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A New Electron Charge Distribution Function of Electron

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Abstract

Here we derive a new charge distribution function for an electron by using as an equation of motion a segment of charge whose self energy interaction is due to electric field potential. Our method is based on the consideration that a charged distribution function should be represented as an eigenfunction of electron mass energy. We compare our electron charge distribution function to that of Weinberg's $\eta(r)$ and our charged electron radius to that obtained by Kim.

Keywords

Charge Distribution Function, Electron

1. Introduction

The theory that particle mass is generated by Higgs mechanism is widely accepted. Especially after the discovery of the Higgs particle [1], the Higgs mechanism has been convinced as the true origin of Bosons. Encouraged by this discovery, many physicists have been trying to show that the Higgs mechanism is also true for fermions, specifically by showing a mass hierarchy for quarks and lepton [2]. However, some physicists are still not convinced that the Higgs mechanism is true for fermions, primarily because of lepton consideration. Quantum Electrodynamics (QED) has been used to describe the interaction between leptons, especially electrons, and it is well known that it describes such electromagnetic phenomena quite well. The only ambiguity appears in the determination of self-energy, and physicists disagree on how to consider this ambiguity. Some physicists argue that to describe leptons, especially electron, QED is sufficient even though this ambiguity exists.

For instance, Weinberg shows an estimated charged radius in his famous book

Quantum Theory of Field I [3]. He recently also published the paper [4] in which he opposed the consideration in some leading approximations that only quarks and leptons with nonzero masses are third generation with the other lepton and quark masses arising from some sort of radiative correction. Weinberg points out that this kind of theory was firstly proposed by S.M. Barr and A. Zee [5] and recently has been revived [6]. The opposition by Weinberg seems to show that he still favors the same scheme mentioned in his book. The motivation of this paper is also based on the work of Weinberg. We investigate an electron charged radius by considering that a charge distribution function is described by a segment of charge function. By constructing a segment of charge function as an eigenfunction of electron mass energy, we find that the electron charge distribution function appears simultaneously when the electron gains mass energy by the absorption of an infinite number of photons, as described by Weinberg.

2. Formalism

In order to obtain a charge distribution function, we introduce a charge matrix state as

$$\rho_{\lambda\eta}\left(t,\vec{r}\right)=\left(-e\right)\left\langle 0\left|q_{\lambda}\left(t,\vec{r}\right)q_{\eta}^{\dagger}\left(t,\vec{r}\right)\right|\text{charge state}\right\rangle=\left(-e\right)\overline{\rho}_{\lambda\eta}\left(t,\vec{r}\right) \tag{1}$$

where λ,η are Dirac indices. This kind of charge matrix (only operator part) is also used by Karnieli [7].

Since electron charge is invariable over time, time derivative of this quantity must be zero.

Dirac equation with mass less field is represented as

$$\gamma^{\mu}D_{\mu}q=0 \tag{2}$$

where $D_{\mu}=\partial_{\mu}-ieA_{\mu}$.

We employ γ matrices following Weinberg way [8] as follows

$$\gamma^{0}=-i\begin{pmatrix}0 & \sigma_{0}\\ \sigma_{0} & 0\end{pmatrix},\ \gamma^{k}=-i\begin{pmatrix}0 & \sigma_{k}\\ -\sigma_{k} & 0\end{pmatrix}$$

Here, σ_{0} is a unit matrix of 2 × 2 matrix and σ_{k} are the 2 × 2 Pauli matrices ($k=1,2,3$), $\alpha^{\mu}=\gamma^{0}\gamma^{\mu}$ ($\mu=0,1,2,3$) and $\beta=i\gamma^{0}$.

From Equation (2) we can obtain following equation as

$$i\frac{\partial q}{\partial t}=-i\alpha^{k}\partial_{k}q-e\alpha^{\mu}A_{\mu}q \tag{3}$$

$$i\frac{\partial q^{\dagger}}{\partial t}=-i\alpha^{k}\partial_{k}q^{\dagger}+e\alpha^{\mu}A_{\mu}q^{\dagger} \tag{4}$$

where $\mu=0,1,2,3$.

Note that we use the same metric system that Weinberg employs in book [8]

$$\eta^{00}=-1,\ \eta^{11}=\eta^{22}=\eta^{33}=1$$

Since we consider the equation of motion for a segment of charge function

$\overline{\rho}_{\lambda\eta}$, we consider the case that $\overline{\rho}_{\lambda\eta}(t,\vec{r}) = \exp(-iP_0 t)\overline{\rho}_{\lambda\eta}(\vec{r})$.

Then from Equation (3) and Equation (4) we obtain

$$P_0\overline{\rho}_{\lambda\eta}(\vec{r}) = -i\alpha^k\partial_k\overline{\rho}_{\lambda\eta}(\vec{r}) \tag{5}$$

Here we use the following notation,

$$-i\alpha^k\partial_k\overline{\rho}_{\lambda\eta}(\vec{r}) = \langle 0|-i\left(\alpha^k\partial_k q_\lambda(\vec{r})\right)q_\eta^\dagger(\vec{r}) + q_\lambda(\vec{r})\left(q_\eta^\dagger\alpha^k\tilde{\partial}_k(-i)\right)|\text{charge state}\rangle$$

Here $|0\rangle$ denotes vacuum state.

We interpolate that Equation (5) implies free motion of a segment of charge. There for to obtain an appropriate equation of motion, we have to also consider self-interaction energy. First though, we decompose $\overline{\rho}_{\lambda\eta}$ to Lorentz invariant form and consider that a segment of charge is depend on only the radius $r = |\vec{r}|$, then $\overline{\rho}_{\alpha\beta}$ is described as

$$\overline{\rho}_{\lambda\eta}(r) = I\overline{\rho}_0(r) + (-i\vec{\alpha}\cdot\hat{r})\overline{\rho}_1(r) + \beta\overline{\rho}_2(r) + \beta(i\vec{\alpha}\cdot\hat{r})\overline{\rho}_3(r) \tag{6}$$

Recalling that actual charge is a scalar quantity, a segment of charge function which is affected by self-interaction energy should be defined as

$$\text{actual charge} = \frac{1}{4}Tr\rho_{\lambda\eta}(r) = (-e)\frac{1}{4}Tr\overline{\rho}_{\lambda\eta}(r) = (-e)\overline{\rho}_0(r) \tag{7}$$

where Tr denotes Trace of matrix.

