Open Access Library Journal
2023, Volume 10, e9734
ISSN Online: 2333-9721
ISSN Print: 2333-9705

Continuous Time Dynamical System with Hidden Attractors under Mathematical Control

Maysoon M. Aziz, Abothar A. Kalalf

Department of Mathematics, College of Computer Sciences and Mathematics, University of Mosul, Mosul, Iraq
Email: aziz_maysoon@uomosul.edu.iq, abudharabdullah5@gmail.com

How to cite this paper: Aziz, M.M. and Kalalf, A.A. (2023) Continuous Time Dynamical System with Hidden Attractors under Mathematical Control. *Open Access Library Journal*, **10**: e9734.
https://doi.org/10.4236/oalib.1109734

Received: January 5, 2023
Accepted: January 27, 2023
Published: January 30, 2023

Abstract

In this paper, a continuous two-dimensional dynamic system is proposed. This system was analyzed by finding the equilibrium points. Also, the stability of the system was analyzed through the roots of the characteristic equation, Roth stability criteria, Hurwitz stability criteria, fractional part stability criteria, and Lyapunov function. It turns out that the system is chaotic at one point of equilibrium and stable at the other point. Also, it was found that the roots of the characteristic equation of the system were in the form of complex numbers, and the real part was relied upon in the stability analysis. And then the system was controlled using adaptive control technology.

Subject Areas

Mathematics

Keywords

Lyapunov Function, Stability, Hopf Bifurcation, Lyapunov Dimension

1. Introduction

Research on chaotic phenomena has been increasingly important in recent years because of the growing range of chaotic applications in scientific and technical systems [1]. Chaotic phenomena arise from the reactivity of adversaries to changes in the structural parameters and initial conditions of some types of dynamic systems. The aperiodicity, broad spectrum, and random-like properties of chaotic signals are characteristics of these phenomena [2]-[7]. The chaotic orbits must be packed in phase space, it is not a transitional topology, and it is sensitive to perturbations in its initial conditions, all of which should lead to unpredictable behavior over time [8] [9]. Studies claim that some of the produced chaos attractors include Chen's [10], the 4-wing attractor [11], Sundarapandian V. Pehlivan [12], and the Rabinovitch system [13] [14] [15]. The fact that a chaotic system has at least one Lyapunov exponent greater than zero is one of its fundamental properties. A system becomes extremely chaotic and sensitive to even the slightest changes in its dynamics when it has a lot of positive Lyapunov exponents [16] [17]. Researchers are paying more and more attention to chaos management because of its synchronizability and controllability, which suggests that it will be helpful in a range of designs, such as biometric identification, artificial intelligence, and secure communications [18] [19]. Dissipative systems can be settled successfully using one of the Lyapunov stability [20]. A stable system will have consistent and predictable behavior, while an unstable system will have behavior that changes significantly over time [21]. If small perturbations in the initial conditions of the system result in only small changes in the long-term behavior of the variables, then the system is considered stable. Conversely, if small perturbations result in large changes in the long-term behavior of the variables, then the system is considered unstable. There are several methods for analyzing the stability of a two-dimensional continuous-time dynamical system, including linear stability analysis, eigenvalue analysis, and Lyapunov stability analysis. Each of these methods involves analyzing the properties of the system's equations and determining how the variables behave over time in response to different initial conditions [22] [23] [24].

2. System Description

Here are the equations that make up the new two-dimensional system:

$$\begin{aligned} \dot{x} &= bx - sxy \\ \dot{y} &= -dy + esxy \end{aligned} \tag{1}$$

x and y are state variables and b, d, e and s are constants.

Were

$$b = 35.5, d = 4.2, e = 31.3, s = 29.4 \tag{2}$$

3. System Analysis

When Equation (1) is set to zero, just one equilibrium point, the origin point, is produced, allowing us to examine a dynamical system's equilibrium points $E_0 = (0,0)$, $E_1 = (0.041, 1.207)$.

