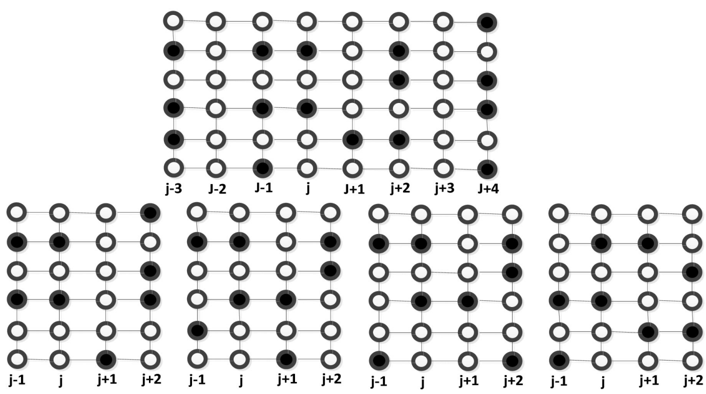

Figure 7. Case c-5.

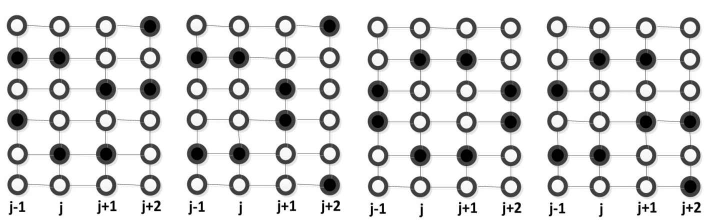

Figure 8. Case c-6.

c-7. If (3, j), (4, j) $\in B$ then all of the $j-1^{\text{th}}$, $j+1^{\text{th}}$, $j+2^{\text{th}}$ columns include at most two of the B set vertices (**Figure 9**).

Case d. $|B_j| = 1$:

We discuss the following cases:

d-1. If (1, j) $\in B$ or (3, j) $\in B$ or (4, j) $\in B$ or (6, j) $\in B$ then the $j-1^{\text{th}}$ column includes at most three of the B set vertices also both of the $j+1^{\text{th}}$, $j+2^{\text{th}}$ columns include at most two of the B set vertices (**Figure 10**).

d-2. If (2, j) $\in B$ or (5, j) $\in B$ then both of the $j-1^{\text{th}}$, $j+1^{\text{th}}$ columns includes at most three of the B set vertices, and the $j+2^{\text{th}}$ column includes at most one of the B set vertices (**Figure 11**).

From the previous cases we conclude $\gamma_s(P_6 \times P_n) \geq 2n$.

To find the upper bound of the signed domination number of ($P_6 \times P_n$) graph, let's define (**Figure 12**).

$$\begin{aligned} B = \Big\{ & (1,1+5i),(6,1+5i) : 0 \leq i \leq \left\lfloor \frac{n-1}{5} \right\rfloor \\ & \cup (3,2+5i),(4,2+5i) : 0 \leq i \leq \left\lfloor \frac{n+2}{5} \right\rfloor \\ & \cup (2,3+5i),(5,3+5i) : 0 \leq i \leq \left\lfloor \frac{n+3}{5} \right\rfloor \\ & \cup (2,4+5i),(5,4+5i) : 0 \leq i \leq \left\lfloor \frac{n+4}{5} \right\rfloor \\ & \cup (3,5+5i),(4,5+5i) : 0 \leq i \leq \left\lfloor \frac{n+5}{5} \right\rfloor \Big\} \end{aligned}$$

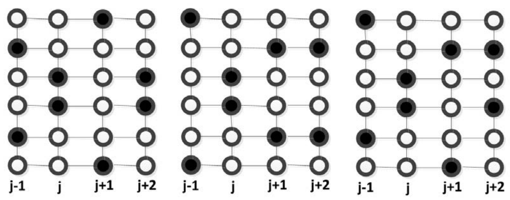

Figure 9. Case c-7.

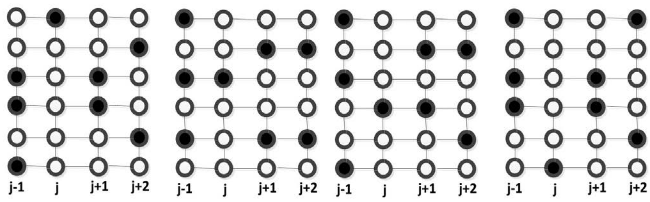

Figure 10. Case d-1.

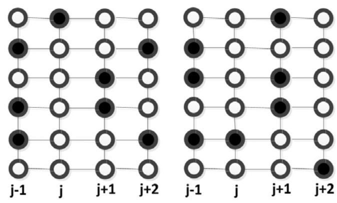

Figure 11. Case d-2.

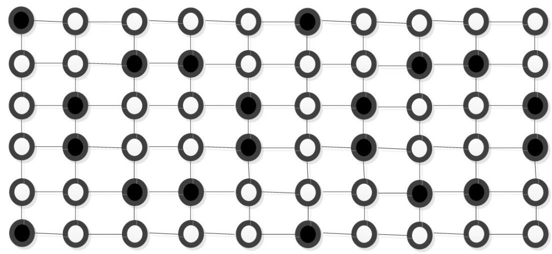

Figure 12. B set.

Case $n \equiv 1 \pmod 5$.