To consider self-interaction energy, we use Gauss's Law as

$$\text{div}\vec{E} = \frac{\rho_0}{\epsilon_0} \tag{8}$$

Since we consider only r dependent case, thus volume integral using Green's theorem gives

$$E_r(r) = \frac{1}{4\pi\epsilon_0}\frac{4\pi\int_0^r \mathrm{d}r' r'^2\rho_0(r')}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{(-e)4\pi\int_0^r \mathrm{d}r' r'^2\overline{\rho}_0(r')}{r^2} \tag{9}$$

Then we obtain an electric field potential as

$$V_r(r) = -\frac{-e}{4\pi\epsilon_0 a_0} + \frac{-e}{4\pi\epsilon_0}\frac{4\pi\int_0^r \mathrm{d}r' r'^2\overline{\rho}_0(r')}{r'} + \frac{-e}{4\pi\epsilon_0}4\pi\int_0^{a_0}\mathrm{d}r' r'\overline{\rho}_0(r') \tag{10}$$

Note that to obtain Equation (8) we use the setting consideration that charge vanishes at and beyond $r = a_0$.

This means that

$$\begin{aligned}\overline{\rho}_0(r) &= 0 \quad (r > a_0)\\ \overline{\rho}_0(r) &\neq 0 \quad (r \le a_0)\end{aligned} \tag{11}$$

Then we have to set the following two conditions.

The first condition is

$$4\pi\int_0^{a_0}\mathrm{d}r r^2\rho_0(r) = -e \tag{12}$$

Rearrangement gives

$$\int_0^{a_0}\mathrm{d}r r^2\overline{\rho}_0(r) = \frac{1}{4\pi} \tag{13}$$

The second condition is

$$\rho_0(a_0) = \bar{\rho}_0(a_0) = 0 \tag{14}$$

From Equation (10), the self-interaction energy is described as

$$\begin{aligned} V_e &= (-e)\bar{\rho}_0(r)V_r(r) \\ &= -\frac{e^2}{4\pi\epsilon_0 a_0}\bar{\rho}_0(r) + \frac{e^2}{4\pi\epsilon_0}\bar{\rho}_0(r)\left[\frac{4\pi\int_0^r \mathrm{d}r' r'^2 \bar{\rho}_0(r')}{r'} + 4\pi\int_r^{a_0} \mathrm{d}r' r' \bar{\rho}_0(r')\right] \end{aligned} \tag{15}$$

Then the equation of motion of a segment of charge function becomes

$$\begin{aligned} P_0\bar{\rho}_0(r) = &-\frac{\partial}{\partial r}\bar{\rho}_1(r) - \frac{2}{r}\bar{\rho}_1(r) + \left[-\frac{e^2}{4\pi\epsilon_0 a_0}\bar{\rho}_0(r)\right. \\ &\left. + \frac{e^2}{4\pi\epsilon_0}\bar{\rho}_0(r)\left[\frac{4\pi\int_0^r \mathrm{d}r' r'^2 \bar{\rho}_0(r')}{r'} + 4\pi\int_r^{a_0} \mathrm{d}r' r' \bar{\rho}_0(r')\right]\right] \end{aligned} \tag{16}$$

$$P_0\bar{\rho}_1(r) = \frac{\partial}{\partial r}\bar{\rho}_0(r) \tag{17}$$

$$P_0\bar{\rho}_2(r) = -\frac{\partial}{\partial r}\bar{\rho}_3(r) - \frac{2}{r}\bar{\rho}_3(r) \tag{18}$$

$$P_0\bar{\rho}_3(r) = \frac{\partial}{\partial r}\bar{\rho}_2(r) \tag{19}$$

Derivation of kinetic term of each equation above is shown in **Appendix**.

Note that because we investigate electron charge distribution function by considering the equation of motion of its segment of charge with self-interaction energy, Equations (16)-(19) represent an electron in the rest frame. Also note that these equations are described in units of $\hbar = c = 1$.

To obtain a proper unit description of the charge distribution of an electron, we rewrite Equations (16)-(19) as follows.

$$\begin{aligned} P_0\bar{\rho}_0(r) = &-\hbar c\frac{\partial}{\partial r}\bar{\rho}_1(r) - \hbar c\frac{2}{r}\bar{\rho}_1(r) + \left[-\frac{e^2}{4\pi\epsilon_0 a_0}\bar{\rho}_0(r)\right. \\ &\left. + \frac{e^2}{4\pi\epsilon_0}\bar{\rho}_0(r)\left[\frac{4\pi\int_0^r \mathrm{d}r' r'^2 \bar{\rho}_0(r')}{r'} + 4\pi\int_r^{a_0} \mathrm{d}r' r' \bar{\rho}_0(r')\right]\right] \end{aligned} \tag{20}$$

$$P_0\bar{\rho}_1(r) = \hbar c\frac{\partial}{\partial r}\bar{\rho}_0(r) \tag{21}$$

$$P_0\bar{\rho}_2(r) = -\hbar c\frac{\partial}{\partial r}\bar{\rho}_3(r) - \hbar c\frac{2}{r}\bar{\rho}_3(r) \tag{22}$$

$$P_0\bar{\rho}_3(r) = \hbar c\frac{\partial}{\partial r}\bar{\rho}_2(r) \tag{23}$$

Because we are interested in only charge distribution, we need only to obtain the solution of $\bar{\rho}_0(r)$, however, this solution must correspond to a certain ei-

genvalue that should be the rest mass of an electron.

To do this, we need to consider only Equation (20) and Equation (21). By inserting the description of $\overline{\rho}_1(r)$ from Equation (21) into Equation (20), we can obtain the equation for $\overline{\rho}_0(r)$ as follows.

$$P_0\overline{\rho}_0(r) = -\frac{(\hbar c)^2}{P_0}\frac{\partial^2\overline{\rho}_0(r)}{\partial r^2} - \frac{(\hbar c)^2}{P_0}\frac{2}{r}\frac{\partial\overline{\rho}_0(r)}{\partial r} + \left[-\frac{e^2}{4\pi\epsilon_0 a_0}\overline{\rho}_0(r) + \frac{e^2}{4\pi\epsilon_0}\overline{\rho}_0(r)\left[\frac{4\pi\int_0^r \mathrm{d}r' r'^2\overline{\rho}_0(r')}{r'} + 4\pi\int_r^{a_0}\mathrm{d}r' r'\overline{\rho}_0(r')\right]\right] \tag{24}$$

Because we did not find out exact treatment of the integral part of Equation (24), we adopt a Tayler expansion around $r=0$. This requires us to make a variable dimensionless.

