Open Access Library Journal
2022, Volume 9, e8347
ISSN Online: 2333-9721
ISSN Print: 2333-9705

Linear Mean Estimates for the 3D Non-Autonomous Brinkman-Forchheimer-Extended-Darcy Equations with Singularly Oscillating Forces

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How to cite this paper: Chen, X.Y. and Zhu, C.S. (2022) Linear Mean Estimates for the 3D Non-Autonomous Brinkman-Forchheimer-Extended-Darcy Equations with Singularly Oscillating Forces. *Open Access Library Journal*, **9**: e8347.
https://doi.org/10.4236/oalib.1108347

Received: January 3, 2022
Accepted: February 25, 2022
Published: February 28, 2022

Abstract

In this paper, we prove the existence of global strong solutions for the three-dimensional nonautonomous Brinkman-Forchheimer-extended-Darcy equation with singularly oscillating and show that the strong solutions are unique. In addition, we also give general estimates for its auxiliary linear equation; finally, we derive the oscillatory averaged estimates of the equation from the results of these general estimates.

Subject Areas

Partial Differential Equation

Keywords

Singularly Oscillating Forces, Brinkman-Forchheimer-Extended-Darcy Equations, Existence and Uniqueness of Solutions, Oscillatory Averaged Estimation

1. Introduction

Let $\rho \in [0,1)$ be a fixed parameter, $\Omega \subset \mathbb{R}^3$ is a bounded domain, and the boundary $\partial\Omega$ is smooth. We study the 3D Non-autonomous Linearization Brinkman-Forchheimer-extended-Darcy Equations with singularly oscillating forces in Ω [1] [2] [3]:

$$\begin{cases} \partial_t u - \mu\Delta u + (u\cdot\nabla)u + au + b|u|^{2\alpha}u + c|u|^{2\beta}u + \nabla p = f_0(t,x) + \varepsilon^{-\rho} f_1(t/\varepsilon, x), \\ \nabla\cdot u = 0, \quad x\in\Omega, \\ u(x,t)\big|_{t=0} = 0, \\ u(x,t)\big|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $a>0$, $b>0$, $c>0$, $\alpha,\beta\in[0,\infty)$, $\mu>0$ is the kinematic viscosity coefficient of the fluid, unknown function $u=u(x,t)=(u_1(x,t),u_2(x,t),u_3(x,t))$ represents the velocity vector field of the fluid, $p=p(x,t)$ indicates pressure. Note that when $b,c=0$, Equation (1.1) is a Navier-Stokes equation with singular oscillatory force.

Combined with Equation (1.1), we study the following averaged Brinkman-Forchheimer-extended-Darcy equation (corresponding to the limit case $\varepsilon=0$):

$$\begin{cases}\partial_t u-\mu\Delta u+(u\cdot\nabla)u+au+b|u|^{2\alpha}u+c|u|^{2\beta}u+\nabla p=f_0(t,x),\\ \nabla\cdot u=0,\quad x\in\Omega,\\ u(x,t)|_{t=0}=0,\\ u(x,t)|_{\partial\Omega}=0.\end{cases}\tag{1.2}$$

Recording function

$$f^{\varepsilon}(t,x)\equiv\begin{cases}f_0(t,x)+\varepsilon^{-\rho}f_1(t/\varepsilon,x), & 0<\varepsilon<1,\\ f_0(t,x), & \varepsilon=0.\end{cases}\tag{1.3}$$

Function $f_0(x,s)$, $f_1(x,s)\in L_b^2(\mathbb{R},H)$, and $L_b^2(\mathbb{R},H)$ is the translation bounded function in $L_{loc}^2(\mathbb{R},H)$, that is, there are two constants $M_0,M_1\ge 0$, making the following formula true:

$$\|f_0\|_{L_b^2}^2\equiv\sup_{t\in R}\int_t^{t+1}\|f_0(s)\|^2\,\mathrm{d}s=M_0^2,$$

$$\|f_1\|_{L_b^2}^2\equiv\sup_{t\in R}\int_t^{t+1}\|f_1(s)\|^2\,\mathrm{d}s=M_1^2.$$

