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Ashtekar-Kodama Gravity as a Classical and Quantum Extension of Loop Quantum Gravity

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Abstract

This paper presents a new theory of gravity, called here Ashtekar-Kodama (AK) gravity, which is based on the Ashtekar-Kodama formulation of loop quantum gravity (LQG), yields in the limit the Einstein equations, and in the quantum regime a full renormalizable quantum gauge field theory. The three fundamental constraints (hamiltonian, gaussian and diffeomorphism) were formulated in 3-dimensional spatial form within LQG in Ashtekar formulation using the notion of the Kodama state with positive cosmological constant Λ. We introduce a 4-dimensional covariant version of the 3-dimensional (spatial) hamiltonian, gaussian and diffeomorphism constraints of LQG. We obtain 32 partial differential equations for the 16 variables E_{mn} (E-tensor, inverse densitized tetrad of the metric) and 16 variables A_{mn} (A-tensor, gravitational wave tensor). We impose the boundary condition: for large distance the E-generated metric $g(E)$ becomes the GR-metric g (normally Schwarzschild-spacetime). The theory based on these Ashtekar-Kodama (AK) equations, and called in the following Ashtekar-Kodama (AK-) gravity has the following properties. • For $\Lambda = 0$ the AK equations become Einstein equations, A-tensor is trivial (constant), and the E-generated metric $g(E)$ is identical with the GR-metric g. • When the AK-equations are developed into a Λ-power series, the Λ-term yields a gravitational wave equation, which has only at least quadrupole wave solutions and becomes in the limit of large distance r the (normal electromagnetic) wave equation. • AK-gravity, as opposed to GR, has no singularity at the horizon: the singularity in the metric becomes a (very high) peak. • AK-gravity has a limit scale of the gravitational quantum region 39 μm, which emerges as the limit scale in the objective wave collapse theory of Gherardi-Rimini-Weber. In the quantum region, the AK-gravity becomes a quantum gauge theory (AK quantum gravity) with the Lie group extended SU(2) = ε-tensor-group(four generators) as gauge group and a corresponding covariant derivative. • AK quantum gravity is fully renormalizable, we derive its Lagrangian, which is dimensionally renormalizable, the normalized one-graviton wave function, the graviton propagator, and demonstrate the calculation of cross-section from Feynman diagrams.

Keywords

Quantum Gravity, Loop Quantum Gravity, General Relativity, Gravitational Wave, Gauge Field Theory, Graviton, Hamiltonian Constraint, Gaussian Constraint, Diffeomorphism Constraint

1. Introduction

The quantum formulation of the four fundamental forces in nature has been one of the main tasks in the physics of the twentieth century.

The quantum version of electrodynamics, Quantum Electrodynamics (QED), emerged in the 1960s based on the work of Feynman, Schwinger, Tomonaga and Dyson [1] [2] [3].

The purely quantum sub-atomic strong interaction was formulated as color SU(3) interaction in Quantum Chromodynamics (QCD) in the 1970s based on the work of Politzer, Coleman, Gross and Wilczek [2] [3].

The purely quantum sub-atomic weak nuclear interaction was formulated as V-A left-handed SU(2) interaction 1957 by Feynman and Gell-Mann, and as SU(2) × SU(1) force within the electroweak force in 1968 by Glashow, Salam, and Weinberg [3].

The quantum version of GR, called quantum gravity theory (QG) is until today one of the major unfulfilled goals in modern physics, despite many efforts [3].

Today, the two most popular approaches to QG are:

- Super-Gravity (Super-Symmetry) [4] [5] [6] or in alternative formulation Super-String-theory [3];

- Loop Quantum Gravity (LQG) [7] [8].

According to literature [3] [4] [9] [10] [11] [12], there are at least seven requirements, which a successful quantum gravity theory has to fulfill:

- It must have a dimensionally renormalizable Lagrangian, *i.e.* the Lagrangian must have the correct dimension without dimensional constants, and a covariant derivative with a gauge-group;

- The static version of the theory must deliver the exact GR, except at singularities;

- The static theory should remove the singularities of GR;

- The time-dependent version of the theory must give a mathematically consistent classical description of gravitational waves (*i.e.* a graviton wave-tensor) with basic quadrupole symmetry (as required by GR);

- The gravitational waves must obey the superposition principle;

- The corresponding energy-momentum tensor must give the Einstein power formula for the gravitational waves and agree with the GR version for small amplitudes;

- The quantum version of the theory must deliver a renormalizable Lagrangian, and a quantum gauge theory, which, within Feynman diagrams yields finite cross-sections in analogy to quantum electrodynamics.

At present, the Ashtekar formulation of loop quantum gravity (LQG) [4] [13] appears as the only viable candidate theory for a QGT. This is the starting point for Ashtekar-Kodama (AK-) gravity.

The AK gravity, which we present here, satisfies all seven requirements; therefore it is a good candidate for the correct classical and quantum gravity theory.

The starting point of the AK gravity is the 3-dimensional AK constraints. They can be derived from the Ashtekar version of the ADM-theory plus Kodama ansatz (chap. 3) or from the Plebanski action of the BF-theory, which is a generalized form of GR (chap. 3).

The 4-dimensional AK equations are 32 partial differential equations for the 16 variables $E^{\mu\nu}$ (E-tensor, inverse densitized tetrad of the metric $g_{\mu\nu}$) and 16 variables $A_\mu{}^\nu$ (A-tensor, gravitational wave tensor).

We show that in the limit Λ→0, for the A-tensor we get the trivial solution $A_\mu{}^\nu$ = *constant half-antisymmetric*, the E-tensor solutions of the remaining 4 gaussian equations (the last 4 vanish identically) is the *Gauss-Schwarzschild tetrad* (or *the Kerr-Schwarzschild tetrad*), which satisfies the Einstein equations.

That means: *in the limit* Λ→0, *the AK gravity becomes GR*.

At the horizon in the limit r→1 the E-tensor becomes very large and the term Λ $E^{\mu\nu}$ not negligible any more, the *horizon singularity is removed* and becomes a peak, we get the corrected outer metric $\left|g_{11,c}\right| = \frac{1}{\left|g_{00,c}\right|} = \frac{1}{1-\frac{r_s - l_0}{r}}$ with the AK correction length $l_0 \approx 64\Lambda r_s^{\ 3}$.

In chap. 6.2 we derive a limit for the *quantum gravitational scale*

$$r_{gr} = \sqrt{l_P \sqrt{\frac{1}{\Lambda}}} = 3.1\times10^{-5}\, m = 31\,\mu\text{m}$$

This quantum gravitational scale scale marks the limit of the quantum coherence length, in other words, it is the *border between quantum and classical regime*.

In chap. 11 we derive the quantum version of AK-gravity.

AK-quantum-gravity is a *renormalizable field gauge theory* with gauge Lie groupgen(SU(2)e) = ε-tensor Lie group, with four generators τ_κ, $\left[\tau_\kappa, \tau_\lambda\right] = i\,\varepsilon_{\kappa\lambda\mu}\tau_\mu$.

The overview of the chapters in and their contents is given below.

In chap. 3, we introduce the Ashtekar version of LQG, and the fundamental Ashtekar-Kodama constraints.

Chap. 4 describes the 4-dimensional AK equations and their properties.

Chap. 5 and chap. 6 deals with the static solutions of the AK-equations, which yield the Schwarzschild, respectively the Kerr metric solving the Einstein equations. At the horizon $r = 1$ the metric has a peak, not a singularity, as in GR.

In chap. 7.2 the resulting gravitational wave equation and its solution are described.

In chap. 7.4 and 7.5 we present the spherical and planar gravitational waves.

Chap. 8 and 9 introduces numerical solutions of AK equations.

In chap. 11 we present the quantum field version of the Ashtekar-Kodama gravity and demonstrate the calculation of cross-sections.

Some chapters contain a schematic diagram, which is a compact representation of the contents.

All derivations and calculations were carried out in Mathematica-programs, so the results can be considered as error-free, the programs are cited in the literature index. The denomination of equations e.g. *eqtoeivnu3b*, refers to the Mathematica code used for calculation.

2. Motivation and Problems

The formulation of quantum gravity encounters several fundamental problems, which stem partly from GR and partly from fundamental differences between classical and quantum interpretation.

1) Quantum vs. classical concept

- Classical and quantum concepts (phase space versus Hilbert space, etc.) are most likely incompatible.

- Semiclassical theory, where gravity stays classical but all other fields are quantum, has failed up to now.

AK gravity is classical in the classical regime $r \gg r_{gr}$, and is a quantum gauge theory in the quantum regime $r \ll r_{gr}$, in between there is a semi-classical WKB approximation. The quantum-likeness is described by the decoherence time t_{dc}, which becomes effectively zero (purely classical) when smaller than Planck-time $t_{dc} \ll t_p$.

2) Cosmology and black-holes

- Initial big-bang state is a quantum state.

- Hawking-Penrose black-holes are quantum objects with a singularity at the horizon, and are no classical objects.

In AK gravity, the singularity of a black-hole becomes a peak, and black-holes are non-singular purely classical objects.

Within GR itself, non-compact solutions of the Tollman-Oppenheimer-Volkov (TOV) equation with equation-of-state of a neutron-Fermi-gas describe shell-stars with radius $R > r_s$ (Schwarzschild radius), which have no singularity, but mimic stellar Hawking-Penrose black-holes with an escape velocity only ~0.1% ... 1% below c [14]. Furthermore, GR numerical calculation of gravitational collapse of a supermassive dust cloud shows that it results in a supermassive shell-star, and not in a Hawking-Penrose black-hole [15].

3) Problem of metric (space-time)

- In quantum theory, metric is a fixed function of coordinates.

- In GR, metric is a variable function of coordinates, a solution of the Einstein equations.

In AK-gravity, metric is generated by the tetrad $E^{\mu\nu}$, which together with the gravitational tensor $A_{\mu}^{\ \nu}$, is a solution of the AK equations, but the metric does not directly enter the AK equations.

4) Superposition principle

- In QM the fundamental equations are linear in the wave function and the operators, the solutions can be combined additively (superposition principle).

- In GR the Einstein equations the Ricci-tensor is explicitly of order 2 in the metric $g_{\mu\nu}$ and of additional order 2 in its inverse $g^{\mu\nu}$, the solutions cannot be combined linearly.

In AK-gravity, the gravitational wave equation is linear in the wave tensor $A_{\mu}^{\ \nu}$, the superposition principle is valid.

5) Action and renormalization

The Einstein-Hilbert action has a dimensional interaction constant $\frac{1}{2\kappa}$, and therefore the action is fundamentally non-renormalizable.

AK-gravity has a dimensionless interaction constant, and is therefore dimensionally renormalizable. Introducing the AK graviton tensor $A_{\mu}^{\ \nu}$ into the Hilbert action makes it dimensionally renormalizable (dimensionless interaction constant) and it still yields the Einstein equations.

6) Gravitational waves

In GR, there is no adequate description of gravitational waves: a spherical gravitational wave is a metric oscillation, and satisfies the Einstein equation only for small amplitudes (linearized Einstein equations).

In AK-gravity, there is a gravitational wave equation, which is satisfied exactly by gravitational waves, gravitational waves satisfy also the linearized Einstein equations. The gravitational wave equation becomes the normal wave equation in the large-distance limit.

7) Singularity in fundamental equations

In GR, solutions of Einstein equations contain singularities, and there is an obvious reason: they contain the inverse metric $g^{\lambda\kappa} = \left(g_{\mu\nu}\right)^{-1}$, which becomes singular when $\det(g) = 0$, and also it "explodes" when $g_{\mu\nu} \ll 1$, which can happen easily.

AK equations on the other hand, are linear in $E^{\mu\nu}$ and quadratic in $A_{\mu}^{\ \nu}$, they have branching points, but no singularities.

3. Loop Quantum Gravity (LQG)

Loop quantum gravity (LQG) is based on the ADM formulation of GR ([16] [17] [18]), and on BF-theory with Palatini and Plebanski action ([19] [20] [21] [22] [23]).

LQG extends both by the introduction of conjugated generalized momenta $A^{a}{}_{i}$ and generalized coordinates (inverse tetrad) $E^{b}{}_{j}$, which satisfy canonical commutation relations.

This was done by Ashtekar in 1986 [24].

Rovelli and Smolin reformulated the theory 1988 [25] in terms of generalized Feynman diagrams on two-dimensional faces (loops). The theory was further extended by Thiemann 2002 [7].

Finally, the three fundamental constraints of the theory were reformulated by Smolin 2002 [13] and Freidel 2003 [26], using the cosmological constant introduced into the theory by Kodama [27].

The ADM formulation of GR as a 3 + 1 decomposition was introduced by Arnowitt & Deser & Misner in 1959 [17]. It became the basis for the GR-based quantum gravity: loop quantum gravity and the Ashtekar formulation. ADM yields the original form of the 3-dimensional hamiltonian and the diffeomorphism constraints.

BF theory in form of Plebanski action formalism ([13] [20] [21] [28]) is a generalization of GR, and in the special case of Palatini action ([22] [29]) it yields the Einstein equations of GR expressed by the tetrad e^i, which serves as generalized coordinates, and replaces the metric g_{ij}. The tetrad (triad in dimension d = 3) formalism is the basis for the introduction of generalized momenta and coordinates in the Ashtekar formulation of LQG.

The evolution of LQG is shown in **Figure 1**.

3.1. Ashtekar Variables

The definition of the (inverse) triad e^a_i is [24]

$$h_{ab}e^a{}_i e^b{}_j = \delta_{ij} \quad h^{ab} = \delta^{ij}e^a{}_i e^b{}_j = \delta_{ij} \quad e_a{}^0 = -n_a = N t_{,a}$$

E^a_i is the inverse densitized triad $E^a{}_i(x) \equiv \sqrt{h(x)}e^a{}_i(x) \quad \sqrt{h(x)} = det\left(e^a{}_i(x)\right)$ the extrinsic curvature $K_a{}^i(x) \equiv K_{ab}(x)e^{bi}(x)$ is the canonical conjugate to E^a_i (1)

$$\begin{aligned} K_a{}^i \delta E^{ia} &= \frac{K_{ab}}{2\sqrt{h}}\delta\left(E^{ia}E^{ib}\right) = \frac{K_{ab}}{2\sqrt{h}}\left(h\delta h^{ab} + h^{ab}\delta h\right) \\ &= -\frac{\sqrt{h}}{2}\left(K^{ab} - Kh^{ab}\right)\delta h_{ab} = -\kappa p^{ab}\delta h_{ab} \end{aligned} \quad \text{with} \quad \kappa = 8\pi G/c^4$$

Resulting *Gauss constraint* is

$$G_i(x) = \varepsilon_{ijk}K_a{}^j(x)E^{ka}(x) \approx 0 \tag{2}$$

The covariant derivative for a vector field $v^a = v^i e_i^a$ is $D_a v^i = \partial_a v^i \omega_a{}^i{}_j v^j$, with the GR connection $\omega_a{}^i{}_j = \Gamma^i{}_{kj} e_a{}^k$, where $\Gamma^i{}_{kj} = e^d{}_k e^f{}_j e_c{}^i \Gamma^c{}_{df} - e^d{}_k e^f{}_j \partial_d e^i{}_f$ are the Christoffel symbols (Levi-Civita connection), the triad is covariant-consistent: $D_a e_b{}^i = 0$ in analogy to $D_a h_{bc} = 0$

Parallel transport is defined by

$$dv^i = -\omega_a{}^i{}_j v^j dx^a \quad \Gamma_a{}^i = -\frac{1}{2}\omega_{ajk}\varepsilon^{ijk} \quad \delta\omega^i = \Gamma_a{}^i dx^a \quad dv^i = \varepsilon^i{}_{jk} v^j \delta\omega^k,$$

and the Riemann curvature components are

$$R^i{}_{ab} = 2\left(\partial_a \Gamma_b{}^i - \partial_b \Gamma_a{}^i\right) + \varepsilon^i{}_{jk}\Gamma_a{}^i \Gamma_b{}^k \quad R^i{}_{ab} e^b{}_i = 0,$$

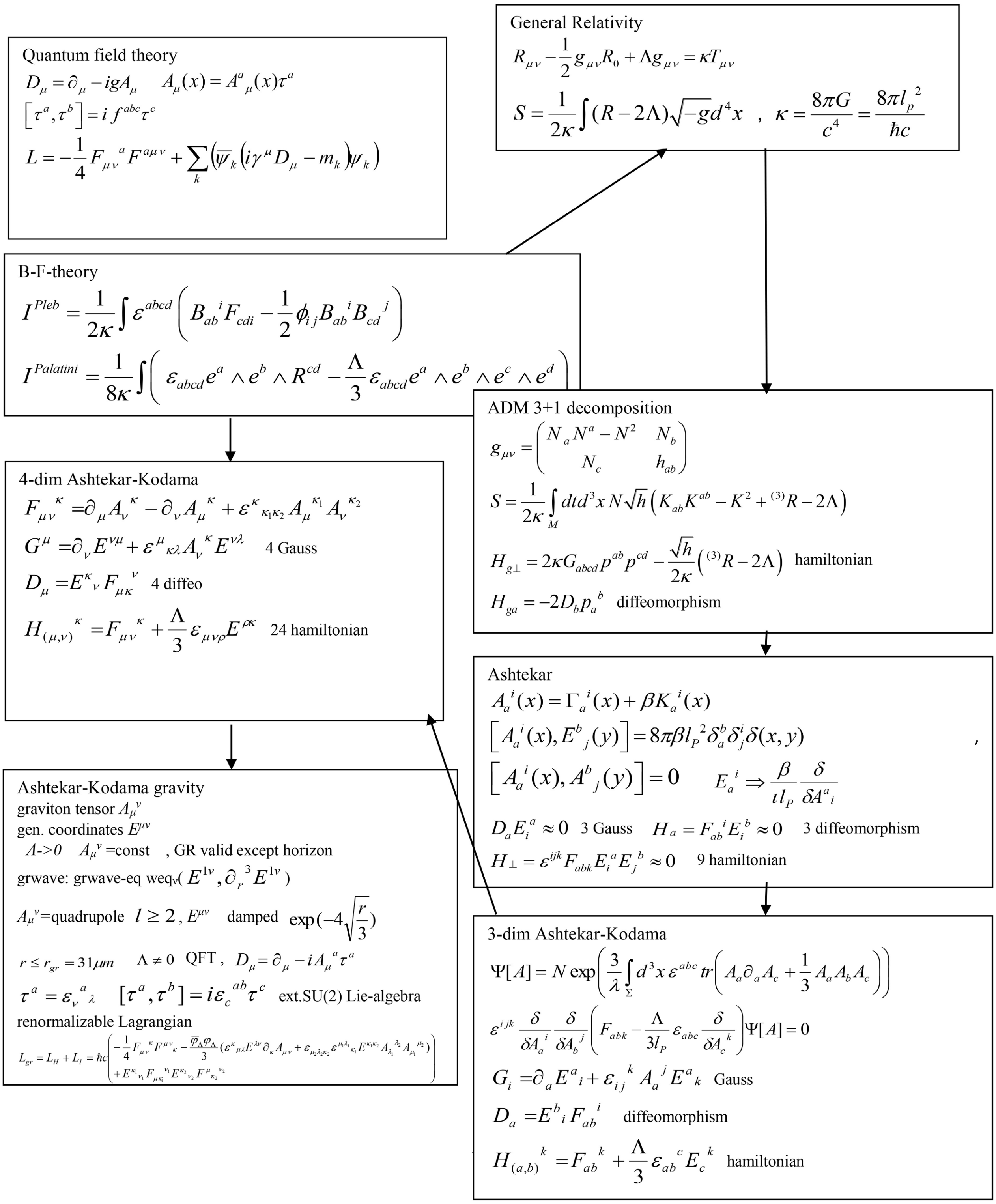


Figure 1. Schematic quantum gravity.

with the Riemann scalar $R[e] = -R^i{}_{ab} \varepsilon_i{}^{jk} e^a{}_j e^b{}_k = -R^i{}_{kab} e^a{}_j e^{bk}$.

