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# Discussion on the Complex Structure of Nilpotent Lie Groups $G_k$

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## Abstract

Consider the real, simply-connected, connected, s-step nilpotent Lie group $G$ endowed with a left-invariant, integrable almost complex structure $J$, which is nilpotent. Consider the simply-connected, connected nilpotent Lie group $G_k$, defined by the nilpotent Lie algebra $g/a_k$, where $g$ is the Lie algebra of $G$, and $a_k$ is an ideal of $g$. Then, $J$ gives rise to an almost complex structure $J_k$ on $G_k$. The main conclusion obtained is as follows: if the almost complex structure $J$ of a nilpotent Lie group $G$ is nilpotent, then $J$ can give rise to a left-invariant integrable almost complex structure $J_k$ on the nilpotent Lie group $G_k$, and $J_k$ is also nilpotent.

## Keywords

Almost Complex Structure, Nilpotent Lie Group, Nilpotent Lie Algebra

## 1. Introduction

In the year 2000, Cordero and others [1] conducted research on nilpotent complex structures on connected simply connected real even-dimensional nilpotent Lie groups $G$ with left-invariant integrable almost complex structures. They provided definitions for an ascending sequence $\{a_k, k \geq 0\}$ compatible with the integrable almost complex structure $J$ of $G$, as well as the definition of nilpotent complex structure. Building upon Cordero *et al.*'s research on nilpotent complex structures, this paper demonstrates that if the left-invariant integrable almost complex structure $J$ on the Lie group $G$ is nilpotent, then $J$ can induce a left-invariant integrable almost complex structure $J_k$ on $G_k$, and $J_k$ is also nilpotent. The study of nilpotent complex structures on the nilpotent Lie group $G_k$ can further investigate topics such as spectral sequences, Dolbeault cohomology groups, and minimal models of compact nilpotent manifolds discussed in references [2] [3]

[4].

The aim of this paper is to investigate the scenario of a connected simply-connected s-step nilpotent Lie group $G$ with a left-invariant integrable almost complex structure $J$, where $J$ is nilpotent. Through the examination of the connected simply-connected nilpotent Lie group $G_k$ defined by the nilpotent Lie algebra $g/a_k$, the objective is to ascertain whether $J$ can induce an almost complex structure $J_k$ on $G_k$, and further demonstrate that $J_k$ is also nilpotent.

In addressing this issue, the paper is divided into two parts. The first part serves as background knowledge, introducing fundamental concepts related to connected simply connected s-step nilpotent Lie groups $G$ with left-invariant integrable almost complex structures. The second part provides evidence that if the left-invariant integrable almost complex structure $J$ is nilpotent, then $J$ can induce a left-invariant integrable almost complex structure $J_k$ on $G_k$, and $J_k$ is also nilpotent.

## 2. Background Knowledge

### 2.1. Integrable Complex Structure

Let $V$ connected simply connected 2n-dimensional real vector space. The so-called complex structure $J$ on $V$ is a linear transformation $J:V\to V$, satisfying:

$$J^2=-\mathrm{id}:V\to V.$$

Let $M$ be a 2n-dimensional smooth manifold, and $J$ be a smooth (1,1)-type tensor field on $M$. For each point $x\in M$, $J_k$ a linear transformation from the tangent space $T_xM$ to itself. If each $J_k(x)(x\in M)$ is a complex structure on the tangent space $T_xM$, then the tensor field $J$ is called a almost complex structure on $M$. The smoothness of the tensor field $J$ implies that if $X$ is a smooth tangent vector field on $M$, then $JX$ is also a smooth tangent vector field on $M$.

Let $G$ be a Lie group with a left-invariant almost complex structure, $g=LieG$. Then, we can define a linear map $J:g\to g$ and $J^2=-\mathrm{id}$. $J$ is called a complex structure on $g$. If $J$ satisfies:

$$[JX,JY]=[X,Y]+J[JX,Y]+J[X,JY]\ \text{ for any }\ (X,Y\in g), \tag{1}$$

then $J$ is integrable. Without distinction, the left-invariant integrable almost complex structure on $G$ and the integrable complex structure on $g$ are both denoted by $J$.

### 2.2. On Sequences of Nilpotent Lie Algebras

Let $g$ be a Lie algebra. Suppose

$$g^0=g,g^1=\left[g^0,g\right],\cdots,g^l=\left[g^{l-1},g\right],\cdots \tag{2}$$

It can be easily proven that $g^i$ is an ideal of $g$, and $g^i\subseteq g^{i-1}$. The sequence $\left\{g^k,k\geq 0\right\}$ is called the descending central series of $g$. If there exists an $s\in N$

such that $g^s = \{0\}$ and $g^{s-1} \neq \{0\}$, then $g$ is called an $s$-step nilpotent Lie algebra [5] [6].

