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The Stephani Universe, K-Essence and Strings in the 5th Dimension

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Abstract

The Stephani universe is an inhomogeneous alternative to ΛCDM. We show that the exotic fluid driving the Stephani exact solution of Einstein's equations is an unusual form of k-essence that is linear in "velocity". Much as the Stephani universe can be embedded into (a section of) flat 5-d Minkowski space-time, we show that the k-essence obtains through dimensional reduction of a 5-d strongly coupled non-linear "electrodynamics" that, in the empty Stephani universe, corresponds to space filling magnetic branes in string/M-theory.

Subject Areas

Modern Physics

Keywords

Inhomogeneous Cosmology, K-Essence

1. Introduction

The discovery [1] [2] that distant supernovae are dimmer than they would be in an Einstein-de Sitter universe which has forced a fresh contemplation of issues almost as old as general relativity itself. If accelerated Hubble expansion driven by a cosmological constant, Λ is the explanation, one is left the difficult task of explaining why Λ is infinitesimally small in natural units of the Planck mass, and just such as to reveal itself in the present epoch [3].

A much discussed alternative is to assume that the cosmological constant vanishes and the acceleration is due to the nonlinearity of the Einstein equations: since the present universe is only homogeneous and isotropic on average, aver-

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aging of the Einstein equations will yield the usual FRW model equations plus corrections from "back-reaction" [4]. It is easy to realise this by constructing spherical LTB metrics that can account for the supernova data [5], however, such models assume a co-moving co-ordinate system and rely upon significant shear. Recalling that the inhomogeneity in the solar system is far larger than the cosmological one, yet is readily treated by post-Newtonian approximation, suggests that corrections to the usual linearly perturbed FRW model will be similarly small, and indeed this proves to be the case [6].

Still, the large-scale homogeneity of the universe is much less observationally secure than its isotropy [7], so suggesting the simplest inhomogeneous but isotropic and shear-free generalization of the FRW metric [7]:

$$\mathrm{d}s^2 = N(t,r)^2\,\mathrm{d}t^2 - R(t,r)^2\,\mathrm{d}\vec{x}^2 \tag{1}$$

Assuming a perfect fluid source, $T_{\mu\nu} = (\rho + p)u_\mu u_\nu - g_{\mu\nu}p$, the resulting Einstein equations[1] were first solved by Wyman [8]; the vanishing of the off-diagonal components of the source in co-moving co-ordinates provides:

$$\frac{R_{,t}}{NR} = \frac{\dot{R}}{R} = H(t),\ N = \frac{R_{,t}}{H(t)R} \tag{2}$$

Herein, e.g. $\dot{R} = N^{-1}R_{,t}$ indicates the proper time derivative, and $H(t)$ is a function of integration. Further, as $T_i^{\,j} = -p\delta_i^j$ implies $G_r^r = G_\theta^\theta = G_\phi^\phi$, one is led to the "pressure isotropy equation":

$$R'' - 2\frac{R'^2}{R} - \frac{R'}{r} = \frac{1}{2}f(r) \tag{3}$$

The "primes" here denote partial derivatives with respect to r and $f(r)$ is another integration function. The remaining Einstein equations can be expressed in the Friedman-like form:

$$\begin{aligned} 3H^2 &= \rho + \frac{1}{R^2}\left[\frac{2R''}{R} - \left(\frac{R'}{R}\right)^2 + \frac{4}{r}\frac{R'}{R}\right] \\ \dot{\rho} &+ 3\frac{\dot{R}}{R}[\rho + p] = 0 \end{aligned} \tag{4}$$

Wyman's objective was to obtain solutions for a barotropic equation of state $p = p(\rho)$, so that he excluded a solution that would later be rediscovered by Stephani [9]

$$f = 0,\quad H = \frac{a_{,t}}{a},\quad R = \frac{a(t)}{1 + ka(t)r^2/4} = aN \tag{5}$$

The corresponding energy density and pressure follow as:

$$\rho = 3\left[H^2 + \frac{k}{a^2}\right],\quad p = -\rho - \frac{1}{3}\frac{1 + ka(t)r^2/4}{H}\rho_{,t} \tag{6}$$

That is to say, the energy density is homogeneous while the pressure is inhomogeneous. Thus, while the Stephani model has been considered as an alterna-

[1]We use units $8\pi G = c = 1$.

tive to the ΛCDM model [10], its viability is obscured by the question: what is the nature of the perfect fluid source having these unusual properties?