The generalized momentum $A^a{}_i$ was introduced by Ashtekar 1986 [24]:

$$\text{Ashtekar variables } A^a{}_i(x){:}\quad A^a{}_i(x) = \Gamma_a{}^i(x) + \beta K_a{}^i(x) \tag{3}$$

with dimension $[A^a{}_i] = 1/\text{cm}$, β Barbero-Immirzi parameter, $A^a{}_i$ and $E^b{}_j$ are canonically conjugate with commutator bracket

$$\left[A_a{}^i(x), E^b{}_j(y)\right] = 8\pi\beta l_P{}^2 \delta_a^b \delta_j^i \delta(x,y) \tag{4}$$

$$\left[A_a{}^i(x), A^b{}_j(y)\right] = 0$$

i.e. we can replace $E_a{}^i$ by the operator

$$E_a{}^i \Rightarrow \frac{\beta}{\iota l_P}\frac{\delta}{\delta A^a{}_i} \tag{5}$$

where $\iota = \sqrt{-1}$.

3.2. Discussion of the Constraints

The *Gauss constraint* is

$$G_i = \partial_a E^a{}_i + \varepsilon_{ijk} A_a{}^j E^{ak} \equiv D_a E^a{}_i \approx 0 \tag{6}$$

where field strength tensor $F_{ab}{}^i = \partial_a A_b{}^i - \partial_b A_a{}^i + \varepsilon_{ijk} A_a{}^j A_b{}^k$, covariant derivative $(D_a)_{ik} = \partial_a + \varepsilon_{ijk} A_a{}^j$ and *hamiltonian constraint* $\sigma = -1$ Lorentzian, $\sigma = 1$ Euclidean (7)

$$\begin{aligned} H_\perp &= -\frac{\sigma}{2}\frac{\varepsilon^{ijk} F_{abk}}{\sqrt{\left|det\left(E^a{}_i\right)\right|}} E^a{}_i E^b{}_j \\ &\quad + \frac{\beta^2\sigma - 1}{\beta^2\sqrt{\left|det\left(E^a{}_i\right)\right|}}\left(E^a{}_i E^b{}_j - E^a{}_j E^b{}_i\right)\left(A_a{}^i - \Gamma_a{}^i\right)\left(A_b{}^j - \Gamma_b{}^j\right) \\ &\approx 0 \end{aligned}$$

and the *diffeomorphism constraint*

$$H_a = F_{ab}{}^i E^b{}_i \approx 0 \tag{8}$$

For $\beta = \iota = \sqrt{-1}$ in the Lorentzian case the hamiltonian constraint simplifies

$$H_\perp = \varepsilon^{ijk} F_{abk} E^a{}_i E^b{}_j \approx 0 \tag{9}$$

5 pdeqs order 1 in (r, θ) non-linear (quadratic) for 6 symmetric $E^a{}_i$ and 6 symmetric $A^a{}_i$.

3.3. Three-Dimensional Ashtekar-Kodama Constraints

We construct a theory based on the densitized inverse tetrad $E^b{}_j(y)$ and the connection $A_a{}^i(x)$ with the commutator

$$\left[A_a{}^i(x), E^b{}_j(y)\right] = -8\pi l_P{}^2 \delta_j^i \delta_a^b \delta(x,y)\beta\iota \quad \text{where} \quad \kappa = \frac{8\pi l_P{}^2}{\hbar c} = \frac{8\pi G}{c^4}$$

the operators act on the wave functional $\Psi[A]$

$$\hat{A}_a{}^i(x)\Psi[A] = A_a{}^i(x)\Psi[A]$$

$$\hat{E}^b{}_j(y)\Psi[A] = -8\pi l_P{}^2 \frac{\beta}{\iota}\frac{\delta\Psi[A]}{\delta A^b{}_j(y)}, \quad \hat{E}^b{}_j(y)\Psi[A] = -8\pi l_P{}^2 \frac{3}{\lambda}\frac{\beta}{\iota}\varepsilon^{bcd} F_{cdj}$$

where $\lambda = 8\pi l_P{}^2\Lambda$, Λ cosmological constant.

The Gauss constraint becomes $D_a \frac{\delta \Psi}{\delta A_a{}^i} = 0$, the diffeomorphism constraint becomes $F_{ab}{}^i \frac{\delta \Psi}{\delta A_b{}^i} = 0$, and the hamiltonian constraint with $\Lambda = 0$ and $\beta = \iota = \sqrt{-1}$

$$\varepsilon^{ijk} F_{kab} \frac{\delta^2 \Psi}{\delta A_a{}^i \delta A_b{}^j} = 0$$

In the case of vacuum gravity with $\Lambda \neq 0$, an exact formal solution in the connection representation was found by Kodama 1990 [27].

The hamiltonian constraint becomes for $\beta = \iota = \sqrt{-1}$

$$\varepsilon^{ijk} \frac{\delta}{\delta A_a{}^i} \frac{\delta}{\delta A_b{}^j} \left(F_{abk} - \frac{\Lambda}{3 l_P} \varepsilon_{abc} \frac{\delta}{\delta A_c{}^k} \right) \Psi[A] = 0,$$

with the global wave function

$$\Psi[A] = N \exp\left(\frac{3}{\lambda} \int_\Sigma d^3 x \, \varepsilon^{abc} \, tr\left(A_a \partial_a A_c + \frac{1}{3} A_a A_b A_c \right) \right)$$

$$\frac{c^3}{G\hbar\Lambda} = \frac{1}{8\pi l_P{}^2 \Lambda} = \frac{1}{\lambda}$$

which is derived from the Chern-Simons action

$$S_{CS}[A] = \int_\Sigma d^3 x \, \varepsilon^{abc} \, tr\left(A_a \partial_b A_c + \frac{1}{3} A_a A_b A_c \right)$$

The equation $\varepsilon_{abc} \frac{\delta \Psi}{\delta A_c^k} = \frac{3}{\Lambda} F_{kab}$ results from the variation of the Chern-Simons covariant Lagrangian

$$L_{CS} = \varepsilon^{\mu\nu\lambda} \left(A_\mu{}^\kappa \partial_\nu A_{\lambda\kappa} + \frac{1}{3} \varepsilon_{\kappa_1 \kappa_2 \kappa_3} A_\mu^{\kappa_1} A_\mu^{\kappa_2} A_\mu^{\kappa_3} \right)$$

$$\frac{\delta L_{CS}}{\delta A_\rho{}^\sigma} = \varepsilon^{\rho\nu\lambda} F_{\nu\lambda\sigma}$$

The resulting constraints are [13] [26]

$$3 \textit{ Gauss constraints } \quad G_i = \partial_a E^a{}_i + \varepsilon_{ij}{}^k A_a{}^j E^a{}_k \tag{10}$$

$$3 \textit{ diffeomorphism constraints } \quad D_a = E^b{}_i F_{ab}{}^i \tag{11}$$

$$3 \times 3 = 9 \textit{ hamiltonian constraints } \quad H_{(a,b)}{}^k = F_{ab}{}^k + \frac{\Lambda}{3} \varepsilon_{ab}{}^c E_c{}^k \tag{12}$$

altogether 15 pdeqs order 1 in (r, θ) nonlinear (quadratic in $E^a{}_i$ and $A^a{}_i$, cubic in both), for 9 $E^a{}_i$ and 9 $A^a{}_i$.

4. Four-Dimensional Ashtekar-Kodama Equations and Their Properties

In the following, we express the location vector $x^\mu = (ct, x_1, x_2, x_2)$ and distance r in units of Schwarzschild radius r_s, and we use the metric signature $(-1, 1, 1, 1)$. Also in the following, we set $c = 1$ and identify time t with $x^0 = ct$.

We can transform the 3-dimensional Ashtekar-Kodama (AK) equations *uniquely* into the 4-dimensional relativistic form by generalizing the ε-tensor from 3 spatial indices (1, 2, 3) to 4 spacetime indices (0, 1, 2, 3), which is mathematically uniquely and well-defined.

with 16 variables $E^{\mu\nu}$: inverse densitized tetrad of the metric $g_{\mu\nu}$

with 16 variables $A_\mu{}^\nu$ gravitational tensor

spacetime curvature tensor (field tensor)

$$F_{\mu\nu}{}^{\kappa} = \partial_\mu A_\nu{}^\kappa - \partial_\nu A_\mu{}^\kappa + \varepsilon^\kappa{}_{\kappa_1\kappa_2} A_\mu{}^{\kappa_1} A_\nu{}^{\kappa_2} \tag{13}$$

$$\text{4 Gauss constraints} \quad G^\mu = \partial_\nu E^{\nu\mu} + \varepsilon^\mu{}_{\kappa\lambda} A_\nu{}^\kappa E^{\nu\lambda} \tag{14}$$

(covariant derivative of $E^{\mu\nu}$ vanishes)

$$\text{4 diffeomorphism constraints} \quad I_\mu = E^\kappa{}_\nu F_{\mu\kappa}{}^\nu \tag{15}$$

$$\text{24 hamiltonian constraints} \quad H_{(\mu,\nu)}{}^\kappa = F_{\mu\nu}{}^\kappa + \frac{\Lambda}{3} \varepsilon_{\mu\nu\rho} E^{\rho\kappa} \tag{16}$$

The expression (μ, ν) in the index of H means that only *pairs* (μ, ν) where $\mu \neq \nu$ in the first index yield different constraints, as the right side is antisymmetric in (μ, ν), that results in 6 × 4 = 24 hamiltonian constraints.

$E_g^{\mu\nu}$ is the solution of the original defining *densitized tetrad* equation

$$E^{\mu\kappa} E^\nu{}_\kappa = g^{\mu\nu} \big/ \left(-\det(g)\right)^{3/4} \tag{17a}$$

or in matrix-notation for $d = 4$:

$$E \eta E^t = g^{-1} \big/ \left(-\det(g)\right)^{3/4} \tag{17b}$$

which is generalized from the densitized tetrad equation for $d = 3$:

$E \eta E^t = g^{-1} \big/ \left(-\det(g)\right)$ with the *scaling behavior* $\det(E) = \dfrac{1}{\det(g)^2}$.

As is easily shown, the densitized tetrad has the same scaling behavior

$\det(E) = \dfrac{1}{\det(g)^2}$ and for the scaling transformation with a scalar α $g \to \alpha g$

follows $E \to \dfrac{E}{\alpha^2}$ for both $d = 3$ and $d = 4$.

As follows immediately, the inverse E-g relation is

$$\left(E \eta E^t\right)^{-1} \det(E)^{3/8} = g \tag{18}$$

and $\det(E) = 1 \big/ \det(g)^2$.

We have 32 *partial differential equations of degree* 1, nonlinear (linear in $E^{\mu\nu}$ and quadratic $A^{\mu\nu}$, cubic in both) in {t, r, θ} *for* 32 *variables*.

In the static case $t = 0$ in the limit $r \to \infty$ the *tetrad-generated metric should be equal to the GR-tetrad* of the mass distribution, $E^{\mu\nu} = tetrad\left(g_{GR}\right)$.

We have the boundary condition in $r = r_{bc}, r_{bc} \gg 1$:

$$\left(E \eta E^t\right)^{-1} \det(E)^{3/8} \left(r = r_{bc}, \theta\right) = g_{GR}\left(r = r_{bc}, \theta\right) \quad \text{(see below).}$$

Furthermore, the field should vanish in the limit: $A_\mu{}^\nu \left(r = r_{bc}, \theta\right) = \dfrac{A_\mu{}^\nu(\theta)}{r_{bc}}$.

Starting with the boundary, we solve for the derivatives $\partial_r E^{\mu\nu}, \partial_r A_\mu{}^\nu$ and using e.g. the Euler procedure on lattice we calculate the variables $E^{\mu\nu}, A_\mu{}^\nu$ successively for values $r > 0$, which *ensures the existence of an analytical solution* for $\Lambda > 0$. The metric generated by the solution tetrad, $g(r,\theta) = \left(E \eta E^t\right)^{-1} \det(E)^{3/8}$, is now the valid metric for $r < r_{bc}$, and is in general *different from the GR-metric* of the mass distribution $g \neq g_{GR}$. The equations-of-motion are the usual relativistic equations

$$\frac{d^2 x^\kappa}{d\lambda^2} + \Gamma^\kappa_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \tag{19}$$

with the usual Christoffel symbols

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} \left(\frac{\partial g_{\kappa\mu}}{\partial x^\nu} + \frac{\partial g_{\kappa\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\kappa} \right) \tag{20}$$

of the solution metric $g_{\mu\nu}(r,\theta)$.

The *solution metric* $g_{\mu\nu}(r,\theta)$ *is analytic* for values $r > 0$, because the solution tetrad $E^{\mu\nu}$ is analytic for $r > 0$.

Why is the solution metric $g_{\mu\nu}(r,\theta)$ analytic, while the corresponding GR-metric $(g_{GR})_{\mu\nu}(r,\theta)$ is not?

Let us compare the Einstein equations [9] [30] [31] in vacuum $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$ to the AK-equations above.

The Ricci tensor $R_{\mu\nu} = \frac{\partial \Gamma^\rho_{\mu\rho}}{\partial x^\nu} - \frac{\partial \Gamma^\rho_{\mu\nu}}{\partial x^\rho} + \Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\sigma\nu} - \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho}$ depends on the Christoffel symbols $\Gamma^\lambda{}_{\mu\nu}$, and these contain the inverse metric $g^{\lambda\kappa} = \left(g_{\mu\nu}\right)^{-1}$, which becomes singular when $\det(g) = 0$, and also it "explodes" when $g_{\mu\nu} \ll 1$, which can happen easily.

In the AK-equations such a behavior is not possible, they are "well-behaved".

Now, what happens for $\Lambda \to 0$? We shall see in chap. 6.1 that in this case the gravitational tensor becomes constant, the hamiltonian and diffeomorphism equations vanish identically, and only the four gaussian equations for the tetrad remain. We show in chap. 6.1, that the tetrad equation $g(r,\theta) = \left(E \eta E^t\right)^{-1} \det(E)^{3/8}$ and the gaussian equations can be satisfied by the tetrad, so *for* $\Lambda = 0$ *the solution metric is identical to the GR-metric* $g\left(E^{\mu\nu}\right) = g_{GR}$.

e.g. for the Schwarzschild metric in spherical coordinates

$$g_{\mu\nu} = diag\left(-\left(1 - \frac{1}{r}\right), 1\Big/\left(1 - \frac{1}{r}\right), r^2, r^2 \sin^2\theta \right)$$

The diagonal tetrad solution is (= diagonal Schwarzschild tetrad E_{ds})

$$\left(E_{ds}\right)^{\mu\nu} = diag\left(\frac{1}{\sqrt{r-1}\, r \sin(\theta)^{3/4}}, \frac{\sqrt{r-1}}{r^2 \sin(\theta)^{3/4}}, \frac{1}{r^{5/2} \sin(\theta)^{3/4}}, \frac{1}{r^{7/2} \sin(\theta)^{3/4}} \right) \tag{21}$$

But, as the tetrad equation has 10 equations for 16 variables, there are 6 degrees of freedom (dof) left.

So we can enforce in addition the validity of the Gauss constraint for the tetrad, and the solution can be calculated in half-analytical form.