If B is the previously defined set and represents the vertices have the weight -1, then every one of the $P_6 \times P_n$ vertices achieves the signed dominating function, and $|B| \geq 2n$, then: $\gamma_s(P_6 \times P_n) \leq 6n - 2(2n) = 2n$. Consequently:
$\gamma_s(P_6 \times P_n) = 2n : n \equiv 1(\mathrm{mod}\,5)$ (Figure 13).

Case $n \equiv 2 \pmod 5$.

In this case, we delete one of the two vertices (3, n) or (4, n) from the previously defined set B vertices, then the signed domination number will increase by 2 than the signed domination number in case of $n \equiv 1 \pmod 5$, and f remains a signed dominating function of the graph. Consequently:
$\gamma_s(P_6 \times P_n) = 2n + 2 : n \equiv 2(\mathrm{mod}\,5)$ (Figure 14).

Case $n \equiv 0, 3, 4 \pmod 5$.

In this case we delete the B set vertices in the last column, then the signed domination number will increase by 4 than signed domination number in case of $n \equiv 1 \pmod 5$. And f remains a signed dominating function of the graph.

Consequently: $\gamma(P_6 \times P_n) = 2n + 4 : n \equiv 0, 3, 4(\mathrm{mod}\,5)$ (Figure 15).

Lemma 2.1.

Let f be a signed domination function of ($P_7 \times P_n$), and B the graph vertices set which having the weight -1, Then for any j were $1 \leq j \leq n - 1$, then $\sum_{k=j}^{j+1} |B_K| \leq 5$. Except the following cases:

(3, j), (5, j) ∈ B, (1, j), (3, j), (5, j) ∈ B, (2, j), (3, j), (5, j) ∈ B or (3, j), (5, j), (7, j) ∈ B. Then $\sum_{k=j}^{j+1} |B_K| \leq 6$ and in this case $|B_{j+2}| + |B_{j+3}| \leq 5$.

Proof:

For any j were $1 \leq j \leq n$ then $|B_j| \leq 4$.

Case a. $|B_j| = 4$:

The $j + 1^{\text{th}}$ column includes at most one of the B set vertices, except case (1, j), (3, j), (5, j), (7, j) ∈ B. then the $j + 1^{\text{th}}$ column includes two of the B set vertices (Figure 16).

Case b. $|B_j| = 3$:

The $j + 1^{\text{th}}$ column includes at most two vertices except in the following cases:

(1, j), (3, j), (5, j) ∈ B, (2, j), (4, j), (6, j) ∈ B, (3, j), (5, j), (7, j) ∈ B. Then $|B_{j+1}|$ = 3 (Figure 17).

Case c. $|B_j| = 2$:

The $j + 1^{\text{th}}$ column includes at most three vertices, except in case (3, j), (5, j) ∈ B, then the $j + 1^{\text{th}}$ column includes four of the B set vertices (Figure 18).

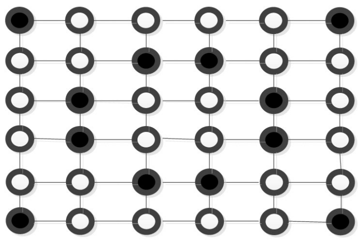

Figure 13. Case $n \equiv 1 \pmod 5$.

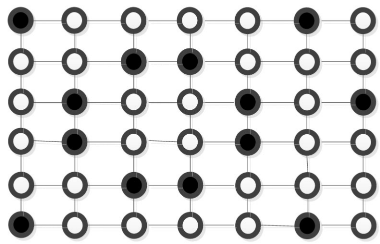

Figure 14. Case $n \equiv 2 \pmod 5$.

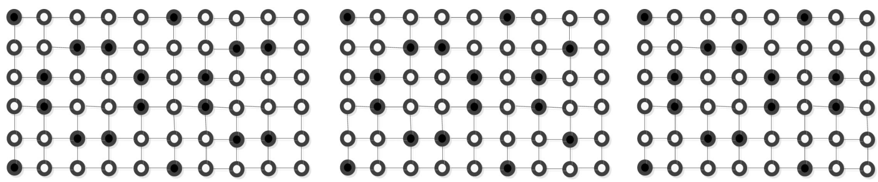

Figure 15. Case a.

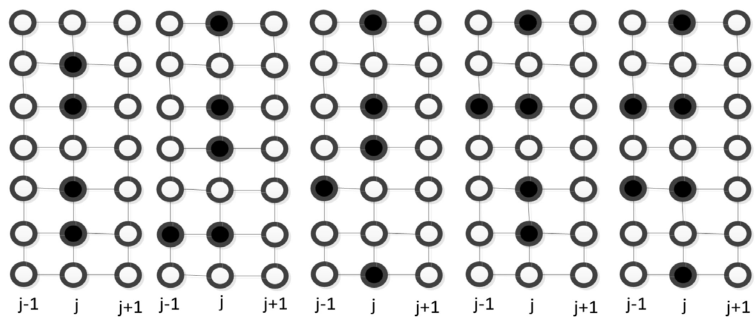

Figure 16. Case a.