Let $G$ be a 2n-dimensional real nilpotent Lie group with a left-invariant integrable almost complex structure, $g = LieG$, and $g^*$ be the dual space of $g$. Let $\{w_1, w_2, \cdots, w_n\}$ denote a complex basis, and $\{w_1, \overline{w}_1, w_2, \overline{w}_2, \cdots, w_n, \overline{w}_n\}$ denote a corresponding real basis. Therefore,

$$dw_i = \sum_{j<k} A_{ijk} w_j \wedge w_k + \sum_{j,k} B_{ijk} w_j \wedge \overline{w}_k + \sum_{j<k} C_{ijk} \overline{w}_j \wedge \overline{w}_k \left(1 \le i \le n\right), \quad (3)$$

because $d\alpha(X,Y) = -\alpha([X,Y]) \left(X, Y \in g^C, \alpha \in \left(g^C\right)^*\right)$, we use the exterior derivative on $g^*$ to describe the Lie bracket on $g$.

Property 1 [7]. Let $G$ be a real nilpotent Lie group, $g = LieG$. $G$ has a left-invariant integrable almost complex structure if and only if $C_{ijk} = 0$, *i.e.*

$$dw_i = \sum_{j<k\le n} A_{ijk} w_j \wedge w_k + \sum_{j,k\le n} B_{ijk} w_j \wedge \overline{w}_k \quad (1 \le i \le n). \quad (4)$$

The structure equation can define connected and simply connected nilpotent Lie groups left-invariant integrable almost complex structure, so we can study some properties of Lie groups through this structure equation.

Definition 1 [8]. Let $G$ be a connected simply connected s-step nilpotent Lie group, $g = LieG$. Define a sequence in $g$ as

$$g_0 = 0, g_1 = \{X \in g \mid [X, g] \subseteq g_0\}, \cdots, g_k = \{X \in g \mid [X, g] \subseteq g_{k-1}\}, \cdots. \quad (5)$$

Then $g_1$ is the center of $g$, where $g_1 \neq \{0\}$. $g_s = g$, $g_k \subseteq g_{k+1}$, and for any $k(0 \le k \le s-1)$ chosen such that $\dim g_k < \dim g_{k+1}$. Thus, there exists an ascending central series

$$g_0 = 0 \subset g_1 \subset g_2 \subset \cdots \subset g_{s-1} \subset g_s = g. \quad (6)$$

Property 2.If the sequence $\{g_k, k \ge 0\}$ satisfies Equation (5), then:

a) $g_1$ is the center of $g$ and $g_1 \neq 0$;

b) For any $k \ge 0$, $g_k \subseteq g_{k+1}$;

c) For any $k(0 \le k \le s-1)$ such that $\dim g_k < \dim g_{k+1}$ $(0 \le k \le s-1)$. We have $g_0 = 0 \subset g_1 \subset g_2 \subset \cdots \subset g_{s-1} \subset g_s = g$.

Proof. a) Since $g_0 = \{0\}$, then $g_1 = \{X \in g \mid [X, g] = 0\} = C(g)$. Moreover, since $g$ is a nilpotent Lie algebra, $C(g) \neq 0$, hence $g_1$ is the center of $g$ and $g_0 \subset g_1$.

b) We use induction to prove $g_k \subseteq g_{k+1}$ $(k \ge 0)$.

For $k = 0$, by a), we have $g_0 \subset g_1$. Assume that when $k = i$ holds, $g_i \subseteq g_{i+1}$. We'll prove that for $k = i+1$, $g_{i+1} \subseteq g_{i+2}$. For any $x_1 \in g_{i+1}$ and $x \in g$, we have $[x_1, x] \in g_i \subseteq g_{i+1}$. According to Equation (5), $x_1 \in g_{i+2}$, thus $g_{i+1} \subseteq g_{i+2}$, hence $g_k \subseteq g_{k+1}$ $(k \ge 0)$.

c) First, we'll prove that there exists an integer $s$ such that $g_s = g$.