4.1. AK Covariant Derivative and Its Gauge Group

Here the covariant derivative (generalized LQG covariant derivative D_a from chap. 3.2) acting on a tensor $t^{\nu\lambda}$ is (see Figure 2)

$$D_\mu t_\nu{}^\lambda = \partial_\mu t_\nu{}^\lambda + \varepsilon^\lambda{}_{\kappa_1\kappa_2} A_\mu{}^{\kappa_1} t^{\nu\kappa_2} \quad \left(D_\mu\right)^\lambda{}_\kappa = \partial_\mu + \varepsilon^\lambda{}_{\kappa_1\kappa} A_\mu{}^{\kappa_1} \tag{22a}$$

where $F_{\mu\nu}{}^\kappa = \left[D_\mu{}^\kappa, D_\nu{}^\kappa\right]$ is the field tensor

4-dimensional Kodama-Ashtekar equations

16 variables $E^{\mu\nu}$: inverse densitized tetrad of the metric $g_{\mu\nu}$

16 variables $A_\mu{}^\nu$ connection tensor

spacetime curvature $F_{\mu\nu}{}^\kappa = \partial_\mu A_\nu{}^\kappa - \partial_\nu A_\mu{}^\kappa + \varepsilon^\kappa{}_{\kappa_1\kappa_2} A_\mu{}^{\kappa_1} A_\nu{}^{\kappa_2}$

4 Gauss constraints $G^\mu = \partial_\nu E^{\nu\mu} + \varepsilon^\mu{}_{\kappa\lambda} A_\nu{}^\kappa E^{\nu\lambda}$

4 diffeomorphism constraints $I_\mu = E^\kappa{}_\nu F_{\mu\kappa}{}^\nu$

24 hamiltonian constraints $H_{(\mu,\nu)}{}^\kappa = F_{\mu\nu}{}^\kappa + \dfrac{\Lambda}{3}\varepsilon_{\mu\nu\rho}E^{\rho\kappa}$

AK covariant derivative and its gauge group

$(D_\mu)^\lambda{}_\kappa = \partial_\mu + \varepsilon^\lambda{}_{\kappa_1\kappa} A_\mu{}^{\kappa_1}$, $D_\mu t_\nu{}^\lambda = \partial_\mu t_\nu{}^\lambda + \varepsilon^\lambda{}_{\kappa_1\kappa_2} A_\mu{}^{\kappa_1} t^{\kappa_2}{}_\nu$

left spin-1/2 representation of the Lorentz-algebra with 4 generators

$$\tau_i = T_+^i\,,\ i = 1,2,3 \qquad \tau_0 = \left(T_+^1 + T_-^1\right) - \left(T_+^2 + T_-^2\right) + \left(T_+^3 + T_-^3\right)$$

or in fundamental matrix representation $\tau_a = i\,\varepsilon_{\nu a\lambda}$

4 extended generators τ_a satisfy the extended SU(2) commutator algebra with spacetime indices {0,1,2,3}

$$\left[\tau_\kappa, \tau_\lambda\right] = i\,\varepsilon_{\kappa\lambda\mu}\tau_\mu$$

Renormalizable Einstein-Hilbert action with the Ashtekar momentum $A_\mu{}^\nu$

Einstein-Hilbert action

$$S = \frac{c}{16\pi l_P{}^2}\int (R - 2\Lambda)\sqrt{-g}\,d^4x \quad , \quad \kappa = \frac{8\pi l_P{}^2}{c}$$

half-asymmetric background $A_\mu{}^\nu = \dfrac{1}{l_P}\begin{pmatrix} 1,1,-1,1 \\ 1,1,-1,1 \\ 1,1,-1,1 \\ 1,1,-1,1 \end{pmatrix} = \dfrac{1}{l_P}\Omega_\mu{}^\nu$

reformulated Einstein-Hilbert action with $\Lambda \approx 0$

$S = \dfrac{c}{\pi}\int (A_\mu{}^\nu A_\nu{}^\mu)\, R\sqrt{-g}\,d^4x$ is dimensionally renormalizable

variation with respect to $g_{\mu\nu}$ yields the Einstein equation as before

variation with respect to $A_\mu{}^\nu$ gives $\dfrac{\partial}{\partial A_\mu{}^\nu}\dfrac{c}{\pi}(A_\mu{}^\nu A_\nu{}^\mu)\, R\sqrt{-g} = -16\dfrac{l_P}{r_s}\Omega_\mu{}^\nu T\sqrt{-g}$

This is ≈ 0 in the classical region, so the eom is satisfied.

Figure 2. Schematic Ashtekar-Kodama equations and Hilbert-Einstein action.

$$F_{\mu\nu}{}^{\kappa} = \partial_\mu A_\nu{}^{\kappa} - \partial_\nu A_\mu{}^{\kappa} + \varepsilon^{\kappa}{}_{\kappa_1\kappa_2} A_\mu{}^{\kappa_1} A_\nu{}^{\kappa_2}$$

$$D_\mu = \partial_\mu - i A_\mu{}^a \tilde{\tau}^a \text{, where } \tilde{\tau}^a = i\,\varepsilon_\nu{}^a{}_\lambda \tag{22b}$$

satisfy the *extended SU*(2) (*ε-tensor*) *Lie-algebra* with four generators

$$\left[\tilde{\tau}^a, \tilde{\tau}^b\right] = i\,\varepsilon^{abc}\tilde{\tau}^c \tag{22c}$$

$\tilde{\tau}^a$ are *antisymmetric*, with the characteristic property $\left(\tilde{\tau}^a\right)^t = -\tilde{\tau}^a$ (transposed matrix changes sign) $\tilde{\tau}^a$ with the commutator [,] *form a Lie-algebra*, because

$$\left(\left[\tilde{\tau}^a, \tilde{\tau}^b\right]\right)^t = \left(\tilde{\tau}^a\tilde{\tau}^b\right)^t - \left(\tilde{\tau}^b\tilde{\tau}^a\right)^t = \left(\tilde{\tau}^b\right)^t\left(\tilde{\tau}^a\right)^t - \left(\tilde{\tau}^a\right)^t\left(\tilde{\tau}^b\right)^t$$
$$= \tilde{\tau}^b\tilde{\tau}^a - \tilde{\tau}^a\tilde{\tau}^b = -\left[\tilde{\tau}^a, \tilde{\tau}^b\right]$$

i.e. the commutator [,] preserves the antisymmetry property, the set $\tilde{\tau}^a$ is closed under the commutator.

$\varepsilon^{\lambda}{}_{\kappa_1\kappa_2}$ are the structure constants of the *extended SU*(2) (*ε-tensor*) *Lie-algebra*.

The above Lie-algebra commutator relation is satisfied setting $\tilde{\tau}^a = i\,\varepsilon_\nu{}^a{}_\lambda$ (fundamental matrix representation):

$\varepsilon_j{}^{ki}\varepsilon_i{}^m{}_n - \varepsilon_j{}^{mi}\varepsilon_i{}^k{}_n = \varepsilon^{km}{}_i\varepsilon_j{}^i{}_n$ is *valid in the* 3*-dimensional Euclidean* space with all indices $i = 1,2,3$, metric $\eta = diag(1,1,1)$, which follows easily from the Levi-Civita-relation $\varepsilon_i{}^k{}_k\varepsilon_i{}^m{}_n = \delta_j{}^m\delta_k{}^n - \delta_j{}^n\delta_k{}^m$.

$\varepsilon_j{}^{ki}\varepsilon_i{}^m{}_n - \varepsilon_j{}^{mi}\varepsilon_i{}^k{}_n = \varepsilon^{km}{}_i\varepsilon_j{}^i{}_n$ is *valid in the* 4*-dimensional Minkowski* space-time with all indices $i = 0,1,2,3$, and Minkowski metric $\eta = diag(-1,1,1,1)$, as is proved by calculation in [32].

A well-known representation of this *extended SU*(2) (*ε-tensor*) *Lie-algebra* are the following 4 × 4 martices

$$\tau_i = T_+{}^i,\ i = 1,2,3 \quad \tau_0 = \left(T_+{}^1 + T_-{}^1\right) - \left(T_+{}^2 + T_-{}^2\right) + \left(T_+{}^3 + T_-{}^3\right)$$

The T_+, T_- are combinations of the 6 generators of the Lorentz group:

$$T_\pm{}^k = \frac{1}{2}\left(J^k \pm K^k\right)$$

of the 3 spatial rotators J^k and the 3 boosts K^k, which are 4 × 4 matrices derived from the 4-tensor generator

$$\left(M^{\mu\nu}\right)^\rho{}_\sigma = -i\left(\eta^{\mu\nu}\delta_\sigma{}^\nu - \eta^{\nu\rho}\delta_\sigma{}^\mu\right),\quad J^k = \frac{1}{2}\varepsilon^{ijk}M^{ij},\quad K^k = M^{0k}$$

where η is the Minkowski metric, e.g.

$$J^1 = M^{23} = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},\quad K^1 = M^{01} = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The $T_+{}^i$ are the generators of the left spin-1/2 representation of the Lorentz-algebra SO(1, 3) and $T_-{}^i$ are the generators of the right spin-1/2 representation of the Lorentz-algebra SO(1, 3), the 3 generators τ^i satisfy with spatial indices $i = 1,2,3$: the ordinary *SU*(2) *algebra*

$$\left[T_+^{\ i},T_+^{\ j}\right]=i\,\varepsilon_{ijk}T_+^{\ k},\quad \left[T_-^{\ i},T_-^{\ j}\right]=i\,\varepsilon_{ijk}T_-^{\ k},\quad \left[T_-^{\ i},T_+^{\ j}\right]=0$$

and the 4 extended generators τ^{μ} satisfy the *extended SU*(2) *algebra* (*ε-tensor algebra*) with spacetime indices $\mu=\{0,1,2,3\}$

$$\left[\tau_{\kappa},\tau_{\lambda}\right]=i\,\varepsilon_{\kappa\lambda\mu}\tau_{\mu}\,.$$

4.2. Renormalizable Einstein-Hilbert Action with the Ashtekar Momentum A_{μ}^{ν}

Starting with semiclassical Einstein equations (see Figure 2)

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}=\kappa\left\langle\Psi\middle|\hat{T}_{\mu\nu}\middle|\Psi\right\rangle \quad \text{([4] (1.37))} \tag{23}$$

Einstein-Hilbert action

$$S=\frac{1}{2\kappa}\int R\sqrt{-g}\,d^4x,\quad \kappa=\frac{8\pi l_P^{\ 2}}{\hbar c}=\frac{8\pi G}{c^4}, \tag{24a}$$

with lambda:

$$S=\frac{1}{2\kappa}\int\left(R-2\Lambda\right)\sqrt{-g}\,d^4x \tag{24b}$$

$$\text{setting}\quad A_{\mu}^{\ \nu}=\frac{1}{l_P}\begin{pmatrix}1,1,-1,1\\1,1,-1,1\\1,1,-1,1\\1,1,-1,1\end{pmatrix}=\frac{1}{l_P}\Omega_{\mu}^{\ \nu}$$

(constant background in the Ashtekar-Kodama equations), one can reformulate the Einstein-Hilbert action with $\Lambda\approx 0$

$$S=\frac{\hbar c}{\pi}\int\left(A_{\mu}^{\ \nu}A_{\nu}^{\ \mu}\right)R\sqrt{-g}\,d^4x, \tag{24c}$$

which makes it dimensionally renormalizable, with the dimensionless interaction constant $g_{gr}=\frac{1}{\pi}$. Variation with respect to $g_{\mu\nu}$ yields then, as before, the Einstein equations:

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\kappa T_{\mu\nu}\quad\text{or equivalent}\quad R_{\mu\nu}=\kappa\left(T_{\mu\nu}-\frac{T}{2}g_{\mu\nu}\right)$$

From this we derive with $\Lambda\approx 0$:

$$R=g^{\nu\mu}R_{\mu\nu}=\kappa\left(T-\frac{T}{2}g^{\nu\mu}g_{\mu\nu}\right)=-\kappa T$$

Now, variation with respect to $A_{\mu}^{\ \nu}$ gives the left side of the equation-of-motion (eom)

$$\frac{\partial}{\partial A_{\mu}^{\ \nu}}\frac{\hbar c}{\pi}\left(A_{\mu}^{\ \nu}A_{\nu}^{\ \mu}\right)R\sqrt{-g}=-\frac{\hbar c}{\pi}2A_{\mu}^{\ \nu}\kappa T\sqrt{-g}=-16l_P\Omega_{\mu}^{\ \nu}T\sqrt{-g}$$

The above expression is calculated, as usually, dimensionless, with correct dimension we have

$$\frac{\partial}{\partial A_{\mu}^{\ \nu}}\frac{\hbar c}{\pi}\left(A_{\mu}^{\ \nu}A_{\nu}^{\ \mu}\right)R\sqrt{-g}=-16\frac{l_P}{r_s}\Omega_{\mu}^{\ \nu}T\sqrt{-g}$$

This is ≈0 in the classical region $l_P \ll r_s$, so the eom is satisfied.

4.3. Relativistic Newtonian Gravitational Force Equation

An ansatz for the exact relativistic Newtonian gravitational force equation is

$$\frac{d}{d\tau} m u_\nu = \partial_\nu \frac{GMm}{\sqrt{x^\mu x_\mu}} = \partial_\nu \frac{m r_s c^2}{2\sqrt{x^\mu x_\mu}} \quad \text{where} \quad x^\mu x_\mu = r^2 - c^2 t^2$$

or $\frac{d}{d\tau} u_\nu = \partial_\nu \frac{c_0}{\sqrt{x^\mu x_\mu}}$ with $c_0 = \frac{r_s}{2}$ with Schwarzschild radius r_s, the space part becomes $\frac{d}{dt} \frac{m\vec{v}}{\sqrt{1-(\vec{v}/c)^2}} = \nabla \frac{GMm}{\sqrt{r^2 - c^2 t^2}} = \nabla \frac{m r_s c^2}{2\sqrt{r^2 - c^2 t^2}}$ with variable transformation $t' = ct$ we eliminate c:

$$\frac{d}{dt'} \frac{m\dot{\vec{r}}}{\sqrt{1-\dot{\vec{r}}^2}} = \nabla \frac{m r_s}{2\sqrt{r^2 - t'^2}},$$

then follows

$$\frac{d}{dt} \frac{\dot{\vec{r}}}{\sqrt{1-\dot{\vec{r}}^2}} = \nabla \frac{c_0}{\sqrt{r^2 - t^2}} = -c_0 \frac{\vec{r}}{\sqrt{r^2 - t^2}^3}$$

and with

$$\int dt \frac{1}{\left(\sqrt{r^2 - t^2}\right)^3} = \frac{t}{r^2 \sqrt{r^2 - t^2}},$$

in integral form

$$d\left(\frac{\dot{\vec{r}}}{\sqrt{1-\left(\dot{\vec{r}}\right)^2}}\right) = -c_0 \int_0^{dt} dt \frac{\vec{r}}{\left(\sqrt{r^2 - t^2}\right)^3} = -c_0 \frac{dt\,\vec{r}}{r^2 \sqrt{r^2 - (dt)^2}},$$

re-inserting c we get

$$d\left(\frac{\dot{\vec{r}}}{c^2 \sqrt{1 - \frac{v^2}{c^2}}}\right) = -c_0 \int_0^{dt} dt \frac{\vec{r}}{\left(\sqrt{r^2 - c^2 t^2}\right)^3} = -\frac{r_s}{2} \frac{dt\,\vec{r}}{r^2 \sqrt{r^2 - c^2 (dt)^2}}$$

without the relativistic addition theorem we get the well-known semi-relativistic Newton equation

$$\frac{d}{dt}\left(\frac{m\dot{\vec{r}}}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = -GMm \frac{\vec{r}}{r^3}, \tag{25}$$

taking into account the relativistic addition theorem $v + dv \to \frac{v + dv}{1 + \frac{v\,dv}{c^2}}$,

$$i.e.\quad dv \to dv\left(1-\frac{v^2}{c^2}\right)$$

$$\text{we get}\quad \frac{d}{dt}\left(\frac{m\dot{\vec{r}}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = -GMm\frac{\vec{r}}{r^3}\frac{1}{\left(1-\frac{v^2}{c^2}\right)},$$

the fully **relativistic gravitational force equation,** and the corresponding **1-order correction** to effective potential becomes $\delta V = V\frac{v^2}{c^2} = -\frac{GMm}{r}\frac{(L/m)^2}{c^2r^2}$, which is **the same as the GR-correction** [9].

Calculation shows [33]: circular orbits exist with $r = r_0$ for angular frequency ω, where $\left(r_0{}^2\omega^2\right)\sqrt{1-\left(r_0\omega\right)^2} = \frac{r_s}{2r_0}$, *i.e.* for $\frac{r_0}{r_s} > \frac{1}{2\times 0.385} = 1.3$, and the horizon (point of no return) of a Newtonian BlackHole is $r = c_0 = \frac{r_s}{2}$, but there is no singularity, as in the GR case.

Still, we must consider **GR as the valid extension of the relativistic Newtonian gravity** in second order $\left(\frac{v}{c}\right)^2$.