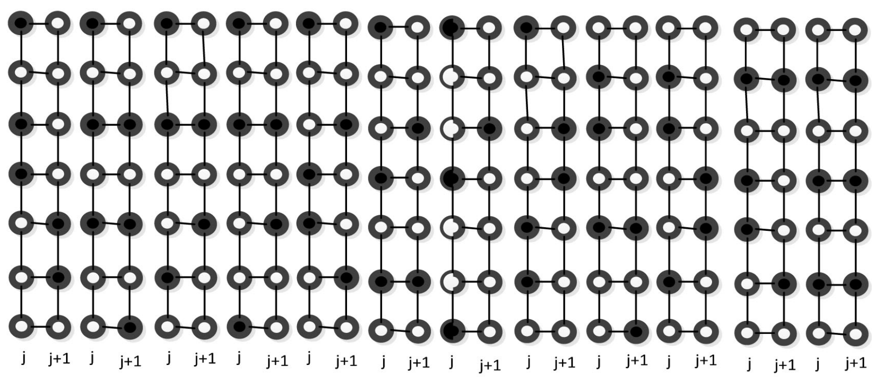

Figure 17. Case b.

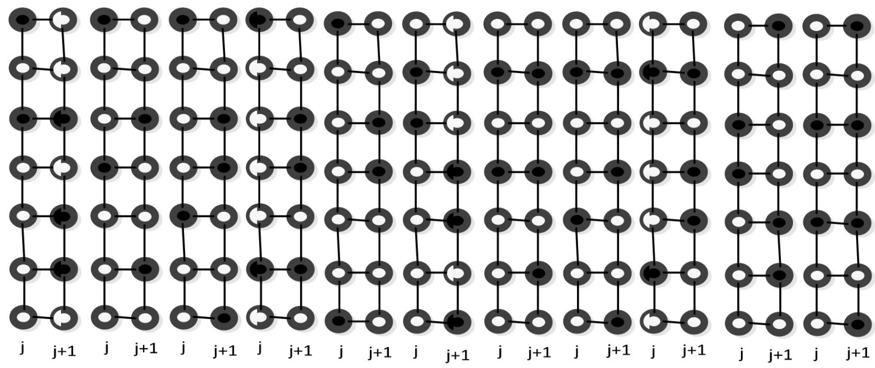

Figure 18. Case c.

In case $|B_j| = 1$ or $|B_j| = 0$ it's proofed easily because $|B_{j+1}| \le 4$.

Lemma 2.2.

Let f be a signed domination function of $(P_7 \times P_n)$ and B the graph vertices set which having the weight -1, then $|B_1| + |B_2| + |B_3| \le 6$. Except for a case (2, 3), (3, 3), (6, 3) $\in B$. Then $|B_1| + |B_2| + |B_3| \le 7$. In this case $|B_4| = 1$.

Proof:

Case a. $|B_2| = 3$:

If (1, 3), (3, 3), (5, 3) $\in B$ or (2, 3), (4, 3), (6, 3) $\in B$ then the second column include three vertices of the B set vertices, and the first column doesn't include any one of the B set vertices (**Figure 19**).

Case b. $|B_2| = 2$:

If (1, 3), (3, 3), (7, 3) $\in B$ or (1, 3), (4, 3), (5, 3) $\in B$ or (1, 3), (4, 3), (6, 3) $\in B$, then the second column include two vertices of the B set vertices, and the first column doesn't include any one of the B set vertices.

If (1, 3), (3, 3), (4, 3) $\in B$ or (1, 3), (3, 3), (6, 3) $\in B$ or (1, 3), (5, 3), (6, 3) $\in B$ or (2, 3), (3, 3), (5, 3) $\in B$ or (2, 3), (4, 3), (5, 3) $\in B$, then the second column include two vertices of the B set vertices, and the first column include one of the B set vertices.

If (2, 3), (3, 3), (6, 3) $\in B$, then the second column include two vertices of the B set vertices, and the first column include two vertices of the B set vertices. In this case the fourth column at most include one of the B set vertices (**Figure 20**).

Case b. $|B_2| = 1$:

If (1, 3), (4, 3), (7, 3) $\in B$, then the second column include one of the B set vertices, and the first column include one of the B set vertices (**Figure 21**).

Remark 2.1. $|B_{n-2}| + |B_{n-1}| + |B_n| \le 6$. Except for a case $(2, n-2)$, $(3, n-2)$, $(6, n-2) \in B$. Then $|B_{n-2}| + |B_{n-1}| + |B_n| \le 7$. In this case $|B_{n-3}| = 1$, and prove as in the lemma (2.2.)

Theorem 2.2. Let n be a positive integer

If $n \equiv 0, 2 \pmod 5$, then $\gamma_s(P_7 \times P_n) = \frac{11n}{5} + 6$;

If $n \equiv 1, 3 \pmod 5$, then $\gamma_s(P_7 \times P_n) = \frac{11n}{5} + 7$;

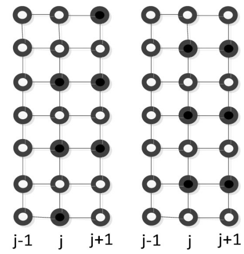

Figure 19. Case b.

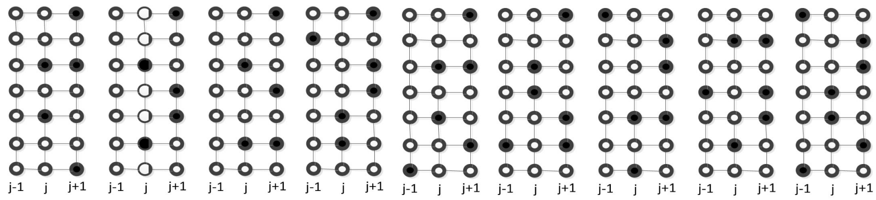

Figure 20. Case b.