Since $g$ is an s-step nilpotent Lie algebra, there exists a descending central series

$$g^0 = g \supset g^1 = \left[g, g^0\right] \supset \cdots \supset g^{k+1} = \left[g, g^k\right] \supset \cdots g^s = \{0\}.$$

Next, we'll use induction to prove $g^{s-i} \subseteq g_i$. When $k=0$, we have $g^s = \{0\} = g_0$. Assuming $k=i$ holds, $g^{s-i} \subseteq g_i$, we'll prove that for $k=i+1$, $g^{s-i-1} \subseteq g_{i+1}$. According to Equation (2), we have $g^{s-i} = \left[g, g^{s-i-1}\right]$. Also, since $g^{s-i} \subseteq g_i$, then $g^{s-i-1} \subseteq g_{i+1}$. Thus, for any $i(0 \le i \le s)$ such that $g^{s-i} \subseteq g_i$, we have $g_s \supseteq g^0 = g$, and since $g_s \subseteq g$, we conclude that $g_s = g$.

Next, we prove that $g_0 = 0 \subset g_1 \subset g_2 \subset \cdots \subset g_{s-1} \subset g_s = g$, namely, for any $k(0 \le k \le s-1)$ such that $\dim g_k < \dim g_{k+1}$, then $g_k \subset g_{k+1}$ is strict.

When $k=0$, by conclusion (a), we have $g_0 \subset g_1$. Assuming for $k=i$ holds, $g_i \subset g_{i+1}$, we'll prove that for $k=i+1$, $g_{i+1} \subset g_{i+2}$. According to Equation (5), we have $g_{i+1} = \{X \in g \mid [X,g] \subseteq g_i\}$, $g_{i+2} = \{X \in g \mid [X,g] \subseteq g_{i+1}\}$, thus $g_{i+1} \subset g_{i+2}$. Also, since $g_s = g$, then for any $k(0 \le k \le s-1)$ such that $\dim g_k < \dim g_{k+1}$, we have $g_k \subset g_{k+1}$ is strict. Thus, we have

$$g_0 = 0 \subset g_1 \subset g_2 \subset \cdots \subset g_{s-1} \subset g_s = g$$

holds.

Before introducing the nilpotent complex structure on *G*, let's first discuss under what conditions *g* is a complex Lie algebra. Let $g^C$ denote the complexification of *g*, and let *J* be the complex structure on the Lie algebra *g*. Then we have $g^C = g^- \oplus g^+$, where $\pm i$ are the eigenvalues of *J*, and $g^- = \{X + iJX \mid X \in g\}$ and $g^+ = \{X - iJX \mid X \in g\}$ are the eigenspaces of *J*.

Property 3 [9]. The eigenspaces $g^\pm$ of *J* are ideals of $g^C$.

Theorem 1. Let *J* be the integrable complex structure on the Lie algebra *g*. If $J[X,Y] = [JX,Y]$ for all $X, Y \in g$, then *g* is a complex Lie algebra.

Proof: Let $g^+ = \{X - iJX \mid X \in g\}$. According to Property 3, we have $[X,Y] \in g^+$ for all $X, Y \in g^+$.

Let $g^- = \{X + iJX \mid X \in g\}$. According to Property 3, we have $[X,Y] \in g^-$ for all $X, Y \in g^-$.

Since $g \subset g^C = g^- \oplus g^+$, any $X, Y \in g$ satisfies $X = X_1 + X_2$ ($X_1 \in g^-$, $X_2 \in g^+$), $Y = Y_1 + Y_2$ ($Y_1 \in g^-$, $Y_2 \in g^+$).

According to Property 3, we have $[X_1, Y_2] \in g^+$ and $[X_1, Y_2] \in g^-$. Since $g^+ \cap g^- = \{0\}$, it follows that $[X_1, Y_2] = 0$ and $[X_2, Y_1] = 0$.

To prove that *g* is a complex Lie algebra, we need to show that the Lie bracket of *g* is *C*-linear, *i.e.*, $[(a+ib)X, Y] = (a+ib)[X,Y]$ ($X, Y \in g$).

