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Homoclinic Bifurcation of a Quadratic Family of Real Functions with Two Parameters

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Abstract

In this work the homoclinic bifurcation of the family $H=\{h_{a,b}(x)=ax^2+b : a\in\mathbb{R}/\{0\}, b\in\mathbb{R}\}$ is studied. We proved that this family has a homoclinic tangency associated to $x=0$ of P_1 for $b=\frac{-2}{a}$. Also we proved that $W^u(P_1)$ does not intersect the backward orbit of P_1 for $b>\frac{-2}{a}$, but has intersection for $b<\frac{-2}{a}$ with $a>0$. So H has this type of the bifurcation.

Subject Areas

Dynamical System

Keywords

Local Unstable Set, Unstable Set, Homoclinic Point, Homoclinic Orbit, Non-Degenerate, Homoclinic Tangency, Homoclinic Bifurcation

1. Introduction

There are various definitions for the homoclinic bifurcation. In the sense of Devaney, the homoclinic bifurcation is a global type of bifurcations, that is, this type of bifurcation is a collection of local and simple types of bifurcations [1] (like, period-doubling and saddle-node of bifurcation [2]).

According to [3] [4] [5] we have another definition of the homoclinic bifurcation via the notions of the unstable sets of a repelling periodic point (fixed point) and the intersection of this set with the backward orbits of this point.

The purpose of this work is to prove the family

$H=\{h_{a,b}(x)=ax^2+b : a\in\mathbb{R}/\{0\}, b\in\mathbb{R}\}$ has homoclinic bifurcation at $b=\frac{-2}{a}$

following the second definition.

2. Definitions and Basic Concepts

2.1. Definition 1: [6]

A fixed point P is said to be **expanding** for a map *f*, if there exists a neighborhood $U(P)$ such that $|f'(x)|>1$ for any $x\in U(P)$.

The neighborhood in the previous definition is exactly **the local unstable set.**

2.2. Definition 2: [7]

Let P be a repelling fixed point for a function $f:I\to I$ on a compact interval $I\subset R$, then there is an open interval about P on which *f* is one-to-one and satisfies the **expansion** property. $|f(x)-f(P)|>|x-P|, \forall x\in I$ where $x\neq P$.

The interval in the previous definition is exactly **the unstable set of *P*.**

2.3. Definition 3: [8]

Let P is fixed point and $f'(P)>1$ for a map $f:\mathbb{R}\to\mathbb{R}$. A point q is called **homoclinic point to *P*** if $q\in w_{loc}^{u}(P)$ and there exists $n>0$ such that $f^{n}(q)=P$.

2.4. Definition 4: [9]

The union of the forward orbit of q with a suitable sequence of preimage of q is called **the homoclinic orbit of *P*.** That is $O(q)=\{P,\cdots,q_{-2},q_{-1},q,q_1,q_2,\cdots,q_m=P\}$ where $q_{i+1}=f(q_i)$ for $i\leq m-1$, $q_m=P$ and $\lim_{i\to-\infty}q_i=P$.

2.5. Definition 5: [10]

The critical x point is **non-degenerate** if $f''(x)\neq 0$. The critical point x is **degenerate** if $f''(x)=0$.

2.6. Definition 6: [11]

Let *f* be a smooth map on $I\subset R$, and let p be a hyperbolic fixed point for the map *f*. If $W^{u}(p)$ intersects the backward orbit of p at a nondegenerate critical point x_{cr} of *f*, then x_{cr} is called a **point of homoclinic tangency associated to *p*.**

2.7. Definition 7: [3]

Let f_{φ} be a smooth map on $I\subset R$, and let p be a hyperbolic fixed point for a map f_{φ}. We say that f_{φ} has **homoclinic bifurcation associated to *p* at** $\varphi=\hat{\varphi}$ if:

1) For $\varphi<\hat{\varphi}$ ($\varphi>\hat{\varphi}$), $W^{u}(p)$ and the backward orbit of p has no intersect.

2) For $\varphi=\hat{\varphi}$, $f_{\hat{\varphi}}$ has a point of homoclinic tangency x_{cr} associated to p.

3) For $\varphi>\hat{\varphi}$ ($\varphi<\hat{\varphi}$), the intersection of $W^{u}(p)$ with the backward orbit of p is nonempty.