Define that

$$Q^{\varepsilon}\equiv\begin{cases}M_0+2M_1\varepsilon^{-\rho}, & 0<\varepsilon<1,\\ M_0, & \varepsilon=0.\end{cases}\tag{1.4}$$

Then, $\|f^{\varepsilon}\|_{L_b^2}\le Q^{\varepsilon}$ can be obtained directly from (1.3). Note that when $\varepsilon\to 0$, the order of magnitude of Q^{ε} is $\varepsilon^{-\rho}$.

Let's introduce the following function space [4] [5]

$$\mathcal{V}=\left\{u\in\left(C_0^{\infty}(\Omega)\right)^3:\nabla u=0\right\},\quad H=cl_{\left(L^2(\Omega)\right)^3}\mathcal{V},\quad V=cl_{\left(H_0^1(\Omega)\right)^3}\mathcal{V},$$

Here cl_X represents the closure in space *X*, obviously, *H*, *V* are separable Hilbert spaces, so that H' is the dual space of *H*, V' is the dual space of *V*, then $V\hookrightarrow H=H'\hookrightarrow V'$, the embedding is continuous and dense, and $\langle\cdot,\cdot\rangle$ represents the dual set between *V* and V'. For *H*, *V* has the following inner product and norm respectively:

$$(u,v)=\sum_{j=1}^{3}\int_{\Omega}u_j(x)v_j(x)\mathrm{d}x,\quad |\cdot|_2=(\cdot,\cdot)^{\frac{1}{2}},\quad \forall u,v\in H,$$

$$((u,v))=\sum_{j=1}^{3}\int_{\Omega}\frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}\mathrm{d}x,\quad \|\cdot\|=((\cdot,\cdot))^{\frac{1}{2}},\quad \forall u,v\in V.$$

Denoted by $|\cdot|_p$ represents the norm in $\mathbf{L}^p(\Omega)$, $|\cdot|_X$ represents the norm in

Banach space *X*. For $L^p(0,T;x)$, $1\le p\le\infty$, define the set of functions $f(t)$ on $(0,T)$ in *X* so that $\int_0^T |f(t)|_X^p \,\mathrm{d}t<\infty$ holds. The letter *C* indicates a positive constant independent of the initial time.

The purpose of this paper is to prove the existence and uniqueness of the solution of the three-dimensional non-autonomous linearized Brinkman-Forchheimer-extended-Darcy equation with singular oscillatory force, and derive some estimates, and obtain the convergence of the corresponding equation compared with the average equation.

2. Main Results and Proof

The first and second equations of (1.1) can be written in the following form:

$$\begin{cases}\partial_t u+\mu Au+au+B(u)+F(u)=f^{\varepsilon}(x,t),\\ \nabla\cdot u=0.\end{cases}\tag{2.1}$$

where $A=-P\Delta$ is the Stokes operator and *P* is the Leray orthogonal projection from $\mathbf{L}^2(\Omega)$ to *H*; $F(u)=P\left(b|u|^{2\alpha}u+c|u|^{2\beta}u\right)$. Define $\langle Au,v\rangle=((u,v))$; $B:V\times V\to V'$ as a bilinear operator, $\langle B(u,v)w\rangle=b(u,v,w)$,

$$B(u)=b(u,u),\quad b(u,v,w)=\sum_{i,j=1}^{3}\int_\Omega u_i\frac{\partial v_j}{\partial x_i}w_j\mathrm{d}x.$$

The following proves the existence and uniqueness of global solutions of Equation (2.1).

