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About Quantum Mechanics without Hamiltonians

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Abstract

An extension of Shrödinger's quantization on the space $(\mathbf{x}, \mathbf{p})$, where the Hamiltonian approach is needed, is made on the space $(\mathbf{x}, \mathbf{v})$ where the Hamiltonian approach is not needed at all. The purpose of this paper is to give a possible extension of the actual formulation of the Quantum Mechanics, and this is achieved through a function $K(\mathbf{x}, \mathbf{v}, t)$ which takes the place of the Hamiltonian on the Shrödinger's equation and has units of energy. This approach allows us to include the quantization of classical velocity depending problems (dissipative) and position depending mass variation problems. Some examples are given.

Keywords

Quantization in the Space $(\mathbf{x}, \mathbf{v})$, Mass Positiondepending, Shrödinger's Equation

1. Introduction

Despite the enormous success of the Hamilton formulation of the Quantum Mechanics [1] [2] [3] [4], there are still some problems in the Hamiltonian Classical Mechanics formalism with ambiguities which become evident when one tries to make their quantization, for example, dissipative [5] and mass variation [6] problems. In addition, even some simple problems [7] can have the ambiguity of having two different Hamiltonians describing the same classical dynamics, but when their quantization is made, they describe different quantum dynamics [8]. These are the main reasons that one would like to study the possibility of making a quantization of systems without the Hamiltonian formulation [9]. Another reason would be that this different formulation could represent a possible extension of the same Quantum Mechanics Theory, and this will be carried out here for the first time.

In this work, it is proposed to make the quantization in terms of the variables $(\mathbf{x},\mathbf{v})$, where $\mathbf{v}=\dot{\mathbf{x}}$, instead of the variables $(\mathbf{x},\mathbf{p})$, where $\mathbf{p}$ is the generalized linear momentum deduced from a Lagrangian of the system, $p_j = \partial L/\partial \dot{x}_j$ $j=1,2,3$, in fact, this was done firstly by Heisenberg at his beginning of the matrix theory quantization. In this way, one can get rid of the Hamiltonian formulation, and the goal is to obtain a function $K(\mathbf{x},\mathbf{v},t)$ (having the energy as units) that can take the place of the Hamiltonian $H(\mathbf{x},\mathbf{p},t)$ in the Schrödinger's equation.

2. Classical Function *K*(x, v, *t*)

In this section, the analysis of several classical examples and cases will be made to obtain a function $K(\mathbf{x},\mathbf{v},t)$ that can be used for quantization of the classical system in terms of its variables $\mathbf{x}$ and $\mathbf{v}$.

2.1. Conservative Systems

Consider a conservative system which describes the motion in the space of a particle of mass position depending, $m(\mathbf{x})$, under a position depending force $\mathbf{F}(\mathbf{x})$. Its Newton's equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_0\mathbf{v}) = \mathbf{F}(\mathbf{x}). \tag{1}$$

This type of systems are invariant under Galileo's transformations, as it is well known, and the so called energy is a constant of motion of the system,

$$K(\mathbf{x},\mathbf{v}) = \frac{1}{2}m_0\mathbf{v}^2 - \int \mathbf{F}(\mathbf{x})\cdot \mathrm{d}\mathbf{x}, \tag{2}$$

where the first term represents the kinetic energy, and the second one is the potential energy.

2.2. Conservative Systems with Position Depending Mass

Consider a conservative system which describes the motion in the space of a particle of mass position depending, $m(\mathbf{x})$, under a position depending force $\mathbf{F}(\mathbf{x})$. Its Newton's equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t}(m(\mathbf{x})\mathbf{v}) = \mathbf{F}(\mathbf{x}). \tag{3}$$