5. The Basic Equations

The Ashtekar-Kodama equations (AKe) consist of (26a)

24 hamiltonian equations with the expression scheme $A\bullet A + \partial A + (\Lambda/3)E$

4 gaussian equations with the expression scheme $A\bullet E + \partial E$

4 diffeomorphism equations with the expression scheme $E\bullet A\bullet A + E\bullet\partial A$

where $\bullet$ represents matrix-multiplicative terms and ∂ means derivatives for covariant coordinates, here the spherical coordinates spacetime $\{t, r, \theta, \varphi\}$

$$\partial^\mu = \left(\partial_t, \partial_r, \frac{1}{r}\partial_\theta, \frac{1}{r\sin\theta}\partial_\varphi\right)$$

We consider here only spacetimes with axial symmetry, *i.e.* $\partial_\varphi = 0$ and the variables $E^{\mu\nu}$ and $A_\mu{}^\nu$ are functions of $\{t, r, \theta\}$.

In the code [32], AKe in above original form in time-dependent form (variables $E^{\mu\nu}(t,r,\theta)$ $A_\mu{}^\nu(t,r,\theta)$) are called *eqtocev*, AKe in static form (variables $E^{\mu\nu}(r,\theta)$ $A_\mu{}^\nu(r,\theta)$) are called *eqtocv*.

Static *eqtocv* have the properties:

derivatives order 1: ∂_r, ∂_θ

∂_r-variables rvars A2i, E1i

∂_θ-variables thvars A1i, E2i

$\partial_r\partial_\theta$-variables rthvars A0i, A3i

Algebraic variables avars E0i, E3i

Time-dependent *eqtocev* have the properties:

derivatives order 1: ∂_t ∂_r, ∂_θ

∂_t-variables tvars A1i, A2i, A3i, E0i

∂_r -variables rvars A2i, E1i

∂_θ -variables thvars A1i, E2i

$\partial_r\partial_\theta$ -variables rthvars A0i, A3i

algebraic variables avars E3i

The static equations *eqtocv* contain algebraic variables {E0i, E3i}, the time-dependent equations *eqtocev* contain algebraic variables {E3i}.

In the static equations, algebraic variables {E0i, E3i} are eliminated and the equations are transformed into pure differential equations.

Corresponding transformations are performed on the time-dependent equations, eliminating pure ∂_θ -variables E2i and pure ∂_r -variables E1i.

This is done in the next chapter.

5.1. The Integrability Conditions

In the static (time-independent) AK equations eq1…4 and eq13…16 contain resp. ∂_rA0i and ∂_θA0i as the only derivative, also eq9…12 and eq17…20 contain resp. ∂_θA3i and ∂_rA3i as the only derivative.

In order to eliminate algebraic variables, we differentiate and subtract static equations *eqtocv*, and use integrability conditions $\partial_\theta\partial_r$A0i = $\partial_r\partial_\theta$A0i and $\partial_\theta\partial_r$A3i = $\partial_r\partial_\theta$A3i, and transform the equations accordingly [32].

This changes the expression scheme in eq9…12, eq13…16: $A \bullet \mathrm{A} + A \bullet \partial A + (\Lambda/3)(\partial E + E)$

Accordingly in the time-dependent *eqtocev* equations eq9…12 and eq21…24 are transformed.

Equations with integrability condition static ($\partial_t = 0$) become *eqtoiv*

Equations with integrability condition time-dependent become *eqtoiev*

Static *eqtoiv* have the properties:

∂_r -variables rvars A2i, E1i

∂_θ -variables thvars A1i, E2i

$\partial_r\partial_\theta$ -variables rthvars A0i, A3i, E0i, E3i

algebraic variables avars: none

Time-dependent *eqtoiev* have the properties:

∂_t -variables tvars A1i, A2i, A3i, E0i, E2i

∂_r -variables rvars A2i, E1i

∂_θ -variables thvars A1i

$\partial_r\partial_\theta$ -variables rthvars A0i, A3i, E0i, E1i

$\partial_t\partial_r\partial_\theta$ -variables trthvars A0i, A3i, E0i, E2i

algebraic variables avars E3i

The static equations *eqtoiv* are 32 pdeq's of first order in r, θ, quadratic in the variables $E^{\mu\nu}$ and $A_\mu{}^\nu$ in the 24 hamiltonian equations and 4 gaussian equations and cubic in the variables $E^{\mu\nu}$ and $A_\mu{}^\nu$ in the last 4 diffeomorphism equations.

The row-variables in the A-tensor and the E-tensor have different derivative behavior:

$A2_i$ and $E1_i$ are pure *r-variables* (only ∂_r derivative present), $A1_i$ and $E2_i$ are pure *θ-variables* (only ∂_θ derivative present), ($A0_i$, $E0_i$, $A3_i$, $E3_i$) are *r-θ-variables* (both ∂_r derivative and ∂_θ derivative present).

The time-dependent equations *eqtoiev* are 32 pdeq's of first order in *t*, *r*, *θ*, quadratic in the variables $E^{\mu\nu}$ and $A_\mu{}^\nu$ in the 24 hamiltonian equations and 4 gaussian equations and cubic in the variables $E^{\mu\nu}$ and $A_\mu{}^\nu$ in the last 4 diffeomorphism equations.

Here $A2_i$ and $E1_i$ are *r-variables*, $A1_i$ are *θ-variables*, ($A0_i$, $E0_i$, $A3_i$, $E2_i$) are *r-θ-variables*, ($A1_i$, $A2_i$, $A3_i$, $E0_i$, $E2_i$) are *t-variables* (∂_t derivative present) and $E3_i$ are algebraic variables (no derivative present).

The overall scheme of the static equations *eqtoiv* becomes (26b)

24 hamiltonian $A \bullet A + A \bullet \partial A + (\Lambda/3)(\partial E + E)$ or $A \bullet A + \partial A + (\Lambda/3)E$

4 gaussian $A \bullet E + \partial E$

4 diffeomorphism $A \bullet A \bullet E + E \bullet \partial A$

The overall scheme of the time-dependent equations *eqtoiev* becomes (26c)

24 hamiltonian $A \bullet A + A \bullet \partial A + (\Lambda/3)(\partial E + E)$ or $A \bullet A + A \bullet \partial A + (\Lambda/3)\partial E$ or $A \bullet A + \partial A + (\Lambda/3)E$

4 gaussian $A \bullet E + \partial E$

4 diffeomorphism $A \bullet A \bullet E + E \bullet \partial A$

The scheme of different forms of AK equations and their solvability is shown in Figure 3.

5.2. Solvability of Static and Time-Dependent Equations

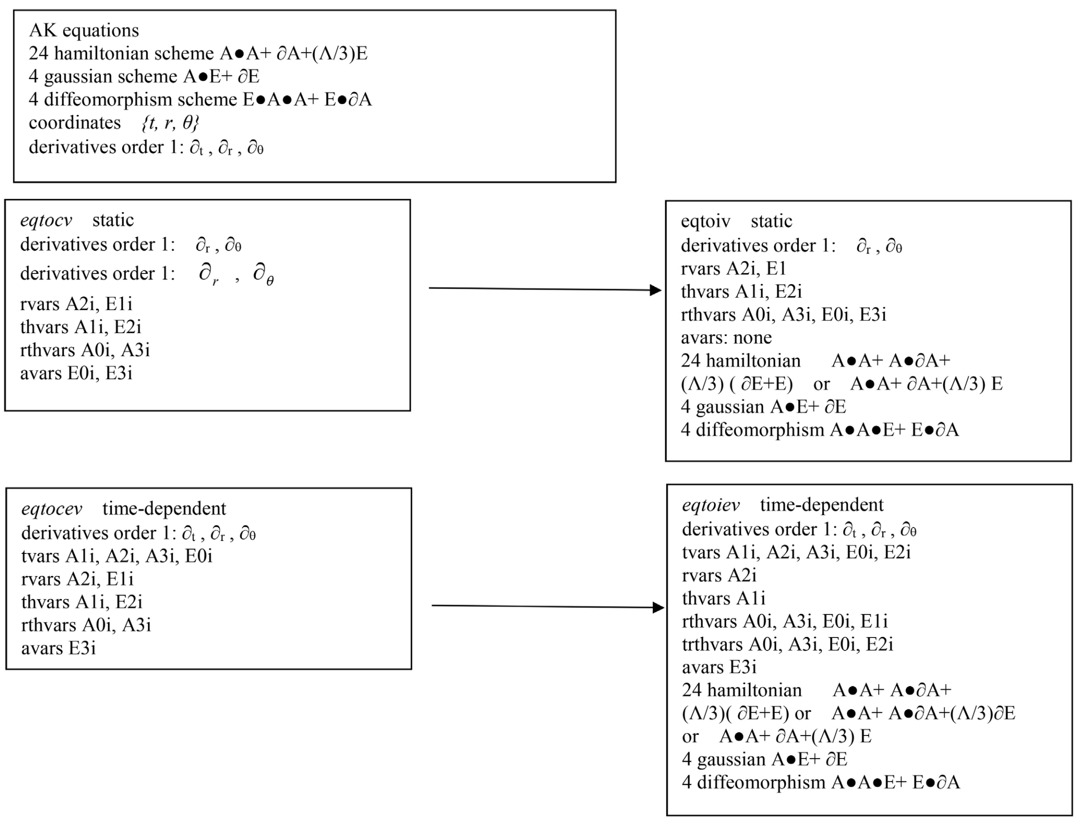


Figure 3. Schematic solvability and form of AK equations.

6. Solutions of Static Equations

6.1. Solution Limit Λ→0

Solution *eqtoiv* Λ→0: Einstein equations valid, metric Schwarzschild or Kerr (**Figure 4**)

eqtoiv static

metric condition for $r\to\infty$

$$E\,\eta\,E' = g^{-1}/(-\det(g))^{3/4}$$

g=Schwarzschild or Kerr

hamiltonian= A●A+ A●∂A

A=unique sol(hamiltonian)

$$A_0 = (1,1,-1,1)\,A_{00c}$$
$$A_1 = (1,1,-1,1)\,A_{10c}$$
$$A_2 = (1,1,-1,1)\,A_{20c}$$
$$A_3 = (1,1,-1,1)\,A_{30c}$$

A= *constant half-antisymmetric background* A_{hab}

equations for E

eqdiff=0

eqgaus=$\partial_\theta E2-\partial_r E1$

metric condition: *solvable for all r*

$$E\,\eta\,E' = g^{-1}/(-\det(g))^{3/4}$$

sol=E_{GS} Gauss-Schwarzschild tetrad for g=Schwarzschild

sol=E_{GK} Gauss-Kerr tetrad for g=Kerr

$$E_{GS}(r,\theta)=\begin{pmatrix} \frac{1}{\sin(\theta)^{3/4} r^{3/2}\sqrt{1-1/r}} & 0 & 0 & 0 \\ 0 & E_{GS,22}(r,\theta) & E_{GS,23}(r,\theta) & E_{GS,24}(r,\theta) \\ 0 & E_{GS,32}(r,\theta) & E_{GS,33}(r,\theta) & E_{GS,34}(r,\theta) \\ 0 & E_{GS,42}(r,\theta) & E_{GS,43}(r,\theta) & E_{GS,44}(r,\theta) \end{pmatrix}$$

Einstein equations satisfied for all *r>1*

GR exactly valid

Exact solution *eqtoiv* Λ→0 : Einstein equations valid, g=Minkowski metric

sol= $E_{GM}(r,\theta)$ Gauss-Minkowski tetrad

metric is the flat Minkowski metric

Einstein equations satisfied for all *r>1*

GR exactly valid

$$E_{GM}(r,\theta)=\begin{pmatrix} \frac{1}{r^{3/2}\sin(\theta)^{3/4}} & 0 & 0 & 0 \\ 0 & \frac{-i\left(f_{hyp}(\theta)\right)^{-3/2}\left(\left(f_{hyp}(\theta)\right)^{3}-1\right)}{2r^{3/2}\sin(\theta)^{3/4}} & \frac{\left(f_{hyp}(\theta)\right)^{-3/2}\left(\left(f_{hyp}(\theta)\right)^{6}-1\right)}{2r^{3/2}\sin(\theta)^{3/4}\left(\left(f_{hyp}(\theta)\right)^{3}-1\right)} & 0 \\ 0 & \frac{\left(f_{hyp}(\theta)\right)^{-3/2}\left(\left(f_{hyp}(\theta)\right)^{3}+1\right)}{r^{3/2}} & \frac{i\left(f_{hyp}(\theta)\right)^{-3/2}\left(\left(f_{hyp}(\theta)\right)^{3}-1\right)}{r^{3/2}} & \frac{i}{r^{3/2}} \\ 0 & \frac{-i\left(f_{hyp}(\theta)\right)^{-3/2}\left(\left(f_{hyp}(\theta)\right)^{3}+1\right)}{2r^{3/2}\sin(\theta)} & \frac{\left(f_{hyp}(\theta)\right)^{-3/2}\left(\left(f_{hyp}(\theta)\right)^{3}-1\right)}{2r^{3/2}\sin(\theta)} & \frac{1}{r^{3/2}\sin(\theta)} \end{pmatrix}$$

Figure 4. Solutions of static AK equations for Λ→0.

6.2. Behavior at Schwarzschild Horizon, Quantum and Classical Region

Approximate AK-correction at the horizon

At the horizon, the Schwarzschild tetrad diverges $E_{dS}{}^{0,0} = \frac{1}{r\sqrt{r-1}\sin^{3/4}(\theta)} \to \infty$, so the term $\frac{\Lambda}{3}E^{\mu\nu}$ becomes significant, the coupling reappears.

When the parameter $dr = r - 1$ becomes $dr = \sqrt{\Lambda}$, we get in the limit $r \to \infty$ for the E-tensor and the A-tensor a r-independent finite solution in the vicinity of $r = 1$:

$$A0_i = A00(\theta)\{1,1,-1,1\}, \quad A1_i = A10c\{1,1,-1,1\},$$

$$A2_i = A20c\{1,1,-1,1\}, \quad A3_i = A30c\{1,1,-1,1\}$$

$$E0_i = E00(\theta)\{1,1,-1,1\}$$

The parameters of the solution are determined by the continuity condition at $r = 1+\sqrt{\Lambda}$, *i.e.* $E00(\theta) = \frac{1}{\Lambda^{1/4}\sin^{3/4}(\theta)}$, the peak in the metric is $g_{1,1} = \frac{1}{\sqrt{\Lambda}}$ (dimensionless, in r_s units), or dimensional $|g_{1,1}| = \frac{1}{r_s\sqrt{\Lambda}} = \frac{r_s}{l_0}$,

$$\text{where} \quad l_0 \approx r_s\left(r_s{}^2\Lambda\right) \tag{27}$$

is the approximate AK correction length (see **Figure 5**).

In GR, the (dimensional) radial equation-of-motion (energy equation) is [9] $\frac{m\dot{r}^2}{2} + V_{eff} = const = E$, where $l = \frac{L}{m}$ is the reduced angular momentum and the effective potential energy is

$$V_{eff} = mc^2\left(-\frac{r_s}{r} + \frac{l^2}{r^2} - \frac{r_s}{r^3}\frac{l^2}{c^2}\right)$$

In Newtonian gravitation, the relativistic third term is missing:

$$V_{eff,N} = mc^2\left(-\frac{r_s}{r} + \frac{l^2}{r^2}\right)$$

So in GR, the pure (free fall) potential energy is $V_{eff} = -\frac{r_s}{r}mc^2$, so at distance r, the redshift (=energy attenuation) is $z(r) = \frac{1}{1-\frac{r_s}{r}} - 1 = \frac{r_s/r}{1-r_s/r}$, where $z(r_s) = \infty$ $z(\infty) = 0$, the escape kinetic energy is $E(r) = -V_{eff} = \frac{r_s}{r}mc^2$ and the escape velocity is $v(r) = c\sqrt{\frac{r_s}{r}}$.

In the AK-gravitation, there is a correction for the metric near the horizon: the redshift is $z_c(r) = \frac{1}{1-\frac{r_s-l_0}{r}} - 1 = \frac{(r_s-l_0)/r}{1-(r_s-l_0)/r}$, where $z_c(r_s) = \frac{r_s-l_0}{l_0} \approx \frac{r_s}{l_0}$

At the horizon, Schwarzschild tetrad diverges

$E_{dS}{}^{0,0} = \frac{1}{r\sqrt{r-1}\sin^{3/4}(\theta)} \to \infty$, so the term $\frac{\Lambda}{3}E^{\mu\nu}$ becomes significant

at $r = 1 + dr$, $dr = \sqrt{\Lambda}$, i.e. $E00(\theta) = \frac{1}{\Lambda^{1/4}\sin^{3/4}(\theta)}$, the (dimensional) peak in the metric is

$\left|g_{11,c}(r = r_s)\right| = \frac{1}{r_s\sqrt{\Lambda}} = \frac{r_s}{l_0}$, $\left|g_{11,c}\right| = \frac{1}{\left|g_{00,c}\right|} = \frac{1}{1-\frac{r_s - l_0}{r}}$ outer metric , and inner metric

$$\left|g_{00,c}(r)\right| = \frac{1}{\left|g_{11,c}\right|} = exp\left(-r_s\int_r^{r_s}\frac{dr_1}{r_1^2}\left(\frac{M(r_1)}{M} + \frac{4\pi r_1^3 p(r_1)}{M}\right)\left(1-\frac{(r_s - f_s(r)l_0)M(r_1)}{r_1 M}\right)^{-1}\right)$$

$\left|g_{11,c}\right| = \frac{1}{\left|g_{00,c}\right|} \approx \frac{r_s}{l_0}$ for $r_s - l_0 \le r \le r_s + l_0$,

where $l_0 \approx r_s{}^3\Lambda$ is the AK correction length

and the escape velocity is $v_c(r) = c\sqrt{1-\frac{l_0}{r_s}} \approx c\left(1-\frac{l_0}{2r_s}\right)$

central Black Hole of the Milky Way $v_c(r) = c\left(1-0.8\times10^{-32}\right)$

$$\left|g_{00,c}(r)\right| = \frac{1}{\left|g_{11,c}\right|} = exp\left(-r_s\int_r^{r_s}\frac{dr_1}{r_1^2}\left(\frac{M(r_1)}{M} + \frac{4\pi r_1^3 p(r_1)}{M}\right)\left(1-\frac{(r_s - f_s(r)l_0)M(r_1)}{r_1 M}\right)^{-1}\right)$$

gravitational limit for the quantum realm becomes $r_{gr} = \sqrt{l_P\sqrt{\frac{1}{\Lambda}}} = 3.1*10^{-5}\,m = 31\,\mu m$

full lambda-correction for the AK-gravity

$\Lambda\, dE_{1i}(r,\theta) = -c_i\Lambda / r$, $g_{11c}(r,\theta) \approx g_{11}(r,\theta)\left(1+2\Lambda dE_{11}(r,\theta)\right) \approx g_{11}(r,\theta)\left(1-64\Lambda/r\right)$

AK correction length is $l_0 \approx 64\,\Lambda r_s^3$

Figure 5. Schematic Schwarzschild horizon and lambda-correction.