3. Homoclinic Bifurcation of the Family $H=\{h_{a,b}(x)=ax^2+b : a\in\mathbb{R}/\{0\}, b\in\mathbb{R}\}$

In this section, we show that the family H has a point of homoclinic tangency associated to P_1 at $b=\frac{-2}{a}$, and H has a homoclinic bifurcation.

We need the following results proved in [12].

At the first, the fixed points of $h_{a,b}(x)$ are

$$P_1=\frac{1+\sqrt{1-4ab}}{2a}, P_2=\frac{1-\sqrt{1-4ab}}{2a}.$$

a) Proposition:

For $h_{a,b}(x)\in H$ with $a>0$ the local unstable set of the fixed point P_1 is $w^u_{loc}(P_1)=\left(\frac{1}{2|a|},\infty\right)$.

b) Lemma:

For $h_{a,b}(x)\in H$, $h^{-1}_{a,b}(P_1)=\mp\sqrt{\frac{P_1-b}{a}}=\mp P_1$ where $P_1>b$ for $a>0$.

c) Theorem:

For $h_{a,b}(x)\in H$ with $a>0$, the unstable set of the fixed point P_1 is $w^u(P_1)=\left(\frac{1}{|a|}-P_1,\infty\right)$.

d) Remark: [13]

The local unstable set of the fixed point P_2 is $w^u_{loc}(P_2)=\left(-\infty,\frac{-1}{2|a|}\right)$, and the unstable set of the fixed point P_2 is $w^u(P_2)=\left(-\infty,\frac{-1}{|a|}-P_2\right)$. In this work we will omit the result about P_2 because ($h'_{a,b}(P_2)<-1$, when $b<\frac{-3}{4a}$ for $a>0$ $b>\frac{-3}{4a}$ for $a<0$). Thus we will not care for the fixed point P_2. (See definition (2.3)).

e) Remark:

For $h_{a,b}(x)\in H$, $h^{-2}_{a,b}(P_1)=\mp\sqrt{\frac{-P_1-b}{a}}$.

f) Proposition:

For $h_{a,b}\in H$, if $b<\frac{-(5+2\sqrt{5})}{4a}$ with $a>0$, then the second preimage of the fixed point P_1 belongs to the local unstable set of P_1.

g) Proposition:

For $h_{a,b}\in H$, if $\frac{-(5+2\sqrt{5})}{4a}\le b\le\frac{-2}{a}$ with $a>0$, then the third preimage of the fixed point P_1 belongs to the local unstable set of P_1.

h) Theorem:

For the family $H=\{h_{a,b}(x)=ax^2+b : a>0\}$, there exist homoclinic points to the fixed point P_1 whenever $b\le\frac{-2}{a}$. Moreover $h_{a,b}^{-2}(P_1)=q_{1,1}, h_{a,b}^{-3}(P_1)=q_{2,1}$ are the first homoclinic points for $b<\frac{-(5+2\sqrt{5})}{4a}$, $\frac{-(5+2\sqrt{5})}{4a}\le b\le\frac{-2}{a}$ respictivelity.

i) Example:

For $h_{1,-6}(x)=x^2-6$, a homoclinic orbit of a homoclinic point $\sqrt{3}$ is: $O(\sqrt{3})=\{3,-3,\sqrt{3},\cdots,3\}$.

The main result:

3.1. Lemma 1

For $h_{a,b}(x)=ax^2+b$ with $a\in\mathbb{R}/\{0\}$, the critical point of $h_{a,b}(x)$ is 0, and it is a non-degenerate critical point.

Proof:

Clearly that the critical point of $h_{a,b}(x)$ is zero.

Since $a\ne 0$, then

$$h''_{a,b}(x)=2a\ne 0.$$

So $h_{a,b}(x)$ has a non-degenerate critical point at $x=0$. ■

3.2. Lemma 2

If $b=0$ of $h_{a,b}(x)\in H$ with $a\in\mathbb{R}/\{0\}$, then the backward orbit of the repelling fixed point P_1 is undefined in $\mathbb{R}$.

Proof:

$h_{a,0}(x)=ax^2$, clearly $P_1=\frac{1}{a}$.