Theorem 2.1. *Suppose* $f^{\varepsilon}\in L^2_{loc}(\mathbb{R},H)$ *translational compactness,* $\int_0^T\left|f^{\varepsilon}(t)\right|_2^2\mathrm{d}t\le C<+\infty$, *any given* $T>0$, *there is a unique strong solution u for the initial boundary value problem of* Equation (2.1),

$$u\in C([0,+\infty);H)\cap L^2(0,T;V)\cap L^{2\alpha+2}\left(0,T;L^{2\alpha+2}(\Omega)\right)\cap L^{2\beta+2}\left(0,T;L^{2\beta+2}(\Omega)\right)$$

and

$$\sup_{0\le t\le T}|u(t)|_2^2+2\mu\int_0^T|\nabla u|_2^2\,\mathrm{d}t^2+2b\int_0^T|u|_{2\alpha+2}^{2\alpha+2}\mathrm{d}t+2c\int_0^T|u|_{2\beta+2}^{2\beta+2}\mathrm{d}t\le\frac{2}{a}\int_0^T\left|f^{\varepsilon}(t)\right|_2^2\mathrm{d}t.$$

Proof. We use Galerkin approximation to prove this theorem. Due to the space *V* is separable and space C^∞ is dense in space *V*, there is a sequence $\omega_1,\omega_2,\cdots,\omega_m$ composed of elements in space C^∞ which is free and completely belongs to space *V*. For any *m*, we define the following approximate solution:

$$u_m=\sum_{i=1}^{m}g_{im}(t)\omega_i(x),$$

make the following formula true

$$\begin{aligned}&\left(u_m'(t),\omega_j\right)+\mu\left(\nabla u_m(t),\omega_j\right)+\left(u_m(t)\cdot\nabla u_m(t),\omega_j\right)+\left(au_m(t),\omega_j\right)\\&+\left(b|u_m|^{2\alpha}(t),\omega_j\right)+\left(b|u_m|^{2\beta}(t),\omega_j\right)=\left(f^{\varepsilon}(t),\omega_j\right),\end{aligned}$$

where $t\in[0,T]$, $j=1,2,\cdots,m$, and in space L^2, $u_{0m}\to0$ as $m\to\infty$. Multiply both sides of the above equation by $g_{im}(t)$ at the same time and sum $j=1,\cdots,m$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_m|_2^2+\mu|\nabla u_m|_2^2+a|u_m|_2^2+b|u_m|_{2\alpha+2}^{2\alpha+2}+c|u_m|_{2\beta+2}^{2\beta+2}$$
$$=\left(f^{\varepsilon}(t),u_m\right)\le\frac{1}{a}\left|f^{\varepsilon}(t)\right|_2^2+a|u_m|_2^2,$$

namely

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_m|_2^2+\mu|\nabla u_m|_2^2+b|u_m|_{2\alpha+2}^{2\alpha+2}+c|u_m|_{2\beta+2}^{2\beta+2}\le\frac{1}{a}\left|f^{\varepsilon}(t)\right|_2^2,$$

for any $u,v\in V$, there are $\left((u\cdot\nabla)v,v\right)=0$. The above equation can be obtained by integrating on $[0,T]$:

$$\sup_{0\le t\le T}|u(t)|_2^2+2\mu\int_0^T|\nabla u|_2^2\,\mathrm{d}t^2+2b\int_0^T|u|_{2\alpha+2}^{2\alpha+2}\,\mathrm{d}t+2c\int_0^T|u|_{2\beta+2}^{2\beta+2}\,\mathrm{d}t\le\frac{2}{a}\int_0^T\left|f^{\varepsilon}(t)\right|_2^2\,\mathrm{d}t. \quad (2.2)$$

From the above formula, it is easy to obtain the existence by using the proof method similar to the Navier-Stokes equation with damping [6].