One must point out that this type of systems are not invariant under Galileo's transformation [10], and Sommerfeld's invariant formulation [11] is not satisfactory [12]. However, one can still keep (3) as the right description of the problem [13]. Therefore, multiplying on both side of this expression by $m(\mathbf{x})\mathbf{v}$, rearranging terms, and integrating with respect the time, one gets

$$\frac{1}{2}(m(\mathbf{x})\mathbf{v})^2 = \int m(\mathbf{x})\mathbf{F}(\mathbf{x})\cdot \mathbf{v}\mathrm{d}t + C, \tag{4}$$

where C is the integration constant. Knowing that one has the relation $\mathbf{v}\mathrm{d}t = \mathrm{d}\mathbf{x}$, and dividing the above expression by a characteristic mass m_0, the following constant of motion is gotten

$$K(\mathbf{x},\mathbf{v}) = \frac{\left(m(\mathbf{x})\mathbf{v}\right)^2}{2m_0} - \frac{1}{m_0}\int m(\mathbf{x})\mathbf{F}(\mathbf{x})\cdot \mathrm{d}\mathbf{x}. \tag{5}$$

The mass m_0 is that one that whenever $m(\boldsymbol{x}) = \text{constant}$, this constant is m_0. For example, for the 1-D harmonic oscillator ($F(x) = -m_0\omega_0 x^2$) with mass variation $m(x) = m_0 + m_1 x$, and at first order in gradient m_1, one would have the 1-D constant of motion

$$K(x,v) = \frac{1}{2}m_0 v^2 + \frac{1}{2}m_0\omega_0^2 x^2 + m_1\left[xv^2 + \frac{m_0\omega_0^2}{3}x^3\right], \tag{6}$$

which will be a good approximation on the region $|m_1 x| \ll m_0$.

2.3. Liniar Dissipation 1-D Case

Consider the 1-D motion of a particle of constant mass m under the Hook's force $-kx$ in a dissipative medium which produces a velocity depending force of the form $-\alpha v$, where k is the spring constant, and α is the dissipative constant. The associated dynamical system is given by the pair of equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v, \quad \frac{\mathrm{d}v}{\mathrm{d}t} = -\omega^2 x - \frac{\alpha}{m}v, \tag{7}$$

where $\omega = \sqrt{k/m}$ represents the natural spring angular frequency. It has been shown [14] that the constant of motion associated to this system is given by

$$K(x,v) = \frac{m}{2}\left(v^2 + 2\omega_\alpha xv + \omega^2 x^2\right)\mathrm{e}^{-2\omega_\alpha G_\alpha(v/x,\omega)}, \tag{8}$$

where ω_α and G_α are defined as

$$\omega_\alpha = \alpha/m, \ \text{and} \ G_\alpha = \begin{cases} \dfrac{1}{2\Omega_\alpha}\ln\left(\dfrac{\omega_\alpha + v/x - \Omega_\alpha}{\omega_\alpha + v/x - \Omega_\alpha}\right), & \text{if } \omega < \omega_\alpha \\ \dfrac{1}{\omega_\alpha + v/x}, & \text{if } \omega = \omega_\alpha \\ \dfrac{1}{\sqrt{\omega^2 - \omega_\alpha^2}}\arctan\left(\dfrac{\omega_\alpha + v/x}{\sqrt{\omega^2 - \omega_\alpha^2}}\right), & \text{if } \omega > \omega_\alpha \end{cases}, \tag{9}$$

where $\Omega_\alpha = \sqrt{\omega_\alpha^2 - \omega^2}$, and corresponding to strong, critical, and weak dissipation cases. Of course, when dissipation is zero ($\alpha = 0$) one gets the usual energy of the harmonic oscillator.