$$\begin{aligned}
[(a+ib)X,Y] &= [(a+ib)(X_1+X_2), Y_1+Y_2] \\
&= [a(X_1+X_2), Y_1+Y_2] + [ib(X_1+X_2), Y_1+Y_2] \\
&= a[X_1+X_2, Y_1+Y_2] + b[i(X_1+X_2), Y_1+Y_2] \\
&= a[X_1+X_2, Y_1+Y_2] + b[JX_1 - JX_2, Y_1+Y_2] \\
&= a[X_1+X_2, Y_1+Y_2] + b[JX_1, Y_1+Y_2] - b[JX_2, Y_1+Y_2]
\end{aligned}$$

$$\begin{aligned}
&= a\left[X_1+X_2,Y_1+Y_2\right]+bJ\left[X_1,Y_1+Y_2\right]-bJ\left[X_2,Y_1+Y_2\right]\\
&= a\left[X_1+X_2,Y_1+Y_2\right]+bJ\left[X_1,Y_1\right]-bJ\left[X_2,Y_2\right]\\
&= a\left[X_1+X_2,Y_1+Y_2\right]+ib\left[X_1,Y_1\right]+ib\left[X_2,Y_2\right]\\
&= (a+ib)\left[X_1+X_2,Y_1+Y_2\right]
\end{aligned}$$

Therefore, the Lie algebra is $C$-linear, proving that $g$ is a complex Lie algebra.

Suppose $G$ is a connected simply connected nilpotent Lie group, $g = LieG$, and $g$ has a complex structure $J$. Then $g$ is a complex vector space, but generally not a complex Lie algebra. According to Theorem 1, if the complex structure satisfies $J[X,Y]=[JX,Y]$ for all $X,Y\in g$, then $g$ is a complex Lie algebra. To study the nilpotent complex structure of the Lie group $G$, we introduce the ascending sequence $\{a_k, k\ge 0\}$ related to the nilpotent Lie algebra $g$.

Definition 2 [10]. Let $G$ be a connected simply connected s-step nilpotent Lie group, and suppose it has a left-invariant integrable almost complex structure. $g = LieG$ and $J$ is the complex structure on the Lie algebra $g$. $g$ has the sequence

$$a_0=\{0\},\cdots,a_k=\left\{X\in g\,|\,[X,g]\subseteq a_{k-1},[JX,g]\subseteq a_{k-1}\right\},\cdots, \tag{7}$$

called the compatible with the integrable almost complex structure $J$ of $G$.

Property 4. If $\{a_k, k\ge 0\}$ is the compatible with the integrable almost complex structure $J$ of $G$, then $a_k$ is an ideal of $g$, $a_k\subseteq a_{k+1}$, and $a_k\subseteq g_k$ $(k\ge 0)$.

Proof: We use induction to prove $a_k\subseteq a_{k+1}$.

When $k=0$, because $[a_1,g]\subseteq a_0=\{0\}$, then $[a_1,g]\subseteq a_0\subseteq a_1$. Suppose that when $k=i$, $a_i\subseteq a_{i+1}$ holds, we need to prove that when $k=i+1$, $a_{i+1}\subseteq a_{i+2}$. According to Equation (7), we have $a_{i+2}=\left\{X\in g\,|\,[X,g]\subseteq a_{i+1},[JX,g]\subseteq a_{i+1}\right\}$, $a_{i+1}=\left\{X\in g\,|\,[X,g]\subseteq a_i,[JX,g]\subseteq a_i\right\}$. Since $a_i\subseteq a_{i+1}$ holds, $a_{i+1}\subseteq a_{i+2}$, hence $a_k\subseteq a_{k+1}$ $(k\ge 0)$.

To prove that $a_k$ is an ideal of $g$, because $[a_k,g]\subseteq a_{k-1}\subseteq a_k$, $a_k$ is an ideal of $g$. By induction, we prove $a_k\subseteq g_k$ $(k\ge 0)$.

When $k=0$, since $a_0=0$, $g_0=0$, then $a_0\subseteq g_0$. Suppose that when $k=i$, $a_i\subseteq g_i$ holds, we need to prove that when $k=i+1$, $a_{i+1}\subseteq g_{i+1}$.

According to Equations (5) and (7), we have

$$a_{i+1}=\left\{X\in g\,|\,[X,g]\subseteq a_i,[JX,g]\subseteq a_i\right\},\quad g_{i+1}=\left\{X\in g\,|\,[X,g]\subseteq g_i\right\}.$$

Since $a_i\subseteq g_i$ holds, $a_{i+1}\subseteq g_{i+1}$, which completes the proof.

Lemma 1 [1]. If there exists $k\ge 0$ such that $a_k=a_{k+1}$, then for any $r\ge k$, $a_r=a_k$.

Now let's look at some properties between the ascending sequence $\{a_k,k\ge 0\}$ and the ascending central sequence $\{g_k,k\ge 0\}$ of $g$.