3.1. Stability Analysis

A necessary and sufficient condition for the stability of the system is that the characteristic equation's eigenvalues have negative real components. Following is the Jacobian matrix for the new system (1) up to $E_0 = (0,0)$:

$$J = \begin{bmatrix} 35.5 & 0 \\ 0 & -4.2 \end{bmatrix}, \tag{3}$$

The characteristic equation is:

$$\lambda^2 - 31.3\lambda - 149.1 = 0, \tag{4}$$

Roots of the characteristic equation:

$$\lambda_1 = 35.5, \lambda_2 = -4.2,$$

Thus, the system is unsteady.

3.2. Routh Stability Criterion

A system meets the Routh requirement for stability (all poles in the half-loop level), if and only if the components in the first column of the Routh row have only positive values for all of their values. The number of sign changes in the first column multiplied by the sum of the non-OLHP columns [25]. Regarding the Roth stability test, see **Table 1**.

$$a_0 = -149.1, a_1 = -31.3, a_2 = 1,$$

The system is unstable because the first column has four negative elements.

3.3. Hurwitz Stability Criteria

Determinants generated from the coefficients of the characteristic equation are used to implement this criterion. System (1) is stable if the tiny minors of its square matrix J are all positive; if not, it is unstable [25].

From Equation (3):

$$\Delta_1 = a_{n-1} = a_1 = -31.3 < 0,$$

$$\Delta_2 = \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix} = \begin{vmatrix} a_1 & 0 \\ a_2 & a_0 \end{vmatrix} = \begin{vmatrix} -31.3 & 0 \\ 1 & 149.1 \end{vmatrix} = -4666.83 < 0,$$

System (1) is unstable because some of the values of the determinants are less than zero.

3.4. Lyapunov Function

Where we assume the Lyapunov function is:

$$\mathcal{V}(x_1, x_2) = \frac{1}{2}\left(x_4^2 + x_2^2\right),$$

$$\dot{\mathcal{V}}(x_1, x_2) = \frac{\partial v}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial v}{\partial x_2}\frac{\partial x_2}{\partial t}, \tag{5}$$

The system is stable, $0 > \dot{\mathcal{V}}$ if

We get: (5) in Equation (1) substituting.

Since $\dot{\mathcal{V}} > 0$, as a result, new system (1) is unstable.

Table 1. Routh array.

λ^2	1	−149.1
λ^1	−31.3	0
λ^0	−149.1	0

3.5. Continued Fraction Stability Criteria

By creating a continuous fraction from the odd and even parts of the equation, the characteristic equation of a continuous system is subjected to this condition. The distinguishing equation:

$$\lambda^2 - 31.3\lambda - 149.1 = 0,$$

By taking the even terms and then the odd terms, respectively, we have:

$$Q_1(\lambda) = \lambda^2 - 149.1 \tag{6}$$

$$Q_2(\lambda) = -31.3\lambda \tag{7}$$

After dividing the even terms by the odd terms and using algebraic steps, we get the following results:

$$h_1 = -0.031, h_2 = -0.209.$$

Since some values of h are negative, the equation of the system has some positive real roots, so system (1) is chaotic.

3.6. Dissipativity

Suppose that

$$f_1 = \frac{\mathrm{d}x}{\mathrm{d}t}, f_2 = \frac{\mathrm{d}y}{\mathrm{d}t}.$$

The obtained vector field,

$$(\dot{x}, \dot{y})^{\mathrm{T}} = (f_1, f_2)^{\mathrm{T}}$$

$$\nabla \cdot (\dot{x}, \dot{y})^{\mathrm{T}} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = b - 1.207s - d + 0.004es = f.$$

Note that, $f = -b - 1.207s - d + 0.004es = -0.504$, for all values that are positive and greater than zero, the system (1) dissipates

Here is the exponential rate:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = f\mathcal{V} \Rightarrow \mathcal{V}(t) = \mathcal{V}_0 \mathrm{e}^{ft} = \mathcal{V}_0 \mathrm{e}^{-0.504t}$$

By flowing into ($V_0 \mathrm{e}^{-0.504t}$), the volume element (V_0) from the previous equation is condensed at the time (t).

