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Reverse Building of Complete (k,r)-Arcs in PG(2,q)

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Abstract

The purpose of this work is to study the construction of complete (k_i,i)-arcs in PG(2,9), where $i = q, q-1, \cdots, 2$ by eliminating points from a complete (k_n,n)-arc to get a complete(k_m,m)-arc, where $m < n$. And we adopted a new sequential way to delete points [1] [2].

Subject Areas

Algebraic Geometry

Keywords

Complete Arcs, Maximal Arcs, Galois Geometry

1. Introduction

A projective plane PG(2,q) over Galois field GF(p), where q is a prime number, consists of q^2+q+1 points and q^2+q+1, every line contains $q+1$ point and every point is on $q+1$ lines [3]. Any point of the plane has the form of a triple, where X_0, X_1, X_2 are elements in PG(q) with the exception of a triple consisting of three zero elements. Two triples (X_0, X_1, X_2) and (Y_0, Y_1, Y_2) represent the same point if there exists j in GF(P)/{0}, s.t. $(Y_0, Y_1, Y_2) = J(X_0, X_1, X_2)$.

The points in PG(2,q) have unique forms which are (1,0,0), (X,1,0), (X,Y,1) for all X, Y in GF(q). There exist one point of the from (1,0,0), q points of the from (X,1,0) and q^2 points of the from (X,Y,1) similarly any line in PG(2,q) has the form (X_0, X_1, X_2), X_0, X_1, X_2 are elements in GF(q) with the exception of a triple consisting of three zero elements. Two triples represent the same line if thereexist J GF(q)\{0}, s.t $(Y_0, Y_1, Y_2) = J(X_0, X_1, X_2)$. A point $P(X_0, X_1, X_2)$ is incident with the line (Y_0, Y_1, Y_2) iff $x_0y_0 + x_1y_1 + x_2y_2 = 0$ [4].

Finally, the points of PG(2,q) can be numerated as follows: the number of the point $(1,0,0)$ is 1, the point is $(X,1,0)$ numerated as X + 2, the point $(X,Y,1)$ is numerated as $(x+(x\cdot q)+q+2)$ [5].

Definition 1: A(K,n)-a in PG(2,q) is asset of K points such that NO n + 1 points them are collinear A(K,2)-arc which is called K-arc is a set of K points such that NO three of them are collinear [6].

Definition 2: A(k,n)-arc is a complete if it is not contained in a (k + 1,n)-arc [6].

Definition 3: The maximum number of points that a (K,2)-arc can have is m(2,q) and a (K,2)-arc with this number of points is an Oval. In the even case Ovals are complete [7].

Theorem 1: $M(2,q)=\begin{cases} q+1 & \text{for } q \text{ is odd} \\ q+2 & \text{for } q \text{ is even} \end{cases}$ [7].

Theorem 2: In PG(2,q), with q odd, every oval is conic [8].

Definition 4: The i-secant of a (K,n)-arc K is a line intersects the arc in exactly i points, a 0-secant is called an external line K, a 1-secant is called a uni secant line, 2-secant is called a bisecant line and 3-secant is called a trisecant line [9].

Corollary 1: A (k,n)-arc K is a maximal if and only if every line PG(2,q) is a 0-secant, or an N-secant [10] [11].

Theorem 3: Let m be a point of a (K,2)-arc K and let t(m) be the number of unisecants through m in PG(2,q) then $t=t(m)=q+2-k$ [12].

Proof: In PG(2,q), there exist exactly q + 1 lines through q. since M is on k then each line passing through M is either unisecant or bisecant of k. There exists exactly k-1 lines joining M with the remaining k-1 points of k which are bisecant of k. then the number of unisecant through M is $q+1-(k-1)=q+2-k$.

$$(i.e.).\ t(m)=q+2-k=t.$$

Corollary 2: If k is an oval then t(m) = 1 [13].

Theorom 4: Let k be a k-arc in PG(2,q) and let T_i be the number of i-secant of k in the plane, that is, T_2 is the number of bisecant, T_1 the number of unisecants, and T_0 the number of external line, then [13]:

1) $T_2=k(k-1)/2$;
2) $T_1=Kt$;
3) $T_0=q(q-1)+t(t-1)/2$.

Proof: 1) We have K points in arc K; we take two points from these k point to find the bisecant of k so.

2) There exist exactly K points on arc K. Each point of K has exactly $t=q+2-k$ line of K. The number of unisecants of K is exactly

$$T_1=Kt=K(q+2-k).$$

3) Each line of the plane PG(2,q) is either bisecant, unisecant or external line. The number of lines is $\text{PG}(2,q)=q^2+q+1=T_0+T_1+T_2$.

$$T_0 = q^2 + q + 1 - T_1 \ ? \ T_2 = \frac{q(q-1)}{2} + \frac{t(t-1)}{2}$$

Corollary: For a $(q+1)$-arcs, $t = 1$, $T_0 = \frac{q(q-1)}{2}$, $T_1 = q+1$, $T_2 = \frac{q(q+1)}{2}$ [13].

Definition 5: Let Q be a point of PG(2,q) not on the K-arc. Let $S_i(Q)$ be the number of i-secants through Q. The number of bisecants $S_2(Q)$ is called the index of Q with respect to K and the number of unisecant $S_1(Q)$ is called the grade of Q with respect to K [14].

Lemma 1: For any point Q in PG(2,q)\K, then $S_1(Q) + 2S_2(Q) = K$ [14].

Proof: Since each unisecant of K. passes through one point of the arc and each bisecant passing through two points of the arc, the number of the points of the arc is K, then $S_1(Q) + 2S_2(Q) = K$.

Lemma 2: Let C_i be the number of points Q of index i. Then

1) $\sum_{\alpha}^{\beta} C_i = q^2 + q + 1 - k$;

2) $\sum_{\alpha}^{\beta} iC_i = k(k-1)(q-1)/2$;

where α is smallest i for Which $C_i \neq 0$, and β is the largest i for Which $C_i \neq 0$.

Proof: 1) $\sum C_i$ represents all the points of the plane not in K. Since the number of point in the plane is $q^2 + q + 1$, then $\sum_{\alpha}^{\beta} C_i = q^2 + q + 1 - k$

2) $\sum_{\alpha}^{\beta} iC_i = C_1 + 2C_2 + 3C_3 + \cdots$

{(Q,l)/Q € l\k, l is a bisecant of K} each bisecant contains $q - 1$ points not in K.

There are $K!/2!(K-2)!$ Bisecant of K. Then there exist

$K(K-1)/2(q-1)$ OF points satisfying the equation

$$\sum_{\alpha}^{\beta} iC_i = K(K-1)/2(q-1)$$

Remark: The (k,n)-arc K is complete if and only if, $C_0 = 0$. Thus, K is complete if every point of PG(2,q) lines on some n-secant of K [15] [16].