$z_c(\infty) = 0$, the escape kinetic energy is $E_c(r) = -V_{eff} = \frac{r_s - l_0}{r}mc^2$, where $E_c(r_s) = \left(1-\frac{l_0}{r_s}\right)mc^2$ and the escape velocity is $v_c(r) = c\sqrt{1-\frac{l_0}{r_s}} \approx c\left(1-\frac{l_0}{2r_s}\right)$,

e.g. for the central Black Hole of the Milky Way with $M = 4\times10^6 M_s$

$r_s = 1.2\times10^{10}$ m $\Lambda = 1.1\times10^{-52}$ m^{-2} we get the AK correction

$\frac{l_0}{r_s} = r_s{}^2\Lambda = 1.6\times10^{-32}$ and the escape velocity

$$v_c(r) = c\left(1-\frac{l_0}{2r_s}\right) = c\left(1-0.8\times10^{-32}\right)$$

GR Schwarzschild metric is

$$-ds^2 = \left(1-\frac{r_s}{r}\right)c^2dt^2 - \frac{dr^2}{1-\frac{r_s}{r}} - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi\right)$$

the AK-corrected finite outer metric for $r \ge r_s$

$$-ds_c^{\ 2}=\left(1-\frac{r_s-l_0}{r}\right)c^2dt^2-\frac{dr^2}{1-\frac{r_s-l_0}{r}}-r^2\left(d\theta^2+\sin^2\theta\,d\phi\right)$$

where for $r\approx r_s$ we have no singularity

$$\left|g_{00,c}\right|=1-\frac{r_s-l_0}{r}\approx\frac{l_0}{r_s}\quad\left|g_{11,c}\right|=\frac{1}{\left|g_{00,c}\right|}=\frac{1}{1-\frac{r_s-l_0}{r}}\approx\frac{r_s}{l_0}$$

For the inner metric $r\le r_s$ we get from the Tolman-Oppenheimer-Volkov (TOV) equation [9]

$$\frac{d\,p(r)}{dr}=-\frac{r_s\rho(r)}{r^2}\frac{M(r)}{M}\left(1+\frac{p(r)}{\rho(r)}\right)\left(1+\frac{4\pi r^3p(r)}{M(r)}\right)\left(1-\frac{r_sM(r)}{rM}\right)^{-1}$$

for the reduced pressure $p(r)=\frac{P(r)}{c^2}$, density $\rho(r)$,mass $M(r)$ and total mass M, the GR equation for g_{00}

$$\left|g_{00}(r)\right|=exp\left(-r_s\int_r^{r_s}\frac{dr_1}{r_1^2}\left(\frac{M(r_1)}{M}+\frac{4\pi r_1^3p(r_1)}{M}\right)\left(1-\frac{r_sM(r_1)}{r_1M}\right)^{-1}\right)$$

which becomes with the AK-correction

$$\left|g_{00,c}(r)\right|=exp\left(-r_s\int_r^{r_s}\frac{dr_1}{r_1^2}\left(\frac{M(r_1)}{M}+\frac{4\pi r_1^3p(r_1)}{M}\right)\left(1-\frac{\left(r_s-f_s(r)l_0\right)M(r_1)}{r_1M}\right)^{-1}\right)$$

with the smooth-step function $f_s(r)=exp\left(-\frac{(r-r_s)^2}{l_0^{\ 2}}\right)$ here again, for $r\le r_s$ $r\approx r_s$ we get the same values as for the outer metric and a continuous transition, and with $p(r)\approx 0\quad \rho(r)\approx 0\quad M(r)\approx M\quad \left|g_{00}\right|\approx 1-\frac{r_s-f_s(r)l_0}{r}$

$\left|g_{11,c}\right|=\frac{1}{\left|g_{00,c}\right|}\approx\frac{1}{1-\frac{r_s-f_s(r)l_0}{r}}$, for $r\approx r_s$: $\left|g_{00}\right|\approx\frac{l_0}{r_s}\quad\left|g_{11,c}\right|=\frac{1}{\left|g_{00,c}\right|}\approx\frac{r_s}{l_0}$

The limit of the classical region

This reappearance of coupling for $dr=\sqrt{\Lambda}r_s$ (dimensionless) results in a new scale, at which the classical character of gravity disappears and the quantum realm begins (see **Figure 5**):

we set the gravitational scale for the quantum realm to be r_{gr} and

$$dr=\frac{l_P}{r_{gr}}=\sqrt{\Lambda}r_{gr}$$

$$\text{so}\quad r_{gr}=\sqrt{l_P\sqrt{\frac{1}{\Lambda}}}=3.9\times10^{-5}\ \text{m}=39\ \mu\text{m}\tag{28}$$

So we can say that gravitation has two scales: in the classical region the Λ-scale $\Lambda=1.1\times10^{-52}\ \text{m}^{-2}$: $R_\Lambda=\frac{1}{\sqrt{\Lambda}}=0.95\times10^{26}\ \text{m}$ and in the quantum region $r_{gr}=3.9\times10^{-5}\ \text{m}$.

The electrodynamics has, in contrast, only one scale, the classical electron radius $r_e = 2.8\times10^{-15}$ m.

The classical and the quantum region

As a consequence of the gravitational quantum scale r_{gr}, we can characterize three regions of gravity (Figure 6).

6.3. Lambda-Correction in AK-Gravity

In the preceding chapter, we obtained an approximate AK correction to Schwarzschild metric $l_0 \approx r_s^2\sqrt{\Lambda}$.

•classical region $\Lambda\approx 0$ $r>>r_{gr}$

AK background equations *eqtoeivnu3b*, where the hamiltonian equations *eqham(Ab,∂Ab)* depend only on *Ab*

eqtoeivnu3b={eqham(Ab,∂Ab), eqgaus (Ab,Eb,∂Eb), eqdiff(Ab, ∂Ab,Eb)}

results in *Einstein equations* valid for the metric $g = (E\eta E^t)^{-1}/(\det E)^{3/2}$ = boundary metric (Schwarzschild or Kerr)

AK wave equations *eqtoeivnu3w* , where in the hamiltonian equations *Λ eqham(As,∂As,Es,Ab) Λ* factors out,

eqtoeivnu3w={ Λ eqham(As,∂As,Es,Ab), eqgaus (Es,∂Es,Ab), eqdiff(Es,Ab,∂Ab)}

$Ab \cong \dfrac{1}{l_P}$ makes EH-action $S = \dfrac{\hbar c}{\pi}\int (A_\mu{}^\nu A_\nu{}^\mu) R\sqrt{-g}d^4x$ dimensionally renormalizable

the metric condition for *Eb* is satisfied for $r\to\infty$

$Eb\bullet\eta\bullet Eb^t = I(g_{\mu\nu}) = g^{-1}/(-\det(g))^{3/4}$, and $Eb=E_{GS}$ resp. $=E_{GK}$ for g=Schwarzschild resp. Kerr

with field AK equations: AK equations

$$H_{(\mu,\nu)}{}^\kappa \equiv F_{\mu\nu}{}^\kappa + \frac{\Lambda}{3}\varepsilon_{\mu\nu\rho}E^{\rho\kappa} = 0 \quad , \quad G^\mu \equiv \partial_\nu E^{\nu\mu} + \varepsilon^\mu{}_{\kappa\lambda}A_\nu{}^\kappa E^{\nu\lambda} = 0 \quad , \quad I_\mu \equiv E^\kappa{}_\nu F_{\mu\kappa}{}^\nu = 0$$

boundary condition at $r\to\infty$ $E^{\mu\kappa}E^\nu{}_\kappa = g^{\mu\nu}/(-\det(g))^{3/4}$, with GR metric g

with equation-of-motion: $\dfrac{d^2x^\kappa}{d\tau^2} + \Gamma^\kappa{}_{\mu\nu}\dfrac{dx^\mu}{d\tau}\dfrac{dx^\nu}{d\tau} = 0$ with Christoffel symbols $\Gamma^\kappa{}_{\mu\nu} = \Gamma^\kappa{}_{\mu\nu}\left((g_{AK})_{\mu\nu}\right)$ from the AK-generated metric $\left(E\,\eta\,E^t\right)^{-1}\det(E)^{3/8} = g_{AK}$

•quantum region $r<<r_{gr}$ $\Lambda<<1$ with gravitational self-interaction

field equations: AK equations

$$H_{(\mu,\nu)}{}^\kappa \equiv F_{\mu\nu}{}^\kappa + \frac{\Lambda}{3}\varepsilon_{\mu\nu\rho}E^{\rho\kappa} = 0 \quad , \quad G^\mu \equiv \partial_\nu E^{\nu\mu} + \varepsilon^\mu{}_{\kappa\lambda}A_\nu{}^\kappa E^{\nu\lambda} = 0 \quad , \quad I_\mu \equiv E^\kappa{}_\nu F_{\mu\kappa}{}^\nu = 0$$

boundary condition at $r = r_{gr}$ $E^{\mu\kappa}E^\nu{}_\kappa = g^{\mu\nu}/(-\det(g))^{3/4}$, with Minkowski metric $g=\eta$

equation-of-motion: Dirac equation $\left(i\hbar\gamma^\mu\bullet D_\mu - mc\right)\Psi = 0$, 4x4-matrix multiplication •,

covariant derivative (matrix) is $(D_\mu)^\lambda{}_\kappa = \partial_\mu + (\varepsilon_a)^\lambda{}_\kappa\sqrt{\alpha_{gr}}A_\mu{}^a$ interaction constant $\sqrt{\alpha_{gr}} = 1.75\times10^{-11}$

In the transition region we have the quasi-classical approximation $E_{gr}(r) = \dfrac{r_\Lambda}{r}mc^2$ for $r \ll r_{gr}$

$r_\Lambda = 3.7\times10^{-49}\,m$ is the quasi-classical gravitational interaction radius

•quantum region $r<<r_{gr}$ with external gravitational field

equation-of-motion: Dirac equation $\left(i\hbar\gamma^\mu g_{\mu\nu}\partial_\nu - mc\right)\Psi = 0$ with external GR-metric g,

in weak-field approximation $\left(i\hbar\gamma^\mu\partial_\mu - mc\dfrac{r_s}{r} - mc\right)\Psi = 0$, with the Newtonian potential energy $V_N(r) = mc^2\dfrac{r_s}{r}$ $r_s = \dfrac{2MG}{c^2}$

Figure 6. Schematic classical and quantm region.

Now, we calculate it in detail.

Similarly as in chap. 6.1, we make an ansatz for the A-tensor

$$\left(A_{0i}\left(r,\theta\right)\right)=\left(1,1,-1,1\right)A_{00c}+\left(dA_{0i}\left(r,\theta\right)\right)$$
$$\left(A_{1i}\left(r,\theta\right)\right)=\left(1,1,-1,1\right)A_{10c}+\left(dA_{1i}\left(r,\theta\right)\right)$$
$$\left(A_{2i}\left(r,\theta\right)\right)=\left(1,1,-1,1\right)A_{20c}+\left(dA_{2i}\left(r,\theta\right)\right)$$
$$\left(A_{3i}\left(r,\theta\right)\right)=\left(1,1,-1,1\right)A_{30c}+\left(dA_{3i}\left(r,\theta\right)\right)$$

with corrections $dA_{ij}\left(r,\theta\right)$ and $dE_{ij}\left(r,\theta\right)$ for the A-tensor and the E-tensor, and insert it into the static AK equations *eqtoiv*.

The result is in schematic form ([32] P1s2s10): the hamiltonian AK equations yield relations between the A-tensor and the E-tensor

$$H^{(1\ldots4)}=\Lambda\left(\frac{\partial dA_{0i}\left(r,\theta\right)}{\partial r}-\frac{1}{3}\left(E_{2i}\left(r,\theta\right)+E_{3i}\left(r,\theta\right)\right)\right)\quad i=\left(0,1,2,3\right)$$
$$H^{(5\ldots8)}=\Lambda\left(\frac{\partial dA_{2i}\left(r,\theta\right)}{\partial r}-\frac{1}{3}\left(E_{0i}\left(r,\theta\right)-E_{3i}\left(r,\theta\right)\right)\right)\quad i=\left(0,1,2,3\right)$$
$$H^{(17\ldots20)}=\Lambda\left(\frac{\partial dA_{3i}\left(r,\theta\right)}{\partial r}-\frac{1}{3}\left(E_{0i}\left(r,\theta\right)+E_{2i}\left(r,\theta\right)\right)\right)\quad i=\left(0,1,2,3\right)$$

the Gauss AK equations yield relations

$$G^{(1\ldots4)}=\frac{1}{r}\frac{\partial dE_{2i}\left(r,\theta\right)}{\partial\theta}+\frac{\partial dE_{1i}\left(r,\theta\right)}{\partial r}+\Lambda C_i\left(E_{kl},dA_{kl}\right)\quad i=\left(0,1,2,3\right)$$

with terms $C_i\left(E_{kl},A_{kl}\right)=\sum_{k_1,k_2,l_1,l_2}c_{ik_1k_2l_1l_2}E_{k_1k_2}dA_{l_1l_2}$ and other equations are of order Λ and can be neglected.

The correct value for the E-tensor in the hamiltonian AK equations would be $E=E_{GS}$, but we insert the approximation $E=E_{dS}$ diagonal Schwarzschild tetrad in order to obtain a closed solution.

$$E_{dS}=\begin{pmatrix}\frac{1}{\sin(\theta)^{3/4}r^{3/2}\sqrt{1-1/r}} & 0 & 0 & 0\\ 0 & \frac{\sqrt{1-1/r}}{\sin(\theta)^{3/4}r^{3/2}} & 0 & 0\\ 0 & 0 & \frac{1}{\sin(\theta)^{3/4}r^{5/2}} & 0\\ 0 & 0 & 0 & \frac{1}{\sin(\theta)^{7/4}r^{5/2}}\end{pmatrix}$$

We can solve the hamiltonian AK equations in the form

$$dA_{0i}\left(r,\theta\right)=\int\frac{1}{3}\left(E_{2i}\left(r,\theta\right)+E_{3i}\left(r,\theta\right)\right)dr\quad i=\left(0,1,2,3\right)$$
$$dA_{1i}\left(r,\theta\right)=0\quad i=\left(0,1,2,3\right)$$
$$dA_{2i}\left(r,\theta\right)=\int\frac{1}{3}\left(E_{0i}\left(r,\theta\right)-E_{3i}\left(r,\theta\right)\right)dr\quad i=\left(0,1,2,3\right)$$

$$dA_{3i}(r,\theta)=\int\frac{1}{3}\left(E_{0i}(r,\theta)+E_{2i}(r,\theta)\right)dr \quad i=(0,1,2,3)$$

then setting $dE_{2i}(r,\theta)=-dE_{1i}(r,\theta)$ and inserting $dA_{ij}(r,\theta)$, we obtain the Gauss AK equations for $dE_{1i}(r,\theta)$

$$0=-\frac{1}{r}\frac{\partial dE_{1i}(r,\theta)}{\partial\theta}+\frac{\partial dE_{1i}(r,\theta)}{\partial r}+\Lambda C_i\left(\left(E_{dS}\right)_{kl},dA_{kl}\right) \quad i=(0,1,2,3)$$

These equations can be solved in the form of an integral ([32] P1s2s10) in schematic form

$$dE_{1i}(r,\theta)=\int_1^r f_i\left(Hypergeometric2F1\left(1/2,7/2,3/2,\cos(\theta+\log r-\log x)^2\right),\cos(\theta+\log r-\log x)\right)dx$$

The integrals are solved numerically on a lattice, and made analytic by interpolation.

