Open Journal of Applied Sciences, 2022, 12, 1232-1242
https://www.scirp.org/journal/ojapps
ISSN Online: 2165-3925
ISSN Print: 2165-3917

Algebra and Coalgebra on Posets

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How to cite this paper: Yuan, R. and Li, H. (2022) Algebra and Coalgebra on Posets. *Open Journal of Applied Sciences*, **12**, 1232-1242.
https://doi.org/10.4236/ojapps.2022.127083

Received: June 21, 2022
Accepted: July 16, 2022
Published: July 19, 2022

Abstract

A lot of combinatorial objects have algebra and coalgebra structures and posets are important combinatorial objects. In this paper, we construct algebra and coalgebra structures on the vector space spanned by posets. Firstly, by associativity and the unitary property, we prove that the vector space with the conjunction product is a graded algebra. Then by the definition of free algebra, we prove that the algebra is free. Finally, by the coassociativity and the counitary property, we prove that the vector space with the unshuffle coproduct is a graded coalgebra.

Keywords

Conjunction Product, Unshuffle Coproduct, Poset, Graded, Free

1. Introduction

A poset is a set with a binary relation satisfying reflexivity, antisymmetry and transitivity. Researches and generalizations on posets are very rich. The most famous result on posets is the decomposition theorem [1] proposed by Dilworth in 1950, also well-known as Dilworth's Theorem, which has great combinatorial and order theoretical value. To learn more about Dilworth's Theorem, please refer to Fulkerson [2], Tverberg [3], Pretzel [4] and Galvin [5].

In 1964, Rota [6] made the Möbius function emerge in clear view as a fundamental invariant, which unifies both enumerative and structural aspects of the theory of partially ordered sets. In 1972, Stanley [7] studied ordered structures and partitions. Later, he proved several identities associated with the binomial posets [8]. In 1977, Trotter and Moore [9] studied the dimension of planar posets and the dimension of trees. In 1988, Stanley [10] first introduced the differential poset with combinatorial and algebraic properties. For more works on differential posets, see [11] [12] [13] [14] [15].

In 2005, Aguiar and Sottile [16] introduced the global descents of permutations in the symmetric group S_n. In 2020, based on the global descents, Zhao and Li [17] studied a new shuffle product $ш_G$ on permutations. Later, they [18] defined a new product $\lozenge$ and a new coproduct Δ^* on permutations, proved that $(\mathbb{K}S, \lozenge, \mu)$ is a graded $\mathbb{K}$-algebra and $(\mathbb{K}S, \Delta^*, \nu)$ is a graded $\mathbb{K}$-coalgebra, where $\mathbb{K}$ is a field, and studied some properties of the structures. In 2021, Liu and Li [19] introduced the super-shuffle product and the cut-box coproduct on permutations and proved that $(\mathbb{K}S, \underline{ш}, \mu)$ is a graded algebra and $(\mathbb{K}S, \Delta_\lozenge, \nu)$ is a graded coalgebra. These papers are helpful for us to study algebra and coalgebra on posets.

In 2020, Aval, Bergeron and Machacek [20] defined a product and a coproduct on posets without proofs. In this paper, we prove that the vector space spanned by posets with these operations is an algebra and a coalgebra, respectively.

We start by recalling some basic definitions of algebra and coalgebra and some notations on posets in Section 2. In Section 3, we introduce the definitions of the conjunction product and the unshuffle coproduct on the vector space spanned by posets. Then we prove the vector space with the conjunction product is a free graded algebra. And the vector space with the unshuffle coproduct is a graded coalgebra. Thus, we construct algebra and coalgebra structures on posets. Finally, we make a summary of this paper in Section 4.

2. Preliminaries

2.1. Basic Definitions

We recall some basic definitions of algebra and coalgebra; see [21] [22] for more details. Let R be an associative commutative ring with identity.

For an R-module A, we call (A, m, μ) an R-*algebra* if there exist two maps $m: A \otimes A \to A$ and $\mu: R \to A$ such that the diagrams in **Figure 1** are commutative. Here m is called a *product* and μ a *unit*.

The R-algebra (A, m, μ) is *graded* if $A = \bigoplus_{i \geq 0} A_i$ and $m(A_s \otimes A_t) \subseteq A_{s+t}$, for all s, t.