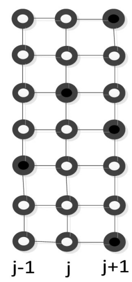

Figure 21. Case c.

If $n \equiv 4 \pmod 5$, then $\gamma_s(P_7 \times P_n) = \frac{11n}{5} + 8$.

Proof:

Case $n \equiv 0 \pmod 5$.

Let f be a signed domination function of the $P_7 \times P_n$. And B the graph vertices set which having the weight −1. Then for any j were $1 \le j \le n-3$ then $\sum_{k=j-1}^{j+3} |B_K| \le 12$.

Case a. $|B_j| = 4$:

Then we discuss the following cases:

a-1. If $(2, j), (3, j), (5, j), (6, j) \in B$ then both of the $j-1^{th}$, $j+1^{th}$ columns don't include any one of the B set vertices, so $|B_{j-1}| + |B_j| + |B_{j+1}| \le 4$. And according to lema1 then $|B_{j+2}| + |B_{j+3}| \le 6$.

a-2. If $(1, j), (3, j), (4, j), (6, j) \in B$ or $(1, j), (3, j), (4, j), (7, j) \in$ B or $(1, j), (3, j), (5, j), (6, j) \in B$. Then one of the $j-1^{th}$ or $j+1^{th}$ column includes one of the B set vertices, as $|B_{j+2}| + |B_{j+3}| \le 6$.

a-3. If $(1, j), (3, j), (5, j), (7, j) \in B$ then both of the $j-1^{th}$, $j+1^{th}$ columns include two of the B set vertices, as $|B_{j+2}| + |B_{j+3}| \le 6$ (**Figure 22**).

Case b. $|B_j| = 3$:

We discuss the following cases:

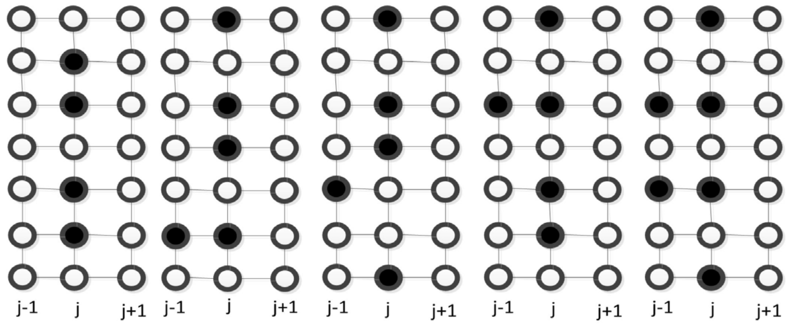

Figure 22. Case a.

b-1. If (1, j), (4, j), (7, j) $\in B$ then at most one of the $j-1^{th}$ columns vertices and also at most one of the $j+1^{th}$ vertices belongs to the B set vertices. Then the number of the vertices from the B set in the five successive columns remains less or equal to 12 (**Figure 23**).

b-2. If (1, j), (3, j), (4, j) $\in B$ or (1, j), (4, j), (5, j) $\in B$ or (1, j), (4, j), (6, j) $\in B$ or (1, j), (5, j), (6, j) $\in B$ or (2, j), (3, j), (5 , j) $\in B$ or (2, j), (3, j), (6, j) $\in B$ or (2, j), (4, j), (5, j) $\in B$. then at most two of the $j-1^{th}$ columns vertices and also at most one of the $j+1^{th}$ vertices belongs to the B set vertices. Then the number of the vertices from the B set in the five successive columns remains less or equal to 12 (**Figure 24**).

b-3. If (2, j), (4, j), (6, j) $\in B$ then at most one of the two vertices (2, $j-1$), (2, $j+1$) and one of the two vertices (4, $j-1$), (4, $j+1$), And one of the two vertices (6, $j-1$), (6, $j+1$) may be of the B set vertices. Then the number of the vertices from the B set in the five successive columns remains less or equal to 12 (**Figure 25**).

b-4. If (1, j), (3, j), (5, j) $\in B$ then the $j-1^{th}$ column includes at most three of the B set vertices. In case $|B_{j-1}| = 3$. Then (3, $j-1$), (5, $j-1$), (7, $j-1$) $\in B$. so (6, $j+1$), (6, $j+2$) $\in B$. Thus it remains in the $j+2^{th}$ column three successive vertices include at most two of the B set vertices, so the $j+3^{th}$ column includes at most two of the B set vertices (**Figure 26**).

b-5. If (1, j), (3, j), (7, j) $\in B$ then both of the $j-1^{th}$, $j+1^{th}$ columns include at most two of the B set vertices.

b-5-1. If (3, $j-1$), (5, $j-1$) $\in B$ then (4, $j+1$), (5, $j+1$)$\in B$ and (2, $j+2$), (6, $j+2$) $\in B$ then three of the $j+3^{th}$ column vertices belongs to the B set vertices.

b-5-2. If (4, $j-1$), (5, $j-1$) $\in B$ then (3, $j+1$), (5, $j+1$) $\in B$, and (2, $j+2$), (5, $j+2$) $\in B$ or (2, $j+2$), (6, $j+2$) $\in B$, then at most three of the $j+3^{th}$ column vertices belong to the B set vertices (**Figure 27**).

b-6. If (1, j), (3, j), (6, j) $\in B$ then the $j-1^{th}$ column includes at most two of the B set vertices, in this case the $j+1^{th}$ column includes at most two of the B set vertices, and the $j+2^{th}$ column includes at most three vertices and the $j+3^{th}$ column includes at most two vertices of the B set vertices (**Figure 28**).