Now the first preimage of $h_{a,0}(x)$ is

$h_{a,0}^{-1}(x)=\mp\sqrt{\frac{x}{a}}$, where $x>0$ for $a>0$.

By Lemma (3-b), we have

$$h_{a,0}^{-1}\left(\frac{1}{a}\right)=\mp\sqrt{\frac{1}{a^2}}=\mp\frac{1}{a}=\mp P_1.$$

But $+P_1$ is a fixed point, and $-P_1=-\frac{1}{a}\notin w_{loc}^{u}(P_1)=\left(\frac{1}{2a},\infty\right)$, see Proposition (3-a).

By Remark (3-e) we have

$$h_{a,0}^{-2}(P_1)=\mp\sqrt{\frac{-P_1}{a}}=\mp\sqrt{\frac{-\frac{1}{q}}{a}}=\mp\sqrt{\frac{-1}{a^2}}\notin\mathbb{R},$$

since $\frac{1}{a^2} > 0$, $\forall a \in \mathbb{R}/\{0\}$.

Therefore $h_{a,0}^{-n}(P_1)$ are undefined in $\mathbb{R}$ with $n \geq 2$.

Thus the backward orbit of the repelling fixed point P_1 is undefined in $\mathbb{R}$ ■

3.3. Theorem 1

For the family $h_{a,b}(x) = ax^2 + b$, 0 belong to the backward orbit of P_1 whenever $b = \frac{-2}{a}$ with $a \in \mathbb{R}/\{0\}$, and the backward orbit of P_1 is:

$$h_{a,\frac{-2}{a}}^{-n}\left(P_1 = \frac{2}{a}\right) = \left\{\frac{2}{a}, -\frac{2}{a}, 0, \frac{\sqrt{2}}{a}, \cdots\right\}.$$

Proof:

We test the values of n which makes $h_{a,b}^{-n}(P_1) = 0$.

By Lemma (3-b), $h_{a,b}^{-1}(P_1) = \pm P_1$.

So $h_{a,b}^{-1}(P_1) \neq 0$.

Now suppose that $h_{a,b}^{-2}(P_1) = 0$, by Remark (3-e) then

$h_{a,b}^{-2}(P_1) = \mp\sqrt{\frac{-P_1 - b}{a}} = 0$, thus

$$\frac{-P_1 - b}{a} = 0$$

$$-P_1 - b = 0$$

$$P_1 = b.$$

Since the fixed point $P_1 = \frac{1+\sqrt{1-4ab}}{2a}$, therefore

$$\frac{1+\sqrt{1-4ab}}{2a} = -b,$$

then

$$1+\sqrt{1-4ab} = -2ab$$

$$\sqrt{1-4ab} = -2ab - 1$$

$$1-4ab = 4a^2b^2 + 4ab + 1$$

$4a^2b^2 + 8ab = 0$, which implies

$4ab(ab+2) = 0$, then either

$b = 0$, but by the above Lemma (3.2) the backward orbit of P_1 is undefined, so we omit this case.

Or $ab + 2 = 0$, thus

$$b = \frac{-2}{a}.$$

Now, $P_1 = \frac{2}{a}$ and to find the backward orbit of P_1, we consider

$$h_{a,\frac{-2}{a}}^{-1}(x) = \pm\frac{\sqrt{ax+2}}{a}.$$

By Lemma (3-b) $h_{a,b}^{-1}(P_1) = \pm P_1$, then

$h_{a,\frac{-2}{a}}^{-1}\left(\frac{2}{a}\right) = \pm\frac{2}{a}$. But $+\frac{2}{a}$ is a fixed point, therefore

$$h_{a,\frac{-2}{a}}^{-1}\left(\frac{2}{a}\right) = -\frac{2}{a}.$$

So

$$h_{a,\frac{-2}{a}}^{-2}\left(\frac{2}{a}\right) = \frac{\sqrt{a\left(\frac{-2}{a}\right)+2}}{a} = 0.$$

$$h_{a,\frac{-2}{a}}^{-3}\left(\frac{2}{a}\right) = \frac{\sqrt{a(0)+2}}{a} = \frac{\sqrt{2}}{a}, \text{ and so on.}$$

Therefore the backward orbit of $P_1 = \frac{2}{a}$ is:

$$h_{a,\frac{-2}{a}}^{-n}\left(P_1 = \frac{2}{a}\right) = \left\{\frac{2}{a}, -\frac{2}{a}, 0, \frac{\sqrt{2}}{a}, \cdots\right\}. \blacksquare$$

3.4. Example

For $h_{1,-2}(x) = x^2 - 2$, 0 belongs to the backward orbit of $P_1 = 2$ (**Figure 1**), and the backward orbit of P_1 is $h_{1,-2}^{-n}(2) = \left\{2, -2, 0, \sqrt{2}, \cdots, 2\right\}$.