Now prove a priori estimate. Multiply both ends of Equation (2.1) by $u_t,-\Delta u$, and then integrate on Ω:

$$\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla u|^2\,\mathrm{d}x+\int_\Omega|u_t|^2\,\mathrm{d}x+\frac{a}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|u|^2\,\mathrm{d}x+\frac{b}{2\alpha+2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|u|^{2\alpha+2}\,\mathrm{d}x$$
$$+\frac{c}{2\beta+2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|u|^{2\beta+2}\,\mathrm{d}x=\left(f^{\varepsilon},u_t\right),$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla u|^2\,\mathrm{d}x+\mu\int_\Omega|\Delta u|^2\,\mathrm{d}x+a\int_\Omega|\nabla u|^2\,\mathrm{d}x+b\int_\Omega|u|^{2\alpha}|\nabla u|^2\,\mathrm{d}x$$
$$+\frac{4b\alpha}{(2\alpha+2)^2}\int_\Omega\left|\nabla|u|^{\frac{2\alpha+2}{2}}\right|^2\mathrm{d}x+c\int_\Omega|u|^{2\beta}|\nabla u|^2\,\mathrm{d}x+\frac{4c\beta}{(2\beta+2)^2}\int_\Omega\left|\nabla|u|^{\frac{\beta+2}{2}}\right|^2\mathrm{d}x$$
$$=-\int_\Omega f^{\varepsilon}\cdot\Delta u\,\mathrm{d}x,$$

adding the two formulas and applying Hölder inequality and Young inequality, we can deduce

$$\frac{\mu+1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla u|^2\,\mathrm{d}x+\int_\Omega|u_t|^2\,\mathrm{d}x+\frac{a}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|u|^2\,\mathrm{d}x+\frac{b}{2\alpha+2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|u|^{2\alpha+2}\,\mathrm{d}x$$
$$+\frac{c}{2\beta+2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|u|^{2\beta+2}\,\mathrm{d}x+\mu\int_\Omega|\Delta u|^2\,\mathrm{d}x+a\int_\Omega|\nabla u|^2\,\mathrm{d}x+b\int_\Omega|u|^{2\alpha}|\nabla u|^2\,\mathrm{d}x$$
$$+\frac{4b\alpha}{(2\alpha+2)^2}\int_\Omega\left|\nabla|u|^{\frac{2\alpha+2}{2}}\right|^2\mathrm{d}x+c\int_\Omega|u|^{2\beta}|\nabla u|^2\,\mathrm{d}x+\frac{4c\beta}{(2\beta+2)^2}\int_\Omega\left|\nabla|u|^{\frac{2\beta+2}{2}}\right|^2\mathrm{d}x$$
$$\le\frac{1}{2}|u_t|_2^2+\frac{\mu}{2}|\Delta u|_2^2+\frac{1}{2}\left|f^{\varepsilon}\right|_2^2+\frac{1}{2\mu}\left|f^{\varepsilon}\right|_2^2,$$

therefore

$$\frac{\mu+1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla u|^2\,\mathrm{d}x+\frac{1}{2}\int_\Omega|u_t|^2\,\mathrm{d}x+\frac{a}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|u|^2\,\mathrm{d}x+\frac{b}{2\alpha+2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|u|^{2\alpha+2}\,\mathrm{d}x$$
$$+\frac{c}{2\beta+2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|u|^{2\beta+2}\,\mathrm{d}x+\frac{\mu}{2}\int_\Omega|\Delta u|^2\,\mathrm{d}x+a\int_\Omega|\nabla u|^2\,\mathrm{d}x+b\int_\Omega|u|^{2\alpha}|\nabla u|^2\,\mathrm{d}x$$
$$+\frac{4b\alpha}{(2\alpha+2)^2}\int_\Omega\left|\nabla|u|^{\alpha+1}\right|^2\mathrm{d}x+c\int_\Omega|u|^{2\beta}|\nabla u|^2\,\mathrm{d}x+\frac{4c\beta}{(2\beta+2)^2}\int_\Omega\left|\nabla|u|^{\beta+1}\right|^2\mathrm{d}x$$
$$\le\frac{1}{2}\left|f^{\varepsilon}\right|_2^2+\frac{1}{2\mu}\left|f^{\varepsilon}\right|_2^2,$$