2. The Additions and Multiplications Operations of GF(9)

To find the addition and multiplication tables, in GF(9), We have the order pier (x_0, x_1) such that x_0, x_1 in GF(3), as follows:

$$0 = (0,0),\ 1 = (1,0),\ 2 = (2,0),\ 3 = (0,1),\ 4 = (1,1),$$
$$5 = (2,1),\ 6 = (0,2),\ 7 = (1,2),\ 8 = (2,2)$$

Put these points in one orbit (1,0) at the first point and by the principle of (0,1) A_i, $i = 0,1,2,\cdots,7$ and

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$(1,0) \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(0,1) (1,1) (1,2) (2,0) (0,2)(1,2) (2,2)

Now, in the left of the following **Table 1** is the operation of multiplication, and in the right n is the operation of addition; in multiplication side, it writes the numeration of points as last, and the addition side take the normal sequence [15] [16].

Mod (8): In addition (Table 2), We have the following relation: $(x_0, x_1) + (y_0, y_1) = (z_0, z_1)$.

In multiplication (Table 3), we have the following relations $M_1 * M_2 = M_3$ $[(1,0) Af(m_1)] Af(m_2) = (1,0) Af(m_1) + f(m_2) \bmod 8 = (X_1, X_2)$. For Example: $3 \times 7 = 2[(1,0)A]A = (1,0)A = (2,0)$ where (2,0) is equal to 2 in multiplication side. Now we have the multiplication (Table 3).

Table 1. Operation of multiplication.

$M(*)$		$(+)n = f(m)$
1	(1,0)	0
2	(2,0)	4
3	(0,1)	1
4	(1,1)	2
5	(2,1)	7
7	(1,2)	3
8	(2,2)	6

Table 2. Point addition.

+	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	0	4	5	3	7	8	6
2	2	0	1	5	3	4	8	6	7
3	3	4	5	6	7	8	0	1	2
4	4	5	3	7	8	6	1	2	0
5	5	3	4	8	6	7	2	0	1
6	6	7	8	0	1	2	3	4	5
7	7	8	6	1	2	0	4	5	3
8	8	6	7	2	0	1	5	3	4

Table 3. Point multiplication.

*	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	6	8	7	3	5	4
3	3	6	4	7	1	8	2	5
4	4	8	7	2	3	5	6	1
5	5	7	1	3	8	2	4	6
6	6	3	8	5	2	4	1	7
7	7	5	2	6	4	1	8	3
8	8	4	5	1	6	7	3	2

2.1. The Construction Complete (k_i,i)-Arc, Where $i = 2,3,\cdots,10$ in PG(2,9) over GF(9)

The projective plane PG(2,q) contains (91) points and (91) lines, every line contains (10) points and every point is on (10) lines. Any line in PG(2,q) can be constructed by means of variety V. let Pi and $L_i, i = 1,2,\cdots,91$ be the points and lines of PG(2,9), respectively. Let i stands for the point $P_i[i]$ stands for the L_i whose coordinates are the same coordinates of the point P_i, and all the points and the lines of PG(2,9) are given in Table 4.

2.2. The Construction of (k_{10},10)-Arc

If i = 10, the M(10,9) = 91 which is the maximal arc, since every line in PG(2,9) is a 10-secan of the (K,10)-arc. This arc contains The construction of (k_9,9)-arc, from the (k_{10},10)-arc: all the points of the plane PG(2,9), So it is a complete arc. Now, we shall construct the (K,m)-arcs as given in Table 4.

2.3. The Construction of (k_9,9)-Arc, from the (k_{10},10)-Arc

We eliminate one line from the (k_{10},10)-arc, say the line L_{I1} = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. On ather hand, in projective plane any two distinct lines are intersected in a unique point, the eliminating line intersects any line of PG(2,9) in exactly one point consequently, we eliminate one point from any line in the plane PG(2,9) The eliminated line is a 0-secant of K_9 and the remaining (90) lines are the 9 secants of the arc. We find:

1) K_9 is a maximal (81,9)-arc in PG(2,9), since every line in PG(2,9) is either 0-secant or a 9-secant of K_9, as given in Table 5.

2) K_9 is a complete (81,9)-arc since there are no point 0f index zero for K_9 *i.e.* $C_0 = 0$.

2.4. The Construction of (k_8,8)-Arc k from k_9

In this section, we construct (K_8,8)-arc K_8 from K_9 by eliminating the line L_2 = [11, 12, 13, 14, 15, 16, 17, 18, 19] and following points [20, 29, 38, 47, 56, 65, 74, 83] then find:

1) K_8 is not a maximal (64,8)-arc in PG(2,9) since every line in PG(2,9) is either 0-secant or a 8-secant of K, as given in Table 6.

2) K_8 is acomplete (64,8)-arc since there are no point of index zero for K_8 *i.e.*, $C_0 = 0$.

2.5. The Construction of (k_7,7)-Arc K from K_8

We construct a (K_7,7)-arc K from K_8 by eliminating one line, the line L_{29}: [20, 21, 22, 23, 24, 25, 26, 27, 28] and the following points [30, 39, 48, 57, 66, 75, 84], then we find:

1) K_7 is not a maximal (49,7)-arc in PG(2,9) since every line in PG(2,9) is either 0-secant or a 7-secant of K_7, as geven in Table 7.

2) K_7 is a complete (49,7)-arc since there are no points of index zero for K_7, *i.e.* $C_0 = 0$.

Table 4. Point and Line of pG(2,9).

i	p_i	L_i									
1	(1,0,0)	2	11	20	29	38	47	56	65	74	83
2	(0,1,0)	1	11	12	13	14	15	16	17	18	19
3	(1,1,0)	4	11	22	30	44	55	63	68	79	87
4	(2,1,0)	3	11	21	31	41	51	61	71	81	91
5	(3,1,0)	9	11	27	34	40	53	60	66	82	86
6	(4,1,0)	6	11	24	37	45	49	59	70	80	84
7	(5,1,0)	8	11	26	32	46	52	58	69	75	90
8	(6,1,0)	7	11	25	36	39	50	64	67	78	89
9	(7,1,0)	5	11	23	35	42	54	57	73	76	88
10	(8,1,0)	10	11	28	33	43	48	62	72	77	85
11	(0,0,1)	1	2	3	4	5	6	7	8	9	10
12	(1,0,1)	2	13	22	31	40	49	58	67	76	85
13	(2,0,1)	2	12	21	30	39	48	57	66	75	84
14	(3,0,1)	2	18	27	36	45	54	63	72	81	90
15	(4,0,1)	2	15	24	33	42	51	60	69	78	87
16	(5,0,1)	2	17	26	35	44	53	62	71	80	89
17	(6,0,1)	2	16	25	34	43	52	61	70	79	88
18	(7,0,1)	2	14	23	32	41	50	59	68	77	86
19	(8,0,1)	2	19	28	37	46	55	64	73	82	91
20	(0,1,1)	1	29	30	31	32	33	34	35	36	37
21	(1,1,1)	4	13	21	29	46	54	62	70	78	86
22	(2,1,1)	3	12	22	29	42	52	59	72	82	89
23	(3,1,1)	9	18	25	29	44	51	58	73	77	84
24	(4,1,1)	6	15	28	29	40	50	63	71	75	88
25	(5,1,1)	8	17	23	29	43	49	64	66	81	87
26	(6,1,1)	7	16	27	29	41	55	57	69	80	85
27	(7,1,1)	5	14	26	29	45	48	60	67	79	91
28	(8,1,1)	10	19	24	29	39	53	61	68	76	90
29	(0,2,1)	1	20	21	22	23	24	25	26	27	28
30	(1,2,1)	3	13	20	30	43	50	60	73	80	90
31	(2,2,1)	4	12	20	31	45	53	64	69	77	88
32	(3,2,1)	7	18	20	34	46	48	59	71	76	87
33	(4,2,1)	10	15	20	37	44	52	57	67	81	86
34	(5,2,1)	5	17	20	32	39	51	63	70	82	85
35	(6,2,1)	9	16	20	36	42	49	62	68	75	91
36	(7,2,1)	8	14	20	35	40	55	61	72	78	84