Thus, we change a variable as $\overline{r} = \frac{r}{a_0}$. Then Equation (24) becomes

$$\left(\frac{P_0}{\frac{\hbar c}{a_0}}\right)^2\overline{\rho}_0(\overline{r}) = -\frac{\partial^2\overline{\rho}_0(\overline{r})}{\partial\overline{r}^2} - \frac{2}{\overline{r}}\frac{\partial\overline{\rho}_0(\overline{r})}{\partial\overline{r}} + \frac{P_0}{\left(\frac{\hbar c}{a_0}\right)^2}\left[-\frac{e^2}{4\pi\epsilon_0 a_0} + \frac{e^2}{4\pi\epsilon_0}\frac{4\pi a_0^2\int_0^{\overline{r}}\mathrm{d}\overline{r}'\overline{r}'^2\overline{\rho}_0(\overline{r}')}{\overline{r}} + \frac{e^2}{4\pi\epsilon_0}\left[4\pi a_0^2\int_0^1\mathrm{d}\overline{r}'\overline{r}'^2\overline{\rho}_0(\overline{r}') + 4\pi a_0^2\int_0^{\overline{r}}\mathrm{d}\overline{r}'\overline{r}'\overline{\rho}_0(\overline{r}')\right]\right] \tag{25}$$

Note that the condition Equation (13) becomes

$$\int_0^1\mathrm{d}\overline{r}\,\overline{r}^2\overline{\rho}_0(\overline{r}) = \frac{1}{4\pi a_0^3} \tag{26}$$

To make dimension correct, we need to set one more condition as

$$\int_0^1\mathrm{d}\overline{r}\,\overline{r}\,\overline{\rho}_0(\overline{r}) = \frac{C_1}{4\pi a_0^2} \tag{27}$$

Here C_1 is constant and is determined after obtaining the corresponding solution $\overline{\rho}_0(\overline{r})$ to the eigenvalue.

To obtain the Tayler expansion, we define these integral as

$$F_1(\overline{r}) = \int_0^{\overline{r}}\mathrm{d}\overline{r}'\overline{r}'^2\overline{\rho}_0(\overline{r}') \tag{28}$$

$$F_2(\overline{r}) = -\int_0^{\overline{r}}\mathrm{d}\overline{r}'\overline{r}'\overline{\rho}_0(\overline{r}') \tag{29}$$

Then Tayler expansion around $\overline{r}=0$ of $F_1(\overline{r})$ and $F_2(\overline{r})$ become

$$F_1(\overline{r}) = \overline{r}\left(\overline{r}^2\overline{\rho}_0(\overline{r})\right) + \frac{1}{2}\overline{r}^2\left(2\overline{r}\rho_0(\overline{r}) + \overline{r}^2\frac{\partial\overline{\rho}_0(\overline{r})}{\partial\overline{r}}\right) \tag{30}$$

$$F_2\left(\bar{r}\right)=-\bar{r}\left(\bar{r}\,\bar{\rho}_0\left(r\right)\right)-\frac{1}{2}\bar{r}^2\left(\bar{\rho}_0\left(\bar{r}\right)+\bar{r}\frac{\partial\bar{\rho}_0\left(\bar{r}\right)}{\partial\bar{r}}\right) \tag{31}$$

Note that usual Tayler expansion around $\bar{r}=0$ involves using the description of the value of *n*th derivative at $\bar{r}=0$ times $\frac{\bar{r}^n}{n!}$. However, we use only the nth derivative term itself because we are considering the case that $\bar{r}$ itself is very close to zero. And note that this consideration restricts the order of singularity at most 1, which means that if $\bar{\rho}_0\left(\bar{r}\right)$ has $\bar{r}^{-n}$ term largest *n* is 1.

For the time being, we neglect derivative expression part in Equation (30) and Equation (31). Then we obtain

$$\frac{1}{\bar{r}}F_1\left(\bar{r}\right)+F_2\left(\bar{r}\right)=\frac{1}{2}\bar{r}^2\bar{\rho}_0\left(\bar{r}\right) \tag{32}$$

We will demonstrate that this neglecting process does not affect the result by checking the corresponding eigenvalue solution $\bar{\rho}_0\left(\bar{r}\right)$.

Then Equation (25) becomes

$$\begin{aligned}\left(\frac{P_0}{\frac{\hbar c}{a_0}}\right)^2\bar{\rho}_0\left(\bar{r}\right)=&-\frac{\partial^2\bar{\rho}_0\left(\bar{r}\right)}{\partial\bar{r}^2}-\frac{2}{\bar{r}}\frac{\partial\bar{\rho}_0\left(\bar{r}\right)}{\partial\bar{r}}-\frac{P_0}{\left(\frac{\hbar c}{a_0}\right)^2}\frac{e^2}{4\pi\epsilon_0 a_0}\left(1-a_0C_1\right)\bar{\rho}_0\left(\bar{r}\right)\\&+\frac{P_0}{\left(\frac{\hbar c}{a_0}\right)^2}\frac{e^2}{4\pi\epsilon_0 a_0}\frac{1}{2}\bar{r}^2\left(\bar{\rho}_0\left(\bar{r}\right)\right)^2\end{aligned} \tag{33}$$

Equation (33) is a homogeneous nonlinear second order differential equation. Because we have not discovered an exact treatment for this equation, we set the following condition to linearize it.

$$\left(\bar{\rho}_0\left(\bar{r}\right)\right)^2=\bar{\rho}_0\left(\bar{r}\right)+f\left(\bar{r}\right) \tag{34}$$

Then Equation (33) becomes the following nonhomogeneous linear second order differential equation.

$$\begin{aligned}&\frac{\partial^2\bar{\rho}_0\left(\bar{r}\right)}{\partial\bar{r}^2}+\frac{2}{\bar{r}}\frac{\partial\bar{\rho}_0\left(\bar{r}\right)}{\partial\bar{r}}+\left(\left(\frac{P_0}{\frac{\hbar c}{a_0}}\right)^2+\alpha\left(\frac{P_0}{\frac{\hbar c}{a_0}}\right)\left(1-a_0C_1\right)\right)\bar{\rho}_0\left(\bar{r}\right)-\frac{1}{2}\alpha\left(\frac{P_0}{\frac{\hbar c}{a_0}}\right)\bar{r}^2\bar{\rho}_0\left(\bar{r}\right)\\&=\frac{1}{2}\alpha\left(\frac{P_0}{\frac{\hbar c}{a_0}}\right)f\left(\bar{r}\right)\end{aligned} \tag{35}$$

To obtain the Equation (35), we used the fact that $\frac{e^2}{4\pi\epsilon_0}=\alpha\hbar c$.