4. Hopf Bifurcation

One of the types of bifurcation that is recognized in mathematics occurs when a modest modification to one of the initial conditions causes a qualitative change in the behavior of the system at an equilibrium point. We take the Equation (3)

$$\lambda^2 - 31.3\lambda - 149.1 = 0,$$

The roots of Equation (3) are:

$$\lambda_1 = 35.5, \lambda_2 = -4.2,$$

Differentiate the Equation (3) and normalize it to zero to find the critical value.

$$2\lambda - 31.3 = 0,$$

So, the critical values are $\lambda = 15.65$.

Derivative at one of the eigenvalues of the equation = −8835774.285

$$\frac{\text{The critical values}}{\text{Derivative at the eigenvalue of the equation}} = \frac{15.65}{39.7} = 0.394 \neq 0$$

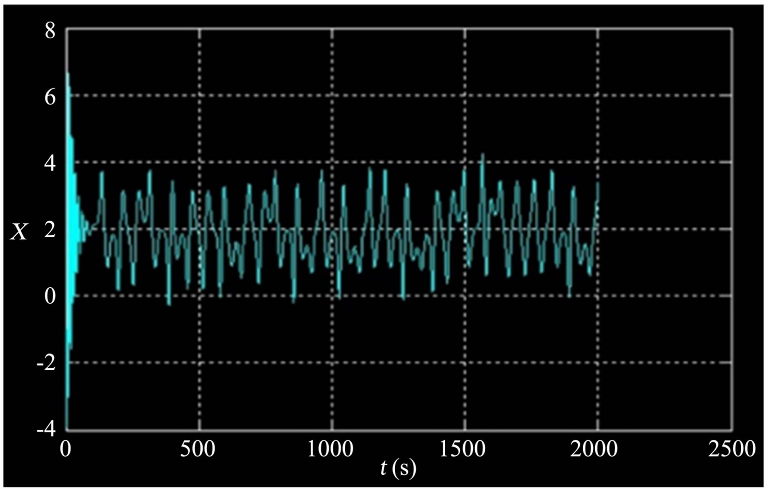
4.1. Numerical and Graphical Analysis

The fourth and fifth order Runge-Kotta method is used to solve the system (1). Initial values included

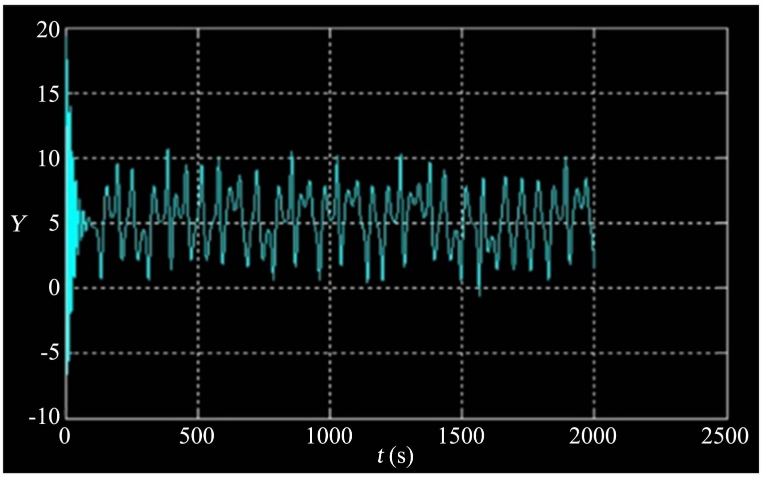
$$x \mid x_{(0)}, y_{(0)} = [0.5, 1]$$

4.2. Waveform of the New System (1)

The waveform exhibits aperiodic structure, the primary defining feature of chaotic systems. $x_{(t)}$ and $y_{(t)}$ for system (1) (as showed in Figure 1).



(a)



(b)

Figure 1. The Waveform of a new system (1). (a): time versus *x*; (b): time versus *y*.

4.3. The System's Phase Portrait (1)

In this paragraph, the strange attractor for the system (1) in (x, y) space is shown along with the chaotic strange attractor for the system (1), shown in Figure 2.

Since the orbit in each graph looks to be dense, the new system exhibits a chaotic attractor.