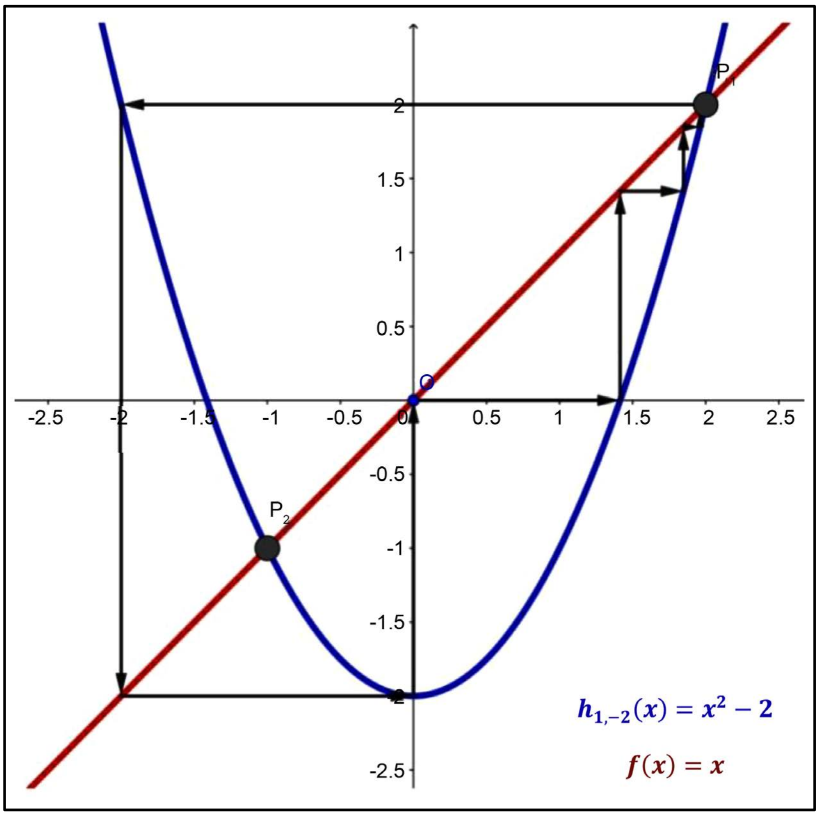

Figure 1. For $h_{1,-2}(x) = x^2 - 2$, the backward orbit of $P_1 = 2$.

3.5. Theorem 2

If $b > \frac{-2}{a}$ for $h_{a,b}(x) \in H$ with $a > 0$, then there is no intersection of the backward orbit with the unstable set of P_1.

Proof:

The backward orbit of P_1

By Lemma (3-b) $h_{a,b}^{-1}(P_1) = \pm P_1$, since $+P_1$ is a fixed point, then we consider

$$h_{a,b}^{-1}(P_1) = -P_1.$$

By Remark (3-e), $h_{a,b}^{-2}(P_1) = \mp\sqrt{\frac{-P_1 - b}{a}}$.

If $-P_1 > b$, then by Theorem (3-h),

$b \le \frac{-2}{a}$ which is a contradiction with $b > \frac{-2}{a}$. Therefore

$-P_1 < b$, which implies

$$h_{a,b}^{-2}(P_1) \notin \mathbb{R}.$$

So $h_{a,b}^{-n}(P_1)$ are undefined in $\mathbb{R}$ with $n \ge 2$.

Thus the backward orbit of P_1 is undefined .

So the intersection of $W^u(P_1)$ with the backward orbit of P_1 is also undefined. ■

3.6. Theorem 3

If $b = \frac{-2}{a}$ for $h_{a,b}(x) \in H$ with $a > 0$, then $h_{a,\frac{-2}{a}}$ has a point of homoclinic tangency at 0 associated to P_1.