To solve Equation (35), we need to solve the following homogeneous equation.

$$\frac{\partial^2 \bar{\rho}_0(\bar{r})}{\partial \bar{r}^2}+\frac{2}{\bar{r}}\frac{\partial \bar{\rho}_0(\bar{r})}{\partial \bar{r}}+\left(\left(\frac{P_0}{\frac{\hbar c}{a_0}}\right)^2+\left(\alpha\left(\frac{P_0}{\frac{\hbar c}{a_0}}\right)\left(1-a_0C_1\right)\right)\right)\bar{\rho}_0(\bar{r})$$
$$-\frac{1}{2}\alpha\left(\frac{P_0}{\frac{\hbar c}{a_0}}\right)\bar{r}^2\bar{\rho}_0(\bar{r})=0 \tag{36}$$

For the simplicity we use the following notation

$$T=\left(\frac{P_0}{\frac{\hbar c}{a_0}}\right)^2+\alpha\left(\frac{P_0}{\frac{\hbar c}{a_0}}\right)\left(1-a_0C_1\right) \tag{37}$$

$$S=\frac{1}{2}\alpha\left(\frac{P_0}{\frac{\hbar c}{a_0}}\right) \tag{38}$$

Then Equation (36) is expressed as

$$\frac{\partial^2 \bar{\rho}_0(\bar{r})}{\partial \bar{r}^2}+\frac{2}{\bar{r}}\frac{\partial \bar{\rho}_0(\bar{r})}{\partial \bar{r}}+\left(T-S\bar{r}^2\right)\bar{\rho}_0(\bar{r})=0 \tag{39}$$

To solve Equation (39), we use $\bar{\rho}_0(\bar{r})=\frac{1}{\bar{r}}\bar{\bar{\rho}}_0(\bar{r})$ and rearrange Equation (39) to become the following after factor out $\frac{1}{\bar{r}}$

$$\frac{\partial^2 \bar{\bar{\rho}}_0(\bar{r})}{\partial \bar{r}^2}+\left(T-S\bar{r}^2\right)\bar{\bar{\rho}}_0(\bar{r})=0 \tag{40}$$

To determine the specific eigenvalue, we use $z=\bar{r}^2$.

Then Equation (40) becomes

$$\frac{\partial^2 \bar{\bar{\rho}}_0}{\partial z^2}+\frac{1}{2z}\frac{\partial \bar{\bar{\rho}}_0}{\partial z}+\left[\frac{T}{4z}-\frac{S}{4}\right]\bar{\bar{\rho}}_0=0 \tag{41}$$

By setting as $\bar{\bar{\rho}}_0(z)=z^{-\frac{1}{4}}\hat{\rho}_0(z)$ and factor out $z^{-\frac{1}{4}}$, Equation (41) becomes

$$\frac{\partial^2 \hat{\rho}_0}{\partial z^2}+\left(-\frac{S}{4}+\frac{T}{4z}+\frac{\frac{3}{16}}{z^2}\right)\hat{\rho}_0=0 \tag{42}$$

To obtain the standard form of Whittaker equation, we use $z=\xi\bar{z}$.

Then Equation (42) becomes

$$\frac{\partial^2 \hat{\rho}_0}{\partial \bar{z}^2}+\left(-\frac{S\xi^2}{4}+\frac{\frac{1}{4}T\xi}{\bar{z}}+\frac{\frac{3}{16}}{\bar{z}^2}\right)\hat{\rho}_0=0 \tag{43}$$

By setting the value of ξ at $S\xi^2 = 1$, Equation (43) becomes

$$\frac{\partial^2 \hat{\rho}_0}{\partial \bar{z}^2} + \left(-\frac{1}{4} + \frac{\frac{1}{4} TS^{-\frac{1}{2}}}{\bar{z}} + \frac{\frac{3}{16}}{\bar{z}^2} \right) \hat{\rho}_0 = 0 \tag{44}$$

The standard form of the Whittaker equation is presented in Ref. [9] as

$$\frac{\partial^2 W}{\partial \bar{z}^2} + \left(-\frac{1}{4} + \frac{\kappa}{\bar{z}} + \frac{-\mu^2 + \frac{1}{4}}{\bar{z}^2} \right) W = 0 \tag{45}$$

By comparison to the standard form of Whittaker equation, we obtain

$$\mu = \pm \frac{1}{4}$$

$$\kappa = \frac{1}{4} TS^{-\frac{1}{2}}$$

Basic solutions of Equation (45) are

$$M_{\kappa,\mu}(\bar{z}) \quad \text{and} \quad M_{\kappa,-\mu}(\bar{z})$$

The definition of $M_{\kappa,\mu}(\bar{z})$ is given in Ref [9] as

$$\begin{aligned} M_{\kappa,\mu}(\bar{z}) &= \bar{z}^{\mu+\frac{1}{2}} \exp\left(-\frac{\bar{z}}{2}\right) F\left(\mu - \kappa + \frac{1}{2}, 2\mu + 1, \bar{z}\right) \\ &= \bar{z}^{\mu+\frac{1}{2}} \exp\left(-\frac{\bar{z}}{2}\right) \sum_{n=0}^{\infty} \frac{\Gamma(2\mu+1)\Gamma\left(\mu - \kappa + n + \frac{1}{2}\right)}{\Gamma(2\mu+n+1)\Gamma\left(\mu - \kappa + \frac{1}{2}\right)} \frac{\bar{z}^n}{n!} \end{aligned} \tag{46}$$

Because we do not want continuous eigenvalue, the series must be terminated. This condition gives

$$\mu - \kappa + n + \frac{1}{2} = 0 \tag{47}$$

The Lowest value ofκ corresponds to lowest eigenvalue comes from the case of $n = 0$ and $\mu = -\frac{1}{4}$.