2.4. Quadratic Dissipation 1-D Case

Consider the motion of a particle with position mass depending $m(x)$ under gravitational force, $-m(x)g$, in a dissipative medium where the force depende quadratically on its velocity, $-\alpha v^2$ (with $v < 0$). The equation of motion is given by

$$\frac{\mathrm{d}\left(m(x)v\right)}{\mathrm{d}t}=-m(x)g-\alpha v^2, \tag{10}$$

where α is a non negative constant. This equation can be written as

$$m(x)\frac{\mathrm{d}v}{\mathrm{d}t}=-m(x)g-\left(\alpha+m_x\right)v^2, \tag{11}$$

where m_x is the variation of the mass with respect the position, $m_x=\mathrm{d}m/\mathrm{d}x$. Integrating this equation, it is not difficult to see that one gets the following function

$$K(x,v)=\frac{\left(m(x)v\right)^2}{2m_0}\mathrm{e}^{2\alpha\int^x\frac{\mathrm{d}\sigma}{m(\sigma)}}+\frac{g}{m_0}\int^x\mathrm{d}\sigma m^2(\sigma)\mathrm{e}^{2\alpha\int^\sigma\frac{\mathrm{d}s}{m(s)}} \tag{12}$$

which is a constant of motion ($\mathrm{d}K/\mathrm{d}t=0$) of the system with energy units. The mass m_0 has been chosen such that if $m(x)=\text{constant}$, this constant has the value m_0.

2.5. Electromagnetic Case

The motion of a charged particle of charge q and mass m under an electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ is governed by Lorentz's equation of motion [12] (non relativistic case and CGS units)

$$\frac{\mathrm{d}(m\mathbf{v})}{\mathrm{d}t}=q\mathbf{E}+\frac{q}{c}\mathbf{v}\times\mathbf{B}, \tag{13}$$

where c is the speed of light. This equation can be written in terms of the scalar potential Φ and vector potential $\mathbf{A}$, where $\mathbf{B}=\nabla\times\mathbf{A}$ and $\mathbf{E}=-\nabla\Phi-\partial\mathbf{A}/\partial(ct)$, as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\mathbf{v}+\frac{q}{c}\mathbf{A}\right)=-\nabla\left(q\Phi-\frac{q}{c}\mathbf{A}\cdot\mathbf{v}\right). \tag{14}$$

Defining the new quantity of motion $m\mathbf{V}$ as

$$m\mathbf{V}=m\mathbf{v}+q\mathbf{A}/c, \tag{15}$$

and knowing that the function K associates to the equation $\mathrm{d}(m\mathbf{V})/\mathrm{d}t=-\nabla\phi$ is just $K=m\mathbf{V}^2/2+\phi$, where $V^2=V_x^2+V_y^2+V_z^2$, one can define the function K for this system as

$$K(\mathbf{x},\mathbf{v},t)=\frac{m}{2}\left(\mathbf{v}+\frac{q}{mc}\mathbf{A}\right)^2+q\Phi-\frac{q}{c}\mathbf{A}\cdot\mathbf{v}. \tag{16}$$

For most of the cases, one has that $\mathbf{A}\cdot\mathbf{v}=0$. So, the function K is given by

$$K(\mathbf{x},\mathbf{v},t)=\frac{m}{2}\left(\mathbf{v}+\frac{q}{mc}\mathbf{A}\right)^2+q\Phi. \tag{17}$$

2.6. Relativistic Conservative Case

This case is given as a good example for completeness of the concept of a constant of motion. The equation of motion of a relativistic particle of constant mass m on a conservative force $\mathbf{F}(\mathbf{x})$, with potential function $V(\mathbf{x})=-\int\mathbf{F}(\mathbf{x})\cdot\mathrm{d}\mathbf{x}$, is given by

$$\frac{\mathrm{d}(\gamma m\mathbf{v})}{\mathrm{d}t} = F(\mathbf{x}), \tag{18}$$

where the function γ is $\gamma = \left(1-|\mathbf{v}|^2/c^2\right)^{-1/2}$. It is well known [15] that this system has the following constant of motion

$$K(\mathbf{x},\mathbf{v}) = (\gamma-1)mc^2 + V(\mathbf{x}), \tag{19}$$

where the function $V(\mathbf{x})$ represents the potential energy of the system, $V(\mathbf{x}) = -\int \mathbf{F}(\mathbf{x})\cdot \mathrm{d}\mathbf{x}$.