integrate on 0 to T

$$\begin{aligned}&(\mu+1)\sup_{0\le t\le T}|\nabla u(t)|_2^2+a\sup_{0\le t\le T}|u(t)|_2^2+\frac{2b}{2\alpha+2}\sup_{0\le t\le T}|u(t)|_{2\alpha+2}^{2\alpha+2}\\&+\frac{2c}{2\beta+2}\sup_{0\le t\le T}|u(t)|_{2\beta+2}^{2\beta+2}+\int_0^T\int_\Omega|u_t|^2\,\mathrm{d}x\mathrm{d}t\\&+\mu\int_0^T\int_\Omega|\Delta u|^2\,\mathrm{d}x\mathrm{d}t+2a\int_0^T\int_\Omega|\nabla u|^2\,\mathrm{d}x\mathrm{d}t\\&+2b\int_0^T\int_\Omega\left||\nabla u||u|^\alpha\right|^2\mathrm{d}x\mathrm{d}t+\frac{8b\alpha}{(2\alpha+2)^2}\int_0^T\int_\Omega\left|\nabla|u|^{\alpha+1}\right|^2\mathrm{d}x\mathrm{d}t\\&+2c\int_0^T\int_\Omega\left||\nabla u||u|^\beta\right|^2\mathrm{d}x\mathrm{d}t+\frac{8c\beta}{(2\beta+2)^2}\int_0^T\int_\Omega\left|\nabla|u|^{\beta+1}\right|^2\mathrm{d}x\mathrm{d}t\\&\le C\int_0^T|f|_2^2\,\mathrm{d}t\le C.\end{aligned}$$

The uniqueness is proved below. Assuming that under the same initial conditions, because of Divergence free, Equation (2.1) has two strong solutions (u,p), $(\bar{u},\bar{p})$ satisfies

$$\begin{aligned}&(u_t,\Phi)+\mu\int_\Omega\nabla u:\nabla\Phi\mathrm{d}x-\int_\Omega(u\cdot\nabla)u\Phi\mathrm{d}x+a(u,\Phi)\\&+b\int_\Omega|u|^{2\alpha}u\Phi\mathrm{d}x+c\int_\Omega|u|^{2\beta}u\Phi\mathrm{d}x=f^\varepsilon(x,t),\end{aligned}\tag{2.3}$$

$$\begin{aligned}&(\bar{u}_t,\Phi)+\mu\int_\Omega\nabla\bar{u}:\nabla\Phi\mathrm{d}x-\int_\Omega(\bar{u}\cdot\nabla)\bar{u}\Phi\mathrm{d}x+a(\bar{u},\Phi)\\&+b\int_\Omega|\bar{u}|^{2\alpha}\bar{u}\Phi\mathrm{d}x+c\int_\Omega|\bar{u}|^{2\beta}\bar{u}\Phi\mathrm{d}x=f^\varepsilon(x,t),\end{aligned}\tag{2.4}$$

where $\Phi\in C^\infty([0,T]\times R^3)$, $A:B=\sum_{i,j=1}^3 a_{ij}b_{ij}$. Subtracting (2.4) from (2.3) and letting $\Phi=u-\bar{u}$, it can be obtained