The 3d-plot for -*dE*10 is (**Figure 7**)

The four functions have the approximate form

$$dE_{10}(r,\theta)=-10/r,\quad dE_{11}(r,\theta)=-32/r,\quad dE_{12}(r,\theta)=-12/r,\quad dE_{10}(r,\theta)=6/r$$

With this result, we obtain the Schwarzschild metric correction

$$g_{11c}(r,\theta)\approx g_{11}(r,\theta)\left(1+2\Lambda dE_{11}(r,\theta)\right)\approx g_{11}(r,\theta)\left(1-64\Lambda/r\right)$$

i.e. the AK correction length is $l_0\approx 64\Lambda r_s^3$ (29)

7. Solutions of Time-Dependent Equations

7.1. The Λ-Scaled Wave Ansatz for the A-Tensor

The covariant derivative of the AK-gravitation is

$$D_\mu t_\nu^{\ \lambda}=\partial_\mu t_\nu^{\ \lambda}+\varepsilon^\lambda_{\ \kappa_1\kappa_2}A_\mu^{\ \kappa_1}t^{\nu\kappa_2} \quad D_\mu^{\ \lambda}=\partial_\mu+\varepsilon^\lambda_{\ \kappa_1\bullet}A_\mu^{\ \kappa_1}$$

The gaussian equations have the form of the covariant derivative acting on the E-tensor

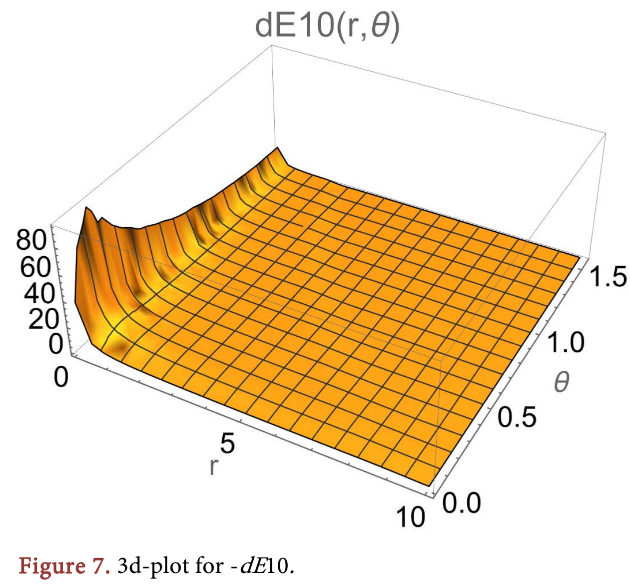


Figure 7. 3d-plot for -*dE*10.

$$G^{\mu} = D_{\nu} E^{\nu\mu} = \partial_{\nu} E^{\nu\mu} + \varepsilon^{\mu}{}_{\kappa\lambda} A_{\nu}^{\kappa} E^{\nu\lambda}$$

One can show, that the second term in the covariant derivative cancels out only if the A-tensor vanishes, *i.e.* the covariant derivative is not background-independent.

Now, if we separate the static background and the wave component in the A-tensor:

$$A = A_{bg} + A_{wave}, \quad E = E_{bg} + E_{wave}$$

we have to take account of the fact that in GR the gravitational wave interacts weakly with the metric, because it interacts through the energy tensor, which appears on the right side of the Einstein equations with the small factor κ [9] [30] [31]:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R_0 + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

Therefore, classically, we have to use some power of Λ as the factor in the Λ-*scaled wave ansatz* above, setting $c = 1$ (see **Figure 8**).

$A_{\mu}^{\ \nu} = Ab_{\mu}^{\ \nu} + \Lambda^{p} \frac{As_{\mu}^{\ \nu}}{r} \exp\left(-ik\left(r-t\right)\right)$, where Ab is the (static) background, As is the wave amplitude

Λ-scaled wave ansatz

$$A_{\mu}^{\ \nu} = Ab_{\mu}^{\ \nu} + \Lambda \frac{As_{\mu}^{\ \nu}}{r} \exp(-ik(r-t))$$

$$E^{\mu\nu} = Eb^{\mu\nu} + \frac{Es^{\mu\nu}}{r} \exp(-ik(r-t))$$

→

dimensions: [A]= 1/cm [E]=1 [As]= cm^2

eqtoiev → static & wave-equations
eqtoeivnu3b={eqham(Ab,∂Ab,Eb), eqgaus (Ab,Eb,∂Eb), eqdiff(Ab, ∂Ab,Eb)}
eqtoeivnu3w={ Λ eqham(As,∂As,Es,Ab), eqgaus (As,Es,∂Es,Ab), eqdiff(As,∂As,Es,Ab,∂Ab)}
linear in As,Es : superposition principle

classical case *Λ≈0* $r >> r_{gr}$
eqtoeivnu3b={eqham(Ab,∂Ab), eqgaus (Ab,Eb,∂Eb), eqdiff(Ab, ∂Ab,Eb)}
eqtoeivnu3w={ Λ eqham(As,∂As,Es,Ab), eqgaus (Es,∂Es,Ab), eqdiff(Es,Ab,∂Ab)}

$Ab \cong \frac{1}{l_P}$ makes EH-action $S = \frac{\hbar c}{\pi} \int (A_{\mu}^{\ \nu} A_{\nu}^{\ \mu}) R \sqrt{-g} d^4 x$ dimensionally renormalizable

metric condition for *Eb* for all *r*

$$Eb \bullet \eta \bullet Eb' = I(g_{\mu\nu}) = g^{-1} / (-\det(g))^{3/4}$$

quantum case $r << r_{gr}$
Λ not zero *Λ<<1* , Ab<<1 A≈As=pure wave graviton
interaction via $(D_{\mu})^{\lambda}{}_{\kappa} = \partial_{\mu} + (\varepsilon_{a})^{\lambda}{}_{\kappa} \sqrt{\alpha_{gr}} (A_{p})_{\mu}^{\ a}$
metric= Minkowski metric
metric condition for *Eb* for *r*→∞ Minkowski *g*=*η* : *Eb=Gauss-Minkowski tetrade*

eqtoeivnu3b={eqham(Ab,∂Ab ,Eb), eqgaus (Ab,Eb,∂Eb), eqdiff(Ab,∂Ab,Eb)}
eqtoeivnu3w={ Λ eqham(As,∂As,Es,Ab), eqgaus (Es,∂Es,Ab), eqdiff(Es,Ab,∂Ab)}

Figure 8. Schematic Λ-scaled wave ansatz.

In order to make As interact with E-tensor in the hamiltonian equations, we have to set $p = 1$, the ansatz becomes (Λ-scaled ansatz for the A-tensor), and correspondingly for the E-tensor:

$$A_\mu{}^\nu = Ab_\mu{}^\nu + \Lambda \frac{As_\mu{}^\nu}{r} \exp\left(-i\,k\left(r-t\right)\right) \tag{30}$$

$$E^{\mu\nu} = Eb^{\mu\nu} + \frac{Es^{\mu\nu}}{r} \exp\left(-i\,k\left(r-t\right)\right)$$

This has some remarkable consequences: in the hamiltonian equations we now have the background part of order 1 for Ab and Eb and the wave part of order Λ for As, and Es.

In the A-tensor and the E-tensor we now have the background part Ab and Eb and the wave part As, and Es.

We insert this into the AK-equations, and separate the static part $eqtoeivnu3b$ in the schematic form

$$eqtoeivnu3b = \{eqham(Ab, \partial Ab, Eb), eqgaus(Ab, Eb, \partial Eb), eqdiff(Ab, \partial Ab, Eb)\}$$

and the wave part $eqtoeivnu3w$ after stripping the wave factor $\exp\left(-i\,k\left(r-t\right)\right)$ in the schematic form

$$eqtoeivnu3w = \{\Lambda eqham(As, \partial As, Es, Ab), \\ eqgaus(Es, \partial Es, Ab), eqdiff(Es, Ab, \partial Ab)\}$$

where $\partial = \{\partial_r, \partial_\theta\}$ ist the differential operator for r and θ.

The wave part is *linear* in As, ∂As, Es, *i.e.* the *superposition principle is valid* for gravitational waves.

As the dimensions are $[A] = [1/r] = 1/\text{cm}$ and $[E] = 1$, we get for the A-amplitude the dimension $[As] = [r^2] = cm^2$ *i.e.* As becomes a cross-section, which is a sensible interpretation in the quantum limit.

In the quantum limit $r < r_{gr}$, the graviton interacts via the covariant derivative, like the photon, and the metric condition for Eb is for the flat Minkowski metric $\Lambda \neq 0$, the Einstein equation and the general covariance are not valid anymore.

In the classical case $\Lambda \approx 0$, the AK-equations separate into the background part for Ab, Eb and the wave-part with the wave factor $\exp\left(-i\,k\left(r-t\right)\right)$ for Es, As, Ab.

$$eqtoeivnu3b = \{eqham(Ab, \partial Ab), eqgaus(Ab, Eb, \partial Eb), eqdiff(Ab, \partial Ab, Eb)\} \tag{31a}$$

$$eqtoeivnu3w = \{\Lambda eqham(As, \partial As, Es, Ab), eqgaus(Es, \partial Es, Ab), eqdiff(Es, Ab, \partial Ab)\} \tag{31b}$$

The background part $eqtoeivnu3b$ has the *metric condition at infinity*, not everywhere, as in the static case. Then other solutions $\{Ab, Eb\}$, other the trivial constant half-antisymmetric background A_{hab} are possible and these influence via Eb the wave part equation $eqtoeivnu3w$: this describes the *interaction of the wave with matter*.

The general solution of $eqtoeivnu3b$ in closed form is not available, but we can get an approximate solution with the ansatz

$$Ab = M_{Ac} + dM_{Abg} \tag{32a}$$

$$Eb = E_{GS,0} + dE_b(r,\theta) \tag{32b}$$

where the constant half-asymmetric background is

$$M_{Ac} = \begin{pmatrix} A_{00c}(1,1,-1,1) \\ A_{10c}(1,1,-1,1) \\ A_{20c}(1,1,-1,1) \\ A_{30c}(1,1,-1,1) \end{pmatrix}$$

and the r-dependent correction

$$dM_{Abg} = \begin{pmatrix} \left(dA_{b00}(r,\theta)/r, dA_{b01}(r,\theta)/r, dA_{b02}(r,\theta)/r, dA_{b03}(r,\theta)/r\right) \\ \left(dA_{b10}(r,\theta)/r, dA_{b11}(r,\theta)/r, dA_{b12}(r,\theta)/r, dA_{b13}(r,\theta)/r\right) \\ dA_{b20}(r,\theta)/r(1,1,-1,1) \\ dA_{b30}(r,\theta)/r(1,1,-1,1) \end{pmatrix}$$

with a general 4 × 4 correction matrix $dEb(r, \theta)$ and with a simplified Gauss-Schwarzschild tetrad [34]

$$E_{GS0} = \begin{pmatrix} \left(\frac{1}{r^{3/2}\sin(\theta)^{3/4}}, 0, 0, 0\right) \\ \left(0, \frac{2E_{21n3}'(\theta)}{3r^{3/2}}, \frac{2E_{21n3}(\theta)E_{21n3}'(\theta)}{3r^{3/2}\sqrt{1-E_{21n3}(\theta)^2}}, 0\right) \\ \left(0, \frac{E_{21n3}(\theta)}{r^{3/2}}, \frac{\sqrt{1-E_{21n3}(\theta)^2}}{r^{3/2}}, \frac{i}{r^{3/2}}\right) \\ \left(0, -\frac{iE_{21n3}(\theta)}{r^{3/2}\sin(\theta)}, -\frac{i\sqrt{1-E_{21n3}(\theta)^2}}{r^{3/2}\sin(\theta)}, \frac{1}{r^{3/2}\sin(\theta)}\right) \end{pmatrix}$$

As we shall see below, the wave As carries the wave energy, and induces locally a tetrad (metric) wave, which is damped exponentially. The gravitational wave energy tensor depends on the wave amplitude As in a similar way as the electromagnetic wave energy depends on the photon vector A_μ. Also, it satisfies the Einstein power formula for the gravitational wave.

7.2. Wave Equation in Schwarzschild Spacetime

In this chapter we derive the gravitational wave equation (**Figure 9**) for the gravitational wave tensor As and the wave tetrad tensor Es, for detailed description see [34].

As described in chap. 7.1., we introduce the Λ-scaled wave ansatz for the A-tensor, and correspondingly for the E-tensor

$$A_\mu{}^\nu(t,r,\theta) = Ab_\mu{}^\nu(r,\theta) + \Lambda\frac{As_\mu{}^\nu(r,\theta)}{r}\exp(-ik(r-t)) \tag{33}$$

$$E^{\mu\nu}(t,r,\theta) = Eb^{\mu\nu}(r,\theta) + \frac{Es^{\mu\nu}(r,\theta)}{r}\exp(-ik(r-t))$$

eqtoiev Λ-scaled wave ansatz
backgrund equation eqtoeivnu3b=eqtoiv
standard solution:
A-tensor = constant background in the half-antisymmetric form
$A0_i = A00c\ \{1,1,-1,1\}$, $A1_i = A10c\ \{1,1,-1,1\}$, $A2_i = A20c\ \{1,1,-1,1\}$, $A3_i = A30c\ \{1,1,-1,1\}$
E-tensor= the Gauss-Schwarzschild-tetrad E_{GS} .

resulting wave equation is linear in $(As, \partial As, Es, \partial Es)$
eqtoeivnu4w={ eqham(As,∂As,Es,∂Es),
eqgaus (Es0i, Es1i,∂Es1i,,∂Es2i), eqdiff=0}
eliminate *(Es0,Es3,As1)*
multipole ansatz *Es(r, θ)= Es(r)exp(i*lx*θ)* , *As(r, θ)= As(r)exp(i*lx*θ)*
eliminate *Es2* and get the gravitational wave equation for *Es1*

gravitational wave equation for E-tensor (*lx*=angular momentum)
eqgravlxEn =

$$r\begin{pmatrix} fs'(r)\left(lx\left(-6k^2r^2+3i\,kr+1\right)+kr\left(-2k^2r^2+i\,kr+2\right)+lx^2\left(-5kr+2i\right)-lx^3\right) \\ +r\left(r\,fs'''(r)\left(kr+lx\right)-i\,fs''(r)\left(5kr\,lx+kr\left(3kr-i\right)+2\,lx^2\right)\right)\end{pmatrix}$$

$$+fs(r)\begin{pmatrix} lx^2\left(3\,i\,k^2r^2+4\,kr-2\,i\right)+lx\left(2\,i\,k^3r^3+4\,k^2r^2-5i\,kr-1\right) \\ +lx^3\left(1+i\,kr\right)-kr\left(2+i\,kr\right)\end{pmatrix}$$

at infinity *eqgravlxEninf*=

$$2ik^2lx\,fs(r)-2k^2r\,fs'(r)-3ikr\,fs''(r)+r\,fs'''(r)$$

solution at infinity: $fs(r)=c_0+c_1\exp(ikr)+c_2\exp(2ikr)$, where the only feasible solution is
$fs(r)=c_0$, i.e. the total solution at infinity is the spherical wave.
For comparison, the radial (electromagnetic) wave equation for the wave factor *fs(r)* from the ansatz

$$fs(t,r,\theta)=\frac{fs(r)}{r}Y_{lx,m}(\theta,\varphi)\exp(-ik(r-t))$$

Helmholtzwr=

$$-lx(1+lx)fs(r)-2ikr^2 fs'(r)+r^2 fs''(r)$$

solution at infinity: $fs(r)=c_0+c_2\exp(2ikr)$, where the only feasible solution is
$fs(r)=c_0$, i.e. the total solution at infinity is the spherical wave.
The gravitational and the electromagnetic wave equation are equivalent at infinity.

In addition, we get the gravitational wave equations for the A-tensor variables *As00* , *As30* depending on *Es10*

$$eqgravlxA0=3(1+i\,lx)\left(lx+kr\right)^2 As00\left(r\right)-r\begin{pmatrix}\left(-1+lx^2+2lx\left(-i+kr\right)\right)Es10\left(r\right) \\ +3\left(lx+kr\right)^2 As00'\left(r\right) \\ +r\left(\left(1+2i\,lx+2i\,kr\right)Es10'\left(r\right)-r\,Es10''\left(r\right)\right)\end{pmatrix}$$

$$eqgravlxA3=\begin{matrix}6k\,lx(1+i\,kr)As30\left(r\right)+\left(1+i\,kr-2k^2r^2+lx\left(i-kr\right)\right)Es10\left(r\right) \\ +r\left(-6k\,lx\,As30'\left(r\right)-i\left(-i+lx+3\,kr\right)Es10'\left(r\right)+r\,Es10''\left(r\right)\right)\end{matrix}$$

Figure 9. Schematic gravitational wave equation.

or schematically:

$$A_i = const. + \Lambda * sphwave * As_i + Ab_i\,, \quad E_i = Eb_i + sphwave * Es_i$$

In the A-tensor and the E-tensor we now have the background part *Ab* and *Eb* and the wave part *As*, and *Es*.

We insert this into the AK-equations, let Λ→0, and separate the static part *eqtoeivnu3b* in the schematic form

$$eqtoeivnu3b = \{eqham(Ab),\ eqgaus(Ab, Eb),\ eqdiff(Ab, Eb)\}$$

and the wave part *eqtoeivnu3w* after stripping the wave factor $\exp\left(-ik\left(r-t\right)\right)$ in the schematic form

$$\begin{gathered} eqtoeivnu3w = \{\Lambda eqham(As, \partial As, Es, \partial Es, Ab), \\ eqgaus(Ab, Es, \partial Es),\ eqdiff(Es, \partial Es, Ab)\} \end{gathered}$$

where $\partial = \{\partial_r, \partial_\theta\}$ ist the differential operator for r and θ.

eqtoeivnu3b is identical with *eqtoiv* the static AK-equations, and the solution is as in 3.1.

for the A-tensor the *constant background* in the *half-antisymmetric form*

$$\begin{gathered} A0_i = A00c\{1,1,-1,1\}\,, \quad A1_i = A10c\{1,1,-1,1\}\,, \\ A2_i = A20c\{1,1,-1,1\}\,, \quad A3_i = A30c\{1,1,-1,1\} \end{gathered}$$

and for the E-tensor the Gauss-Schwarzschild-tetrad E_{GS}.