We reverse all the arrows in **Figure 1** to get the definition of coalgebra since algebra and coalgebra are dual concepts.

For an R-module A, we call (A, Δ, ν) an R-*coalgebra* if there exist two maps $\Delta: A \to A \otimes A$ and $\nu: A \to R$ such that the diagrams in **Figure 2** are commutative. Here Δ is called a *coproduct* and ν a *counit*.

The R-coalgebra (A, Δ, ν) is *graded* if $A = \bigoplus_{i \geq 0} A_i$ and $\Delta(A_n) \subseteq \bigoplus_i (A_i \otimes A_{n-i})$, for all n.

A coalgebra A is *cocommutative* if the diagram in **Figure 3** commutes, where $\tau(a \otimes b) = b \otimes a$, for all a, b in A.

Let V be a vector space over field $\mathbb{K}$. Denote the tensor algebra on V by $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$, where $V^{\otimes 0} = \mathbb{K}$ and $V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}$.

Define the multiplication on $T(V)$ by the concatenation product and unit

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "m\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes m"'] & A\otimes A \arrow[d, "m"] \\ A\otimes A \arrow[r, "m"] & A \end{tikzcd}$$

(a)

$$\begin{tikzcd} & A\otimes A \arrow[d, "m"] & \\ R\otimes A \arrow[ur, "\mu\otimes \mathrm{id}"] \arrow[r, "\cong"] & A & A\otimes R \arrow[ul, "\mathrm{id}\otimes\mu"'] \arrow[l, "\cong"'] \end{tikzcd}$$

(b)

Figure 1. Associative law and unitary property. (a) Associative law; (b) Unitary property.

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A\otimes A \arrow[d, "\Delta\otimes \mathrm{id}"] \\ A\otimes A \arrow[r, "\mathrm{id}\otimes\Delta"] & A\otimes A\otimes A \end{tikzcd}$$

(a)

$$\begin{tikzcd} & A \arrow[dl, "\cong"'] \arrow[d, "\Delta"] \arrow[dr, "\cong"] & \\ R\otimes A & A\otimes A \arrow[l, "\nu\otimes \mathrm{id}"'] \arrow[r, "\mathrm{id}\otimes\nu"] & A\otimes R \end{tikzcd}$$

(b)

Figure 2. Coassociative law and counitary property. (a) Coassociative law; (b) Counitary property.

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}"'] & A\otimes A \arrow[d, "\tau"] \\ A \arrow[r, "\Delta"] & A\otimes A \end{tikzcd}$$

Figure 3. Commutativity.

$\mu:\mathbb{K}\to T(V)$ on $T(V)$ by $\mu(a)=a$, for $a\in\mathbb{K}$. Then the algebra $T(V)=\bigoplus_{n\geq 0}V^{\otimes n}$ is *free* on V since it satisfies the following universal property: for each $\mathbb{K}$-algebra A and each linear map $f:V\to A$, there exists a unique algebra homomorphism $g:T(V)\to A$ such that $g\iota=f$ where $\iota:V\to T(V)$ is the inclusion map.

2.2. Basic Notations

Now let's recall some notations on posets; see [23] [24] for more details.

A *partial order relation* is a binary relation satisfying reflexivity, antisymmetry and transitivity. A set P together with a partial order relation $\leq_P$ is called a *poset*, denoted by $(P,\leq_P)$. The set P is called the *ground set* of poset $(P,\leq_P)$. We denote the number of elements of P by $|P|$. When the ground set is empty, we have an empty poset, denoted by ϵ. When the partial order relation is obvious, P can represent both the ground set and the poset.

For distinct elements x,y in poset P, if $x\leq_P y$ and there is no element z that differs from x,y and satisfies $x\leq_P z\leq_P y$, then we say that y *covers* x, denoted by $x\lessdot_P y$, and we also call (x,y) a *cover relation* in P. Define $\mathcal{A}(P)$ to be the set of all cover relations in P by

$$\mathcal{A}(P)=\{(x,y)\mid x\lessdot y, x,y\in P\}.$$

If $\mathcal{A}(P)$ is given in poset P, then we can get the partial order relation $\leq_P$

corresponding to the cover relations through reflexivity, antisymmetry and transitivity. Obviously, $\mathcal{A}(P)$ and $\leq_P$ are uniquely determined by each other.