3. Quantization in the Space (x, v)

The main idea is to assign a linear operator to the energy function $K(\mathbf{x},\mathbf{v},t)$ in order to have an equation that can be identify with Schrödinger's equation. Firstly one needs to point out that in the case of Hamilton approach, if the generalized linear momentum, $p_j = \partial L/\partial v_j$ with $L(\mathbf{x},\mathbf{v},t)$ being a Lagrangian of the system, is related with the velocity of the form $p_j = mv_j$ (m is the constant mass), the function K is exactly a Hamiltonian of the system, $H(\mathbf{x},\mathbf{p},t)$. Therefore, the quantization (Schrödinger's equation) done with the function K can represent an extension of the theory of Quantum Mechanics. To do this, operators associated to the variables "$\mathbf{x}$" and "$\mathbf{v}$" are introduced, and as one could expect, the operators associated to these variables are postulated as

$$\mathbf{x} \to \hat{\mathbf{x}} \quad \text{and} \quad \mathbf{v} \to \hat{\mathbf{v}} = -i\frac{\hbar}{m(\mathbf{x})}\nabla, \tag{20}$$

where $\hbar$ is as usual the Planck's constant divided by 2π. In addition, one has the following commutation relation between the components of these operators

$$[x_k, \hat{v}_l] = -i\frac{\hbar}{m(\mathbf{x})}\delta_{kl} I, \tag{21}$$

where "I" is the identity operator and the component $\hat{v}_l$ is

$$\hat{v}_l = -i\frac{\hbar}{m(\mathbf{x})}\frac{\partial}{\partial x_l}. \tag{22}$$

One needs to point out that the operator (20) would be an Hermitian operator if and only if the mass does not depend on the position. The square, or self composition of this operator, has the following expression

$$\hat{\mathbf{v}}^2 = -\frac{\hbar^2}{m(\mathbf{x})}\left(-\frac{1}{m^2(\mathbf{x})}\nabla m\cdot\nabla + \frac{1}{m(\mathbf{x})}\nabla^2\right). \tag{23}$$

Then, using the above definitions, one associates and operator to the energy function $K(\mathbf{x},\mathbf{v},t)$ as $\hat{K}(\mathbf{x},\hat{\mathbf{v}},t) = i\hbar\,\partial/\partial t$ and defines the Schrödinger-like equation in the space $(\mathbf{x},\mathbf{v})$ as

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{K}(\mathbf{x},\hat{\mathbf{v}},t)\Psi, \tag{24}$$

where $\Psi = \Psi(\mathbf{x}, t)$ is the wave function. If one has position depending mass problem, the operator $\hat{K}$ may be not Hermitian ($\hat{K}^\dagger \neq \hat{K}$) and there would not be conservation of the probability since the system is not invariant under Galileo's transformation,

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Re^3} \left|\Psi(\mathbf{x}, t)\right|^2 \mathrm{d}^3\mathbf{x} = \left\langle \Psi, \hat{K}\Psi \right\rangle - \left\langle \hat{K}^\dagger \Psi, \Psi \right\rangle, \tag{25}$$

where the inner product has the usual definition, $\langle \phi, \psi \rangle = \int \phi^\dagger(\mathbf{x}) \psi(\mathbf{x}) \mathrm{d}\mathbf{x}$.

However, for the case of position depending mass problems, one can construct a Schrödinger equation with Hermitian operator in the following way

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[\hat{K}(\mathbf{x}, \hat{\mathbf{v}}, t) + \hat{K}^\dagger(\mathbf{x}, \hat{\mathbf{v}}, t)\right]\Psi. \tag{26}$$

In addition, although there might be some doubts where the function $K(\mathbf{x}, \mathbf{v}, t)$ can not be a constant of motion [16], this statement is not necessarily a request, but it is necessary that this function must have units of energy. Let us see few examples of Schrödinger-like equations for the classical systems shown before. However, one needs to point out that the dissipation force on a motion of a body in the classical system appears as the result of the average collisions of the body with the particles of the medium where this body is moving, meanwhile in quantum mechanics this collisions are much more complicated and depends strongly of the energy of the particle and the particle itself ($e^-, e^+, p, \overline{p}, \mu^-, \mu^+, n, \Lambda$, etc.) [17], therefore, it is not so direct to make the identification and transition of the classical problem to the quantum problem. However, from the mathematical point of view, the quantum theory must be able to address these types of problems. The same situation is presented with the mass position depending problem.