Continued

37	(8,2,1)	6	19	20	33	41	54	58	66	79	89
38	(0,3,1)	1	74	75	76	77	78	79	80	81	82
39	(1,3,1)	8	13	28	34	45	51	57	68	74	89
40	(2,3,1)	5	12	24	36	43	55	58	71	74	86
41	(3,3,1)	4	18	26	37	42	50	61	66	74	85
42	(4,3,1)	9	15	22	35	41	48	64	70	74	90
43	(5,3,1)	10	17	25	30	40	54	59	69	74	91
44	(6,3,1)	3	16	23	33	46	53	63	67	74	84
45	(7,3,1)	6	14	27	31	39	52	62	73	74	87
46	(8,3,1)	7	19	21	32	44	49	60	72	74	88
47	(0,4,1)	1	47	48	49	50	51	52	53	54	55
48	(1,4,1)	10	13	27	32	42	47	64	71	79	84
49	(2,4,1)	6	12	25	35	46	47	60	68	81	85
50	(3,4,1)	8	18	24	30	41	47	62	67	82	88
51	(4,4,1)	4	15	23	34	39	47	58	72	80	91
52	(5,4,1)	7	17	22	33	45	47	61	73	75	86
53	(6,4,1)	5	16	28	31	44	47	59	66	78	90
54	(7,4,1)	9	14	21	37	43	47	63	69	76	89
55	(3,4,1)	3	19	26	36	40	47	57	70	77	87
56	(4,4,1)	1	65	66	67	68	69	70	71	72	73
57	(1,5,1)	9	13	26	33	39	55	59	65	81	88
58	(2,5,1)	7	12	23	37	40	51	62	65	79	90
59	(3,5,1)	6	18	22	32	43	53	57	65	78	91
60	(4,5,1)	5	15	27	30	46	49	61	65	77	89
61	(5,5,1)	4	17	28	36	41	52	60	65	76	84
62	(6,5,1)	10	16	21	35	45	50	58	65	82	87
63	(7,5,1)	3	14	24	34	44	54	64	65	75	85
64	(8,5,1)	8	19	25	31	42	48	63	65	80	86
65	(0,6,1)	1	56	57	58	59	60	61	62	63	64
66	(1,6,1)	5	13	25	37	41	53	56	72	75	87
67	(2,6,1)	8	12	27	33	44	50	56	70	76	91
68	(3,6,1)	3	18	28	35	39	49	56	69	79	86
69	(4,6,1)	7	15	26	31	43	54	56	68	82	84
70	(5,6,1)	6	17	21	34	42	55	56	67	77	90
71	(6,6,1)	4	16	24	32	40	48	56	73	81	89
72	(7,6,1)	10	14	22	36	46	51	56	66	80	88
73	(8,6,1)	9	19	23	30	45	52	56	71	78	85

Continued

74	(0,7,1)	1	38	39	40	41	42	43	44	45	46
75	(1,7,1)	7	13	24	35	38	52	63	66	77	91
76	(2,7,1)	9	12	28	32	38	54	61	67	80	87
77	(3,7,1)	10	18	23	31	38	55	60	70	75	89
78	(4,7,1)	8	15	21	36	38	53	59	73	79	85
79	(5,7,1)	3	17	27	37	38	48	58	68	78	88
80	(6,7,1)	6	16	26	30	38	51	64	72	76	86
81	(7,7,1)	4	14	25	33	38	49	57	71	82	90
82	(8,7,1)	5	19	22	34	38	50	62	69	81	84
83	(0,8,1)	1	83	84	85	86	87	88	89	90	91
84	(1,8,1)	6	13	23	36	44	48	61	69	82	83
85	(2,8,1)	10	12	26	34	41	49	63	73	78	83
86	(3,8,1)	5	18	21	33	40	52	64	68	80	83
87	(4,8,1)	3	15	25	32	45	55	62	66	76	83
88	(5,8,1)	9	17	24	31	46	50	57	72	79	83
89	(6,8,1)	8	16	22	37	39	54	60	71	77	83
90	(7,8,1)	7	14	28	30	42	53	58	70	81	83
91	(8,8,1)	4	19	27	35	43	51	59	67	75	83

Table 5. Point and Line of pG(2,9).

i	pi									Li
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3	(1,1,0)	11	22	30	44	55	63	68	79	87
4	(2,1,0)	11	21	31	41	51	61	71	81	91
5	(3,1,0)	11	27	34	40	53	60	66	82	86
6	(4,1,0)	11	24	37	45	49	59	70	80	84
7	(5,1,0)	11	26	32	46	52	58	69	75	90
8	(6,1,0)	11	25	36	39	50	64	67	78	89
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16	(5,0,1)	17	26	35	44	53	62	71	80	89
17	(6,0,1)	16	25	34	43	52	61	70	79	88

Continued

18	(7,0,1)	14	23	32	41	50	59	68	77	86
19	(8,0,1)	19	28	37	46	55	64	73	82	91
20	(0,1,1)	29	30	31	32	33	34	35	36	37
21	(1,1,1)	13	21	29	46	54	62	70	78	86
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30	(1,2,1)	13	20	30	43	50	60	73	80	90
31	(2,2,1)	12	20	31	45	53	64	69	77	88
32	(3,2,1)	18	20	34	46	48	59	71	76	87
33	(4,2,1)	15	20	37	44	52	57	67	81	86
34	(5,2,1)	17	20	32	39	51	63	70	82	85
35	(6,2,1)	16	20	36	42	49	62	68	75	91
36	(7,2,1)	14	20	35	40	55	61	72	78	84
37	(8,2,1)	19	20	33	41	54	58	66	79	89
38	(0,3,1)	74	75	76	77	78	79	80	81	82
39	(1,3,1)	13	28	34	45	51	57	68	74	89
40	(2,3,1)	12	24	36	43	55	58	71	74	86
41	(3,3,1)	18	26	37	42	50	61	66	74	85
42	(4,3,1)	15	22	35	41	48	64	70	74	90
43	(5,3,1)	17	25	30	40	54	59	69	74	91
44	(6,3,1)	16	23	33	46	53	63	67	74	84
45	(7,3,1)	14	27	31	39	52	62	73	74	87
46	(8,3,1)	19	21	32	44	49	60	72	74	88
47	(0,4,1)	47	48	49	50	5	52	53	54	55
48	(1,4,1)	13	27	32	42	47	64	71	79	84
49	(2,4,1)	12	25	35	46	47	60	68	81	85
50	(3,4,1)	18	24	30	41	47	62	67	82	88
51	(4,4,1)	15	23	34	39	47	58	72	80	91
52	(5,4,1)	17	22	33	45	47	61	73	75	86
53	(6,4,1)	16	28	31	44	47	59	66	78	90
54	(2,4,1)	14	21	37	43	47	63	69	76	89