The two elements x and y in poset P are called *comparable*, if either $x \leq_P y$ or $y \leq_P x$. In a poset, it is not necessary that any two elements are comparable. When x and y are two elements of P such that neither $x \leq_P y$ nor $y \leq_P x$, they are called *incomparable*.

To study posets more intuitively, we can represent posets by *Hasse diagrams*. A Hasse diagram is a graphical rendering of a poset displayed by the cover relations of the poset with an implied upward orientation. Drawing line segments between these elements of a poset follows these two rules:

1) If $x \leq_P y$ in the poset, then the point representing x is lower in the drawing than the point representing y.

2) Drawing line segment between the points representing elements x and y of the poset if y covers x.

In addition, incomparable elements can be drawn on the same layer.

For example, for set $P = \{2,3,4\}$ with $\mathcal{A}(P) = \{(2,3),(3,4)\}$, *i.e.*, 3 covers 2 and 4 covers 3, according to transitivity, we have $2 \leq_P 4$. Further, we get poset $P = \begin{smallmatrix}4\\|\\3\\|\\2\end{smallmatrix}$. Similarly, if $Q = \{1,2,3\}$ with $\mathcal{A}(Q) = \{(1,3),(2,3)\}$, then poset $Q = \begin{smallmatrix}&3&\\&\wedge&\\1&&2\end{smallmatrix}$.

In poset P, an element x is called *maximal* in P, if there is no other element y in P satisfying $x \leq_P y$. Similarly, x is *minimal* if no other element y in P satisfing $y \leq_P x$. A partially ordered set may have more than one maximal or minimal elements. Hence, we denote $\max(P)$ as the set containing all maximal elements in P and $\min(P)$ containing all minimal elements in P. From the above example, we have $\max(P) = \{4\}$, $\min(P) = \{2\}$, $\max(Q) = \{3\}$ and $\min(Q) = \{1,2\}$.

Let P and Q be disjoint sets with partial orders $\leq_P$ and $\leq_Q$, respectively. Define T to be the union of P and Q with partial order $\leq_T$ given by the cover relations

$$\mathcal{A}(T) = \mathcal{A}(P) \cup \mathcal{A}(Q) \cup \{(x,y) \mid x \in \max(P), y \in \min(Q)\}.$$

We denote the poset T as $P < Q$, which means $<$ is an operation for posets. Obviously, $<$ satisfies the associative law. For example, Let $P = \begin{smallmatrix}&2&\\&\wedge&\\4&&5\end{smallmatrix}$ and $Q = 1\ 3$, then $\mathcal{A}(P) = \{(4,2),(5,2)\}$, $\mathcal{A}(Q) = \{\varnothing\}$, $\max(P) = \{2\}$ and $\min(Q) = \{1,3\}$. Hence we have $\mathcal{A}(T) = \{(4,2),(5,2),(2,1),(2,3)\}$, *i.e.*,

$T = \begin{smallmatrix}1&&3\\&\vee&\\&2&\\&\wedge&\\4&&5\end{smallmatrix}$.

Denote $\mathcal{P}_n$ to be the set of all posets on $[n] = \{1,2,\cdots,n\}$, where $\mathcal{P}_0 = \{\varepsilon\}$. For example, when $n = 3$,