3.1. Quantization of Mass Variation of Conservative Systems

Using (20) and (24) in (5), and assigning to the function $f(\mathbf{x})^2 v^2$ (for any arbitrary function *f*) the operator

$$\widehat{f^2 v^2} = \frac{1}{3}\left(\hat{v}^2 f^2 + f^2 \hat{v}^2 + \hat{v} f^2 \hat{v}\right), \tag{27}$$

one can get the operator for the function $m^2 \mathbf{v}^2$ as

$$\widehat{m^2 \mathbf{v}^2} = -\frac{\hbar^2}{3m}\left[\left(-\frac{1}{m^2}\nabla m \cdot \nabla + \frac{1}{m}\nabla^2\right)m^2 + 2m\nabla^2\right]. \tag{28}$$

Therefore, it follows that the Scrödinger-like equation is

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{6m_0 m(\mathbf{x})}\left[\left(-\frac{1}{m(\mathbf{x})}\nabla m(\mathbf{x}) \cdot \nabla + \frac{1}{m(\mathbf{x})}\nabla^2\right)m^2(\mathbf{x}) + 2m(\mathbf{x})\nabla^2\right]\Psi - \left[\frac{1}{m_0}\int m(\mathbf{x})\mathbf{F}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x}\right]\Psi. \tag{29}$$

If one makes the approximation to an Hermitian operator $\hat{\mathbf{v}} = -(i\hbar/m_0)\nabla$, the operatior $\widehat{m^2 \mathbf{v}^2}$ would be given instead by

$$\widehat{m^2(\mathbf{x})v^2} = -\frac{\hbar^2}{3m_0^2}\Big[3m(\mathbf{x})^2\nabla^2 + 5m(\mathbf{x})\nabla m(\mathbf{x})\cdot\nabla + 2\nabla m(\mathbf{x})\cdot\nabla m(\mathbf{x}) + 2m(\mathbf{x})\left(\nabla^2 m(\mathbf{x})\right)\Big], \tag{30}$$

and the wave equation would be given by

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{6m_0^3}\Big[3m(\mathbf{x})^2\nabla^2 + 5m(\mathbf{x})\nabla m(\mathbf{x})\cdot\nabla + 2\nabla m(\mathbf{x})\cdot\nabla m(\mathbf{x}) + 2m(\mathbf{x})\left(\nabla^2 m(\mathbf{x})\right)\Big]\Psi - \left[\frac{1}{m_0}\int m(\mathbf{x})\mathbf{F}(\mathbf{x})\cdot \mathrm{d}\mathbf{x}\right]\Psi. \tag{31}$$

3.2. Quantization of a Charged Particle Motion under Electromagnetic Forces and Pauli-Like Equation

From the expression (17) and since the mass is constant, the quantization of these systems can be carried out with the following Shrödinger-like equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \left\{\frac{m}{2}\left(-\frac{i\hbar}{m}\nabla + \frac{q}{mc}\mathbf{A}\right)^2 + q\Phi\right\}\Psi, \tag{32}$$

where $\Psi = \Psi(\mathbf{x},t)$ is an scalar function. Here again there is conservation of probability, and if the system has dipole electric and magnetic moments $\mathbf{P}$ and $\mathbf{m}$, the interaction with the electric and magnetic fields can be added in the usual way [18] by adding the terms $-\mathbf{P}\cdot\mathbf{E}$ and $-\mathbf{m}\cdot\mathbf{B}$. Now, it is well known Pauli's matrix, σ_k, properties [19], and their relation with the spin-1/2, $\mathbf{S}$ of a charged particle,