Then we obtain the determining equation for eigenvalue as

$$\kappa = \frac{1}{4} = \frac{1}{4} TS^{-\frac{1}{2}} = \frac{1}{4} \left(\left(\frac{P_0}{\frac{\hbar c}{a_0}} \right)^2 + \alpha \left(\frac{P_0}{\frac{\hbar c}{a_0}} \right) (1 - a_0 C_1) \right) \left(\frac{2}{\alpha} \right)^{\frac{1}{2}} \left(\frac{\frac{\hbar c}{a_0}}{P_0} \right)^{\frac{1}{2}} \tag{48}$$

Rearrangement yields the following for the eigenvalue determining equation.

$$\left(\left(\frac{P_0}{\frac{\hbar c}{a_0}} \right)^2 + \alpha \left(\frac{P_0}{\frac{\hbar c}{a_0}} \right) (1 - a_0 C_1) \right) \left(\frac{2}{\alpha} \right)^{\frac{1}{2}} \left(\frac{\frac{\hbar c}{a_0}}{P_0} \right)^{\frac{1}{2}} = 1 \tag{49}$$

Recalling that $\frac{1}{\bar{r}}$ and $\bar{z}^{-\frac{1}{4}}$, as well as $\bar{z}=\frac{z}{\xi}$ and $z=\bar{r}^2$, can be factored out, the corresponding eigenfunction becomes

$$\bar{\rho}_0^{(1)}\left(\bar{r}\right)=const\frac{1}{\bar{r}}\exp\left(-\frac{\bar{r}^2}{2\xi}\right) \tag{50}$$

This solution has maximum, actually it is a singularity, at $\bar{r}=0$, so it has an appropriate solution from Tayler expansion around $\bar{r}=0$. The condition for the order of singularity that we mentioned before is also satisfied. And we can obtain the same factor of $\frac{1}{2}$ when calculating the second order Tayler series with the derivative term that was neglected before. Thus, this solution is very acceptable.

Another basic solution is

$$\bar{\rho}_0^{(2)}=const\frac{\xi^{\frac{1}{2}}}{\bar{r}^{\frac{3}{2}}}M_{\frac{1}{4},\frac{1}{4}}\left(\frac{\bar{r}^2}{\xi}\right) \tag{51}$$

Note that $\bar{\rho}_0^{(2)}$ does not have a singularity at $\bar{r}=0$. We need this solution also because in Equation (35) we are dealing with nonhomogeneous equation and we must construct a particular solution to complete it.

We adopt $\mu=-\frac{1}{4}$ solution because in the case the solution corresponds to eigenvalue. If we take $\mu=+\frac{1}{4}$ in Equation (47), the determining equation is $\kappa=\frac{3}{4}$ $(n=0)$ instead $\kappa=\frac{1}{4}$, which has no singularity at $\bar{r}=0$. Therefore the type of solution in Equation (50) is more appropriate for the consideration besides the determined eigenvalue is larger than $\mu=-\frac{1}{4}$ case.

To obtain $f\left(\bar{r}\right)$, we describe $\bar{\rho}_0\left(\bar{r}\right)$ as a function of $f\left(\bar{r}\right)$ by solving the following equation

$$\left(\bar{\rho}_0\left(\bar{r}\right)\right)^2=\bar{\rho}_0\left(\bar{r}\right)+f\left(\bar{r}\right) \tag{52}$$

We insert the solution of Equation (52), that is a function of $f\left(\bar{r}\right)$, into Equation (33) and obtain the following equation.

$$\begin{aligned}&\frac{\partial^2 f}{\partial\bar{r}^2}-\frac{2\left(\frac{\partial f}{\partial\bar{r}}\right)^2}{1+4f}+\frac{2}{\bar{r}}\frac{\partial f}{\partial\bar{r}}+T\left(\frac{1\pm\sqrt{1+4f}}{2}\right)\frac{\sqrt{1+4f}}{2}\\&-S\bar{r}^2\left(\frac{1\pm\sqrt{1+4f}}{2}+f\right)\frac{\sqrt{1+4f}}{2}=0\end{aligned} \tag{53}$$

We could not find the exact solution of Equation (54), so instead we seek a plausible solution by recalling that the equation we are dealing with is under the condition that $\bar{r}$ is very small (near 0). Then as a trial solution, we set the following form.

$$1+4f\left(\bar{r}\right)=a\bar{r}^2+b\bar{r}+c \tag{54}$$

Recalling that $\bar{r}$ is small, and noticing the fact that

$$\frac{2}{\bar{r}}\frac{\partial f}{\partial \bar{r}}=2a+\frac{\frac{b}{4}}{\bar{r}}$$

the $\frac{1}{\bar{r}}$ term does not appear separately from the other terms, thus $b$must be 0.

Then, we can use the following approximations to describe Equation (53) second term of $1+4f=c$

$$\sqrt{1+4f}=\sqrt{c}\left(1+\frac{a\bar{r}^2}{\sqrt{c}}\right)^{\frac{1}{2}}=\sqrt{c}\left(1+\frac{a\bar{r}^2}{2\sqrt{c}}\right)$$

Comparing each $\bar{r}^0$ (order o term), $\bar{r}^2$ term, we obtain the following equations.

$$\bar{r}^0 \quad \text{term} \quad 0=\pm\left(\frac{3a}{2}\right)+\frac{\sqrt{c}\pm c}{2}T \tag{55}$$

$$\bar{r}^2 \quad \text{term} \quad 0=\pm\left(-\frac{a^2}{2c}\right)+\frac{1}{2}\left(\frac{1}{\sqrt{c}}\pm 1\right)a^2T-S\left(\frac{c\pm c\sqrt{c}}{2}+\frac{c\sqrt{c}-\sqrt{c}}{4}\right) \tag{56}$$

The remaining terms are only S multiplies of order $\bar{r}^4$ and $\bar{r}^6$ terms, and because we are dealing with Equation (35) under the condition that $\bar{r}$ is small, we neglect them.

For the + sign, we obtain the determining equation for c as

$$\begin{aligned}0=&\left(\frac{3}{4}S+\frac{T}{2}\right)\left(\sqrt{c}\right)^4+\left(\frac{S}{2}+\frac{3}{2}T\right)\left(\sqrt{c}\right)^3\\&+\left(\frac{S}{4}-\frac{T^2}{9}+T\right)\left(\sqrt{c}\right)^2-\frac{T^2}{18}\sqrt{c}-\frac{T^2}{18}\end{aligned} \tag{57}$$

Once the value of c is obtained, a is given from Equation (55) as

$$a=-\frac{\sqrt{c}+c}{3} \tag{58}$$

Here, $T$and $S$are defined by Equation (37) and Equation (38).

Note that the obtained $f\left(r\right)$ is only an approximation and includes uncertainty. However, this uncertainty does not affect the eigenvalue because it is only involved in the ambiguity for a_0, as shown in later.

Then our $f\left(\bar{r}\right)$ is expressed as

$$f\left(\bar{r}\right)=\bar{a}\bar{r}^2+\bar{c} \tag{59}$$

where $\bar{a},\bar{c}$ denote the values by Equation (58) and Equation (57).