$$\mathcal{P}_3 = \left\{ \begin{smallmatrix}3\\|\\2\\|\\1\end{smallmatrix}, \begin{smallmatrix}2\\|\\3\\|\\1\end{smallmatrix}, \begin{smallmatrix}3\\|\\1\\|\\2\end{smallmatrix}, \begin{smallmatrix}1\\|\\3\\|\\2\end{smallmatrix}, \begin{smallmatrix}2\\|\\1\\|\\3\end{smallmatrix}, \begin{smallmatrix}1\\|\\2\\|\\3\end{smallmatrix}, \begin{smallmatrix}2&&3\\&\vee&\\&1&\end{smallmatrix}, \begin{smallmatrix}1&&3\\&\vee&\\&2&\end{smallmatrix}, \begin{smallmatrix}1&&2\\&\vee&\\&3&\end{smallmatrix}, \begin{smallmatrix}&1&\\&\wedge&\\2&&3\end{smallmatrix}, \begin{smallmatrix}&2&\\&\wedge&\\1&&3\end{smallmatrix}, \begin{smallmatrix}&3&\\&\wedge&\\1&&2\end{smallmatrix}, \begin{smallmatrix}&3\\&|\\1&2\end{smallmatrix}, \begin{smallmatrix}&2\\&|\\1&3\end{smallmatrix}, \begin{smallmatrix}&3\\&|\\2&1\end{smallmatrix}, \begin{smallmatrix}&1\\&|\\2&3\end{smallmatrix}, \begin{smallmatrix}&2\\&|\\3&1\end{smallmatrix}, \begin{smallmatrix}&1\\&|\\3&2\end{smallmatrix}, 1\,2\,3 \right\}.$$

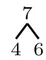
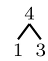
Let $\mathcal{P} = \biguplus_{n\geq 0} \mathcal{P}_n$ be the disjoint union of $\mathcal{P}_n$, and $\mathbb{K}\mathcal{P} = \bigoplus_{n\geq 0} \mathbb{K}\mathcal{P}_n$, where $\mathbb{K}\mathcal{P}_n$ is the linear space spanned by $\mathcal{P}_n$ over field $\mathbb{K}$.

For a positive integer n, define $\left(P^{\uparrow n}, \leq_{P^{\uparrow n}}\right)$ as a poset by increasing each element in P by n satisfying

$$(x+n) \leq_{P^{\uparrow n}} (y+n) \Leftrightarrow x \leq_P y,$$

for x, y in P. Similarly, define $\left(P^{\downarrow n}, \leq_{P^{\downarrow n}}\right)$ as a poset by reducing each element in P by n satisfying

$$(x-n) \leq_{P^{\downarrow n}} (y-n) \Leftrightarrow x \leq_P y,$$

for x, y in P. For example, let $P =$ , then $P^{\uparrow 2} =$  and $P^{\downarrow 1} =$ .

For a poset P in $\mathcal{P}_n$, we call n a *global split* of P if

$$P = P_{[i]} < P_{[n]\setminus[i]},$$

where $0 \leq i \leq n$. If a nonempty poset has no global splits except 0 and n, we call it an *indecomposable* poset. We denote $\bar{\mathcal{P}}_n$ as the subset of $\mathcal{P}_n$ containing all indecomposable posets in $\mathcal{P}_n$, and $\bar{\mathcal{P}} = \biguplus_{n\geq 1} \bar{\mathcal{P}}_n$.

For a nonempty set of intergers $S = \{s_1, s_2, \cdots, s_n\}$ with $s_1 < s_2 < \cdots < s_n$, denote st_S as a mapping from S to $[n]$ by $\mathrm{st}_S(s_i) = i$ for each $1 \leq i \leq n$.

Let $(P, \leq_P)$ be a poset, where P is an interger set. We define $\mathrm{st}(P, \leq_P) = \left(\mathrm{st}(P), \leq_{\mathrm{st}(P)}\right)$, where $\mathrm{st}(P) = \left[|P|\right]$ and $\leq_{\mathrm{st}(P)}$ is the partial order on $\left[|P|\right]$ satisfying

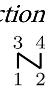
$$\mathrm{st}_P(i) \leq_{\mathrm{st}(P)} \mathrm{st}_P(j) \Leftrightarrow i \leq_P j,$$

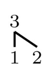
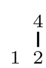
for any i, j in P. For convenience, we denote the $\mathrm{st}(P, \leq_P)$ as $\mathrm{st}(P)$ when the partial order $\leq_P$ is obvious. We call $\mathrm{st}(P)$ the *standard form* of poset P.

For example, $\mathrm{st}\left(\right.$  $\left.\right) =$ . Obviously, $\mathrm{st}(\varepsilon) = \varepsilon$.