Lemma 2 [1]. If $\{a_k,k\ge 0\}$ and $\{g_k,k\ge 0\}$ are respectively the ascending sequences of $g$, then the following three conclusions hold.

i) If there exists $k\ge 0$ such that $a_k=g_k$, then $g_k=J(g_k)$;

ii) If there exists $k>0$ such that $a_{k-1}=g_{k-1}$, then $a_k=g_k$ if and only if

$g_k = J(g_k)$;

iii) If $a_{k-1} = g_{k-1}$, then $a_k$ is the largest *J*-invariant subspace of $g_k$.

Under the conditions of Lemma 2 and according to Equations (5), (7), we know $a_0 = g_0 = 0$, then

a) If there exists $k \ge 0$ such that $J(g_k) \not\subset g_k$, then $a_k \subset g_k$ is strict;

b) If there exists an integer *s* such that $a_{s-1} = g_{s-1}$, then $a_s = g_s = g$;

c) $a_1$ is the largest *J*-invariant subspace of $g_1$;

d) If any term $g_k$ of $\{g_k, k \ge 0\}$ is *J*-invariant, then for any $k \ge 0$, $a_k = g_k$ and $a_s = g_s = g$.

Next, consider some related properties between the ascending sequence $\{a_k, k \ge 0\}$ and the descending central sequence $\{g^l, l \ge 0\}$ of *g*. Under the conditions of Lemma 2, suppose $\{g^l, l \ge 0\}$ is the descending central sequence of *g*. Then we have

① If for some $k \ge 0$ and some $l \ge 0$, $g^l \subset a_k$ and $J(g^{l-1}) = g^{l-1}$, then $g^{l-1} \subset a_{k+1}$;

② If for some $k \ge 0$, $[g, g] \subseteq a_k$, then $a_{k+1} = g$;

③ If any $g^l \in \{g^l, l \ge 0\}$ is *J*-invariant, then $a_s = g_s = g$ [1].

### 2.3. Nilpotent Complex Structure

Definition 3 [11]. Let *G* be a connected simply connected s-step nilpotent Lie group, and suppose it has a left-invariant integrable almost complex structure. $g = LieG$ and *J* is the complex structure on the Lie algebra *g*. If there exists $t > 0$ such that $a_t = g$, then the left-invariant integrable almost complex structure *J* is called a nilpotent left-invariant complex structure.

Lemma 3 [1]. Let $\{w_i, 1 \le i \le n\}$ be a (1,0)-type left-invariant form complex basis for $g^*$, satisfying the structural equation

$$\sum_{j<k<i} A_{ijk} w_j \wedge w_k + \sum_{j,k<i} B_{ijk} w_j \wedge \overline{w}_k \quad (1 \le i \le n).$$

If $\{Z_i, \overline{Z}_i, 1 \le i \le n\}$ is a basis for *g* and dual to the basis $\{w_i, \overline{w}_i, 1 \le i \le n\}$, let $X_i = \mathrm{Re}(Z_i)$, $Y_i = \mathrm{Im}(Z_i)$ $(1 \le i \le n)$, then any term $a_l$ $(1 \le l \le n)$ in the ascending sequence $\{a_l, l \ge 0\}$ contains at least generators $X_{n-l+1}, Y_{n-l+1}, \cdots, X_n, Y_n$.

Proposition 1. Under the conditions of Lemm3, we have

(i) If $a_k$ is a member of the sequence $\{a_k, k \ge 0\}$, and $a_k \ne g$, then $\dim a_{k+1} \ge 2 + \dim a_k$;

(ii) If $\{g_k, k \ge 0\}$ is the ascending central sequence of *g*, then $\dim g_k \ge \dim a_k \ge 2k$ $(1 \le k \le n)$;

(iii) There exists a unique integer *t* such that $\dim a_{t-1} < \dim a_t$ and $a_t = g$ $(s \le t \le n)$.

Theorem 2 [1]. Let $\{a_k, k \ge 0\}$ be the compatible with the integrable almost complex structure *J* of *G*. If there exists a(1,0)-type left-invariant form complex basis $\{w_i, 1 \le i \le n\}$ for $g^*$ such that the basis satisfies the structural equation

$$dw_i = \sum_{j<k<i} A_{ijk} w_j \wedge w_k + \sum_{j,k<i} B_{ijk} w_j \wedge \overline{w}_k \quad (1 \le i \le n),$$

then the left-invariant integrable almost complex structure is nilpotent if and only if it is almost nilpotent.

Raghunathan [12] concludes: let $G$ be a connected simply connected nilpotent Lie group with Lie algebra $g = LieG$. $G$ has a lattice $D$ if and only if $g$ admits a basis with rational structure constants. By applying Theorem 2, Theorem 3 can be obtained.