To obtain a particular solution of Equation (35), we use the Wronskian method. Recalling that Equation (36) is homogeneous and that its solutions are obtained in Equation (50) and Equation (51), the Wronskian becomes

$$W(\bar{r})=\begin{vmatrix} \frac{1}{\bar{r}}\mathrm{e}^{-\frac{\bar{r}^2}{2\xi}} & \frac{\partial}{\partial\bar{r}}\left(\frac{1}{\bar{r}}\mathrm{e}^{-\frac{\bar{r}^2}{2\xi}}\right) \\ \frac{\xi^{\frac{1}{2}}}{\bar{r}^{\frac{3}{2}}}M_{\frac{1}{4},\frac{1}{4}}\left(\frac{\bar{r}^2}{\xi}\right) & \frac{\partial}{\partial\bar{r}}\left(\frac{\xi^{\frac{1}{2}}}{\bar{r}^{\frac{3}{2}}}M_{\frac{1}{4},\frac{1}{4}}\left(\frac{\bar{r}^2}{\xi}\right)\right) \end{vmatrix} \tag{60}$$

Then a particular solution of Equation (35) is constructed as

$$\bar{\rho}_0^P(\bar{r})=\frac{\xi^{\frac{1}{2}}}{\bar{r}^{\frac{3}{2}}}M_{\frac{1}{4},\frac{1}{4}}\left(\frac{\bar{r}^2}{\xi}\right)\int_0^{\bar{r}}\mathrm{d}r'\frac{f(r')\frac{1}{r'}\mathrm{e}^{-\frac{\bar{r}^2}{2\xi}}}{W(r')}-\frac{1}{\bar{r}}\mathrm{e}^{-\frac{\bar{r}^2}{2\xi}}\int_0^{\bar{r}}\mathrm{d}r'\frac{f(r')\frac{\xi^{\frac{1}{2}}}{r'^{\frac{3}{2}}}M_{\frac{1}{4},\frac{1}{4}}\left(\frac{\bar{r}^2}{\xi}\right)}{W(r')} \tag{61}$$

This formula is presented as an example in book of Ince [10].

Thus, the general solution of Equation (35) is written as

$$\bar{\rho}_0(\bar{r})=\bar{\rho}_0^P(\bar{r})+A\frac{1}{\bar{r}}\exp\left(-\frac{\bar{r}^2}{2\xi}\right)+B\left(\frac{\xi^{\frac{1}{2}}}{\bar{r}^{\frac{3}{2}}}\right)M_{\frac{1}{4},\frac{1}{4}}\left(\frac{\bar{r}^2}{\xi}\right) \tag{62}$$

where A and B are arbitrary constants determined by condition Equation (26) and Equation (27).

The corresponding eigenvalue is given in Equation (49). Equation (49) is re-written as

$$\left(\frac{P_0}{\frac{\hbar c}{a_0}}\right)^{\frac{3}{2}}\left(\frac{2}{\alpha}\right)^{\frac{1}{2}}+(2\alpha)^{\frac{1}{2}}(1-a_0C_1)\left(\frac{P_0}{\frac{\hbar c}{a_0}}\right)^{\frac{1}{2}}=1 \tag{63}$$

To see whether this scheme works or not, we use the following form for $M_{\frac{1}{4},\frac{1}{4}}\left(\frac{\bar{r}^2}{\xi}\right)$.

$$M_{\frac{1}{4},\frac{1}{4}}\left(\frac{\bar{r}^2}{\xi}\right)=\left(\frac{\bar{r}}{\xi}\right)^{\frac{3}{2}}\exp\left(-\frac{\bar{r}^2}{2\xi}\right) \tag{64}$$

Then the second basic solution becomes

$$\bar{\rho}_0^{(2)}(\bar{r})=\exp\left(-\frac{\bar{r}^2}{2\xi}\right) \tag{65}$$

Then Wronskian becomes $W=\frac{1}{\bar{r}^2}\mathrm{e}^{-\frac{\bar{r}^2}{\xi}}$. Thus we obtain the particular solution

$$\bar{\rho}_P(\bar{r})=\left[c+\frac{5}{2}|a|\sqrt{\frac{2}{\alpha}}\right]\bar{r}^2 \tag{66}$$

Then the general solution of Equation (35) is expressed as

$$\bar{\rho}_0(\bar{r}) = \bar{f}_P + A\left(\frac{1}{\bar{r}}\exp\left(-\frac{\bar{r}^2}{2\xi}\right)\right) + B\exp\left(-\frac{\bar{r}^2}{2\xi}\right) \tag{67}$$

Inserting Equation (67) to the condition equations Equation (26) and Equation (27), we obtain arbitrary constants A and B as

$$A = 6\left(\frac{1}{4\pi a_0^3} - \frac{\eta}{5} + \frac{\eta}{3}\mathrm{e}^{\frac{1}{2\xi}}\right) \tag{68}$$

$$B = -3\eta\mathrm{e}^{\frac{1}{2\xi}} + \frac{6\eta}{5} - \frac{6}{4\pi a_0^3} \tag{69}$$

where $\eta = c + \frac{5}{2}|a|\sqrt{\frac{2}{\alpha}}$. Then, the solution of this system is

$$\begin{aligned}\bar{\rho}_0(\bar{r}) = \eta\bar{r}^2 + \frac{6}{4\pi a_0^3}\left(\frac{1}{\bar{r}} - 1\right)\exp\left(-\frac{\bar{r}^2}{2\xi}\right) + \left(-\frac{6\eta}{5} + 2\eta\mathrm{e}^{\frac{1}{2\xi}}\right)\frac{1}{\bar{r}}\exp\left(-\frac{\bar{r}^2}{2\xi}\right) \\ + \left(-3\eta\mathrm{e}^{\frac{1}{2\xi}} + \frac{6\eta}{5}\right)\exp\left(-\frac{\bar{r}^2}{2\xi}\right) \quad \left(\bar{r} \in [0,1]\right)\end{aligned} \tag{70}$$

To calculate C_1, we use only the second term because it is proportional to $\frac{1}{a_0^3}$ and reminding the fact that a_0 is very small quantity.

Then we obtain $a_0 C_1 = 3$. Note that the second term is not dependent on η, so we can calculate anal most exact eigenvalue without knowing its value.