Let K be a subset of P. Define $(P, \leq_P)_K$ as poset $(K, \leq_K)$, where $\leq_K$ is the partial order on K satisfying

$$k_1 \leq_K k_2 \Leftrightarrow k_1 \leq_P k_2,$$

for any k_1, k_2 in K. For convenience, we denote $(P, \leq_P)_K$ by P_K when the partial order $\leq_P$ is obvious, and call P_K the *restriction* of P on K, and P_K a *subposet* of poset P. For example, let poset $P =$ , then

$P_{\{1,2,3\}} =$ , $P_{\{1,2,4\}} =$ , $P_{\{1,3\}} =$.

Obviously, $P_{\{1,2,3\}}$, $P_{\{1,2,4\}}$ and $P_{\{1,3\}}$ are subposets of poset P.

In particular, $P_{[n]} = P$ and $P_{\varnothing} = \epsilon$, for P in $\mathcal{P}_n$.

3. Conjunction Product and Unshuffle Coproduct

In 2020, Aval, Bergrron and Machacek [20] defined a product and coproduct on posets, which are called the conjunction product and the unshuffle coproduct, respectively, without proofs. Here, we prove that the vector space spanned by posets with these operations is an algebra and a coalgebra.

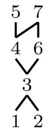
Define the *conjunction product* $*$ on $\mathbb{K}\mathcal{P}$ by

$$P * Q = P < Q^{\uparrow m},$$

for P in $\mathcal{P}_m$ and Q in $\mathcal{P}_n$. Obviously, $P*Q$ in $\mathcal{P}_{m+n}$, and the conjunction product $*$ is not commutative.

Define the *unit* $\mu : \mathbb{K} \to \mathbb{K}\mathcal{P}$ by $\mu(1) = \epsilon$.

Example 1. Let $P =$  and $Q =$ , then

$$P * Q = P < Q^{\uparrow 3} =$$  .

Theorem 1. $(\mathbb{K}\mathcal{P}, *, \mu)$ is a graded algebra.

Proof It is easy to verify that μ is a unit.

Let P_i be in $\mathcal{P}$ with $|P_i| = n_i$, $1 \le i \le 3$. By the definition of conjunction product $*$, we have $P_1 * P_2 = P_1 < P_2^{\uparrow n_1}$ and $P_2 * P_3 = P_2 < P_3^{\uparrow n_2}$. Obviously, $P_1 * P_2$ is in $\mathcal{P}_{n_1+n_2}$ and $P_2 * P_3$ is in $\mathcal{P}_{n_2+n_3}$. Furthermore,

$$(P_1 * P_2) * P_3 = \left(P_1 < P_2^{\uparrow n_1}\right) < P_3^{\uparrow n_1+n_2}$$

and

$$P_1 * (P_2 * P_3) = P_1 < \left(P_2 < P_3^{\uparrow n_2}\right)^{\uparrow n_1}$$

are both in $\mathcal{P}_{n_1+n_2+n_3}$. We have

$$\begin{aligned} P_1 * (P_2 * P_3) &= P_1 < \left(P_2 < P_3^{\uparrow n_2}\right)^{\uparrow n_1} \\ &= P_1 < \left(P_2^{\uparrow n_1} < P_3^{\uparrow n_1+n_2}\right) \\ &= \left(P_1 < P_2^{\uparrow n_1}\right) < P_3^{\uparrow n_1+n_2} \\ &= (P_1 * P_2) * P_3. \end{aligned}$$

From above, $*$ satisfies the associative law. Hence, $(\mathbb{K}\mathcal{P}, *, \mu)$ is an algebra.

From the definitions of conjunction product $*$ and unit μ, we have $\mathbb{K}\mathcal{P}_m * \mathbb{K}\mathcal{P}_n \subseteq \mathbb{K}\mathcal{P}_{m+n}$ and $\mu(\mathbb{K}) = \mathbb{K}\mathcal{P}_0$. Hence, the algebra $(\mathbb{K}\mathcal{P}, *, \mu)$ is graded. □

Lemma 1. Define a linear mapping $h : T(\mathbb{K}\overline{\mathcal{P}}) \to \mathbb{K}\mathcal{P}$ by

$$h\left(\bigoplus_{k=0}^{n} x_{k1} \otimes \cdots \otimes x_{kk}\right) = \sum_{k=0}^{n} x_{k1} * \cdots * x_{kk}, \tag{1}$$

where $x_{00} \in (\mathbb{K}\overline{\mathcal{P}})^{\otimes 0} = \mathbb{K}$, $x_{ki} \in \mathbb{K}\overline{\mathcal{P}}$ for $i = 1, \cdots, k$ and $h(1_{\mathbb{K}}) = \epsilon$. Denote

$$\overline{\mathcal{P}}^{\otimes n} = \left\{x_1 \otimes x_2 \otimes \cdots \otimes x_n \mid x_1, x_2, \cdots, x_n \in \overline{\mathcal{P}}\right\}.$$