Continued

55	(3,4,1)	19	26	36	40	47	57	70	77	87
56	(4,4,1)	65	66	67	68	69	70	71	72	73
57	(1,5,1)	13	26	33	39	55	59	65	81	88
58	(2,5,1)	12	23	37	40	51	62	65	79	90
59	(3,5,1)	18	22	32	43	53	57	65	78	91
60	(4,5,1)	15	27	30	46	49	61	65	77	89
61	(5,5,1)	17	28	36	41	52	60	65	76	84
62	(6,5,1)	16	21	35	45	50	58	65	82	87
63	(7,5,1)	14	24	34	44	54	64	65	75	85
64	(8,5,1)	19	25	31	42	48	63	65	80	86
65	(0,6,1)	56	57	58	59	60	61	62	63	64
66	(1,6,1)	13	25	37	41	53	56	72	75	87
67	(2,6,1)	12	27	33	44	50	56	70	76	91
68	(3,6,1)	18	28	35	39	49	56	69	79	86
69	(4,6,1)	15	26	31	43	54	56	68	82	84
70	(5,6,1)	17	21	34	42	55	56	67	77	90
71	(6,6,1)	16	24	32	40	48	56	73	81	89
72	(7,6,1)	14	22	36	46	51	56	66	80	88
73	(8,6,1)	19	23	31	45	52	56	71	78	85
74	(0,7,1)	38	39	40	41	42	43	44	45	46
75	(1,7,1)	13	24	35	38	52	63	66	77	91
76	(2,7,1)	12	28	32	38	54	61	67	80	87
77	(3,7,1)	18	23	31	38	55	60	70	75	89
78	(4,7,1)	15	21	37	38	53	59	73	79	85
79	(5,7,1)	17	27	30	38	48	58	68	78	88
80	(6,7,1)	16	26	33	38	51	64	72	76	86
81	(7,7,1)	14	25	34	38	49	57	71	82	90
82	(8,7,1)	19	22	37	38	50	62	69	81	84
83	(0,8,1)	83	84	85	86	87	88	89	90	91
84	(1,8,1)	13	23	36	44	48	61	69	82	83
85	(2,8,1)	12	26	34	41	49	63	73	78	83
86	(3,8,1)	18	21	33	40	52	64	68	80	83
87	(4,8,1)	15	25	32	45	55	62	66	76	83
88	(5,8,1)	17	24	31	46	50	57	72	79	83
89	(6,8,1)	16	22	37	39	54	60	71	77	83
90	(7,8,1)	14	28	30	42	53	58	70	81	83
91	(8,8,1)	19	27	35	43	51	59	67	75	83

Table 6. Point and Line of pG(2,9).

	pi	Li							
1	(1,0,0)								
3	(1,1,0)	22	30	44	55	63	68	79	87
4	(2,1,0)	21	31	41	51	61	71	81	91
5	(3,1,0)	27	34	40	53	60	66	82	86
6	(4,1,0)	24	37	45	49	59	70	80	84
7	(5,1,0)	26	32	46	52	58	69	75	90
8	(6,1,0)	25	36	39	50	64	67	78	89
9	(7,1,0)	23	35	42	54	57	73	76	88
10	(8,1,0)	28	33	43	48	62	72	77	85
12	(1,0,1)	22	31	40	49	58	67	76	85
13	(2,0,1)	21	30	39	48	57	66	75	84
14	(3,0,1)	27	36	45	54	63	72	81	90
15	(4,0,1)	24	33	42	51	60	69	78	87
16	(5,0,1)	26	35	44	53	62	71	80	89
17	(6,0,1)	25	34	43	52	61	70	79	88
18	(7,0,1)	23	32	41	50	59	68	77	86
19	(8,0,1)	28	37	46	55	64	73	82	91
20	(0,1,1)	30	31	32	33	34	35	36	37
21	(1,1,1)	21		46	54	62	70	78	86
22	(2,1,1)	22		42	52	59	72	82	89
23	(3,1,1)	25		44	51	58	73	77	84
24	(4,1,1)	28		40	50	63	71	75	88
25	(5,1,1)	23		46	49	64	66	81	87
26	(6,1,1)	27		42	55	57	69	80	85
27	(7,1,1)	26		45	48	60	67	79	91
28	(8,1,1)	24		39	53	61	68	76	90
29	(0,2,1)	21	22	23	24	25	26	27	28
30	(1,2,1)		30	43	50	60	73	80	90
31	(2,2,1)		31	45	53	64	69	77	88
32	(3,2,1)		34	46	48	59	71	76	87
33	(4,2,1)		37	44	52	57	67	81	86
34	(5,2,1)		32	39	51	63	70	82	85
35	(6,2,1)		36	42	49	62	68	75	91
36	(7,2,1)		35	40	55	61	72	78	84
37	(8,2,1)		33	41	54	58	66	79	89
38	(0,3,1)	75	76	77	78	79	80	81	82
39	(1,3,1)	28	34	45	51	57	68		89

Continued

40	(2,3,1)	24	36	43	55	58	71		86
41	(3,3,1)	26	37	42	50	61	66		85
42	(4,3,1)	22	35	41	48	64	70		90
43	(5,3,1)	25	30	40	54	59	69		91
44	(6,3,1)	23	33	46	53	63	67		84
45	(7,3,1)	27	31	39	52	62	73		87
46	(8,3,1)	21	32	44	49	60	72		88
47	(0,4,1)	48	49	50	5	52	53	54	55
48	(1,4,1)	27	32	42		64	71	79	84
49	(2,4,1)	25	35	46		60	68	81	85
50	(3,4,1)	24	30	41		62	67	82	88
51	(4,4,1)	23	34	39		58	72	80	91
52	(5,4,1)	22	33	45		61	73	75	86
53	(6,4,1)	28	31	44		59	66	78	90
54	(2,4,1)	21	37	43		63	69	76	89
55	(3,4,1)	26	36	40		57	70	77	87
56	(4,4,1)	66	67	68	69	70	71	72	73
57	(1,5,1)	26	33	39	55	59		81	88
58	(2,5,1)	23	37	40	51	62		79	90
59	(3,5,1)	22	32	43	53	57		78	91
60	(4,5,1)	27	30	46	49	61		77	89
61	(5,5,1)	28	36	41	52	60		76	84
62	(6,5,1)	21	35	45	50	58		82	87
63	(7,5,1)	24	34	44	54	64		75	85
64	(8,5,1)	25	31	42	48	63		80	86
65	(0,6,1)	57	58	59	60	61	62	63	64
66	(1,6,1)	25	37	41	53		72	75	87
67	(2,6,1)	27	33	44	50		70	76	91
68	(3,6,1)	28	35	39	49		69	79	86
69	(4,6,1)	26	31	43	54		68	82	84
70	(5,6,1)	21	34	42	55		67	77	90
71	(6,6,1)	24	32	40	48		73	81	89
72	(7,6,1)	22	36	46	51		66	80	88
73	(8,6,1)	23	31	45	52		71	78	85
74	(0,7,1)	39	40	41	42	43	44	45	46
75	(1,7,1)	24	35		52	63	66	77	91
76	(2,7,1)	28	32		54	61	67	80	87
77	(3,7,1)	23	31		55	60	70	75	89