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u-\bar{u}|_2^2+\mu|\nabla(u-\bar{u})|_2^2+a|u-\bar{u}|_2^2+b\left||u|^\alpha|u-\bar{u}|\right|_2^2+c\left||u|^\beta|u-\bar{u}|\right|_2^2\\&\le\int_\Omega|u-\bar{u}|_2^2|\nabla\bar{u}|\mathrm{d}x+b\int_\Omega|u-\bar{u}||\bar{u}|\left||u|^{2\alpha}-|\bar{u}|^{2\alpha}\right|\mathrm{d}x\\&\quad+c\int_\Omega|u-\bar{u}||\bar{u}|\left||u|^{2\beta}-|\bar{u}|^{2\beta}\right|\mathrm{d}x\\&\equiv I_1+I_2+I_3,\end{aligned}\tag{2.5}$$

where we use $((u\cdot\nabla)u,v)=0$, $u\in V$, $v\in V$. Then we use Hölder inequality and Sobolev inequality

$$\begin{aligned}I_1&\le|u-\bar{u}|_4^2|\nabla\bar{u}|_2\\&\le C\left(|\nabla(u-\bar{u})|_2^{\frac{3}{4}}|u-\bar{u}|_2^{\frac{1}{4}}\right)^2|\nabla\bar{u}|_2\\&\le C|\nabla(u-\bar{u})|_2^{\frac{3}{2}}|u-\bar{u}|_2^{\frac{1}{2}}|\nabla\bar{u}|_2\\&\le\varepsilon|\nabla(u-\bar{u})|_2^2+C|u-\bar{u}|_2^2|\nabla\bar{u}|_2^4,\end{aligned}$$

$$\begin{aligned} I_2 &\le b\int_\Omega |u-\bar{u}||\bar{u}|\left||u|^{2\alpha}-|\bar{u}|^{2\alpha}\right|\mathrm{d}x \\ &\le C(2\alpha)\int_\Omega |u-\bar{u}|\left||u|^{2\alpha-1}+|\bar{u}|^{2\alpha-1}\right||u-\bar{u}||\bar{u}|\mathrm{d}x \\ &\le C|u-\bar{u}|_4^2|\bar{u}|_6\left||u|^{2\alpha-1}+|\bar{u}|^{2\alpha-1}\right|_3 \\ &\le C\left(|\nabla(u-\bar{u})|_2^{\frac{3}{4}}|u-\bar{u}|_2^{\frac{1}{4}}\right)^2|\bar{u}|_6\left\||u|+|\bar{u}|\right\|_{3(2\alpha-1)}^{2\alpha-1} \\ &\le C|\nabla(u-\bar{u})|_2^{\frac{3}{2}}|u-\bar{u}|_2^{\frac{1}{2}}|\bar{u}|_6\left\||u|+|\bar{u}|\right\|_{3(2\alpha-1)}^{2\alpha-1} \\ &\le \varepsilon|\nabla(u-\bar{u})|_2^2+C|u-\bar{u}|_2^2|\bar{u}|_6^4\left\||u|+|\bar{u}|\right\|_{3(2\alpha-1)}^{4(2\alpha-1)}. \end{aligned}$$

Similarly, we can get the estimate of I_3. Substitute the estimate I_1, I_2, I_3 into the inequality (2.4), select $\varepsilon=\frac{\mu}{6}$, it can be obtained

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}|u-\bar{u}|_2^2+\mu|\nabla(u-\bar{u})|_2^2+2a|u-\bar{u}|_2^2+2b\left\||u|^\alpha|u-\bar{u}|\right\|_2^2+2c\left\||u|^\beta|u-\bar{u}|\right\|_2^2 \\ &\le C|u-\bar{u}|_2^2\left(|\nabla\bar{u}|_2^4+|\bar{u}|_6^4\left[|u|_{3(2\alpha-1)}^{4(2\alpha-1)}+|\bar{u}|_{3(2\alpha-1)}^{4(2\alpha-1)}+|u|_{3(2\beta-1)}^{4(2\beta-1)}+|\bar{u}|_{3(2\beta-1)}^{4(2\beta-1)}\right]\right). \end{aligned} \tag{2.6}$$