Proof:

By Theorem (3.3), $h_{a,\frac{-2}{a}}^{-n}\left(P_1 = \frac{2}{a}\right) = \left\{\frac{2}{a}, -\frac{2}{a}, 0, \frac{\sqrt{2}}{a}, \cdots\right\}$.

By Theorem (3-c), $W^u(P_1) = \left(\frac{1}{a} - P_1, \infty\right)$, then

$W^u\left(P_1 = \frac{2}{a}\right) = \left(\frac{1}{a} - \frac{2}{a}, \infty\right)$, *i.e.*

$W^u\left(P_1 = \frac{2}{a}\right) = \left(-\frac{1}{a}, \infty\right)$. Now

$h_{a,\frac{-2}{a}}^{-n}\left(\frac{2}{a}\right)$ intersects $W^u\left(P_1 = \frac{2}{a}\right)$ at 0.

By Lemma (3.1) 0 is a non-degenerate critical point. So $h_{a,\frac{-2}{a}}$ has a point of homoclinic tangency at 0 associated to P_1. ■

3.7. Theorem 4

If $b < \frac{-2}{a}$ for $h_{a,b}(x) \in H$ with $a > 0$, then the backward orbit of P_1 crosses the unstable set $W^u(P_1)$.

Proof:

First consider the backward orbit of P_1.

By Lemma (3-b) $h_{a,b}^{-1}(P_1) = \pm P_1$.

But + P_1 is a fixed point, therefore we consider

$$h_{a,b}^{-1}(P_1) = -P_1.$$

By Remark (3-e), $h_{a,b}^{-2}(P_1) = \mp\sqrt{\dfrac{-P_1 - b}{a}}$.

Since $b < \dfrac{-2}{a}$, then by Theorem (3-h)

$$h_{a,b}^{-2}(P_1) \in \mathbb{R}.$$

Let $h_{a,b}^{-2}(P_1) = q_{1,1}$, $h_{a,b}^{-3}(P_1) = q_{2,1}$.

By Proposition (3-f), if $b < \dfrac{-(5+2\sqrt{5})}{4a}$, then $q_{1,1} \in W_{loc}^{u}(P_1)$.

By Proposition (3-g), if $\dfrac{-(5+2\sqrt{5})}{4a} \le b < \dfrac{-2}{a}$, then $q_{2,1} \in W_{loc}^{u}(P_1)$.

Now since the local unstable set of the repelling fixed point contained in the unstable set of the repelling fixed point. Therefore

$$h_{a,b}^{-n}(P_1) \cap W^{u}(P_1) \neq \varnothing. \blacksquare$$

Following examples explain the cases for $b > \dfrac{-2}{a}$, $b = \dfrac{-2}{a}$ and $b < \dfrac{-2}{a}$ (with $a > 0$) respectively.

3.8. Example 1

For $h_{1,-1}(x) = x^2 - 1$, we have no intersection of the backward orbit of P_1 with the unstable set of P_1.

Solution:

Consider the fixed point of $h_{1,-1}(x)$ is $P_1 = \dfrac{1+\sqrt{5}}{2}$, and

$$h_{1,-1}^{-1}(x) = \pm\sqrt{x+1}.$$

The backward orbit of $P_1 = \dfrac{1+\sqrt{5}}{2}$

$h_{1,-1}^{-1}\left(\dfrac{1+\sqrt{5}}{2}\right) = \pm\dfrac{1+\sqrt{5}}{2}$, where $+\dfrac{1+\sqrt{5}}{2}$ is a fixed point, therefore we consider $h_{1,-1}^{-1}\left(\dfrac{1+\sqrt{5}}{2}\right) = -\dfrac{1+\sqrt{5}}{2}$. Now

$$h_{1,-1}^{-2}\left(\dfrac{1+\sqrt{5}}{2}\right) = \mp\sqrt{-\dfrac{1+\sqrt{5}}{2}+1} \notin \mathbb{R}.$$

So $h_{1,-1}^{-n}\left(\dfrac{1+\sqrt{5}}{2}\right)$ are undefined in $\mathbb{R}$ with $n \ge 2$.

Thus the backward orbit of P_1 is undefined.