$$\left[\sigma_k,\sigma_j\right] = i2\varepsilon_{kj}^{l}\sigma_l,\quad \sigma_j^2 = I,\quad (\vec{\sigma}\cdot\mathbf{a})(\vec{\sigma}\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}I + i(\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\sigma},\quad \mathbf{S} = \frac{\hbar}{2}\vec{\sigma}, \tag{33}$$

where I is the 2×2 identity matrix, $\vec{\sigma} = (\sigma_x,\sigma_y,\sigma_z)$, $a,b\in\Re^3$ are arbitrary vectors, and Einstein's convention was used. Thus, the Pauli's equation in the quantum space $(\boldsymbol{x},\boldsymbol{v})$ for a charged particle of spin one-half can be written as

$$i\hbar\frac{\partial\Psi}{\partial t} = \left\{\frac{m}{2}\left[\vec{\sigma}\cdot\left(-\frac{i\hbar}{m}\nabla + \frac{q}{mc}\mathbf{A}\right)\right]^2 + q\Phi\right\}\Psi, \tag{34}$$

where Ψ is an spinor (a two components vector of scalar complex functions)

$$\Psi(\mathbf{x},t) = \begin{pmatrix}\psi_1(\mathbf{x},t)\\ \psi_2(\mathbf{x},t)\end{pmatrix}. \tag{35}$$

3.3. Quantization of 1-D Dissipative-Mass Variable Problem

From the expression (12), one notices that it will appear the product of $f(x)v^2$ where the this function is given by

$$f(x) = m^2(x)\mathrm{e}^{2\alpha\int \mathrm{d}\sigma/m(\sigma)}. \tag{36}$$

So, using the same expression (27) for 1-D, the Shcrödinger-like equation can be given as

$$i\hbar\frac{\partial\Psi}{\partial t}=\left\{-\frac{\hbar^2}{6m_0}\left[\left(\frac{-1}{m^2(x)}\frac{\partial}{\partial x}m(x)\frac{\partial}{\partial x}+\frac{1}{m(x)}\frac{\partial^2}{\partial x^2}\right)m^2(x)\lambda(x)\right.\right.$$
$$\left.+\lambda(x)\left(-\frac{\partial}{\partial x}m(x)\frac{\partial}{\partial x}+m(x)\frac{\partial^2}{\partial x^2}\right)+\frac{\partial}{\partial x}m(x)\lambda(x)\frac{\partial}{\partial x}\right] \tag{37}$$
$$\left.+\frac{g}{m_0}\int \mathrm{d}\sigma m^2(\sigma)\lambda(\sigma)\right\}\Psi,$$

where the function $\lambda(x)$ has been defined as

$$\lambda(x)=\mathrm{e}^{2\alpha\int \mathrm{d}s/m(s)}. \tag{38}$$

If the mass of the system is constant ($m(x)=m_0$), one would have that $\hat{v}=-(i\hbar/m_0)\partial/\partial x$ is an Hermitian operator, the function λ would be $\lambda(x)=\mathrm{e}^{2\alpha x/m_0}$, and the wave equation would be

$$i\hbar\frac{\partial\Psi}{\partial t}=\left\{-\frac{\hbar^2\lambda(x)}{3m_0}\left[3\frac{\partial^2}{\partial x^2}+\left(\frac{6\alpha}{m_0}\right)\frac{\partial}{\partial x}+\left(\frac{2\alpha}{m_0}\right)^2\right]+\frac{gm_0^2}{2\alpha}\left(\lambda(x)-1\right)\right\}\Psi. \tag{39}$$

3.4. Quantization of the Relativistic Scalar Case

For completeness, for the system characterized by the expression (19), let us make first some algebraic manipulation. Let us write this expression in the form

$$K-V(\mathbf{x})+mc^2=\gamma mc^2,$$

let us take the square of this expression and pass the velocity dependence to the left hand side. So, one gets

$$\left(1-\frac{v^2}{c^2}\right)\left(K-V(\mathbf{x})+mc^2\right)^2=m^2c^4. \tag{40}$$

In this way, using the identification of the operators for different variables and the function K, it follows that

$$\left(1+\frac{\hbar^2}{m^2c^2}\nabla^2\right)\left(-i\hbar\frac{\partial}{\partial t}-V(\mathbf{x})+mc^2\right)^2\Psi=m^2c^4\Psi. \tag{41}$$

4. Some Particular Solutions on the Space (x, v)

In this section, two simple solutions of the above approach are presented for illustration.