Continued

78	(4,7,1)	21	37		53	59	73	79	85
79	(5,7,1)	27	30		48	58	68	78	88
80	(6,7,1)	26	33		51	64	72	76	86
81	(7,7,1)	25	34		49	57	71	82	90
82	(8,7,1)	22	37		50	62	69	81	84
83	(0,8,1)	84	85	86	87	88	89	90	91
84	(1,8,1)	23	36	44	48	61	69	82	
85	(2,8,1)	26	34	41	49	63	73	78	
86	(3,8,1)	21	33	40	52	64	68	80	
87	(4,8,1)	25	32	45	55	62	66	76	
88	(5,8,1)	24	31	46	50	57	72	79	
89	(6,8,1)	22	37	39	54	60	71	77	
90	(7,8,1)	28	30	42	53	58	70	81	
91	(8,8,1)	27	35	43	51	59	67	75	

Table 7. Point and Line of pG(2,9).

	ipi	Li							
1	(1,0,0)								
3	(1,1,0)			44	55	63	68	79	87
4	(2,1,0)		31	41	51	61	71	81	91
5	(3,1,0)		34	40	53	60		82	86
6	(4,1,0)		37	45	49	59	70	80	
7	(5,1,0)		32	46	52	58	69		90
8	(6,1,0)		36		50	64	67	78	89
9	(7,1,0)		35	42	54		73	76	88
10	(8,1,0)		33	43		62	72	77	85
12	(1,0,1)		31	40	49	58	67	76	85
13	(2,0,1)								
14	(3,0,1)		36	45	54	63	72	81	90
15	(4,0,1)		33	42	51	60	69	78	87
16	(5,0,1)		35	44	53	62	71	80	89
17	(6,0,1)		34	43	52	61	70	79	88
18	(7,0,1)		32	41	50	59	68	77	86
19	(8,0,1)		37	46	55	64	73	82	91
20	(0,1,1)		31	32	33	34	35	36	37
21	(1,1,1)			46	54	62	70	78	86
22	(2,1,1)			42	52	59	72	82	89

Continued

23	(3,1,1)		44	51	58	73	77	
24	(4,1,1)		40	50	63	71		88
25	(5,1,1)		46	49	64		81	87
26	(6,1,1)		42	55		69	80	85
27	(7,1,1)		45		60	67	79	91
28	(8,1,1)			53	61	68	76	90
30	(1,2,1)		43	50	60	73	80	90
31	(2,2,1)	31	45	53	64	69	77	88
32	(3,2,1)	34	46		59	71	76	87
33	(4,2,1)	37	44	52		67	81	86
34	(5,2,1)	32		51	63	70	82	85
35	(6,2,1)	36	42	49	62	68		91
36	(7,2,1)	35	40	55	61	72	78	
37	(8,2,1)	33	41	54	58		79	89
38	(0,3,1)	76	77	78	79	80	81	82
39	(1,3,1)	34	45	51		68		89
40	(2,3,1)	36	43	55	58	71		86
41	(3,3,1)	37	42	50	61			85
42	(4,3,1)	35	41		64	70		90
43	(5,3,1)		40	54	59	69		91
44	(6,3,1)	33	46	53	63	67		
45	(7,3,1)	31		52	62	73		87
46	(8,3,1)	32	44	49	60	72		88
47	(0,4,1)	49	50	51	52	53	54	55
48	(1,4,1)	32	42		64	71	79	87
49	(2,4,1)	35	46		60	68	81	85
50	(3,4,1)		41		62	67	82	88
51	(4,4,1)	34		58	72	80	91	
52	(5,4,1)	33	45		61	73	86	
53	(6,4,1)	31	44		59		78	90
54	(2,4,1)	37	43		63	69	76	89
55	(3,4,1)	36	40			70	77	87
56	(4,4,1)	67	68	69	70	71	72	73
57	(1,5,1)	33		55	59		81	88
58	(2,5,1)	37	40	51	62		79	90
59	(3,5,1)	32	43	53			78	91
60	(4,5,1)		46	49	61		77	89
61	(5,5,1)	36	41	52	60		76	

Continued

62	(6,5,1)	35	45	50	58		82	87
63	(7,5,1)	34	44	54	64			85
64	(8,5,1)	31	42		63		80	86
65	(0,6,1)	58	59	60	61	62	63	64
66	(1,6,1)	37	41	53		72		87
67	(2,6,1)	33	44	50		70	76	91
68	(3,6,1)	35		49		69	79	86
69	(4,6,1)	31	43	54		68	82	
70	(5,6,1)	34	42	55		67	77	90
71	(6,6,1)	32	40			73	81	89
72	(7,6,1)	36	46	51		66	80	88
73	(8,6,1)	31	45	52		71	78	85
74	(0,7,1)	40	41	42	43	44	45	46
75	(1,7,1)	35		52	63		77	91
76	(2,7,1)	32		54	61	67	80	87
77	(3,7,1)	31		55	60	70		89
78	(4,7,1)	37		53	59	73	79	85
79	(5,7,1)				58	68	78	88
80	(6,7,1)	33		51	64	72	76	86
81	(7,7,1)	34		49		71	82	90
82	(8,7,1)	37		50	62	69	81	
83	(0,8,1)	85	86	87	88	89	90	91
84	(1,8,1)	36	44		61	69	82	
85	(2,8,1)	34	41	49	63	73	78	
86	(3,8,1)	33	40	52	64	68	80	
87	(4,8,1)	32	45	55	62		76	
88	(5,8,1)	31	46	50		72	79	
89	(6,8,1)	37		54	60	71	77	
90	(7,8,1)		42	53	58	70	81	
91	(8,8,1)	35	43	51	59	67		

2.6. The Construction of (K_6,6)-Arc k_6 from k_7

WE construct a (K_6,6)-arc K_6 from K_7 by eliminating one line, the line L_{20} = [29, 30, 31, 32, 33, 34, 35, 36, 37] and the following points [40, 49, 58, 67, 76, 85], then we find:

1) K_6 is not a maximal (36,6)-arc in PG(2,9) since every line in PG(2,9) is either 0-secant or a 6-secant of K_6 as given in Table 8.

2) K_6 is a complete (36,6)-arc since there are no points of index zero for K_6, *i.e.*, $C_0 = 0$.

Table 8. Point and Line of pG(2,9).