For any

$$x = p_{11} \otimes \cdots \otimes p_{1m} \in \overline{\mathcal{P}}^{\otimes m}, \tag{2}$$

and

$$y = p_{21} \otimes \cdots \otimes p_{2n} \in \overline{\mathcal{P}}^{\otimes n}, \tag{3}$$

if $x \neq y$, then $h(x) \neq h(y)$.

Proof If $x \neq y$, then there exists z such that $p_{1i} = p_{2i}$ for $1 \leq i \leq z-1$ but $p_{1z} \neq p_{2z}$. We denote $r = \sum_{i=1}^{z-1} |p_{1i}|$, $h(x) = P_1$ and $h(y) = P_2$. If $|p_{1z}| = |p_{2z}|$, then

$$\mathrm{st}\left((P_1)_{[r+1, r+|p_z|]}\right) = p_{1z} \neq p_{2z} = \mathrm{st}\left((P_2)_{[r+1, r+|p_z|]}\right).$$

So $h(x) \neq h(y)$. If $|p_{1z}| < |p_{2z}|$, then $r + |p_{1z}|$ is a split of $h(x)$ but not a split of $h(y)$. So $h(x) \neq h(y)$. Similarly, if $|p_{1z}| > |p_{2z}|$, we also have $h(x) \neq h(y)$. □

Theorem 2. The algebra $(\mathbb{K}\mathcal{P}, *, \mu)$ is free on $\mathbb{K}\overline{\mathcal{P}}$.

Proof It is sufficient to prove $(\mathbb{K}\mathcal{P}, *, \mu)$ is isomorphic to the tensor algebra $T(\mathbb{K}\overline{\mathcal{P}})$ through the mapping h in (1). Obviously, h is an algebra homomorphism. For any nonempty P in $\mathcal{P}$, let $0 = i_0 < i_1 < \cdots < i_t = n$ be all splits of P, then

$$h\left(\mathrm{st}\left(P_{[i_0+1, i_1]}\right) \otimes \cdots \otimes \mathrm{st}\left(P_{[i_{t-1}+1, i_t]}\right)\right) = P.$$

Hence, h is surjective.

For $x = x_{00} \oplus x_{11} \oplus x_{21} \otimes x_{22} \oplus \cdots \oplus x_{n1} \otimes x_{n2} \otimes \cdots \otimes x_{nn}$ in $T(\mathbb{K}\overline{\mathcal{P}})$, where $x_{00} \in (\mathbb{K}\overline{\mathcal{P}})^{\otimes 0} = \mathbb{K}$ and $x_{ki} \in \mathbb{K}\overline{\mathcal{P}}$ for $i = 1, \cdots, k$, $1 \leq k \leq n$, suppose

$$h(x) = 0, \tag{4}$$

i.e.,

$$\begin{aligned} & h\left(x_{00} \oplus x_{11} \oplus x_{21} \otimes x_{22} \oplus \cdots \oplus x_{n1} \otimes x_{n2} \otimes \cdots \otimes x_{nn}\right) \\ & = h(x_{00}) + h(x_{11}) + h(x_{21} \otimes x_{22}) + \cdots + h(x_{n1} \otimes \cdots \otimes x_{nn}) \\ & = 0. \end{aligned} \tag{5}$$

Obviously, any two terms in (5) are linearly independent because they have different numbers of splits. It means $h(x_{00}) = 0$ and

$$h(x_{k1} \otimes \cdots \otimes x_{kk}) = 0, \tag{6}$$

for all $1 \leq k \leq n$. By the associative law of tensor product, we have

$$x_{k1} \otimes \cdots \otimes x_{kk} = \sum_{j=1}^{m} l_j x_j,$$

for some $x_j \in \overline{\mathcal{P}}^{\otimes k}$, $l_j \in \mathbb{K}$ and $m > 0$, where $x_i \neq x_j$ for $i \neq j$. Then

$$h(x_{k1} \otimes \cdots \otimes x_{kk}) = h\left(\sum_{j=1}^{m} l_j x_j\right) = \sum_{j=1}^{m} l_j h(x_j). \tag{7}$$