4.1. 1-D harmonic Oscillator with Position Depending Mass

Using the approximated constant of motion (6), one can write this expression of the form

$$K(x,v)=K_0(x,v)+W(x,v), \tag{42}$$

where K_0 represents the usual harmonic oscillator with constant mass m_0, and W represents the term of the variation of mass at first approximation in

Taylor expansion,

$$K_0(x,v)=\frac{1}{2}m_0v^2+\frac{1}{2}m_0\omega_0^2x^2,\quad W(x,v)=m_1\left[xv^2+\frac{m_0\omega_0^2}{3}x^3\right]. \tag{43}$$

Of course, for $m_1=0$, one has that $p=mv$, and K_0 represents the Hamiltonian $H(x,p)=p^2/2m_0+m_0\omega_0 2x^2/2$, and one knows that the solution of the Schrödinger's equation would be $\Psi_0(x,t)=\sum_{n=0}c_n\exp(-iE_nt/\hbar)\Phi_n(x)$, where the set $\{E_n=\hbar\omega_0(n+1/2),\Phi_n(x)\}$ is the solution of the eigenvalue problem $\hat{H}\Phi=E\Phi$ [2]. This solution is exactly the solution of Schrödinger-like equation

$$i\hbar\frac{\partial\Psi_0}{\partial t}=\hat{K}(x,\hat{v})\Psi_0. \tag{44}$$

For $m_1\neq 0$, one considers that W is a perturbation of the system and uses perturbative theory to find the modification of the energy levels of the system. Since W contains odd monomials order, there is not contribution a first order perturbation, $\langle n|\hat{W}|n\rangle$ with $\langle x|n\rangle=\Phi_n(x)$. It is not difficult to calculate that up to second order in perturbation theory, the eigenvalues are of the form

$$E_n\approx\hbar\omega_0(n+1/2)+\frac{m_1^2\hbar^2}{2m_0^3}\left(\frac{39(2n^2+2n)}{18}+\frac{7}{6}\right). \tag{45}$$

4.2. Free Relativistic Particle

Just to have some idea what the relativistic case would be, let us consider the quantization in the space (x,v) of the relativistic free particle motion. In this case, one makes $V(\mathbf{x})=0$ on the expression (41) and propose a plane wave solution of the form

$$\psi(\mathbf{x},t)\sim \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{46}$$

on the resulting equation

$$\left(1+\frac{\hbar^2}{m^2c^2}\nabla^2\right)\left(-i\hbar\frac{\partial}{\partial t}+mc^2\right)^2\Psi=m^2c^4\Psi. \tag{47}$$

Thus, one gets the dispersion relation given by

$$\omega(k)=\frac{mc^2}{\hbar}\left[\frac{1}{\sqrt{1-\frac{\hbar^2k^2}{m^2c^2}}}-1\right]. \tag{48}$$

See following figure (**Figure 1**) where it has been plotted this relativistic dispersion relation.