Lipi							
1	(1,0,0)						
3	(1,1,0)	44	55	63	68	79	87
4	(2,1,0)	41	51	61	71	81	91
5	(3,1,0)		53	60		82	86
6	(4,1,0)	45		59	70	80	
7	(5,1,0)	46	52		69		90
8	(6,1,0)		50	64		78	89
9	(7,1,0)	42	54		73		88
10	(8,1,0)	43		62	72	77	
12	(1,0,1)						
13	(2,0,1)						
14	(3,0,1)	45	54	63	72	81	90
15	(4,0,1)	42	51	60	69	78	87
16	(5,0,1)	44	53	62	71	80	89
17	(6,0,1)	43	52	61	70	79	88
18	(7,0,1)	41	50	59	68	77	86
19	(8,0,1)	46	55	64	73	82	91
21	(1,1,1)	46	54	62	70	78	86
22	(2,1,1)	42	52	59	72	82	89
23	(3,1,1)	44	51		73	77	
24	(4,1,1)		50	63	71		88
25	(5,1,1)	46		64		81	87
26	(6,1,1)	42	55		69	80	
27	(7,1,1)	45		60		79	91
28	(8,1,1)		53	61	68		90
30	(1,2,1)	43	50	60	73	80	90
31	(2,2,1)	45	53	64	69	77	88
32	(3,2,1)	46		59	71		87
33	(4,2,1)	44	52			81	86
34	(5,2,1)		51	63	70	82	
35	(6,2,1)	42		62	68		91
36	(7,2,1)		55	61	72	78	
37	(8,2,1)	41	54			79	89
38	(0,3,1)	77	78	79	80	81	82
39	(1,3,1)	45	51		68		89
40	(2,3,1)	43	55		71		86
41	(3,3,1)	42	50	61			

Continued

42	(4,3,1)	41		64	70		90
43	(5,3,1)		54	59	69		91
44	(6,3,1)	46	53	63			
45	(7,3,1)		52	62	73		87
46	(8,3,1)	44		60	72		88
47	(0,4,1)	50	51	52	53	54	55
48	(1,4,1)	42		64	71	79	
49	(2,4,1)	46		60	68	81	
50	(3,4,1)	41		62		82	88
51	(4,4,1)				72	80	91
52	(5,4,1)	45		61	73		86
53	(6,4,1)	44		59		78	90
54	(7,4,1)	43		63	69	89	
55	(8,4,1)	40		70	77	87	
56	(4,4,1)	68	69	70	71	72	73
57	(1,5,1)		55	59		81	88
58	(2,5,1)		51	62		79	90
59	(3,5,1)	43	53			78	91
60	(4,5,1)	46		61		77	89
61	(5,5,1)	41	52	60			
62	(6,5,1)	45	50			82	87
63	(7,5,1)	44	54	64			
64	(8,5,1)	42		63		80	86
65	(0,6,1)	59	60	61	62	63	64
66	(1,6,1)	41	53		72		87
67	(2,6,1)	44	50		70		91
68	(3,6,1)				69	79	86
69	(4,6,1)	43	54		68	82	
70	(5,6,1)	42	55			77	90
71	(6,6,1)		48		73	81	89
72	(7,6,1)	46	51			80	88
73	(8,6,1)	45	52		71	78	
74	(0,7,1)	41	42	43	44	45	46
75	(1,7,1)		52	63		77	91
76	(2,7,1)		54	61		80	87
77	(3,7,1)		55	60	70	89	
78	(4,7,1)		53	59	73	79	
79	(5,7,1)				68	78	88

Continued

80	(6,7,1)		51	64	72		86
81	(7,7,1)				71	82	90
82	(8,7,1)		50	62	69	81	
83	(0,8,1)	86	87	88	89	90	91
84	(1,8,1)	44		61	69	82	
85	(2,8,1)	41		63	73	78	
86	(3,8,1)		52	64	68	80	
87	(4,8,1)	45	55	62			
88	(5,8,1)	46	50		72	79	
89	(6,8,1)		54	60	71	77	
90	(7,8,1)	42	53		70	81	
91	(8,8,1)	43	51	59			

2.7. The Construction of (k_5,5)-Arc k_5 from k_6

We construct a (k_5,5)-arc K_5 from K_6 by eliminating one line, the line L_{74} = [38, 39, 40, 41, 42, 43, 44, 45, 46] and following points [50, 59, 68, 77, 86], then we find:

1) K_5 is not a maximal (25,5)-arc in PG(2,9), since every line in PG(2,9) is either 0-secant or a 5-secant of K_5 as given in Table 9.

2) K_5 is a complete (25,5)-arc since there are no point of index zero for K_5, *i.e.*

$$C_0 = 0 .$$

2.8. The Construction of (k_4,4)-Arc k_4 from k_5

We construct a (k_4,4)-arc from k_5 by eliminating one line, the line L_{47} = [47, 48, 49, 50, 51, 52, 53, 54, 55] and following points [60, 69, 78, 87], then we find:

1) K_4 is not a maximal (16,4)-arc inPG(2,9), since every line in PG(2,9) is either 0-secant or a 4-secant of K4 as given in Table 10.

2) K_4 is a complete (16,4)-arc since there are no points index zero for K_4, *i.e.*

$$C_0 = 0 .$$

2.9. The Constrcution of (k_3,3)-Arc k_3 from k_4

We construct (K_3,3)-arc K_3 from K_4 by eliminting one line, the line L_{65} = [56, 57, 58, 59, 60, 61, 62, 63, 64] and following points [70, 79, 88], we find:

1) K_3 is not a maximal (9,3)-arc in PG(2,9), since every line in PG(2,9) is either 0-secant or a 3-secant of K_3, as given in Table 11.

2) K_3 is a complete (9,3)-arc since there are no points of index zero for K_3, *i.e.*

$$C_0 = 0 .$$

2.10. The Construction of (k_2,2)-Arc k_2 from k_3

We construct a (k_2,2)-arc K_2 from K_3 by eliminating one lines the line L_{56} = [65, 66, 67, 68, 69, 70, 71, 72, 73] and following points [80, 89], then we find:

1) K_2 is not a maximal (4,2)-arc in PG(2,9), since some line in PG(2,9) which are 0-secant, 1-secants and 2-secant of K_2, as given in Table 12.

Table 9. Point and Line of pG(2,9).

i	pi	Li				
1	(1,0,0)					
3	(1,1,0)	55	63		79	87
4	(2,1,0)	51	61	71	81	91
5	(3,1,0)	53	60		82	
6	(4,1,0)			70	80	
7	(5,1,0)	52		69		90
8	(6,1,0)		64		78	89
9	(7,1,0)	54		73		88
10	(8,1,0)		62	72		
12	(1,0,1)					
13	(2,0,1)					
14	(3,0,1)	54	63	72	81	90
15	(4,0,1)	51	60	69	78	87
16	(5,0,1)	53	62	71	80	89
17	(6,0,1)	52	61	70	79	88
18	(7,0,1)					
19	(8,0,1)	55	64	73	82	91
21	(1,1,1)	54	62	70	78	
22	(2,1,1)	52		72	82	89
23	(3,1,1)	51		73		
24	(4,1,1)		63	71		88
25	(5,1,1)		64		81	87
26	(6,1,1)	55		69	80	
27	(7,1,1)		60		79	91
28	(8,1,1)	53	61			90
30	(1,2,1)		60	73	80	90
31	(2,2,1)	53	64	69		88
32	(3,2,1)			71		87
33	(4,2,1)	52			81	
34	(5,2,1)	51	63	70	82	
35	(6,2,1)		62			91