By Lemma 1, $\{h(x_j)\}_{j=1}^{m}$ are linear independent. So $l_j = 0$ for all $1 \leq j \leq m$ from (6) and (7). Then

$$x_{k1} \otimes \cdots \otimes x_{kk} = \sum_{j=1}^{m} l_j x_j = 0,$$

for all $1 \leq k \leq m$. Hence, $x = 0$ in (4), *i.e.*, h is injective. Then $\mathbb{K}\mathcal{P} \cong T(\mathbb{K}\overline{\mathcal{P}})$ is a free algebra on $\mathbb{K}\overline{\mathcal{P}}$. □

Define the *unshuffle coproduct* Δ on $\mathbb{K}\mathcal{P}$ by

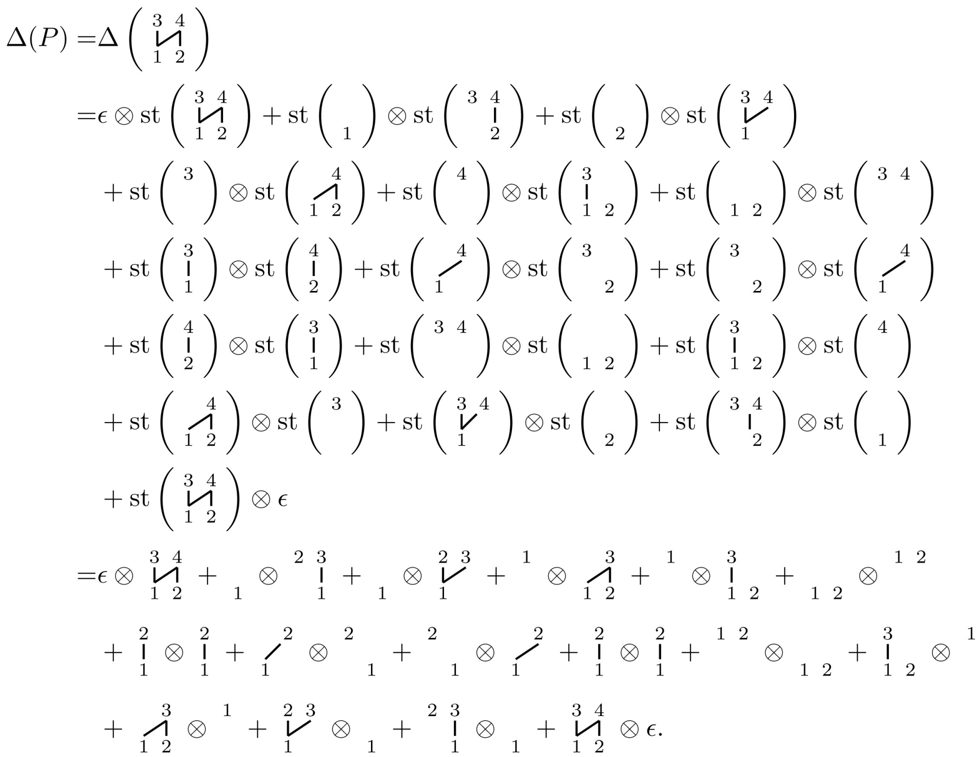
$$\Delta(P)=\sum_{I\subseteq[n]}\mathrm{st}(P_I)\otimes\mathrm{st}\left(P_{[n]\setminus I}\right),$$

where I traverses all subsets of $[n]$, for any non-empty poset P in $\mathcal{P}_n$ and $\Delta(\epsilon)=\epsilon\otimes\epsilon$. Obviously, the unshuffle coproduct Δ is cocommutive. Define the *counit* $\nu:\mathbb{K}\mathcal{P}\to\mathbb{K}$ by

$$\nu(P)=\begin{cases}1, & P=\epsilon,\\ 0, & \text{otherwise,}\end{cases}$$

for P in $\mathcal{P}$.