Then Equation (49) (the determining equation for eigenvalue) becomes as

$$\left(\frac{\frac{P_0}{\hbar c}}{a_0}\right)^{\frac{3}{2}} - 2\alpha\left(\frac{\frac{P_0}{\hbar c}}{a_0}\right)^{\frac{1}{2}} = \sqrt{\frac{\alpha}{2}} \tag{71}$$

The solution of Equation (71) is

$$\frac{\frac{P_0}{\hbar c}}{a_0} = 4.74^2\alpha \tag{72}$$

Thus P_0 is expressed as

$$P_0 = \frac{\alpha\hbar c}{a_c}\frac{4.74^2 a_c}{a_0}$$

where a_c denotes classical electron radius.

Thus if $a_0 = 4.74^2 a_c$, then must be $P_0 = m_e c^2$ (electron rest mass energy).

Using Equation (72) we can determine $\sqrt{c}$ to be 0.02284. With Equation (37), we can find $c = 0.000522$ and with Equation (38), we can find $a = -0.00779$. From these values η becomes as $\eta = 0.322822$.

We describe this scheme in terms of choosing only the first term of $M_{\frac{1}{4},\frac{1}{4}}$, but we can also obtain $P_0 = m_e c^2$ when using the exact form of $M_{\frac{1}{4},\frac{1}{4}}$. This is

because we can absorb the difference caused by including infinite series terms of $M_{\frac{1}{4},\frac{1}{4}}$ into a_0 used to obtain the electron mass energy eigenvalue. Our viewpoint is that the result using exact form of $M_{\frac{1}{4},\frac{1}{4}}$ is not very different from the result shown in this paper because the maximum value of the nth term of $M_{\frac{1}{4},\frac{1}{4}}$ is

$$\frac{1}{(2n+1)n!}\left(\frac{\alpha}{2}\right)^{\frac{n}{2}} \approx \frac{1}{(2n+1)n!}\left(\frac{1}{17}\right)^{n}$$

3. Results

Using the equation of motion for a segment of charge, we obtain the charge wave function for an electron as Equation (63). We call $\bar{\rho}_0(\bar{r})$ as a wave function because it appears as the eigenfunction corresponding to eigenvalue. An important point is that this is interpreted as a charge distribution function. To make this clearer, we performed a calculation example by choosing only the first term of the infinite series of $M_{\frac{1}{4},\frac{1}{4}}$. This result in an obtained charge distribution function that comes from an eigenfunction of electron mass energy. Also, its dominant function is $\frac{1}{\bar{r}}\exp\left(-\frac{\bar{r}^2}{2\xi}\right)$, which can be favorably compared to Weinberg's charge distribution function relating $\eta(\vec{r})$ to $-\frac{const}{r^3}\int_0^1 dx x(1-x)\left(1+\frac{mr}{\sqrt{x(1-x)}}\right)\exp\left(-\frac{mr}{\sqrt{x(1-x)}}\right)$. We can say that the characteristic form of this expression is $-\frac{1}{r^3}\exp(-mr)$, which has a minus sign that, as pointed out by Weinberg, becomes plus sign after volume integration. This is exactly the same as our $\bar{\rho}_0(r)$ expression. Weinberg adds the $\delta(\vec{r})$ term to satisfy this condition. This modification seems to be inconsistent because it creates dipole moment, which is unlikely for reasons described later. However, apart from this and sign issue, we think that the behavior of Weinberg expression as a charge distribution function carries real physical meaning. When we compare our $\bar{\rho}_0(r)$ to the characteristic form of Weinberg's $\eta(\vec{r})$, both functions have a singularity at the origin, although the order of them is different and the corresponding term of the exponential part is Gaussian for our case. Thus we can say that the overall behavior is similar. The most interesting point is that our derivation is totally different from that of Weinberg even though both results are similar. Thus, we can say that our result may reflect Weinberg's comment that by absorbing an infinite number of photons, an electron has mass while charge is simultaneously spread out to finite volume [3] because our result is obtained as an eigenfunction of electron mass energy.

Our result has another interesting property. It shows a charged electron radius, denoted by a_0, that is almost 23 times as much as the classical electron ra-

dius. This result is consistent with the result obtained by Kim *et al.* [11], who reformulate an electron equation of motion of electron by using nonrelativistic quantum mechanics. Anequation of motion of electron was first founded by Abraham [12] and Lorentz [13] more than century ago considering an electron as a point-like which caused run-aways solutions and no causal behavior. Later-Caldiola [14] extended it, still in classical theory, showed that no run-aways and causality solution occur when the electron charged radius is more than $\frac{2}{3}$ times classical electron radius. Kim *et al.* used nonrelativistic quantum mechanics consideration based on neglecting magnetic term to this problem and obtained the result that are alistic electron with no run-aways and no causality must satisfy the condition that the charged radius is smaller than 1.75 times Compton wave lengths. However, it has also pointed out that nonrelativistic quantum mechanics cannot strictly valid in point-particle limit because of vacuum polarization. In addition, QED calculation in this matter has not been reported. This is presumably because no physicists know how to regularize soft photons. Even Weinberg showed his electron charged radius had some remaining divergence [3]. The ACME collaboration report that they could not find a dipole moment inside electron electric field suggests that regularization based on the idea that creation and annihilation of tiny plus charge and minus charge in electric field is unlikely [15]. Therefore, Weinberg's expression of an electron charged radius is still meaningful. This suggests that the lower bound of an electron charged radius according to quantum theory remains unclear. Thus, at the present time, we can say safely that the realistic electron satisfy the condition that its charged radius is more than $\frac{2}{3}$ times the classical electron radius and less than 1.75 times the Compton wave lengths. As we mentioned in sec. 2, our obtained a_0 is only approximation, but even using full expression of $M_{\frac{1}{4},\frac{1}{4}}$, would not yield a very different result. Thus, with respect to the electron charged radius, our result is also plausible.