4.4. Lyapunov Exponent and Lyapunov Dimension

The typical exponential growth rates of almost divergent trajectories in phase space are frequently referred to as the Lyapunov exponent. The new system is regarded as chaotic if it has at least one positive Lyapunov exponent. Values of the Lyapunov exponent are:

$$\left(L_1 = 1.523, L_2 = -2.388\right).$$

As a result, the system's "Kaplan-Yorke dimension" or Lyapunov dimension is as follows:

$$D_L = 1 + \frac{L_1}{|L_2|} = 1.637$$

Figure 3 shows that system (1) is very Chaotic.

5. Adaptive Controller Technique

5.1. Theoretical Results

To stabilize a chaotic system (1) use the sufficiency control law generalized with an unknown parameter c as follows:

$$\begin{aligned} \dot{x} &= 35.5x - 29.4xy + u_1 \\ \dot{y} &= -4.2y + 29.4exy + u_2 \end{aligned} \tag{8}$$

where $[u_1, u_2]^{\mathrm{T}}$ are feedback controllers.

We now consider the following adaptive control procedures to make sure the managed system (7) converges asymptotically to the origin.

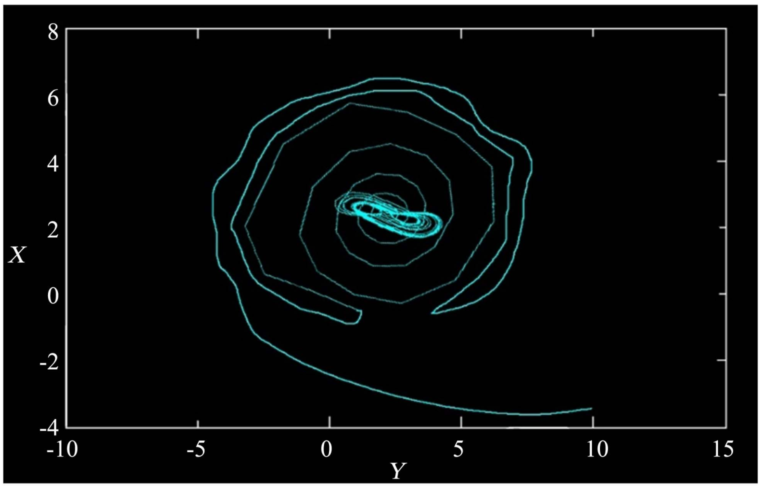


Figure 2. The system attractor in (x, y).

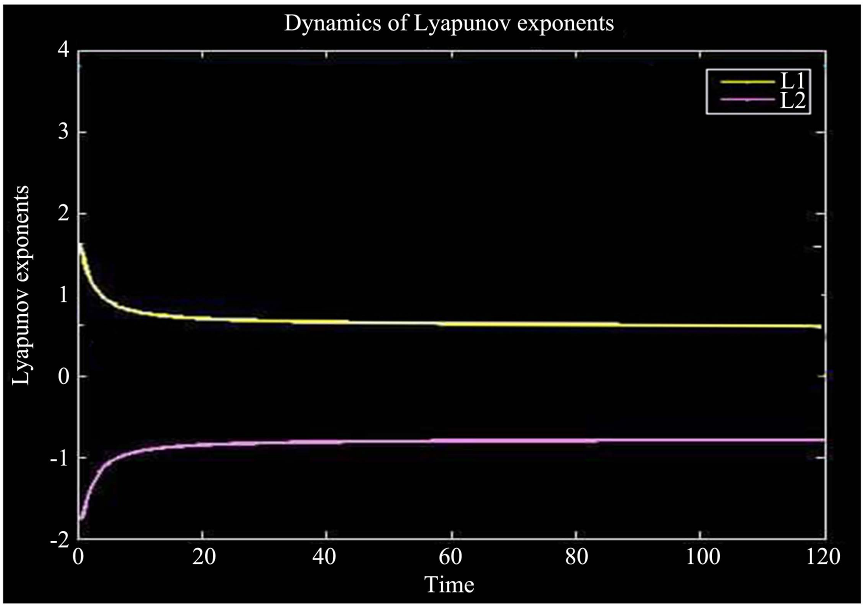


Figure 3. Lyapunov exponent of the system (1).

$$\begin{aligned} u_1 &= -35.5x + 29.4xy - \mu_1 x \\ u_2 &= 4.2y - 29.4\hat{e}xy - \mu_2 y \end{aligned} \tag{9}$$

where μ_1, μ_2 are constants, $\hat{c}$ is an estimator of the parameter c.