In this paper we will provide an answer to the aforementioned question: the source is a particular case of "k-essence" [11], having a Lagrangian density that is linear in the "velocity". Moreover, just as the Stephani metric is exceptional in that it can be embedded into 5-dimensional Minkowski space [9], so too can the k-essence source be lifted to a 5-dimensional nonlinear "electrodynamics".

The remainder of this paper is organised as follows: in Section 2 we briefly review and reformulate k-essence in a way that makes the choice of Lagrangian density yielding (6) self-evident. Then in Section 3 we show how general k-essence models can be obtained by dimensional reduction from 5-dimensional nonlinear electrodynamics. Finally, our conclusions are presented in Section 4.

2. K-Essence and the Stephani Universe

K-essence [11] is simply the most general model Lagrangian density for a scalar field φ involving its first covariant derivative $\varphi_{,\mu}$. We take the derivative to be time-like so

$$\mathcal{L} = \mathcal{L}\left(\varphi, Y \equiv \sqrt{g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}}\right) \tag{7}$$

The use of the "velocity" Y instead of the usual $X = g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}$ as the kinematic variable considerably simplifies and clarifies the subsequent treatment, e.g. the stress-energy tensor takes the perfect fluid form $T_{\mu\nu} = (\rho + p)u_\mu u_\nu - g_{\mu\nu}p$ with the identifications

$$u_\mu = \varphi_{,\mu}/Y, \quad p = \mathcal{L}, \quad \rho = Y\mathcal{L}_{,Y} - \mathcal{L} \tag{8}$$

Indeed, in co-moving coordinates $Y = \dot{\varphi}$ and $u_\mu = \delta_\mu^0/N$, while the energy density is evidently just the Hamiltonian. The φ field equation here reads

$$\left(\mathcal{L}_{,Y} g^{\mu\nu}\varphi_{,\nu}/Y\right)_{;\mu} = \mathcal{L}_{,\varphi} \tag{9}$$

Imposing the nominal requirements of stability and causality, the adiabatic speed of sound squared is given by[2]

$$0 \le c_s^2 = \frac{p_{,Y}}{\rho_{,Y}} = \frac{\mathcal{L}_{,Y}}{Y\mathcal{L}_{,YY}} \le 1 \tag{10}$$

That is to say, the Lagrangian density must satisfy the inequalities: $\mathcal{L}_{,Y} \ge 0$ & $\mathcal{L}_{,YY} \ge 0$.

Particular classes of k-essence are factorizable models, $\mathcal{L}(\varphi, Y) = -V(\varphi)F(Y)$ (which includes tachyon models [12] [13] for $F(Y) = \sqrt{1-Y^2}$ and the Chaplygin gas [14] in the subcase of constant potential), and purely kinetic models (such as Scherrer's [15] model, $\mathcal{L}(Y) \simeq \mathcal{L}(Y_0) + \mathcal{L}''(Y_0)(Y - Y_0)^2/2$). Herein we are interested in models of the type [16]:

$$\mathcal{L}(\varphi, Y) = F(Y) - V(\varphi) = p \Rightarrow \rho = YF'(Y) - F(Y) + V(\varphi) \tag{11}$$

[2]For a derivation and further discussion see [12].

This is because in co-moving co-ordinates φ is a function of $x^0 = t$ only, so that the energy density will be homogeneous if $YF'(Y) - F(Y) = 0$, *i.e.* for some constant K

$$\mathcal{L}(\varphi, Y) = KY - V(\varphi) \tag{12}$$

Note that in this linear velocity model the pressure is nonetheless inhomogeneous via the lapse function $Y = \varphi_{,t} / N(t, x^i)$.