Case c. $|B_j| = 2$:

c-1. If (1, j), (4, j) $\in B$ or (1, j), (7, j) $\in B$ then both of the $j-1^{th}$, $j+1^{th}$ columns include at most two of the B set vertices, then the $j-1^{th}$, jth, $j+1^{th}$ columns

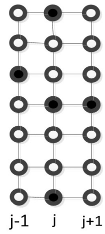

Figure 23. Case b-1.

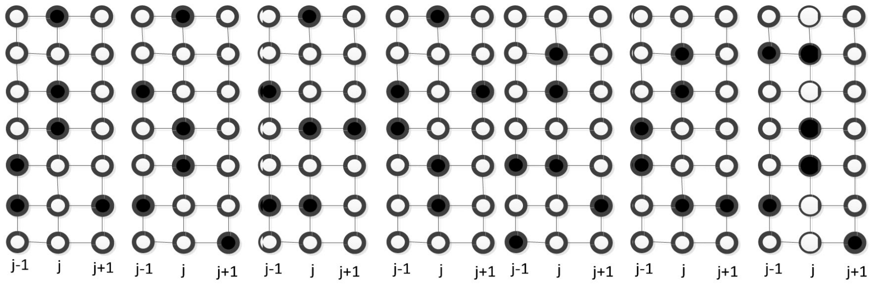

Figure 24. Case b-2.

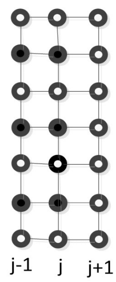

Figure 25. Case b-3.

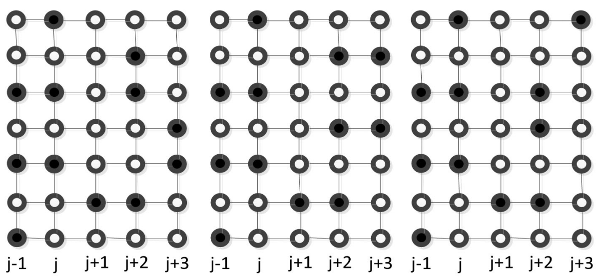

Figure 26. Case b-4.

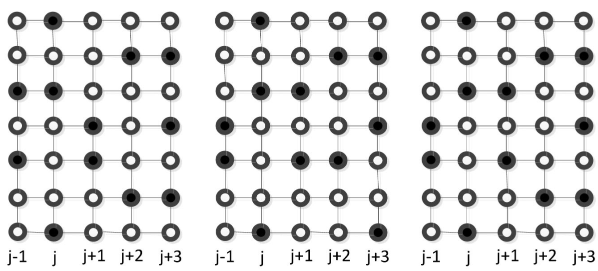

Figure 27. Case b-5.

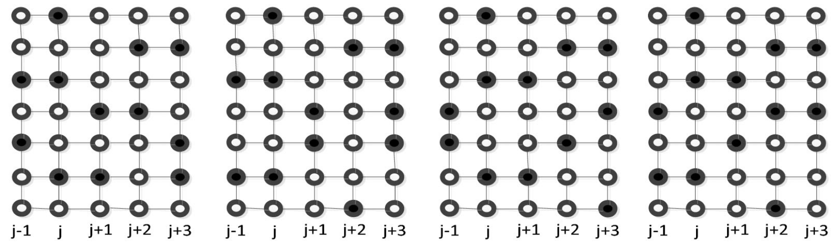

Figure 28. Case b-6.

include at most six of the B set vertices, as any two columns include at most six vertices (**Figure 29**).

c-2. If $(1, j)$, $(3, j) \in B$ then the $j-1^{th}$ column includes at most three vertices, because one of the two vertices $(3, j-1) \in B$ or $(4, j-1) \in B$ and either the two vertices $(5, j-1)$ and $(6, j-1)$ or $(5, j-1)$ and $(7, j-1)$ belong to the B set vertices.

c-2-1. If $(3, j-1) \in B$ the $j+1^{th}$ column includes at most three of the B set vertices, in this case the $j+2^{th}$ column includes at most one of the B set vertices, and the $j+3^{th}$ column includes at most three vertices. Or the $j+2^{th}$ column includes two of the B set vertices and the $j+3^{th}$ column includes at most three vertices.

c-2-2. If $(4, j-1) \in$ B then the $j+1^{th}$ column includes at most three of the B set vertices, in this case $(3, j+1)$, $(5, j+1)$, $(6, j+1) \in B$ and $(2, j+2) \in B$, so $(2, j+3)$, $(4, j+3)$, $(5, j+3)$, $(7, j+3) \in B$, then the $j-2^{th}$ column includes at most one of the B set vertices, then $\sum_{k=j-2}^{j+2}|B_K| \leq 12$. Also the $j+4^{th}$ column doesn't include any one of the B set vertices, so $\sum_{k=j}^{j+4}|B_K| \leq 12$. And according to lemma 2-1 note $|B_{j+5}| + |B_{j+6}| \leq 6$, so $|B_{j+7}| \leq 6$. Then every ten successive columns include at most twenty four of the B set vertices (**Figure 30**).