4. Discussion

To this point, there has not been a decisive experiment result to determine the size of a charged electron, the few results we do have illustrate why this problem is difficult. Let us consider the example of $e^+ - e^-$ collision. Approaches to using these data can be divided into two groups. One group says that an electron is a point-like because no resolution data show up even though collision energy is very high, very much larger than the mass energy of electron. The other group uses data for extrapolating to low energy, ($q^2 \approx 0$), then claims that the electron charged radius is a finite and not point-like. The consideration that the size of a charged electron is obtained at low energy, (almost $q^2 \approx 0$) during a $e^+ - e^-$ collision experiment is based on Cabbibo's paper [16]. Also, this consideration corresponds to the determination method of pion charged radius [17] together

with the determination equation of $|F_\pi|^2$ from cross section of pion-electron collision data given by NA7 Collaboration (CERN) [18]. However, there is a paper that might suggest an answer to this problem. Karnieli *et al.* suggest that the temporal duration of Cherenkov radiation, envisioned for almost a century as a shock wave, is limited by underlying entanglement between particle and light defined current operator as $\hat{J}(\vec{r},t)=ec\psi^{\dagger}(\vec{r},t)\vec{\alpha}\psi(\vec{r},t)$ where $\psi(\vec{r},t)$ is a delocalized wave function instead of being limited by $\vec{J}(\vec{r},t)=e\vec{v}\delta(\vec{r}-\vec{v}t)$ which uses current density and emits shock wave [7]. This means that a free charged electron is not a point-like but spread out in space-time. However, a complete and accurate description of this property depends importantly on whether or not the Higgs mechanism generates the masses of leptons, or at least electrons.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Cho, A. (2012) *Science*, **338**, 1524-1525. https://doi.org/10.1126/science.338.6114.1524

[2] Gonzalez, E., Kane, G., Nguyen, K.D. and Perry, M. (2021) *Physical Review D*, **103**, Article ID: 126027. https://doi.org/10.1103/PhysRevD.103.126027

[3] Weinberg, S. (1995) The Quantum Theory of Field I. Cambridge University Press, Cambridge.

[4] Weinberg, S. (2020) *Physical Review D*, **101**, Article ID: 035020. https://doi.org/10.1103/PhysRevD.101.035020

[5] Barr, S.M. and Zee, A. (1978) *Physical Review D*, **17**, 1854. https://doi.org/10.1103/PhysRevD.17.1854

[6] Dobrescu, B.A. and Fox, P. (2008) *Journal of High Energy Physics*, **8**, 100. https://doi.org/10.1088/1126-6708/2008/08/100

[7] Karnieli, A., Rivera, N. and Kaminer, I. (2021) *Science Advances*, **7**. https://doi.org/10.1126/sciadv.abf8096

[8] Weinberg, S. (2000) The Quantum Theory of Field III. Cambridge University Press, Cambridge.

[9] Moriguchi, S., Udagawa, K. and Hitotsumatsu, S. (1975) Mathematics Formula III. Iwanami.

[10] Ince, E.L. (1956) Ordinary Differential Equations. Dover Publication, Mineola.

[11] Kim, K.J. and Sessler, A.M. (1998) 8th Workshop on Advanced Acceleration Concepts July 5-11, 1998.

[12] Abraham, M. (1905) "Theorie der Elektrizitat" Vol. II: Elektromagnetische Theorie der Strahlung. Teubner, Leipzig.

[13] Lorentz, H.A. (1909) The Theory of Electrons. Teubner, Leipzig.

[14] Caldirola, P. (1956) *Nuovo Cimento*, **3**, 297. https://doi.org/10.1007/BF02743686

[15] Andreev, V., Ang, D.G., DeMille, D., *et al.* (2018) *Nature*, **562**, 355-360. https://doi.org/10.1038/s41586-018-0599-8

[16] Cabbibo (1961) *Physical Review*, **124**, 1577. https://doi.org/10.1103/PhysRev.124.1577

[17] Cui, Z.F. and Schmidt, S.M. (2021) *Physics Letters B*, **882**, Article ID: 136631. https://doi.org/10.1016/j.physletb.2021.136631

[18] Amendolia, S.R., *et al.* (1986) *Nuclear Physics B*, **277**, 168-196.

Appendix

Considering polar coordinate each x_1, x_2 and x_3 are represented as

$$x_1 = r\sin\theta\cos\theta, x_2 = r\sin\theta\sin\theta, x_3 = \cos\theta$$

$$\begin{aligned}-i\alpha^k\partial_k\left(1q_0(r)\right) &= i\left(\alpha^1\frac{\partial}{\partial x_1}+\alpha^2\frac{\partial}{\partial x_2}+\alpha^3\frac{\partial}{\partial x_3}\right)\left(1q_0(r)\right)\\ &= -i\left(\alpha^1\sin\theta\cos\theta\frac{\partial}{\partial r}+\alpha^2\sin\theta\sin\theta\frac{\partial}{\partial r}+\alpha^3\cos\theta\frac{\partial}{\partial r}\right)\left(1q_0(r)\right)\\ &= -i\left(\alpha^1\frac{x_1}{r}+\alpha^2\frac{x_2}{r}+\alpha^3\frac{x_3}{r}\right)\frac{\partial}{\partial r}\left(1q_0(r)\right)\\ &= -i\left(\vec{\alpha}\cdot\hat{r}\right)\frac{\partial}{\partial r}q_0(r)1\end{aligned}$$

where 1 denote unit matrix.

Note that because q_μ is a function only of *r*, the $\frac{\partial}{\partial\theta},\frac{\partial}{\partial\varphi}$ terms become 0.

$-i\alpha^k\partial_k\left(-i\vec{\alpha}\cdot\hat{r}\right)q_1(r)$ case

Recalling that $\left(\alpha^1\right)^2=\left(\alpha^2\right)^2=\left(\alpha^3\right)^2=1$ (case 1) $\alpha^k\alpha^l=-\alpha^l\alpha^k\left(k\neq l\right)$ (case 2), we obtain the following.

From case 1 term, multiplying by factor $(-i)^2=-1$ yields

$$\begin{aligned}&\left(\frac{3}{r}-\frac{x_1^2+x_2^2+x_3^2}{r^3}\right)q_1+\left(\left(\sin\theta\cos\varphi\right)^2+\left(\sin\theta\sin\varphi\right)^2+\left(\cos\theta\right)^2\right)\frac{\partial}{\partial r}q_1\\ &=\frac{2}{r}q_1+\frac{\partial}{\partial r}q_1\end{aligned}$$

From case 2 term,

We show only the $\alpha^1\alpha^2+\alpha^2\alpha^1$ part. Recalling that $\alpha^2\alpha^1=-\alpha^1\alpha^2$

$$\begin{aligned}&\frac{\partial}{\partial x_1}\left(\frac{x_2}{r}\right)+\frac{x_2}{r}\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\left(\frac{x_1}{r}\right)-\frac{x_1}{r}\frac{\partial}{\partial x_2}\\ &=-\frac{x_2x_1}{r^3}+\frac{x_1x_2}{r^3}+\left(\sin\theta\right)^2\sin\varphi\cos\varphi\frac{\partial}{\partial r}-\left(\sin\theta\right)^2\cos\varphi\sin\varphi\frac{\partial}{\partial r}=0\end{aligned}$$