So the intersection of $W^u\left(\frac{1+\sqrt{5}}{2}\right)$ with the backward orbit of P_1 is also undefined. ■

3.9. Example 2

For $h_{1,-2}(x)=x^2-2$, then $h_{1,-2}$ has a point of tangency at 0 associated to P_1.

Solution:

Consider the fixed point of $h_{1,-2}(x)$ is $P_1=2$.

By Example (3.4), The backward orbit of $P_1=2$ is

$$h_{1,-2}^{-n}(2)=\{2,-2,0,\sqrt{2},\cdots,2\}.$$

On the other hand, the unstable set of $P_1=2$ is

$W^u(2)=(-1,\infty)$, (see Theorem (3-c)). Now

$h_{1,-2}^{-n}(2)$ intersects $W^u(2)$ at 0.

By Lemma (3.1), 0 is a non-degenerate critical point. So $h_{1,-2}$ has a point of tangency at 0 associated to $P_1=2$. ■

3.10. Example 3

For $h_{1,-6}(x)=x^2-6$, the backward orbit of P_1 crosses the unstable set $W^u(P_1)$.

Solution:

First consider the fixed point $P_1=3$.

The backward orbit of 3 is:

$h_{1,-6}^{-n}(3)=\{3,-3,\sqrt{3},\cdots,3\}$ (see Example (3-i)), with

$h_{1,-6}^{-1}(3)=h_{1,-2}^{2}(\sqrt{3})$, and $h_{1,-6}^{-2}(3)=\sqrt{3}$.

Since $\sqrt{3}$ is a homoclinic point of $P_1=3$, then

$$\sqrt{3}\in W_{loc}^{u}(3).$$

Now since the local unstable set of the repelling fixed point $P_1=3$ contained in the unstable set of the repelling fixed point $P_1=3$. Therefore

$$h_{1,-6}^{-n}(3)\cap W^u(3)\neq\varnothing.\ ■$$

Note , the main theorem in the work :

3.11. Theorem 5

$h_{a,b}(x)=ax^2+b, a>0$ has a homoclinic bifurcation associated to the repelling fixed point of $h_{a,b}$, $P_1=\frac{1+\sqrt{1-4ab}}{2a}$, at $b=\frac{-2}{a}$.

Proof:

1) For $b>\frac{-2}{a}$, by Theorem (3.5) the intersection of the backward orbit of P_1 and the unstable set of P_1 is undefined.

2) For $b=\frac{-2}{a}$, by Theorem (3.6) $h_{a,\frac{-2}{a}}$ has a point of homoclinic tangency

associated to P_1 at $x=0$.

3) For $b<\frac{-2}{a}$, by Theorem (3.7) the backward orbit of P_1 crosses the unstable set of P_1, $W^u(P_1)$.

Therefore $h_{a,b}$ has a homoclinic bifurcation associated to P_1 at $b=\frac{-2}{a}$. ■

3.12. Example

$h_{1,-2}(x)=x^2-2$ has a homoclinic bifurcation associated to the repelling fixed point of $h_{1,-2}$, $P_1=2$, at $b=-2$.

$h_{1,-2}(x)$ has a homoclinic bifurcation associated to the repelling fixed point of $h_{1,-2}$, $P_1=2$, at $b=-2$. See examples (3.8), (3.9), (3.10).

3.13. Remark

For $a<0$, we have same results which proved above for $a>0$. In fact, we can prove in similar ways, that: $h_{a,b}(x)=ax^2+b, a<0$ has a homoclinic bifurcation associated to the repelling fixed point of $h_{a,b}$, $P_1=\frac{1+\sqrt{1-4ab}}{2a}$, at $b=\frac{-2}{a}$.

4. Conclusion

We conclude that the family $H=\left\{h_{a,b}(x)=ax^2+b:a\in\mathbb{R}/\{0\},b\in\mathbb{R}\right\}$ has homoclinic tangency associated to P_1 at the critical point $x=0$. Also for $b>\frac{-2}{a}$ we have no intersection between the backward orbit of P_1 and the unstable set of P_1, and the backward orbit of P_1 crosses the unstable set of P_1 for $b<\frac{-2}{a}$. So we have homoclinic bifurcation at $b=\frac{-2}{a}$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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