Substituting (8) into (7), we get:

$$\begin{aligned} \dot{x} &= -\mu_1 x \\ \dot{y} &= 29.4xy\left(e - \hat{e}\right) - \mu_2 y \end{aligned} \tag{10}$$

Let the estimation error of the parameter be:

$$e_c = c - \hat{c} \tag{11}$$

Using (10), system (9) can be written as:

$$\begin{aligned} \dot{x} &= -\mu_1 x, \\ \dot{y} &= 29.4e_e xy - \mu_2 y, \end{aligned} \tag{12}$$

The parameter estimates $\hat{c}$ is changed using the Lyapunov method of obtaining the updated law. It is thought that the quadratic Lyapunov function:

$$\mathcal{V}\left(x_1, x_2\right) = \frac{1}{2}\left(x_1^2 + x_2^2 + e_e^2\right), \tag{13}$$

Which definite, positive-in $\mathbb{R}^3$.

Also

$$\dot{e}_c = -\dot{\hat{c}} \tag{14}$$

Differentiate V & substituting (11) and (13), we get:

$$\dot{\mathcal{V}} = -\mu_1 x^2 - \mu_2 y^2 + e_e\left(29.4xy^2 - \dot{\hat{e}}\right)$$

Assume that:

$$\dot{\hat{e}} = xy + \mu_3 e_e \tag{15}$$

where μ_3 is higher than 0 in value.

Substitute (14) into $\dot{V}$, we get:

$$\dot{V} = -\mu_1 x^2 - \mu_2 y^2 - \mu_3 e_e^2 + 29.4 e_e y^2 x - e_e xy \tag{16}$$

Which is negative-definite on $\mathbb{R}^3$.

The outcome is as follows because of Lyapunov stability, Eigenvalues, and the Routh array criteria.

Proposition 1. Byadaptive control (10), where $\dot{\hat{c}} = xy + \mu_3 e_c$ and μ_1, μ_2, μ_3 are positive constants, The chaotic system (8) is stabilized for $x(0) \in \mathbb{R}^2$.

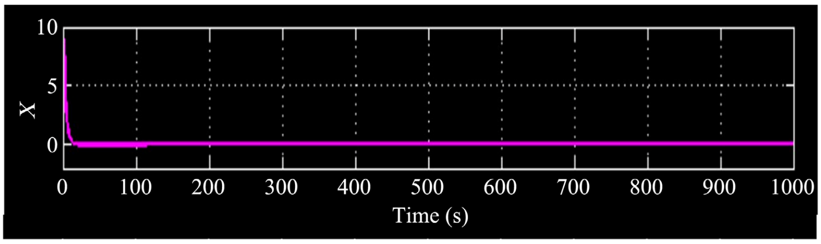
5.2. Simulation and Numerical Results

The controlled extremely chaotic system (8) was simulated using

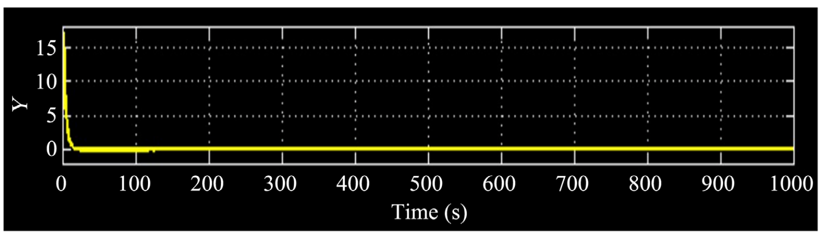
$$x \mid x_{1_{(0)}}, x_{2_{(0)}} = [3,9]$$

$$\mu_2, \mu_1 = [30,15] \text{ and } e_c = 25.3.$$

The new system (1)'s-controlled state trajectories are displayed in Figure 4.



(a)



(b)

Figure 4. The behavior of state variables *x*, *y* for the controlled (8).