Note that from the Gagliardo-Nirenberg inequality and a priori estimation, we can derive:

$$\begin{aligned} \int_0^T \|u\|_{3(2\alpha-1)}^{4(2\alpha-1)}\mathrm{d}t &\le \int_0^T |u|_{2\alpha+2}^{\frac{4(2\alpha+1)(\alpha+1)}{\alpha+4}}|\Delta u|_2^{\frac{4(4\alpha-5)}{\alpha+4}}\mathrm{d}t \\ &\le C\sup_{0\le t\le T}|u|_{2\alpha+2}^{\frac{4(2\alpha+1)(\alpha+1)}{\alpha+4}}\left(\int_0^T|\Delta u|_2^2\mathrm{d}t\right)^{\frac{2(4\alpha-5)}{\alpha+4}}T^{\frac{28-14\alpha}{\alpha+4}}\le C, \end{aligned} \tag{2.7}$$

using $\bar{u}$ instead of u in (2.7) also applies to the above estimation. In Equation (2.6), we have a limit on α making $0\le\frac{4(4\alpha-5)}{\alpha+4}\le 2$ establish, namely $\frac{5}{4}\le\alpha\le 2$, similarly, we limit $\frac{5}{4}\le\beta\le 2$. Substitute (2.7) into (2.6) and apply Gronwall inequality, it can be obtained that under the restriction of the above formula, for almost everywhere $(x,t)\in\Omega\times[0,T]$, there are $u=\bar{u}$. Theorem 2.1 proved. □

Next we consider the auxiliary linear equation of Equation (2.1):

$$Y_t+\mu AY+aY=K(t),\quad Y|_{t=0}=0, \tag{2.8}$$

we get the following theorem.

Theorem 2.2. *Suppose* $K\in L_{loc}^2(\mathbb{R},H)$, *then the problem* (2.8) *has a unique strong solution*:

$$Y\in C([0,T];V)\cap L^2(0,T;V),\quad Y_t\in C([0,T];V').$$

and for $\forall t\ge 0$, *the following inequality is satisfied*:

$$\|Y(t)\|^2\le C\int_0^t \mathrm{e}^{-C\mu(t-s)}|K(s)|_2^2\mathrm{d}s, \tag{2.9}$$

$$\int_t^{t+1}|Y(s)|_2^2\mathrm{d}s\le C\left(|Y(t)|_2^2+\int_t^{t+1}|K(s)|_2^2\mathrm{d}s\right). \tag{2.10}$$

Proof. First, the inner product of Equation (2.8) and Y can be obtained:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|Y|_2^2+\mu\lambda_1|Y|_2^2+a|Y|_2^2=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|Y|_2^2+\mu\|Y\|^2+a|Y|_2^2$$
$$=\left(K(t),Y\right)\le\frac{1}{a}|K|_2^2+a|Y|_2^2,$$

namely

$$\frac{\mathrm{d}}{\mathrm{d}t}|Y|_2^2+2\mu\lambda_1|Y|_2^2\le\frac{2}{a}|K|_2^2.$$

integrate the above formula on $[t,t+1]$, and then use Poincaré inequality:

$$2\mu\lambda_1\int_t^{t+1}|Y(s)|^2\,\mathrm{d}s\le-|Y(t+1)|_2^2+|Y(t)|_2^2+\frac{2}{a}\int_t^{t+1}|K(s)|_2^2\,\mathrm{d}s$$
$$\le|Y(t)|_2^2+\frac{2}{a}\int_t^{t+1}|K(s)|_2^2\,\mathrm{d}s,$$

namely

$$\int_t^{t+1}|Y(s)|_2^2\,\mathrm{d}s\le C\left(|Y(t)|_2^2+\int_t^{t+1}|K(s)|_2^2\,\mathrm{d}s\right).$$

take Equation (2.8) and AY as inner product to obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Y\|^2+\mu|AY|_2^2+a\|Y\|^2=\left(K(t),AY\right)\le\frac{1}{\mu}|K|_2^2+\mu|AY|_2^2,$$

thus, it can be obtained:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Y\|^2=-a\|Y\|^2+\frac{1}{\mu}|K|_2^2.$$

apply Gronwall inequality to the above formula in the interval $[0,t]$:

$$\|Y(t)\|^2\le C\int_0^t \mathrm{e}^{-C\mu(t-s)}|K(s)|_2^2\,\mathrm{d}s.$$

therefore, the existence of solutions can be deduced by Galerkin approximation method. Theorem 2.2 is proved. □