The general solution would be the superposition of all the solutions,

$$\Psi(\mathbf{x},t)=\int_{\Re^3}A(\mathbf{k})\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{x}-\omega(k)t)}\mathrm{d}^3\mathbf{k}, \tag{49}$$

where, due to conservation of probability, the function $A(\mathbf{k})$ is such a

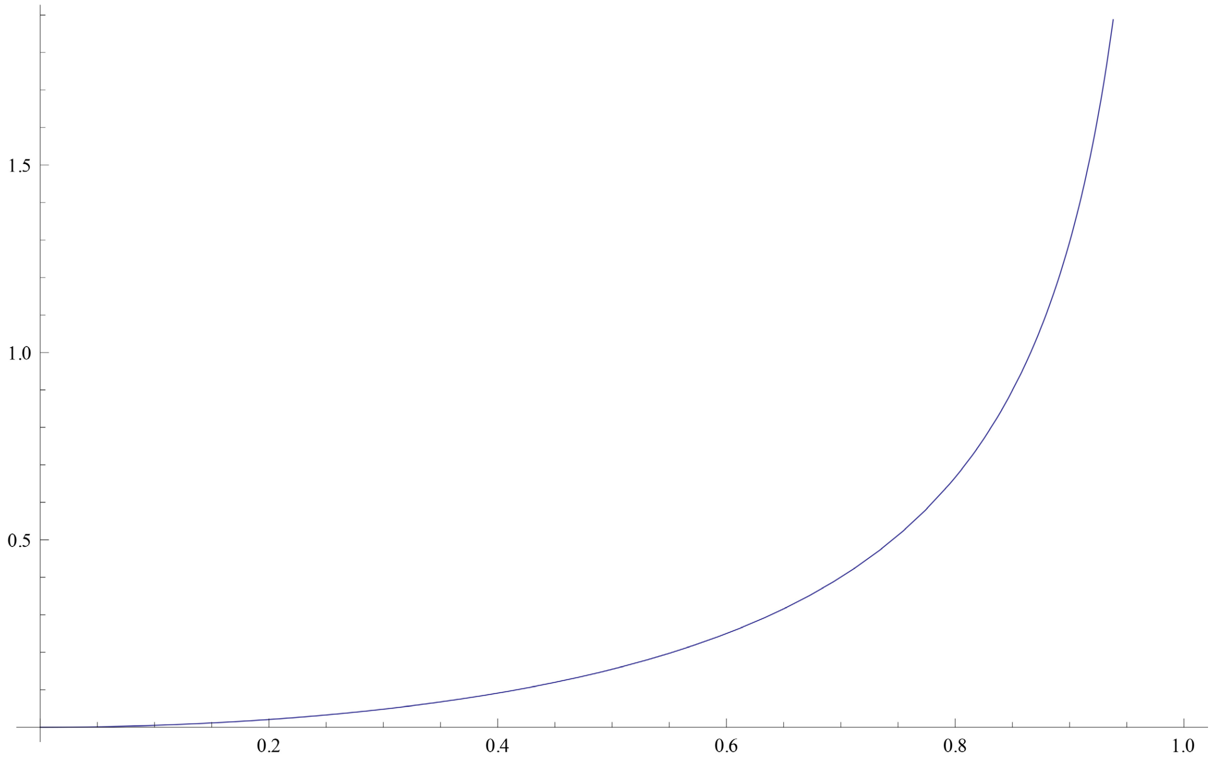

Figure 1. $\omega/(mc^2/\hbar)$ (vertical) vs. $k^2c^2/(mc^2/\hbar)^2$ (horizontal) with $mc^2/\hbar = 1.42\times 10^{24}$ sec.

$$\int_{\Re^3} |A(\mathbf{k})|^2 \, \mathrm{d}^3\mathbf{k} = 1. \tag{50}$$

5. Conclusions and Comments

It has been done an extension of the Shrödinger's quantization approach to the quantization on the space $(\mathbf{x}, \mathbf{v})$ through the function $K(\mathbf{x}, \mathbf{v}, t)$ which has energy units. Within this approach, the Hamiltonian notion is not needed, and the quantization of conservative systems is the same with this approach and the Hamiltonian approach (in fact, it must be the same whenever the generalized linear momentum is of the form $\mathbf{p} = m\mathbf{v}$). The possibility to include the quantization of mass variation problems and velocity depending problems (dissipation) is clearly stablished. In addition, the quantization of non relativistic interaction of charged particles with electromagnetic field is also stablished.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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