Example 2. Let $P=$ , then



Theorem 3. $(\mathbb{K}\mathcal{P},\Delta,\nu)$ is a graded coalgebra.

Proof It is easy to verify that ν is a counit. For P in $\mathcal{P}_n$,

$$\Delta(P)=\sum_{I\subseteq[n]}\mathrm{st}(P_I)\otimes\mathrm{st}\left(P_{[n]\setminus I}\right).$$

We have

$$\begin{aligned}(\mathrm{id}\otimes\Delta)\circ\Delta(P)&=(\mathrm{id}\otimes\Delta)\circ\left(\sum_{I\subseteq[n]}\mathrm{st}(P_I)\otimes\mathrm{st}\left(P_{[n]\setminus I}\right)\right)\\&=\sum_{I\subseteq[n]}\mathrm{st}(P_I)\otimes\Delta\left(\mathrm{st}\left(P_{[n]\setminus I}\right)\right)\\&=\sum_{I\subseteq[n]}\mathrm{st}(P_I)\otimes\left(\sum_{J\subseteq[n]\setminus I}\mathrm{st}(P_J)\otimes\mathrm{st}\left(P_{([n]\setminus I)\setminus J}\right)\right)\end{aligned}$$

and

$$\begin{aligned}(\Delta\otimes\mathrm{id})\circ\Delta(P) &= (\Delta\otimes\mathrm{id})\circ\left(\sum_{I\subseteq[n]}\mathrm{st}(P_I)\otimes\mathrm{st}\left(P_{[n]\setminus I}\right)\right)\\ &= \sum_{I\subseteq[n]}\Delta\left(\mathrm{st}(P_I)\right)\otimes\mathrm{st}\left(P_{[n]\setminus I}\right)\\ &= \sum_{I\subseteq[n]}\left(\sum_{A\subseteq I}\mathrm{st}(P_A)\otimes\mathrm{st}(P_{I\setminus A})\right)\otimes\mathrm{st}\left(P_{[n]\setminus I}\right)\\ &= \sum_{A\subseteq I}\sum_{I\subseteq[n]}\mathrm{st}(P_A)\otimes\mathrm{st}(P_{I\setminus A})\otimes\mathrm{st}\left(P_{[n]\setminus I}\right)\\ &= \sum_{A\subseteq[n]}\mathrm{st}(P_A)\otimes\left(\sum_{B\subseteq[n]\setminus A}\mathrm{st}(P_B)\otimes\mathrm{st}\left(P_{([n]\setminus A)\setminus B}\right)\right).\end{aligned}$$

Therefore

$$(\mathrm{id}\otimes\Delta)\circ\Delta(P) = (\Delta\otimes\mathrm{id})\circ\Delta(P).$$

From above, Δ satisfies the coassociative law. Hence, $(\mathbb{K}\mathcal{P},\Delta,\nu)$ is a coalgebra.

From the definitions of Δ and ν, we have $\Delta(\mathbb{K}\mathcal{P}_n)\subseteq\bigoplus_{i=0}^{n}(\mathbb{K}\mathcal{P}_i\otimes\mathbb{K}\mathcal{P}_{n-i})$ and $\nu(\mathbb{K}\mathcal{P}_0)=\mathbb{K}$. Hence, the coalgebra $(\mathbb{K}\mathcal{P},\Delta,\nu)$ is graded. $\square$

4. Conclusions

Let $\mathbb{K}\mathcal{P}$ be the vector space spanned by posets. Firstly, we give the definitions of conjunction product $*$ and unshuffle coproduct Δ on $\mathbb{K}\mathcal{P}$. Then we prove that the conjunction product $*$ satisfies the associativity. So $(\mathbb{K}\mathcal{P},*,\mu)$ is an algebra. Futhermore, we prove that $(\mathbb{K}\mathcal{P},*,\mu)$ is graded and free on $\mathbb{K}\overline{\mathcal{P}}$, where $\overline{\mathcal{P}}$ contains all indecomposable posets in $\mathcal{P}$. Finally, we prove that unshuffle coproduct Δ satisfies the coassociativity and $(\mathbb{K}\mathcal{P},\Delta,\nu)$ is a graded coalgebra.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (Nos. 11701339, 12071265 and 11771256).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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