Theorem 3 [1]. Given the structure equations of Theorem 2:

$$dw_i = \sum_{j<k<i} A_{ijk} w_j \wedge w_k + \sum_{j,k<i} B_{ijk} w_j \wedge \overline{w}_k \quad (1 \le i \le n),$$

a connected simply connected nilpotent Lie group $G$ with a left-invariant almost complex structure that is nilpotent can be defined, and its left-invariant complex structure is nilpotent. Then a complex structure, which is also nilpotent, can be defined on the compact homogeneous nilpotent manifold $G/D$. Conversely, if the left-invariant integrable almost complex structure of a connected simply connected nilpotent Lie group $G$ is nilpotent, with structure equations

$$dw_i = \sum_{j<k<i} A_{ijk} w_j \wedge w_k + \sum_{j,k<i} B_{ijk} w_j \wedge \overline{w}_k \quad (1 \le i \le n),$$

then $G$ has a left-invariant complex structure that is nilpotent.

## 3. Exploring Complex Structures on Nilpotent Lie Group $G_k$

This section mainly discusses that if the left-invariant integrable almost complex structure $J$ on a Lie group $G$ is nilpotent, then the nilpotent Lie group $G_k$ has a left-invariant integrable almost complex structure $J_k$, and $J_k$ is nilpotent (where $k < t$, and $t$ is the smallest integer such that $a_t = g$). By Property 4, $a_k$ is an ideal of $g$, and since $g$ is a nilpotent Lie algebra, $g/a_k$ is also a nilpotent Lie algebra. Let $G_k$ be the connected simply connected nilpotent Lie group defined by the nilpotent Lie algebra $g/a_k$.

### 3.1. Complex Structure of Nilpotent Lie Group $G_k$

Definition 4. Let $G$ be a connected simply connected nilpotent Lie group with a left-invariant integrable almost complex structure, $g = LieG$, and $J$ be the complex structure on $g$. Define the mapping on $g/a_k$:

$$J_k : g/a_k \to g/a_k$$

$$\overline{x} \mapsto \tilde{J}(\overline{x}) = J(x) + a_k, \quad x \in g, \overline{x} \in g/a_k.$$

Lemme 5. Suppose $G$ is a connected simply connected nilpotent Lie group with a left-invariant integrable almost complex structure that is nilpotent, then $G_k$ has a left-invariant integrable almost complex structure.

Proof: First, we prove that $J_k$ is a complex structure. Since $J$ is linear, and according to Definition 4,

$$J_k(\overline{x}) = J(x) + a_k, \quad x \in g, \overline{x} \in g/a_k,$$

$J_k$ is linear. Next, we prove that $J_k$ is a complex structure. Since

$$\begin{aligned} J_k\left(J_k\left(\bar{x}\right)\right) &= J_k\left(Jx + a_k\right) \\ &= J\left(Jx + a_k\right) + a_k \\ &= J^2 x + a_k = -\bar{x} \end{aligned}$$

thus $J_k^2 = -id : g/a_k \to g/a_k$, so $J_k$ is a complex structure on $g/a_k$.

For any $\bar{X}, \bar{Y} \in g/a_k$, we have $\bar{X} = X + a_k$, $\bar{Y} = Y + a_k$ ($X, Y \in g$),

$$\begin{aligned} \left[J_k\left(\bar{X}\right), J_k\left(\bar{Y}\right)\right] &= \left[JX + a_k, JY + a_k\right] \\ &= \left[JX, JY\right] + a_k \\ &= \left[\bar{X}, \bar{Y}\right] + J\left[JX, Y\right] + J\left[X, JY\right] + a_k \\ &= \left[\bar{X}, \bar{Y}\right] + J_k\left(\left[JX, Y\right] + a_k\right) + J_k\left(\left[X, JY\right] + a_k\right) \\ &= \left[\bar{X}, \bar{Y}\right] + J_k\left(\left[J_k\left(\bar{X}\right), \bar{Y}\right]\right) + J_k\left(\left[\bar{X}, J_k\left(\bar{Y}\right)\right]\right) \end{aligned}$$

so $G_k$ has a left-invariant integrable almost complex structure. □

We know that the left-invariant integrable almost complex structure $J$ of the Lie group $G$ induces a left-invariant integrable almost complex structure $J_k$ on the Lie group $G_k$. Next, we first give a sequence $\{b_k, k \ge 0\}$ on $g/a_k$, and then use this sequence to prove that if $J$ is nilpotent, then $J_k$ is also nilpotent.