Now we fix the angular momentum of the wave by setting

$$Es1i(r, \theta) = Es1i(r)exp(i * lx * \theta)$$

and correspondingly for $\{Es2i, As2i, As3i, As0i\}$,

where $lx = 0, 1, 2, \ldots$ is the angular momentum of the wave: $lx = 0$ for a spherical wave, $lx = 1$ for a dipole wave, $lx = 2$ for a quadrupole wave .

In GR one can show that the gravitational wave must be at least quadrupole waves, there are no spherical and dipole waves.

We get the **gravitational wave equation** for the variable $Es10 = fs$,

$$\begin{aligned} &eqgravlxEn \\ &= r\Big(fs'(r)\Big(lx\left(-6k^2r^2+3ikr+1\right)+kr\left(-2k^2r^2+ikr+2\right)+lx^2\left(-5kr+2i\right)-lx^3\Big) \\ &\quad + r\Big(r\, fs'''(r)\left(kr+lx\right)-i\, fs''(r)\Big(5kr\, lx+kr\left(3kr-i\right)+2lx^2\Big)\Big)\Big) \\ &\quad + fs(r)\Big(lx^2\left(3ik^2r^2+4kr-2i\right)+lx\left(2ik^3r^3+4k^2r^2-5ikr-1\right) \\ &\quad + lx^3\left(1+ikr\right)-kr\left(2+ikr\right)\Big) \end{aligned} \tag{34a}$$

At infinity

$$eqgravlxEninf = 2ik^2lx\, fs(r)-2k^2r\, fs'(r)-3ikr\, fs''(r)+r\, fs'''(r) \tag{34b}$$

For bounded polynomial $fs(r)$ we can neglect the first term and we obtain the equation $-2k^2r\, fs'(r)-3ikr\, fs''(r)+r\, fs'''(r)=0$, which has the solution $fs(r)=c_0+c_1\exp(ikr)+c_2\exp(2ikr)$, and c_2 must be zero, because otherwise we would get an incoming wave, and c_1 must also be zero, because otherwise we would get a simple oscillation, so the wave factor $fs(r)=c_0$ and we have a spherical wave as the only solution at infinity.

For comparison, the radial (electromagnetic) Helmholtz wave equation for the wave factor $fs(r)$ from the ansatz

$$fs(t,r,\theta)=\frac{fs(r)}{r}Y_{lx,m}(\theta,\varphi)\exp(-ik(r-t))$$
$$Helmholtzwr = \tag{35}$$
$$-lx(1+lx)\,fs(r)-2ikr^2\,fs'(r)+r^2\,fs''(r)$$

with the solution $c_1\sqrt{r}\exp(ikr)J_{lx+1/2}(kr)+c_2\sqrt{r}\exp(ikr)Y_{lx+1/2}(kr)$, where J_l and Y_l are Bessel functions of the first and the second kind.

Solutions of the Gravitational Wave Equation

We derive the solutions for different values of angular momentum lx (Figure 10):

For $lx = 0$ (spherical wave) we get the solution for $Es10$:

$$Es10(r)=r\,exp(2ir)C1+r\,exp(2ir)ExpIntegralEi(-ir)C2$$

which has the limit at infinity:

$$Es10(r)\to\frac{C1exp(2ir)(-2+ir+r^2)}{\pi r^2}$$

The factor e^{2ir} generates an incoming wave, which is not feasible, therefore $C1 = 0$ and $Es10 = 0$, and consequently $As20 = As30 = Es20 = 0$, there is only the zero solution: there are no spherical gravitational waves.

For $lx = 1$ (dipole wave) we get the solution for $Es10$:

$$Es10(r)=\frac{2}{3}ir\,exp(2ir)Hypergeometric1F1\left(1-i,2,\frac{2}{3}ir\right)C1$$
$$+MeijerG\left(((\ ),(1+i)),((0,1),(\ )),-\frac{2}{3}ir\right)C2$$

with hypergeometric and Meijer functions, the limit at infinity is in Mathematica notation

$$Series\left(r\,Hypergeometric1F1\left(1-i,2,\frac{2}{3}ir\right)\right)$$
$$=r^{-i}\left(exp\left(\frac{2ir}{3}\right)\left(\frac{(2i/3)^{-(1+i)}}{Gamma(1-i)}+\frac{(1+i)i^{-(1+i)}2^{-(2+i)}3^{(2+i)}}{r\,Gamma(1-i)}+O\left(\frac{1}{r^2}\right)\right)\right)$$
$$+r^{2i}\left(\frac{(-2i/3)^{-(1+i)}}{Gamma(1-i)}+\frac{(1+i)i^{-(1+i)}2^{-(2+i)}3^{(2+i)}}{r\,Gamma(1-i)}+O\left(\frac{1}{r^2}\right)\right)$$

$$Series\left(MeijerG\left(((\ ),(1+i)),((0,1),(\ )),-\frac{2}{3}ir\right)\right)$$
$$=r^{-i}exp\left(\frac{2ir}{3}-\frac{\pi}{2}\right)\left((2/3)^{-i}+\frac{(1+i)2^{-(1+i)}}{r}+O\left(\frac{1}{r^2}\right)\right)$$

which diverges, therefore $Es10 = 0$, and, as before, there is only the zero solution: there are no dipole gravitational waves.

solution lx=0 spherical wave:

$$Es10(r) = r\,exp(2ir)C1 + r\,exp(2ir)\,ExpIntegralEi(-ir)C2$$

generates an incoming wave, which is not feasible, therefore *C1=0* and *Es10=0*,
solution lx=1 dipole wave:

$$Es10(r) = \frac{2}{3} i r\,exp(2ir)\,Hypergeometric1F1\left(1-i,2,\frac{2}{3} i r\right)C1$$

$$+MeijerG\left(\left((\ ),(1+i)\right),\left((0,1),(\ )\right),-\frac{2}{3} i r\right)C2$$

diverges, therefore *Es10=0*

solution lx=2 quadrupole wave:

$$\mathrm{Re}(Es10(r)) = C3 - \frac{i r^3}{3\sqrt{2}} HypergeometricPFQ\left(\left(\frac{3}{2}\right),\left(2,\frac{5}{2}\right),\frac{r^2}{2}\right)C1$$

$$+\frac{r^2}{2} MeijerG\left(\left((0),(-1)\right),\left(\left(-\frac{1}{2},\frac{1}{2}\right),(-1,-1)\right),-\frac{ir}{2},\frac{1}{2}\right)C2$$

$$\mathrm{Im}(Es10(r)) = -2\left(6^{1/3} r^{2/3}\right) BesselI\left(-\frac{4}{3},\frac{4\sqrt{r}}{\sqrt{3}}\right) Gamma\left(\frac{2}{3}\right)C1$$

$$+\frac{8}{3}\frac{2^{1/3}}{3^{2/3}} r^{2/3} BesselI\left(\frac{4}{3},\frac{4\sqrt{r}}{\sqrt{3}}\right) Gamma\left(\frac{4}{3}\right)C2$$

at infinity; $Es10(r) = i r^{5/12} exp\left(-\frac{4\sqrt{r}}{\sqrt{3}}\right)C1$ exponentially damped

$As20(r,\theta) = \frac{As20c}{r} exp(2i\theta)$ linearly damped, $As00(r,\theta) = -\frac{As20c}{2+r}\left(1+\frac{r}{2}\right) exp(2i\theta)$

quadrupole wave amplitude $\frac{As20c}{2}$, $As10(r,\theta) = -\frac{As20c}{2r}(-i+r) exp(2i\theta)$, again a quadrupole wave

$As30(r) = i\ exp\left(-\frac{4\sqrt{r}}{\sqrt{3}}\right) C1 \frac{-49-64\sqrt{3}\sqrt{r}+96r+288r^2}{1728 r^{7/12}}$ exponentially damped wave

The overall result is:

-the E-tensor is exponentially damped with $\exp(-\frac{4\sqrt{r}}{\sqrt{3}})$

-the A-tensor components *As0i* and *As1i* are pure quadrupole waves, *As2i* is a linearly damped quadrupole wave,

As3i is exponentially damped with $\exp(-\frac{4\sqrt{r}}{\sqrt{3}})$

This means that a classical wave source generates gravitational waves *As* via the metric , the energy is carried away by the As-tensor and , when the wave is absorbed, it dissipates energy and generates again a (locally damped) metric oscillation *Es* .

Figure 10. Schematic solution of gravitational wave equation.

For $lx = 2$ (quadrupole wave) we get the solution for the real part $Re(Es10)$:

$$\mathrm{Re}(Es10(r)) = C3 - \frac{i r^3}{3\sqrt{2}} HypergeometricPFQ\left(\left(\frac{3}{2}\right),\left(2,\frac{5}{2}\right),\frac{r^2}{2}\right)C1$$

$$+\frac{r^2}{2} MeijerG\left(\left((0),(-1)\right),\left(\left(-\frac{1}{2},\frac{1}{2}\right),(-1,-1)\right),-\frac{ir}{2},\frac{1}{2}\right)C2$$

and for the imaginary part $Im(Es10)$:

$$\operatorname{Im}\left(Es10(r)\right) = -2\left(6^{1/3}\, r^{2/3}\right) BesselI\left(-\frac{4}{3}, \frac{4\sqrt{r}}{\sqrt{3}}\right) Gamma\left(\frac{2}{3}\right) C1 + \frac{8}{3}\frac{2^{1/3}}{3^{2/3}} r^{2/3} BesselI\left(\frac{4}{3}, \frac{4\sqrt{r}}{\sqrt{3}}\right) Gamma\left(\frac{4}{3}\right) C2$$

calculation of the limit at infinity yields

$$Es10(r) = i\, r^{5/15}\, exp\left(-\frac{4\sqrt{r}}{\sqrt{3}}\right) C1 \tag{36a}$$

i.e. $Es10$ is purely imaginary and exponentially damped with $\exp\left(-\frac{4\sqrt{r}}{\sqrt{3}}\right)$, the same is valid for $Es20$, for $As20$ we get

$$As20(r,\theta) = \frac{As20c}{r} exp(2i\theta) \tag{36b}$$

i.e. a linearly damped quadrupole wave, for $As00$ we get

$$As00(r,\theta) = -\frac{As20c}{2+r}\left(1+\frac{r}{2}\right) exp(2i\theta) \tag{36c}$$

i.e. $As00$ is a quadrupole wave with the amplitude $\frac{As20c}{2}$, for $As10$ we get

$$As10(r,\theta) = -\frac{As20c}{2r}(-i+r)\, exp(2i\theta) \tag{36d}$$

again a quadrupole wave, for $As30$ we get an exponentially damped wave again:

$$As30(r) = i\, exp\left(-\frac{4\sqrt{r}}{\sqrt{3}}\right) C1 \frac{-49 - 64\sqrt{3}\sqrt{r} + 96r + 288r^2}{1728 r^{7/12}} \tag{36e}$$

The overall result is:

- The **E-tensor is exponentially damped** with $\exp\left(-\frac{4\sqrt{r}}{\sqrt{3}}\right)$.

- The **A-tensor components *As*0 and *As*1 are pure quadrupole waves**, $As2$ is a linearly damped quadrupole wave,

***As*3 is exponentially damped** with $\exp\left(-\frac{4\sqrt{r}}{\sqrt{3}}\right)$.

This means that a classical wave source generates gravitational waves As via the metric, the energy is carried away by the As-tensor and, when the wave is absorbed, it dissipates energy and generates again a (locally damped) metric oscillation Es.

7.3. Gravitational Waves in General Relativity

The gravitational waves in GR are metric waves, *i.e.* a disturbance of the metric tensor $g_{\mu\nu}$, for a plane wave [9]:

$$h_{\mu\nu} = e_{\mu\nu} exp\left(-i k_\lambda x^\lambda\right)$$

They satisfy the wave equation and, with the additional gauge condition

$$2\partial_\mu h^\mu{}_\nu = \partial_\nu h^\mu{}_\mu \tag{37a}$$

they satisfy the *linearized Einstein equations* for small amplitudes (but not the full Einstein equations)

$$\Box h_{\mu\nu} = \partial^\lambda \partial_\lambda h_{\mu\nu} = \frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2}\eta_{\mu\nu}\right) \tag{37b}$$

They can be transformed by coordinate transformations

$$x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x)$$

into the standard form for a plane wave in x-direction

$$\left(h_{\mu\nu}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_{11} & e_{12} \\ 0 & 0 & e_{12} & -e_{11} \end{pmatrix} \exp\left(ik\left(x^1 - ct\right)\right)$$

where $e_{12} = 0$ and $e_{11} = 0$ determine the two polarizations of a tensor (spin = 2) wave (**Figure 11**).

The Newtonian gravitation emerges from GR using the ansatz for the relative Newton potential energy of mass *m* in the field of a large mass *M*

$$\Phi_N = \frac{E_{pot}}{mc^2} = -\frac{MG_N}{r} = -\frac{r_s}{2r},$$

where r_s is the Schwarzschild radius of *M* and

$$\Phi = \sqrt{g_{00}} - 1 = \sqrt{1 - \frac{r_s}{r}} - 1 \approx -\frac{r_s}{2r} = \Phi_N$$

So a Newtonian wave caused by an oscillation of Φ_N is approximately the component h_{00} of a GR metric wave. Correspondingly, a small static distortion of the metric component g_{00} caused by a wave causes a *radial* force.

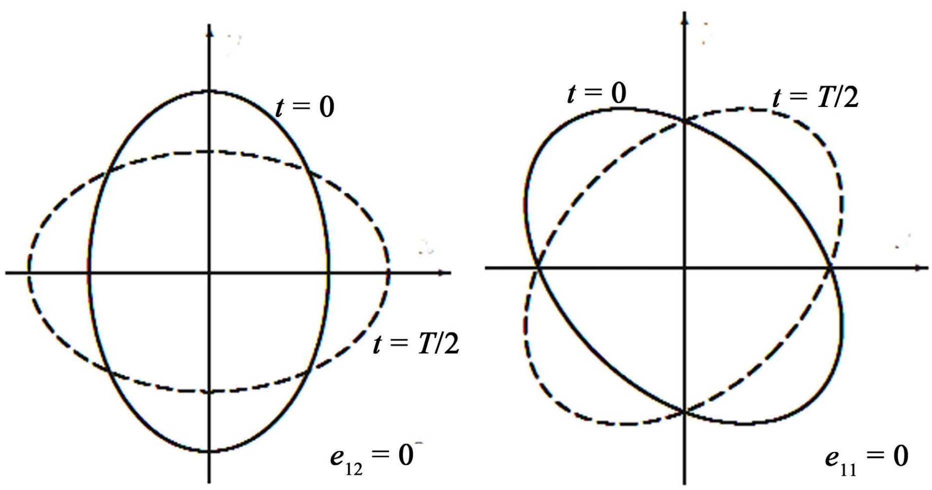


Figure 11. Polarizations of gravitational waves.

7.4. Spherical Gravitational Waves in AK-Gravity

Gravitational waves are quadrupolar (or of higher multipolarity), as was shown in chap. 7.3., and so are the resulting metric waves, in agreement with GR. Spherical gravitational waves (**Figure 12**) are a valid approximation for small intervals of the polar angle θ or large radii r, and planar waves an approximation for very large radii r and small θ.

We start, as usual, with the Λ-scaled wave ansatz:

$$A_\mu{}^\nu = \frac{dAb_\mu{}^\nu}{r} + \Lambda \frac{As_\mu{}^\nu}{r} \exp\left(-ik\left(r-t\right)\right)$$

$$E^{\mu\nu} = Eb_{GS}{}^{\mu\nu}\left(r,\theta\right) + \frac{dEb^{\mu\nu}\left(\theta\right)}{r^{3/2}} + \left(\frac{Es^{\mu\nu}}{r} + \frac{Er^{\mu\nu}}{r^2}\right)\exp\left(-ik\left(r-t\right)\right)$$

$Eb_{GM}{}^{\mu\nu}\left(r,\theta\right)$ is the Gauss-Minkowski tetrad, which represents the background metric of the (empty) Minkowski spacetime, $\frac{dEb^{\mu\nu}\left(\theta\right)}{r^{3/2}}$ is the change in the tetrad induced by the wave, *i.e.* the *interaction of the wave with matter*.

$As_\mu{}^\nu$ and $Es^{\mu\nu}$ are the (constant) wave factors of first order, and $Er^{\mu\nu}$ is the wave factor of the tetrad of second order.