c-3. If $(1, j)$, $(5, j) \in B$ then the $j-1^{th}$ column includes at most three of the B set vertices, so the $j+1^{th}$ and $j+2^{th}$ columns includes at most two of the B set vertices, and the $j+3^{th}$ column includes at most three vertices (**Figure 31**).

c-4. If $(1, j)$, $(6, j) \in B$ then the $j-1^{th}$ column includes at most three vertices, in this case the $j+1^{th}$ column includes at most two of the B set vertices, also the $j+2^{th}$ column includes three of the B set vertices, and the $j+3^{th}$ column includes at most two vertices (**Figure 32**).

c-5. If $(2, j)$, $(3, j) \in B$ then the $j-1^{th}$ column includes at most three of the B set vertices, then the $j+1^{th}$ column includes two of the B set vertices which are $(5, j+1)$, $(6, j+1)$, also $(1, j+2)$, $(3, j+2)$, $(4, j+2) \in B$, and the $j+3^{th}$ column includes only one of the B set vertices (**Figure 33**).

c-6. If $(2, j)$, $(4, j) \in B$ then the $j-1^{th}$ column includes at most three of the B set vertices, so the $j+1^{th}$ column includes at most two of the B set vertices, in this case the $j+2^{th}$ column includes at most three of the B set vertices, and the $j+3^{th}$ column includes at most two vertices (**Figure 34**).

c-7. If $(2, j)$, $(5, j) \in$ B then the $j-1^{th}$ column includes at most three of the B set vertices, and the $j+1^{th}$ column includes two of the B set vertices, then the $j+2^{th}$, $j+3^{th}$ columns include at most five of the B set vertices (**Figure 35**).

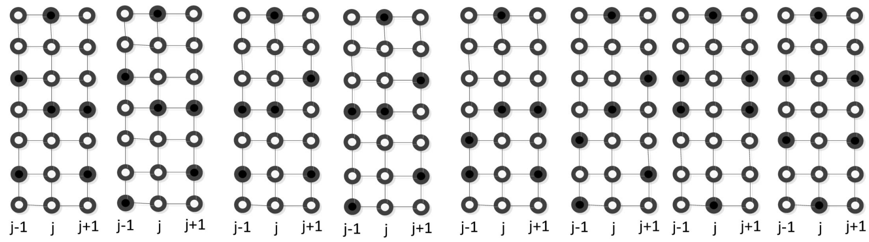

Figure 29. Case c-1.

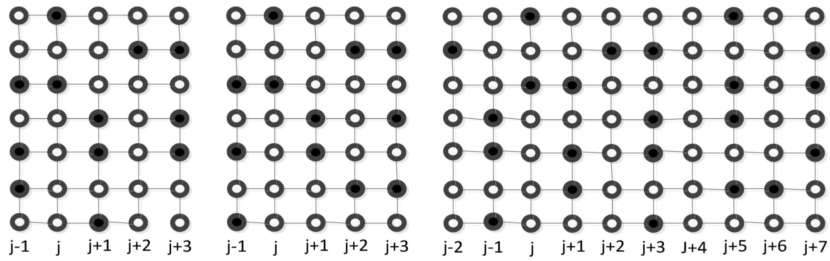

Figure 30. Case c-2.

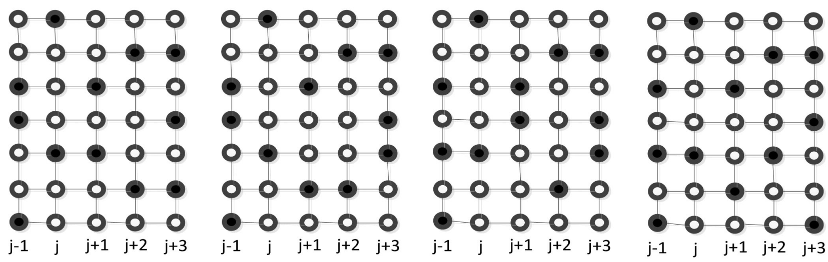

Figure 31. Case c-3.

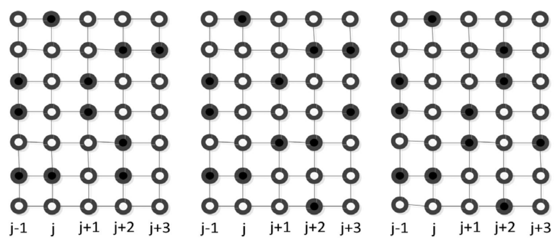

Figure 32. Case c-4.

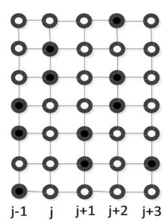

Figure 33. Case c-5.

c-8. If (2, j), (6, j) ∈ B then both of the $j-1^{\text{th}}$, $j+1^{\text{th}}$ columns include at most three of the B set vertices, so the $j+2^{\text{th}}$ column includes at most one of the B set vertices, and the $j+3^{\text{th}}$ column includes at most three vertices (**Figure 36**).