Definition 5. Let $G$ be a connected simply connected nilpotent Lie group with a left-invariant integrable almost complex structure, $g = LieG$, $J$ be the complex structure on $g$, and $J_k$ be the complex structure on $g/a_k$. $g/a_k$ has a sequence

$$b_0 = \{0\}, \cdots, b_k = \left\{\bar{X} \in g/a_k \mid \left[\bar{X}, g/a_k\right] \subseteq b_{k-1}, \left[\tilde{J}\left(\bar{X}\right), g/a_k\right] \subseteq b_{k-1}\right\}, \cdots. \quad (8)$$

## 3.2. Properties of the Complex Structure of Nilpotent Lie Group $G_k$

Property 5. $g/a_k$ has a sequence $\{b_k, k \ge 0\}$ satisfying Equation (8), which implies that

(1) For any $k \ge 0$, $b_k \subseteq b_{k+1}$;

(2) If there exists $k \ge 0$ such that $b_k = b_{k+1}$, then for any $r \ge k$, $b_r = b_k$.

Proof: We use induction to prove $b_k \subseteq b_{k+1}$.

When $k = 0$, according to Equation (8), we have $b_0 = \{0\}$ and $b_0 \subseteq b_1$.

Assume that when $k = i$, $b_i \subseteq b_{i+1}$ holds. When $k = i+1$, $b_{i+1} \subseteq b_{i+2}$ holds. According to Equation(8), we have

$$b_{i+2} = \left\{\bar{X} \in g/a_k \mid \left[\bar{X}, g/a_k\right] \subseteq b_{i+1}, \left[J_k\left(\bar{X}\right), g/a_k\right] \subseteq b_{i+1}\right\},$$

$$b_{i+1} = \left\{\bar{X} \in g/a_k \mid \left[\bar{X}, g/a_k\right] \subseteq b_i, \left[J_k\left(\bar{X}\right), g/a_k\right] \subseteq b_i\right\},$$

and since $b_i \subseteq b_{i+1}$ holds, $b_{i+1} \subseteq b_{i+2}$ holds. Thus, $b_k \subseteq b_{k+1}$ ($k \ge 0$).

Next, we prove property (2). According to Equation (8), we have

$$\begin{aligned} b_{i+2} &= \left\{\bar{X} \in g/a_k \mid \left[\bar{X}, g/a_k\right] \subseteq b_{i+1}, \left[J_k\left(\bar{X}\right), g/a_k\right] \subseteq b_{i+1}\right\} \\ &= \left\{\bar{X} \in g/a_k \mid \left[\bar{X}, g/a_k\right] \subseteq b_i, \left[J_k\left(\bar{X}\right), g/a_k\right] \subseteq b_i\right\} \\ &= b_{k+1} \end{aligned}$$

which means that for any $r \ge k$, $b_r = b_{k+1} = b_k$. □

Definition 6. Let $G$ be a connected simply connected nilpotent Lie group with a left-invariant integrable almost complex structure, $g = LieG$, $J_k$ be the complex structure on $g/a_k$, and $\{b_k, k \ge 0\}$ be an ascending sequence on $g/a_k$. If there exists $t > 0$ such that $b_t = g/a_k$, then the left-invariant integrable almost complex structure of $G_k$ is called nilpotent left-invariant complex structure.

Theorem 4. Suppose a connected simply connected s-step real nilpotent Lie group $G$ has a left-invariant integrable almost complex structure $J$, and the sequence $\{a_k, k \ge 0\}$ is a ascending sequence of $g$. If the left-invariant integrable almost complex structure $J$ is nilpotent, then $b_n = a_{n+k}/a_k$ and the left-invariant integrable almost complex structure of $G_k$ is nilpotent.

Proof: To prove that the left-invariant integrable almost complex structure of $G_k$ is nilpotent, we only need to prove that there exists $t$ such that $b_t = g/a_k$. We will prove by induction that there exists $n$ such that $b_n = g/a_k$.

$b_0 = a_k/a_k = \{0\}$, Next, we prove $b_1 = a_{k+1}/a_k$.