Continued

36	(7,2,1)	55	61	72	78	
37	(8,2,1)	54			79	89
38	(0,3,1)	78	79	80	81	82
39	(1,3,1)	51				89
40	(2,3,1)	55		71		
41	(3,3,1)		61			
42	(4,3,1)		64	70		90
43	(5,3,1)	54		69		91
44	(6,3,1)	53	63			
45	(7,3,1)	52	62	73		87
46	(8,3,1)		60	72		88
48	(1,4,1)		64	71	79	
49	(2,4,1)		60		81	
50	(3,4,1)		62	67	82	88
51	(4,4,1)			72	80	91
52	(5,4,1)		61	73		
53	(6,4,1)				78	90
54	(2,4,1)		63	69		89
55	(3,4,1)			70		87
56	(4,4,1)	69	70	71	72	73
57	(1,5,1)	55			81	88
58	(2,5,1)	51	62		79	90
59	(3,5,1)	53			78	91
60	(4,5,1)		61			89
61	(5,5,1)	52	60			
62	(6,5,1)				82	87
63	(7,5,1)	54	64			
64	(8,5,1)		63		80	
65	(0,6,1)	60	61	62	63	64
66	(1,6,1)	53		72		87
67	(2,6,1)			70		91
68	(3,6,1)			69	79	86
69	(4,6,1)	54			82	
70	(5,6,1)	55				90
71	(6,6,1)			73	81	89
72	(7,6,1)	51			80	88
73	(8,6,1)	52		71	78	
75	(1,7,1)	52	63			91

Continued

76	(2,7,1)	54	61		80	87
77	(3,7,1)	55	60	70		89
78	(4,7,1)	53		73	79	
79	(5,7,1)				78	88
80	(6,7,1)	51	64	72		
81	(7,7,1)			71	82	90
82	(8,7,1)		62	69	81	
83	(0,8,1)	87	88	89	90	91
84	(1,8,1)		61	69	82	
85	(2,8,1)		63	73	78	
86	(3,8,1)	52	64		80	
87	(4,8,1)	55	62			
88	(5,8,1)			72	79	
89	(6,8,1)	54	60	71		
90	(7,8,1)	53		70	81	
91	(8,8,1)		51			

Table 10. Point and Line of pG(2,9).

Li	i	pi				
1	(1,0,0)					
3	(1,1,0)		63		79	
4	(2,1,0)		61	71	81	91
5	(3,1,0)				82	
6	(4,1,0)			70	80	
7	(5,1,0)					90
8	(6,1,0)		64			89
9	(7,1,0)			73		88
10	(8,1,0)		62	72		
12	(1,0,1)					
13	(2,0,1)					
14	(3,0,1)		63	72	81	90
15	(4,0,1)					
16	(5,0,1)		62	71	80	89
17	(6,0,1)		61	70	79	88
18	(7,0,1)					
19	(8,0,1)		64	73	82	91
21	(1,1,1)		62	70		
22	(2,1,1)			72	82	89

Continued

23	(3,1,1)		73		
24	(4,1,1)	63	71		88
25	(5,1,1)	64		81	
26	(6,1,1)			80	
27	(7,1,1)			79	91
28	(8,1,1)	61			90
30	(1,2,1)		73	80	90
31	(2,2,1)	64			88
32	(3,2,1)		71		
33	(4,2,1)			81	
34	(5,2,1)	63	70	82	
35	(6,2,1)	62			91
36	(7,2,1)	61	72		
37	(8,2,1)			79	89
38	(0,3,1)	79	80	81	
39	(1,3,1)				89
40	(2,3,1)		71		
41	(3,3,1)	61			
42	(4,3,1)	64	70		90
43	(5,3,1)				91
44	(6,3,1)	63			
45	(7,3,1)	62	73		
46	(8,3,1)		72		88
48	(1,4,1)	64	71	79	
49	(2,4,1)			81	
50	(3,4,1)	62		82	88
51	(4,4,1)		72	80	91
52	(5,4,1)	61	73		
53	(6,4,1)				90
54	(2,4,1)	63			89
55	(3,4,1)		70		
56	(4,4,1)	70	71	72	73
57	(1,5,1)			81	88
58	(2,5,1)	62		79	90
59	(3,5,1)				91
60	(4,5,1)	61			89
61	(5,5,1)				
62	(6,5,1)			82	

Continued

63	(7,5,1)	64			
64	(8,5,1)	63		80	
65	(0,6,1)	61	62	63	64
66	(1,6,1)		72		
67	(2,6,1)		70		91
68	(3,6,1)			79	86
69	(4,6,1)			82	
70	(5,6,1)				90
71	(6,6,1)		73	81	89
72	(7,6,1)			80	88
73	(8,6,1)		71		
75	(1,7,1)	63			91
76	(2,7,1)	61		80	
77	(3,7,1)		70		89
78	(4,7,1)		73	79	
79	(5,7,1)				88
80	(6,7,1)	64	72		
81	(7,7,1)		71	82	90
82	(8,7,1)	62		81	
83	(0,8,1)	88	89	90	91
84	(1,8,1)	61		82	
85	(2,8,1)	63		73	
86	(3,8,1)	64		80	
87	(4,8,1)	62			
88	(5,8,1)		72	79	
89	(6,8,1)		71		
90	(7,8,1)		70	81	
91	(8,8,1)				

Table 11. Point and Line of pG(2,9).

i	pi	Li			
1	(1,0,0)				
3	(1,1,0)				
4	(2,1,0)		71	81	91
5	(3,1,0)			82	
6	(4,1,0)			80	
7	(5,1,0)				90
8	(6,1,0)				89
9	(7,1,0)		73		

Continued

10	(8,1,0)	72		
12	(1,0,1)			
13	(2,0,1)			
14	(3,0,1)	72	81	90
15	(4,0,1)			
16	(5,0,1)	71	80	89
17	(6,0,1)			
18	(7,0,1)			
19	(8,0,1)	73	82	91
21	(1,1,1)			
22	(2,1,1)	72	82	89
23	(3,1,1)	73		
24	(4,1,1)	71		
25	(5,1,1)		81	
26	(6,1,1)		80	
27	(7,1,1)			91
28	(8,1,1)			90
30	(1,2,1)	73	80	90
31	(2,2,1)			
32	(3,2,1)	71		
33	(4,2,1)		81	
34	(5,2,1)		82	
35	(6,2,1)			91
36	(7,2,1)	72		
37	(8,2,1)			89
38	(0,3,1)	80	81	82
39	(1,3,1)			89
40	(2,3,1)	71		
41	(3,3,1)			
42	(4,3,1)			90
43	(5,3,1)			91
44	(6,3,1)			
45	(7,3,1)	73		
46	(8,3,1)	72		
48	(1,4,1)	71		
49	(2,4,1)		81	
50	(3,4,1)		82	
51	(4,4,1)	72	80	91

Continued

52	(5,4,1)	73		
53	(6,4,1)			90
54	(2,4,1)			89
55	(3,4,1)			
56	(4,4,1)	71	72	73
57	(1,5,1)		81	
58	(2,5,1)			90
59	(3,5,1)			91
60	(4,5,1)			89
61	(5,5,1)			
62	(6,5,1)		82	
63	(7,5,1)			
64	(8,5,1)		80	
66	(1,6,1)	72		
67	(2,6,1)			91
68	(3,6,1)			
69	(4,6,1)		82	
70	(5,6,1)			90
71	(6,6,1)	73	81	89
72	(7,6,1)		80	
73	(8,6,1)	71		
75	(1,7,1)			91
76	(2,7,1)		80	
77	(3,7,1)			89
78	(4,7,1)	73		
79	(5,7,1)			
80	(6,7,1)	72		
81	(7,7,1)	71	82	90
82	(8,7,1)		81	
83	(0,8,1)	89	90	91
84	(1,8,1)		82	
85	(2,8,1)	73		
86	(3,8,1)		80	
87	(4,8,1)			
88	(5,8,1)	72		
89	(6,8,1)	71		
90	(7,8,1)		81	
91	(8,8,1)			

Table 12. Point and Line of pG(2,9).