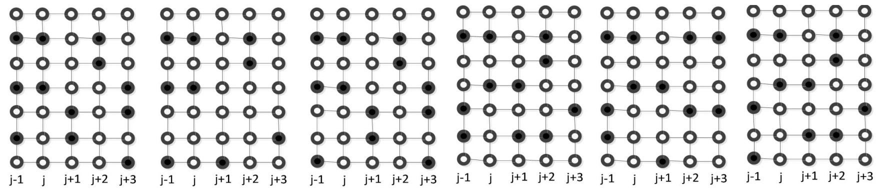

Figure 34. Case c-6.

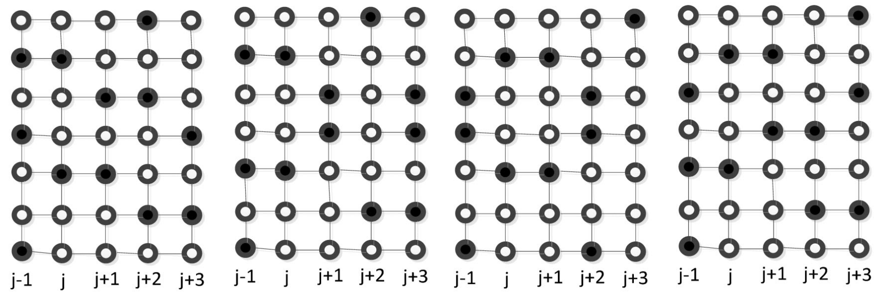

Figure 35. Case c-7.

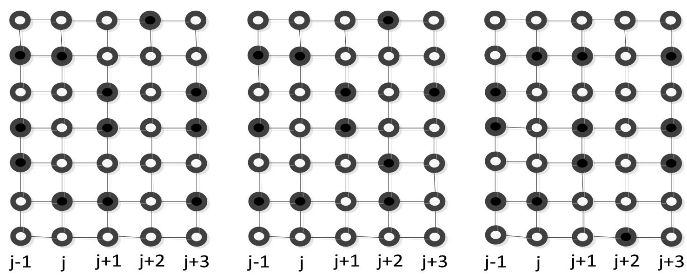

Figure 36. Case c-8.

c-9. If (3, j), (4, j) $\in$ B then the $j-1^{th}$ column includes at most three of the B set vertices, then the $j+1^{th}$ column includes at most three of the B set vertices, then the $j+2^{th}$ column includes only one of the B set vertices, and the $j+3^{th}$ column includes at most three vertices (Figure 37).

c-10. If (3, j), (5, j) $\in B$ then the $j-1^{th}$ column includes at most four of the B set vertices, so the $j+1^{th}$ column includes at most two vertices, then the $j+2^{th}$ column includes at most three of the B set vertices, and the $j+3^{th}$ column includes at most one vertex (Figure 38).

Case d. $|B_j| = 1$:

In this case the $j+1^{th}$, $j+2^{th}$ columns include at most five of the B set vertices, so if the $j+3^{th}$, $j+4^{th}$ columns include six of the B set vertices, then the number of the vertices in the five columns is less or equal to 12 (Figure 39).

We note from all the previous cases $|B| \le \frac{12n}{5}$. Then $\gamma_s(P_7 \times P_n) \ge 7n - 2\left(\frac{12n}{5}\right) = \frac{11n}{5}$.

To find the upper bound of the signed domination number of ($P_7 \times P_n$) graph, let's define (Figure 40).

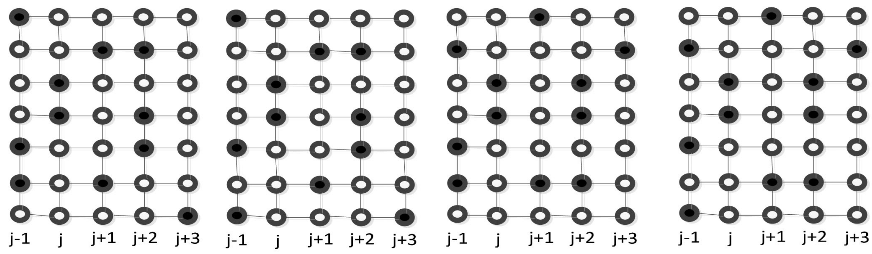

Figure 37. Case c-9.

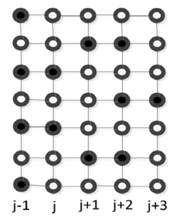

Figure 38. Case c-10.

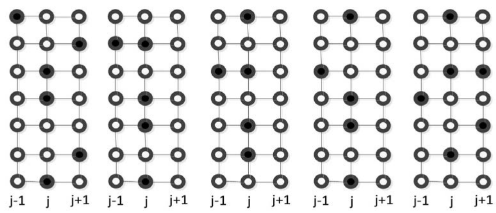

Figure 39. Case d.