For any $\bar{X} \in a_{k+1}/a_k$, we have $\bar{X} = X + a_k \left(X \in a_{k+1}\right)$, then $\left[\bar{X}, g/a_k\right] = [X, g] + a_k$ and $\left[J_k\left(\bar{X}\right), g/a_k\right] = [JX, g] + a_k$ According to Equation (7), we know that $[X, g] \subseteq a_k$ and $\left[J\left(X\right), g\right] \subseteq a_k$, so

$$\left[J_k\left(\bar{X}\right), g/a_k\right] = 0 \text{ and } \left[\bar{X}, g/a_k\right] = 0.$$

Therefore $\bar{X} \in b_1$, implying $b_1 \supseteq a_{k+1}/a_k$.

For any $\bar{X} \in b_1$, we have $\bar{X} = X + a_k \left(X \in g\right)$, according to Equation (8),

$$b_1 = \left\{\bar{X} \in g/a_k \mid \left[\bar{X}, g/a_k\right] \subseteq b_0 = \{0\}, \left[J_k\left(\bar{X}\right), g/a_k\right] \subseteq b_0 = \{0\}\right\},$$

which means $[X, g] + a_k \subseteq \{0\}$ and $[JX, g] + a_k \subseteq \{0\}$, so $[X, g] \subseteq a_k$ and $[JX, g] \subseteq a_k$, hence $X \in a_{k+1}$, so $\bar{X} \in a_{k+1}/a_k$, which means $b_1 \subseteq a_{k+1}/a_k$.

So $b_1 = a_{k+1}/a_k$.

Assume that when $n = i$, $b_i = a_{k+i}/a_k$ holds.

Next, we prove that when $n = i + 1$, $b_{i+1} = a_{k+i+1}/a_k$ holds.

For any $\bar{X} \in b_{i+1}$ we have $\bar{X} = X + a_k \left(X \in a_{k+i+1}\right)$, then $\left[\bar{X}, g/a_k\right] = [X, g] + a_k$ and $\left[J_k\left(\bar{X}\right), g/a_k\right] = [JX, g] + a_k$. according to Equation (8), we know that $[X, g] \subseteq a_{k+i}$ and $[JX, g] \subseteq a_{k+i}$. thus $\left[J_k\left(\bar{X}\right), g/a_k\right] \subseteq a_{k+i}/a_k = b_i$ and $\left[\bar{X}, g/a_k\right] \subseteq a_{k+i}/a_k = b_i$. Therefore $\bar{X} \in b_{i+1}$, implying $b_{i+1} \supseteq a_{k+i+1}/a_k$.

For any $\bar{X} \in b_{i+1}$, we have $\bar{X} = X + a_k \left(X \in g\right)$, according to Equation (8), $b_{i+1} = \left\{\bar{X} \in g/a_k \mid \left[\bar{X}, g/a_k\right] \subseteq b_i = a_{k+i}/a_k, \left[J_k\left(\bar{X}\right), g/a_k\right] \subseteq b_i = a_{k+i}/a_k\right\}$, which means $[X, g] + a_k \subseteq a_{k+i}/a_k$ and $[JX, g] + a_k \subseteq a_{k+i}/a_k$, so $[X, g] \subseteq a_{k+i}$ and $[JX, g] \subseteq a_{k+i}$, hence $X \in a_{k+i+1}$, so $\bar{X} \in a_{k+i+1}/a_k$, which means $b_{i+1} \subseteq a_{k+i+1}/a_k$.

So $b_{i+1} = a_{k+i+1}/a_k$.

Next, we prove that the complex structure $J_k$ on $g/a_k$ is nilpotent. Since the complex structure $J$ on the Lie group $G$ is nilpotent, there exists $t > 0$ such that $a_t = g$, then $b_{t-k} = a_t / a_k = g / a_k$, thus $J_k$ is nilpotent.

According to Theorem4, if a connected simply connected s-step real nilpotent Lie group $G$ has a left-invariant integrable almost complex structure $J$ that is nilpotent, then $J$ can induce a nilpotent left-invariant integrable almost complex structure $J_k$ on the Lie group $G_k$.

## 4. Summary

Let $g$ be a Lie algebra. If $g^*$ has a (1,0)-type left-invariant complex structure with complex basis $\{w_1, 1 \le i \le n\}$, satisfying the structural equation

$$dw_i = \sum_{j<k<i} A_{ijk} w_j \wedge w_k + \sum_{j,k<i} B_{ijk} w_j \wedge \overline{w}_k \quad (1 \le i \le n),$$

then we can define a connected simply connected nilpotent Lie group $G$. Its left-invariant integrable almost complex structure $J$ on $G$ is nilpotent, and $J$ induces a left-invariant integrable almost complex structure $J_k$ on the nilpotent Lie group $G_k$, and $J_k$ is nilpotent.

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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