For the model (12)

$$V = 3\left[H^2 + \frac{k}{a^2}\right] \tag{13}$$

Combining (2), (9) and (12) implies the Hubble expansion is directly related to the potential:

$$3KH = -\partial V / \partial \varphi \tag{14}$$

Taking the partial time derivative of (13), and using (14),

$$\varphi_{,t} = \frac{2}{K}\left[\frac{k}{a^2} - H_{,t}\right] \tag{15}$$

Hence, given $a(t)$ equations (13) and (15) allow one to reconstruct the potential (at least in parametric form). For the power law expansion $a(t) = (t/t_0)^n$

$$\begin{aligned} V(t) &= 3\left[\frac{n^2}{t^2} + k\left(\frac{t_0}{t}\right)^{2n}\right] \\ \varphi(t) &= -\frac{2}{K}\left[\frac{n}{t} + \frac{kt_0}{2n-1}\left(\frac{t_0}{t}\right)^{2n-1}\right] \end{aligned} \tag{16}$$

In the de Sitter-like case $a(t) = \mathrm{e}^{H(t-t_0)}$

$$\begin{aligned} V(t) &= 3\left[H^2 + k\mathrm{e}^{-2H(t-t_0)}\right] \\ \varphi(t) &= -\frac{k}{K}\mathrm{e}^{-2H(t-t_0)} \\ \therefore V(\varphi) &= 3\left(H^2 - K\varphi\right) \end{aligned} \tag{17}$$

3. K-Essence from 5-D and Non-Linear "Electrodynamics"/Branes

Albeit the model (12) has the requisite properties to serve as the source in the Stephani universe, one seems to have traded one mystery for another: how is one to understand the linear dependence on *Y*? To answer this we recall that long before Kaluza and Klein, Nordstrom [17] proposed to obtain a scalar gravity theory from 5-dimensional "electrodynamics" by applying a "cylinder condition". More specifically, let the 5-dimensional co-ordinates be denoted by $x^M = (x^\mu, y)$ and for the 5-vector potential $A_M = (A_\mu, \varphi)$; assuming A_M is independent of the fifth co-ordinate, the 5-dimensional field strength $F^{(5)}_{MN} = A_{N,M} - A_{M,N}$ decomposes as $F^{(5)}_{\mu\nu} = F_{\mu\nu}$, $F^{(5)}_{\mu 5} = \varphi_{,\mu}$. Then taking the 5-dimensional metric $g^{(5)}_{MN}$ of the form $g^{(5)}_{\mu\nu} = g_{\mu\nu}$, $g^{(5)}_{\mu 5} = 0$, $g^{(5)}_{55} = -1$, we have:

$$-\frac{1}{2}F^{(5)}\cdot F^{(5)}\equiv-\frac{1}{2}F^{(5)}_{MN}F^{(5)MN}=-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}+g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}=-\frac{1}{2}F\cdot F+Y^2 \quad (18)$$

As $[A\bullet A]^{(5)}=A_M A^M=A_\mu A^\mu-\varphi^2=A\bullet A-\varphi^2$, for compact y it follows that *any* k-essence model $\mathcal{L}(\varphi,Y)$ can be obtained from a 5-dimensional model[3] $\mathcal{L}^{(5)}\left(\sqrt{-[A\bullet A]^{(5)}},\sqrt{-\frac{1}{2}F^{(5)}\cdot F^{(5)}}\right)$ by dimensional reduction provided we also set $A_\mu=0$.

Taking the range of the fifth co-ordinate as $0\le y\le l_5$, for our model source in the Stephani universe

$$l_5\mathcal{L}^{(5)}=K\sqrt{-\frac{1}{2}F^{(5)}\cdot F^{(5)}}-V\left(\sqrt{-[A\bullet A]^{(5)}}\right) \quad (19)$$

Similar kinetic terms appear in the context of nonlinear Born-Infeld electrodynamics and D-branes in string/M-theory. Of particular note is that Nielson and Oleson [18] proposed a Lagrangian density of the form $\sqrt{-\frac{1}{2}F\cdot F}$ as a field theory for closed dual strings identified as magnetic field lines. In our case $F^{(5)}_{05}\neq 0$ is electric but its dual is the Kalb-Ramond field strength H_{ijk} that is purely magnetic [19]. We thus suggest that the kinetic part of (19) be understood as originating in string/M-theory space filling magnetic branes.

4. Conclusion

In this paper, we have considered the issue of the matter source in the Stephani universe as an inhomogeneous alternative to the FRW model with a cosmological constant. We have shown that a form of k-essence has the requisite properties to be that source, and that this k-essence can be obtained by dimensional reduction of a 5-dimensional model truncation of string/M-theory.

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Conflicts of Interest

The authors declare no conflicts of interest.

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