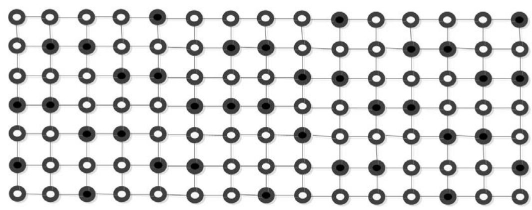

Figure 40. Case B.

$$
\begin{aligned}
B = \Big\{ & (4,5j),(6,5j) : 0 \le j \le \left\lfloor \frac{n}{5} \right\rfloor \\
& \cup (2,5j+1),(4,5j+1),(6,5j+1) : 0 \le j \le \left\lfloor \frac{n-1}{5} \right\rfloor \\
& \cup (2,5j+2),(5,5j+2),(7,5j+2) : 0 \le j \le \left\lfloor \frac{n-2}{5} \right\rfloor \\
& \cup (3,5j+3),(5,5j+3) : 0 \le j \le \left\lfloor \frac{n-3}{5} \right\rfloor \\
& \cup (1,5j+4),(3,5j+4),(6,5j+4) : 0 \le j \le \left\lfloor \frac{n-4}{5} \right\rfloor \Big\}
\end{aligned}
$$

If B the graph vertices set which having the weight -1, then every one of the $P_7 \times P_n$ graph vertices achieves the signed domination function and $|B| \ge \left\lfloor \frac{12n}{5} \right\rfloor$.

According to lemma 2-2 we deleted the vertex (4, 1) from the previously defined set B vertices in all cases, then $\gamma_s(P_7 \times P_n) \geq \left\lfloor \frac{11n}{5} \right\rfloor + 2$.

Case $n \equiv 0, 2 \pmod 5$.

According to lemma 2-2, then in case $n \equiv 0 \pmod 5$, we delete the vertices (3, n), (6, n), so in case $n \equiv 2 \pmod 5$, we delete the vertex (4, n). Then the signed domination number will increase by 4.

Consequently: $\gamma_s(P_7 \times P_n) = \left\lfloor \frac{11n}{5} \right\rfloor + 2 + 4 = \left\lfloor \frac{11n}{5} \right\rfloor + 6 : n \equiv 0, 2 \pmod 5$ (Figure 41).

Case $n \equiv 1, 3 \pmod 5$.

When we add one column on case $n \equiv 0 \pmod 5$, note that the number of vertices will increase by 7, and the number of set B vertices will increase by 2, in this case

$$\gamma_s(P_7 \times P_n) = \left\lfloor \frac{11(n-1)}{5} \right\rfloor + 2 + 7 = \left\lfloor \frac{11n}{5} \right\rfloor + 7 : n \equiv 1 \pmod 5.$$

When we add three columns on case $n \equiv 0 \pmod 5$, note that the number of vertices will increase by 21, and the number of set B vertices will increase by 5, in this case

$$\gamma_s(P_7 \times P_n) = \left\lfloor \frac{11(n-3)}{5} \right\rfloor + 2 + 21 - 2 \times 5 = \left\lfloor \frac{11n}{5} \right\rfloor + 7 : n \equiv 3 \pmod 5.$$

Consequently: $\gamma_s(P_7 \times P_n) \geq \left\lfloor \frac{11n}{5} \right\rfloor + 7 : n \equiv 1, 3 \pmod 5$. (Figure 42)

Case $n \equiv 4 \pmod 5$.

When we add four column on case $n \equiv 0 \pmod 5$, note that the number of vertices will increase by 28, and the number of set B vertices will increase by 9, in this case (Figure 43)

$$\gamma_s(P_7 \times P_n) = \left\lfloor \frac{11(n-4)}{5} \right\rfloor + 2 + 28 - 2 \times 7 = \left\lfloor \frac{11n}{5} \right\rfloor + 8 : n \equiv 4 \pmod 5.$$

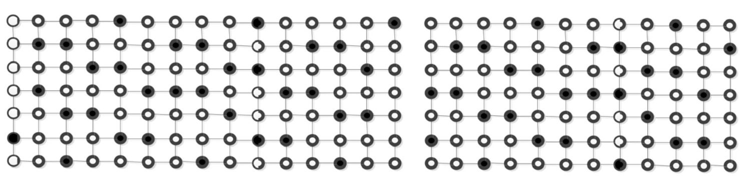

Figure 41. Case $n \equiv 0, 2$ (mode 5).

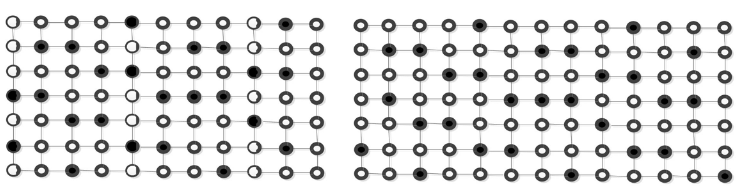

Figure 42. Case $n \equiv 1, 3$ (mode 5).

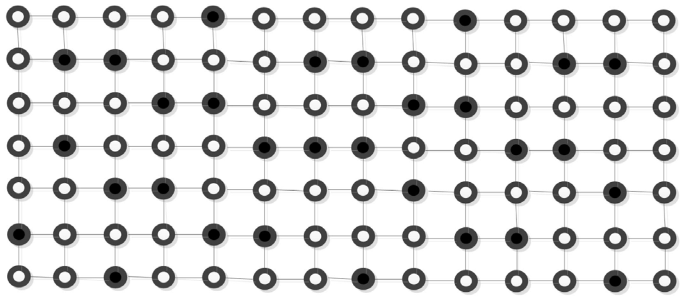

Figure 43. Case $n \equiv 4$ (mode 5).

3. Conclusion

In this paper, we studied the signed domination numbers of the Cartesian product of two paths P_m and P_n for $m = 6, 7$ and arbitrary n. We will work to find the signed domination numbers of the Cartesian product of two paths P_m and P_n for arbitraries m and n, and special graphs.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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