		Li	pii
1	(1,0,0)		
3	(1,1,0)		
4	(2,1,0)	81	91
5	(3,1,0)	82	
6	(4,1,0)		
7	(5,1,0)		90
8	(6,1,0)		
9	(7,1,0)		
10	(8,1,0)		
12	(1,0,1)		
13	(2,0,1)		
14	(3,0,1)	81	90
15	(4,0,1)		
16	(5,0,1)		
17	(6,0,1)		
18	(7,0,1)		
19	(8,0,1)	82	91
21	(1,1,1)		
22	(2,1,1)	82	
23	(3,1,1)		
24	(4,1,1)		
25	(5,1,1)	81	
26	(6,1,1)		
27	(7,1,1)		91
28	(8,1,1)		90
30	(1,2,1)		90
31	(2,2,1)		
32	(3,2,1)		
33	(4,2,1)	81	
34	(5,2,1)	82	
35	(6,2,1)		91
36	(7,2,1)		
37	(8,2,1)		
38	(0,3,1)	81	82
39	(1,3,1)		
40	(2,3,1)		

Continued

41	(3,3,1)		
42	(4,3,1)		90
43	(5,3,1)		91
44	(6,3,1)		
45	(7,3,1)		
46	(8,3,1)		
48	(1,4,1)		
49	(2,4,1)	81	
50	(3,4,1)	82	
51	(4,4,1)		91
52	(5,4,1)		
53	(6,4,1)		90
54	(2,4,1)		
55	(3,4,1)		
57	(1,5,1)	81	
58	(2,5,1)		90
59	(3,5,1)		91
60	(4,5,1)		
61	(5,5,1)		
62	(6,5,1)	82	
63	(7,5,1)		
64	(8,5,1)		
66	(1,6,1)		
67	(2,6,1)		91
68	(3,6,1)		
69	(4,6,1)	82	
70	(5,6,1)		90
71	(6,6,1)	81	
72	(7,6,1)		
73	(8,6,1)		
75	(1,7,1)		91
76	(2,7,1)		
77	(3,7,1)		
78	(4,7,1)		
79	(5,7,1)		
80	(6,7,1)		
81	(7,7,1)	82	90

Continued

82	(8,7,1)		81	
83	(0,8,1)		90	91
84	(1,8,1)		82	
85	(2,8,1)			
86	(3,8,1)			
87	(4,8,1)			
88	(5,8,1)			
89	(6,8,1)			
90	(7,8,1)		81	
91	(8,8,1)			

2) Since there are no points of index zero for K_2, *i.e.* $C_0 = 0$, (4,2)-arc is a complete arc and it is oval.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Yahya, N.Y.K. and Sulaimaan, A.E.M. (2019) New Structural Engineering Construction Methods by Remove q Points in PG (2, q). *Applied Mathematical Science*, **13**, 1048-1041. https://doi.org/10.12988/ams.2019.99134

[2] Kasm, N.Y. and Ibrahim, H.S. (2019) The Optimal Size of {b, t}-Blocking Set When t = 3, 4 By Intersection the Tangents in PG (2, q). *Modern Applied Science*, **13**, 15. https://doi.org/10.5539/mas.v13n7p15

[3] Kasm, N.Y. and Ibrahim, H.S. (2019) The Possibility of Applying Rumen Research at the Projective Plane PG (2, 17). *Modern Applied Science*, **13**, 150. https://doi.org/10.5539/mas.v13n8p150

[4] Yahya, N.Y.K. and Sulaimaan, A.E.M. (2019) Linear Code Related *Ω*-Blocking Sets. *Applied Mathematical Science*, **13**, 1033-1040. https://doi.org/10.12988/ams.2019.99133

[5] Yahya, N.Y.K. and Hamad, A.Z. (2019) A Geometric Construction of a (56, 2)-Blocking Set in PG (2, 19) and on Three Dimensional Linear [325,3,307]19 Griesmer Code. *Rafidain Journal of Computer Sciences and Mathematics*, **14**, 13-27.

[6] Yahya, N.Y.K. (2014) A Non PGL (3, q) k-arcs in the Projective Plane of Order 37. *Tikrit Journal of Pure Science*, **19**, 135-145.

[7] Yahya, N.Y.K. (2018) A Geometric Construction of Complete (k, r)-arc in PG (2, 7) and the Related Projective [n,3,d]7 Codes. *Rafidain Journal of Computer Sciences and Mathematics*, **12**, 24-40.

[8] Yahya, N.Y.K. and Salim, M.N. (2019) New Geometric Methods for Prove Existence Three-Dimensional Linear [97,3,87]11 and [143,3,131]13 Codes. *Journal of Education and Science*, **28**, 312-333. https://doi.org/10.33899/edusj.2019.161057

[9] Yahya, N.Y.K. and Sulaimaan, A.E.M. (2019) Linear Code Originates from Com-

plete Arc by Engineering Building Methods. *Applied Mathematical Science*, **13**, 1013-1020. https://doi.org/10.12988/ams.2019.99132

[10] Yahya, N.Y.K. (2012) Existence of Minimal Blocking Sets of Size 31 in the Projective Plane PG (2, 17). *Journal of University of Babylon/Pure and Applied Sciences*, **20**, 1138-1146.

[11] Yahya, N.Y.K. (2014) The Use of 7-Blocking Sets in Galois Geometries. *Journal of University of Babylon/Pure and Applied Sciences*, **22**, 1229-1235.

[12] Yahya, N.Y.K. and Salim, M.N. (2019) 17 New Existences Linear [n,3,d]19 Codes by Geometric Structure Method in PG (2, 19). *Rafidain Journal of Computer Sciences and Mathematics*, **13**, 61-86.

[13] Kasm, N.Y. and Hamad, A.Z. (2019) Applications of Algebraic Geometry in Cryptography. *Modern Applied Science*, **13**, 130. https://doi.org/10.5539/mas.v13n5p130

[14] Yahya, N.Y.K. and Salim, M.N. (2018) The Geometric Approach to Existences Linear [n,k,d]13 Code. *International Journal of Enhanced Research in Science, Technology and Engineering*.

[15] Hirschfeld, J.W.P. (1979) Projective Geometries over Finite Fields. Oxford Clarendon Press, New York.

[16] Kareem, F.F. (2010) Maximum Arc in Projective Plane PG (2, 9) over Galois Field. *Journal of College of Education*, No. 3, 306-317.