Let $K(t,0)=\int_0^t k(s)\,\mathrm{d}s$, $t\ge0$, next, we prove that the solution of the linear equation of singular oscillatory force converges to the solution of the average equation.

Theorem 2.3. *Let* $k\in L_{loc}^2(\mathbb{R},H)$, *suppose there is a constant* $l\ge0$ *satisfying*

$$\sup_{t\ge0}\left\{|K(t,0)|_2^2+\int_t^{t+1}|K(s,0)|_2^2\,\mathrm{d}s\right\}\le l^2, \tag{2.11}$$

then the solution $Y(t)$ *of the linear equation with singular oscillation force*

$$Y_t+\mu AX+aY=k(t/\varepsilon),\quad Y|_{t=0}=0 \tag{2.12}$$

with $\varepsilon\in(0,1)$, *for* $\forall t\ge0$, *satisfy the following inequality*

$$\|Y(t)\|^2+\int_t^{t+1}|Y(s)|_2^2\,\mathrm{d}s\le Cl^2\varepsilon^2,$$

where $C>0$ *and not related to K.*

Proof. The proof of this theorem is similar to literature [7], for the convenience of readers, a summary of the proof is given here. Firstly

$$K_\varepsilon(t)=\int_0^t k(s/\varepsilon)\mathrm{d}s=\varepsilon\int_{0/\varepsilon}^{t/\varepsilon}k(s)\mathrm{d}s=\varepsilon K(t/\varepsilon,0/\varepsilon),$$

then the following estimate of $K_\varepsilon(t)$ can be derived from Equation (2.11): $\sup_{t\geq 0}|K_\varepsilon(t)|_2\leq l\varepsilon$ and

$$\int_t^{t+1}|K_\varepsilon(s)|_2^2\,\mathrm{d}s\leq C\varepsilon^2\sup_{t\geq 0}\int_t^{t+1}|K(s,0)|_2^2\,\mathrm{d}s\leq Cl^2\varepsilon^2,$$

from Theorem 2.2, we can deduce that:

$$\int_0^t \mathrm{e}^{-C\mu(t-s)}|K_\varepsilon(s)|_2^2\,\mathrm{d}s\leq\frac{1}{1-\mathrm{e}^{-C\mu}}\sup_{t\geq 0}\int_t^{t+1}|K_\varepsilon(s)|_2^2\,\mathrm{d}s\leq Cl^2\varepsilon^2. \qquad (2.13)$$

therefore, by (2.9), (2.10), (2.13) and Poincaré inequality we can know:

$$\|Y(t)\|^2\leq Cl^2\varepsilon^2,\quad \int_t^{t+1}|Y(s)|_2^2\,\mathrm{d}s\leq C\left(|Y(t)|_2^2+\int_t^{t+1}|K(s)|_2^2\,\mathrm{d}s\right)\leq Cl^2\varepsilon^2. \qquad (2.14)$$

Equation (2.12) integrates time from 0 to *t*, $\partial_t Y+\mu AY+aY=K_\varepsilon(t)$, $Y|_{t=0}=0$. It can be derived from (2.14):

$$|Y(t)|_2^2+\|Y(t)\|^2+\int_t^{t+1}|Y(s)|_2^2\,\mathrm{d}s\leq Cl^2\varepsilon^2.$$

Theorem 2.3 proved. □

Conflicts of Interest

The authors declare no conflicts of interest.

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