6. A Table of Comparisons before and after the Control

More results can be found in Tables 2-5. A comparison before and after control of system (1) was done, for eigenvalues given in Table 2, Routh array criterion values in Table 3, calculated values of Hurwitz stability criteria in Table 4, and calculated values of continued fraction in Table 5, all shows that system(1) is stable after control.

Table 2. Eigenvalues of a new system (1).

Equilibrium point	Before Control	After Control
(0, 0)	$\lambda_1 = 35.5$	$\lambda_1 = -57$
	$\lambda_2 = -2.617$	$\lambda_2 = -2$

Table 3. Calculated Routh array criterion values for a new system (1).

Equilibrium point	λ	Before Control		After Control	
(0, 0)	λ^2	1	−149.1	1	24
	λ^1	−31.3	0	14	0
	λ^0	−149.1	0	24	0

Table 4. Calculated values of Hurwitz stability criteria of a new system (1).

Equilibrium point	Before Control	After Control
(0, 0)	$\Delta_1 = -31.3$	$\Delta_1 = 14$
	$\Delta_2 = -4666.83$	$\Delta_2 = 366$

Table 5. Calculated values of continued fraction stability criteria of new.

Equilibrium point	Before Control	After Control
(0, 0)	$h_1 = -0.031$	$h_1 = 0.071$
	$h_2 = -0.209$	$h_2 = 0.583$

7. Conclusion

In this study, a two-dimensional model of continuous dynamical systems was taken. The permissible equilibrium points for the analysis of this system were found, and the parameters of stability were evaluated in various ways, which are:

- Roots of the characteristic equation.
- Roth stability criterion.
- The criterion of the stability of Hurwitz.
- Lyapunov function.
- Fractional stability criterion.

The Lyapunov exponentially was examined, and the system was found to be chaotic. The proposed system dissipation detected Hopf bifurcation, and then the system was regulated using an adaptive control approach. Finally, for the system under study, the numerical and morphological results before and after the control were compared.

Acknowledgements

Mosul University/College of Computer Sciences and Mathematics' support, which enhanced the caliber of this work, is greatly appreciated by the authors.

Conflicts of Interest

There are no conflicts of interest reported by the authors.

References

[1] Aziz, M.M. and Hamid, M. (2019) The Possibility of Increasing the Predictability Indices after Control of 3D-Continuous-Time System. 2019 *International Confe-*

rence on Computing and Information Science and Technology and Their Applications (*ICCISTA*), Kirkuk, 3-5 March 2019, 1-5. https://doi.org/10.1109/ICCISTA.2019.8830650

[2] Jihad, O.M. and Aziz, M.M. (2021) Stability, Chaos Tests with Adaptive and Feedback Control Methods for 3D Discrete-Time Dynamical System. *International Journal of Electronics Communication and Computer Engineering*, **12**, 31-42.

[3] Aziz, M.M. and Jihad, O.M. (2021) Stability & Chaos Tests of 2D Discrete Time Dynamical System with Hidden Attractors. *Open Access Library Journal*, **8**, e7501. https://doi.org/10.4236/oalib.1107501

[4] Aziz, M.M. and Faraj, M.N. (2015) Difficulty of Predicting Earthquakes in Mosul Dam. *International Journal of Technical Research and Applications*, **3**, 29-36.

[5] Jihad, O.M. and Aziz, M.M. (2021) Stability, Chaos Diagnose and Adaptive Control of Two Dimensional Discrete-Time Dynamical System. *Open Access Library Journal*, **8**, e7270. https://doi.org/10.4236/oalib.1107270

[6] Jovic, B. (2011) Synchronization Techniques for Chaotic Communication Systems. Springer-Verlag, Berlin. https://doi.org/10.1007/978-3-642-21849-1

[7] Aziz, M.M.A. and Faraj, M.N. (2012) Numerical and Chaotic Analysis of Chua's Circuit. *Journal of Emerging Trends in Computing and Information Science*, **3**, 783-791.

[8] Abdullah Hamad, A., Lellis Thivagar, M., Bader Alazzam, M., Alassery, F., Mahmood Khalil, M., Ramesh, V. and Kumar, V. (2021) Dynamic Systems Enhanced by Electronic Circuits On 7D. *Advances in Materials Science and Engineering*, **2021**, Article ID: 8148772. https://doi.org/10.1155/2021/8148772

[9] Zhang, W., Tang, Y., Wu, X. and Fang, J.A. (2013) Synchronization of Nonlinear Dynamical Networks with Heterogeneous Impulses. *IEEE Transactions on Circuits and Systems I*: *Regular Papers*, **61**, 1220-1228. https://doi.org/10.1109/TCSI.2013.2286027

[10] Chen, G. and Ueta (1999) Yet Another Chaotic Attractors. *International Journal of Bifurcation and Chaos*, **9**, 1465-1466. https://doi.org/10.1142/S0218127499001024

[11] Yu, F. and Wang, C. (2013) Generation of a New Three-Dimensional Autonomous Chaotic Attractor and Its Four Wing Type. *Engineering Technology and Applied Science Research*, **3**, 352-358. https://doi.org/10.48084/etasr.190

[12] Sundarapandian, V. and Pehlivan (2012) Analysis, Control, Synchronization and Circuit Design of a Novel Chaotic System. *Mathematical and Computer Modelling*, **55**, 1904-1915. https://doi.org/10.1016/j.mcm.2011.11.048

[13] Pikovsky, A.S., *et al.* (1978) Onset of Stochasticity in Decay Confinement of Parametric Instability. *Soviet Physics—JETP*, **47**, 715-719.

[14] Liu, Y., Yang, Q. and Pang, G. (2010) A Hyperchaotic System from the Rabinovich System. *Journal of Computational and Applied Mathematics*, **234**, 101-113. https://doi.org/10.1016/j.cam.2009.12.008

[15] He, J.M. and Chen, F.Q. (2017) A New Fractional Order Hyperchaotic Rabinovich System and Its Dynamical Behaviors. *International Journal of Non-Linear Mechanics*, **95**, 73-81. https://doi.org/10.1016/j.ijnonlinmec.2017.05.013

[16] Aziz, M.M. and Merie, D.M. (2020) A Three-Dimensional Chaotic System and Its New Proposed Electronic Circuit. *Open Access Library Journal*, **7**, e6555. https://doi.org/10.4236/oalib.1106555

[17] Aziz, M.M. and Merie, D.M. (2020) Stability and Chaos with Mathematical Control of 4-d Dynamical System. *Indonesian Journal of Electrical Engineering and Com-*

puter Science, **20**, 1242-1251. https://doi.org/10.11591/ijeecs.v20.i3.pp1242-1251

[18] Cuomo, K.M., *et al.* (1993), Synchronization of Lorenz-Based Chaotic Circuits and Application to Secure Communication, *IEEE Transaction on Circuits and Systems-II*: *Analogue and Digital Processing*, **40**, 626-633. https://doi.org/10.1109/82.246163

[19] Saeed, M.Gh., *et al.* (2018) Multiple Models of Binary-Support-Vector-Machine for Face Verification Using Histogram Orientation Gradient Features. *Journal of Theoretical and Applied Information technology*, **96**, 6349-6360.

[20] Aziz, M.M. (2018) Stability Analysis of the Mathematical Model. *International Journal of Science and Research* (*IJSR*), **7**, 147-148.

[21] Malallah, F.L., *et al.* (2016) Off-Line Arabic (Indian) Numbers Recognition Using Expert System. *International Journal of Advanced Computer Science and Applications*, **7**, 397-406. https://doi.org/10.14569/IJACSA.2016.070453

[22] Hahn, W. (1967) The Stability of Motion, Springer, New York. https://doi.org/10.1007/978-3-642-50085-5

[23] Aziz, M.M. and Merie, D.M. (2021) A Four-Dimensional Chaotic System with Hidden Attractor and Its New Proposed Electronic Circuit. *Open Access Library Journal*, **8**, e8101. https://doi.org/10.4236/oalib.1108101

[24] Aziz, M.M. and Al-Nuaimi (2013) Stability and Hop-Bifurcation for Diabetes Model. *International Journal of Electronics Communication and Computer Engineering*, **4**, 805-809.

[25] Joseph, J. and Ivan (1990) Feedback and Control Systems. McGraw-Hill, New York.