The AK-equations separate into the background and the wave part

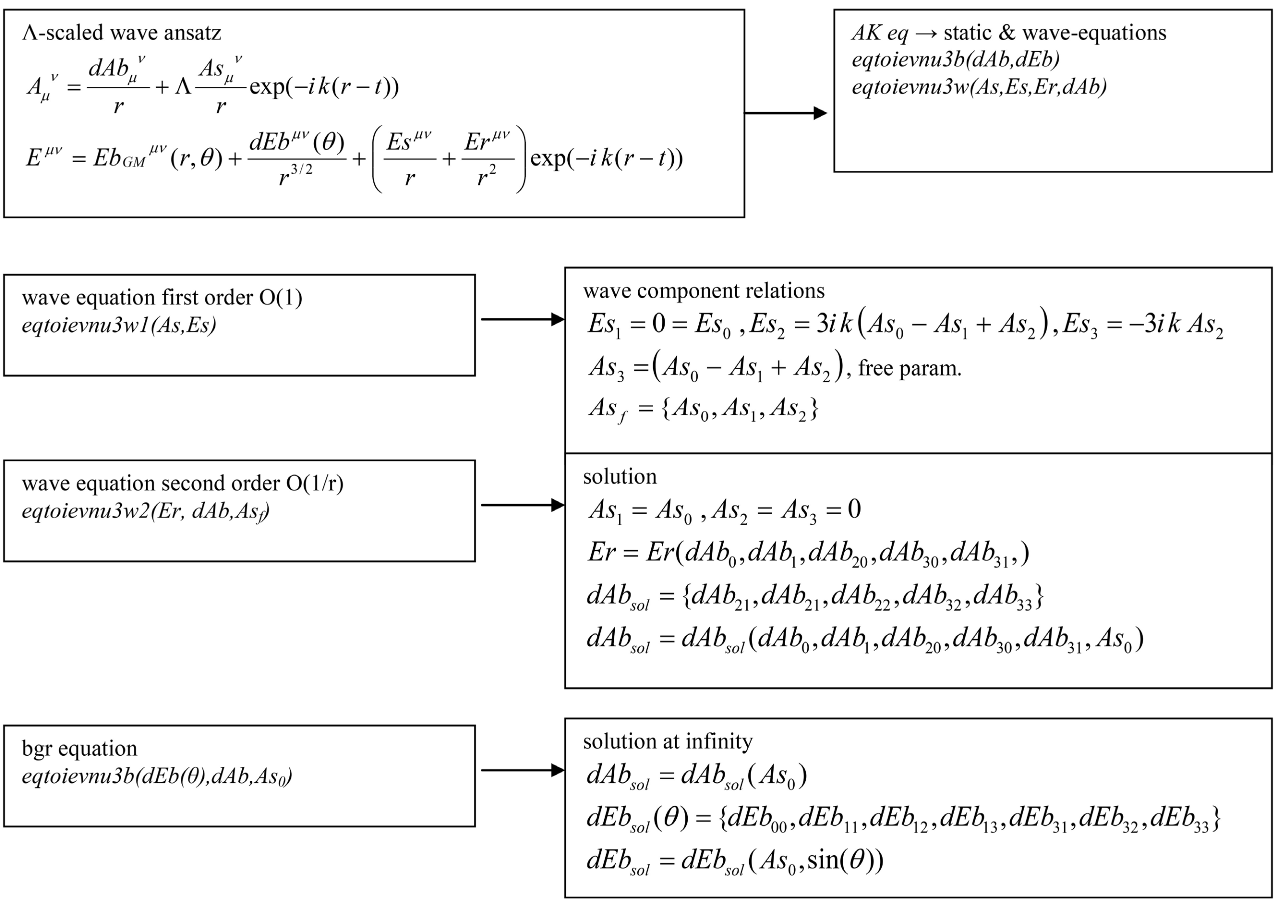


Figure 12. Schematic spherical gravitational waves.

$$eqtoievnu3b(dAb, dEb)$$

$$eqtoievnu3w(As, Es, Er, dAb)$$

The wave equation first order O(1), *i.e.* at infinity, is $eqtoeivnu3w1(As, Es)$ and the solution are the *wave component relations*

$$Es_1 = 0 = Es_0, Es_2 = 3ik\left(As_0 - As_1 + As_2\right), Es_3 = -3ik\,As_2$$

$As_3 = \left(As_0 - As_1 + As_2\right)$, with the 12 free parameters $As_f = \left\{As_0, As_1, As_2\right\}$

We insert the solution into the wave equation of second order O(1/r) and get

$$eqtoeivnu3w2(Er, dAb, As_f)$$

We know from the gravitational wave equation in chap. 7.3 that all Es are exponentially damped, so we get from the wave component relations the *vanishing tetrad condition* $As_1 = As_0, As_2 = As_3 = 0$, which reduces the number of parameters As to 4.

With this condition, $eqtoeivnu3w2$ has 28 equations for 16 Er and 16 dAb with the parameters As_f, we eliminate 5 dAb and all Er:

$$Er = Er\left(dAb_0, dAb_1, dAb_{20}, dAb_{30}, dAb_{31}\right)$$

$$dAb_{sol} = \left\{dAb_{21}, dAb_{21}, dAb_{22}, dAb_{32}, dAb_{33}\right\}$$

$$dAb_{sol} = dAb_{sol}\left(dAb_0, dAb_1, dAb_{20}, dAb_{30}, dAb_{31}, As_f\right)$$

The result is inserted into the background equation giving

$$eqtoeivnu3b(dEb(\theta), dAb, As_f)$$

It has 19 independent equations of order 4 in $dAb * As_f$ with the variables:

7 dEb $dEb_{sol}\left(\theta\right) = \left\{dEb_{00}, dEb_{11}, dEb_{12}, dEb_{13}, dEb_{31}, dEb_{32}, dEb_{33}\right\}$,

11 dAb, and the parameters As_f and $\{E21c, E21cs\}$ from $Eb_{GM}{}^{\mu\nu}\left(r,\theta\right)$.

In principle, it is possible to solve the equations algebraically, but it is hopelessly complicated.

So we make a Ritz-Galerkin linear ansatz in As_{0i} and $\{1/\sin(\theta), 1/\sin^{3/4}(\theta)\}$, and minimize the equations.

We get a half-analytic solution linear in As_{0i}

$$dAb_{sol} = dAb_{sol}\left(As_0\right)$$

$$dEb_{sol} = dEb_{sol}\left(As_0, \sin\left(\theta\right)\right)$$

dEb_{sol} represents the *interaction of the wave with matter*:

dEb_{00} and dEb_{11} generate a potential of a radial force, *i.e.* the wave *exerts pressure* in direction of movement, the remaining components represent shear-stress tensor components in xy(=13), xz(=12) and yz(=32) directions. As those forces are linearly dependent on the wave components As_{0i}, they are normally unmeasurably small, but they should exist.

7.5. Planar Gravitational Waves in AK-Gravity

Planar gravitational waves (**Figure 13**) are on Earth of course the only realistic

Λ-scaled wave ansatz

$$A_\mu^{\ \nu} = dAb_\mu^{\ \nu} + \Lambda As_\mu^{\ \nu} \exp(-i\,k(x-t))$$

$$E^{\mu\nu} = Eb_{GM}^{\ \mu\nu}(x,\theta) + \frac{dEb^{\mu\nu}}{x^{3/2}} + Es^{\mu\nu} \exp(-i\,k(x-t))$$

→

eqtoiev → static & wave-equations
eqtoievnu3b(dAb,dEb)
eqtoievnu3w(As,Es, dAb)

wave equation first order in O(1)
eqtoievnu3w1(As,Es)

→

wave component relations

$$Es_1 = 0 = Es_0\ , Es_2 = 3i\,k\left(As_0 - As_1 + As_2\right), Es_3 = -3i\,k\,As_2$$

$As_3 = \left(As_0 - As_1 + As_2\right)$, free param.

$$As_f = \{As_0, As_1, As_2\}$$

wave equation second order O(1/r)
eqtoievnu3w2(dAb,As$_f$)

→

solution

$$dAb_i = dAb_i(As_f)$$

bgr equation
eqtoievnu3b(dEb$_{sol}$,dAb$_0$, dAb$_1$)
dEb$_{sol}$={dEb$_{00}$, dEb$_{11}$, dEb$_{12}$, dEb$_{13}$}

→

solution at infinity

$$dEb_{sol} = dEb_{sol}(As_0)$$

wave form planar wave in x-direction

$$As = \begin{pmatrix} As_0 \\ As_1 \\ 0 \\ 0 \end{pmatrix} \quad Es = \begin{pmatrix} 0 \\ 0 \\ Es_2 = 3i\,k\left(As_0 - As_1 + As_2\right) \\ Es_3 = -3i\,k\,As_2 \end{pmatrix}$$

tetrad Es has only transversal components $(2,3) = (\theta,\varphi) \equiv (z,y)$

metric wave has also only transversal components

$$Es \bullet \eta \bullet Es' = gs\,, \quad gs = \begin{pmatrix} 0 & 0 \\ 0 & gs_{22} \end{pmatrix}$$

$$gs_{22} = \begin{pmatrix} Es_2^{\ 2} & Es_2 \bullet Es_3 \\ Es_2 \bullet Es_3 & Es_3^{\ 2} \end{pmatrix} = \begin{pmatrix} As_2^{\ 2} & -\left(As_0 - As_1\right)^2/2 \\ -\left(As_0 - As_1\right)^2/2 & -As_2^{\ 2} \end{pmatrix}$$

gauge cond. $2As_2 \bullet \left(As_2 + As_0 - As_1\right) + \left(As_0 - As_1\right)^2 = 0$

i.e. gs has the normal form of a GR metric wave
gs satisfies the linearized Einstein equations

generated metric wave $Es^{\mu\kappa} Es^{\nu}{}_{\kappa} = gs^{\mu\nu} = h^{\mu\nu}$ is the metric wave of GR

reflection and absorption of gravitational waves:

at matter boundary the relative potential changes $\Phi \approx -\dfrac{r_s}{2r} \to \widetilde{\Phi} = \Phi + \delta\Phi \quad \delta\Phi = -\dfrac{r_s(M)}{2r}$

$r_s(M)$ =Schwarzschild radius of the interacting matter M

k has a jump δk : with $k = \dfrac{\sqrt{r_s}}{\sqrt{2r_0^{\ 3}}}$, $\dfrac{\delta k}{k} = \dfrac{r_s(M)}{2r_s}$

so the reflected and absorbed amplitude ratio is approximately

$$\frac{\delta A_r}{A} = \frac{\delta k}{k} = \frac{r_s(M)}{2r_s}\ , \quad \frac{\delta A_a}{A} = \sqrt{\frac{r_s(M)}{r_s}}$$

Figure 13. Schematic planar gravitational waves.

form to be measured. As the LIGO observation show, their scale is around $r_s \propto 100\ \text{km}$ and the induced relative metric shift $\varepsilon \propto 10^{-21}$, so for the tetrad

$\frac{\delta E}{E} \approx \varepsilon/2 \propto 10^{-21}$.

We begin, as before, with the Λ-scaled wave ansatz

$$A_\mu{}^\nu = dAb_\mu{}^\nu + \Lambda As_\mu{}^\nu \exp\left(-i\,k\left(r-t\right)\right)$$

$$E^{\mu\nu} = Eb_{GM}{}^{\mu\nu}\left(x,\theta\right) + \frac{dEb^{\mu\nu}}{x^{3/2}} + Es^{\mu\nu}\exp\left(-i\,k\left(r-t\right)\right),$$

where we use the same variables as before, except the second-order wave factor Er.

Again, the AK-equations separate into the background and the wave part

$$eqtoievnu3b(dAb,\ dEb)$$

$$eqtoievnu3w(As,\ Es,\ Er,\ dAb)$$

and the wave equation first order O(1), *i.e.* at infinity, is $eqtoeivnu3w1(As,\ Es)$ and the solution are the *same* wave component relations, as for the spherical case:

$$Es_1 = 0 = Es_0,\ Es_2 = 3i\,k\left(As_0 - As_1 + As_2\right),\ Es_3 = -3i\,k\,As_2$$

$As_3 = \left(As_0 - As_1 + As_2\right)$, with the 12 free parameters $As_f = \left\{As_0, As_1, As_2\right\}$

The solution is inserted into the wave equation of second order O(1/*r*) giving $eqtoeivnu3w2(dAb,\ As_f)$.

Now we get a (trivial) solution if we set dAb to a *half-antisymmetric background*:

$$dAb_{hab} = \left\{dAb_0\left(1,1,-1,1\right), dAb_1\left(1,1,-1,1\right), dAb_2\left(1,1,-1,1\right), dAb_3\left(1,1,-1,1\right)\right\},$$

which furthermore enforces the vanishing tetrad condition

$$As_1 = As_0,\ As_2 = As_3 = 0.$$

This is undesirable, since then the background equations are identically zero.

So we demand that the solution deviates from dAb_{hab} and introduce the *penalty-factor* $\frac{1}{\left\|dAb - dAb_{hab}\right\|}$ in the equations. Now again make a linear Ritz-Galerkin ansatz $dAb_{ij} = \sum_k \alpha_{ijk} As_{f,k}$, minimize an get the solution

$$dAb_i = dAb_i\left(As_f\right)$$

Finally after insertion, we get the bgr equation in O(1) (at infinity)

$eqtoievnu3b(dEb_{sol},\ d2Ab_0,\ d2Ab_1)$ where

$d2Ab_0 = \left\{dAb_{00} - dAb_{01}, dAb_{01} + dAb_{02}, dAb_{02} + dAb_{03}, dAb_{03} - dAb_{01}\right\}$ are the *half-antisymmetric background differences* and $dEb_{sol} = \{dEb_{00},\ dEb_{11},\ dEb_{12},\ dEb_{13}\}$, which depends on dAb_0, dAb_1 only.

We minimize again with Ritz-Galerkin and get the solution

$$dEb_{sol,ij} = \sum_k \alpha_{ijk} As_{0k}$$

With planar waves, there is again the radial pressure dEb_{00}, dEb_{11}, and the shear-stress in *xy*(=13) and *xz*(=12), but no other directions, as is to be expected.

We calculate now the form of planar waves. Originally, we have 3 × 4 free parameters As_f, so we get for a planar wave in x-direction

$$As = \begin{pmatrix} As_0 \\ As_1 \\ 0 \\ 0 \end{pmatrix} \quad Es = \begin{pmatrix} 0 \\ 0 \\ Es_2 = 3i\,k\left(As_0 - As_1 + As_2\right) \\ Es_3 = -3i\,k\,As_2 \end{pmatrix}$$

the tetrad Es has only transversal components $(2,3) = (\theta, \varphi) \equiv (z, y)$, as expected.

The form of the metric wave follows from the defining equation of the (inversed densitized) tetrad E

$$Es \bullet \eta \bullet Es^t = gs \quad \text{or} \quad Es^{\mu\kappa} Es^{\nu}{}_{\kappa} = gs^{\mu\nu} = h^{\mu\nu}$$

$gs = \begin{pmatrix} 0 & 0 \\ 0 & gs_{22} \end{pmatrix}$ and the wave exponential is $\exp\left(2ik\left(x^1 - ct\right)\right)$, *i.e.* the metric frequency is the double of the source frequency $\omega_g = 2\omega_E$, as is well known.

$$gs_{22} = \begin{pmatrix} Es_2{}^2 & Es_2 \bullet Es_3 \\ Es_2 \bullet Es_3 & Es_3{}^2 \end{pmatrix} = \begin{pmatrix} As_2{}^2 & -\left(As_0 - As_1\right)^2 / 2 \\ -\left(As_0 - As_1\right)^2 / 2 & -As_2{}^2 \end{pmatrix}$$

Now if we impose the gauge condition $2As_2 \bullet \left(As_2 + As_0 - As_1\right) + \left(As_0 - As_1\right)^2 = 0$, then gs has the normal form of a GR metric wave

$$\left(h^{\mu\nu}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_{11} & e_{12} \\ 0 & 0 & e_{12} & -e_{11} \end{pmatrix} \exp\left(2ik\left(x^1 - ct\right)\right) \tag{38}$$

and therefore *gs satisfies the linearized Einstein equations*, see chap. 7.4 above.

The contravariant instead of covariant indices in h do not change the equations apart from the sign, so physically it is the same.

Now it is clear, why the metric wave is not superposable (because quadratic in Es) and why it has the double frequency ω0 of the generating binary black-hole (As and Es have the frequency ω0 and h is quadratic in Es).

The use of contravariant indices in h does not change the equations (apart from the sign), so the physical meaning is the same.

Now in principle we can detect the wave tensor $As_\mu{}^\nu$ by

1) Measuring the generated metric wave h with a gravitational interferometer.

2) By measuring the (minuscule) energy of the absorbed (by the Earth) part of the wave tensor As, which at the moment is rather hopeless.

Finally, let us examine the reflection and absorption of gravitational waves. Let us assume, as a simplification, that there is a sharp edge of the interacting matter, and at this boundary there is a jump of the potential.

This is not true, of course, and in fact we have to calculate the background tetrad from the real backgound metric.

But under this assumption, at the boundary the relative potential changes:

$\Phi \approx -\frac{r_s}{2r} \to \tilde{\Phi} = \Phi + \delta\Phi \quad \delta\Phi = -\frac{r_s(M)}{2r}$, where $r_s(M)$ = Schwarzschild radius of the interacting matter *M*.

Then *k* has a jump δk : with $k = \frac{\sqrt{r_s}}{\sqrt{2r_0^3}}$ (see chap. 7.6), $\frac{\delta k}{k} = \frac{r_s(M)}{2r_s}$

If we consider the wave component relations, and require that the tetrad and the metric be coitinuous, the A-tensor will have a jump so the reflected amplitude ratio is approximately

$$\frac{\delta A_r}{A} = \frac{\delta k}{k} = \frac{r_s(M)}{2r_s}$$

The absorption ratio results from the energy balance (*T* = energy, A_0 = amplitude): $T_{in} = A_0^2 + (\Delta A_0)^2 = T_{out} = (A_0 + \Delta A_0)^2$, so the absorbed energy is $2A_0\Delta A_0$ and $\frac{\delta A_a}{A} = \sqrt{\frac{2A_0\Delta A_0}{A_0^2}} = \sqrt{\frac{r_s(M)}{r_s}}$.

The approximation is only valid if $r_s(M) \ll r_s$. If we apply it to the Earth with $r_s(M) = 9\ \text{mm}$ and the first LIGO black hole merger event GW140915 with $r_s = r_s(60M_{sun}) = 180\ \text{km}$, we see that the effect is currently unmeasurable. But recently, echoes of reflection from the originating black hole in GW140915 have been discussed [35] with an assessed amplitude ratio of $\alpha_r \approx 0.05$, which is consistent with a reflection from a debris-sphere with a mass of $M \approx 3M_{sun}$.

7.6. Wave Equation in Binary Rotator Spacetime

The binary gravitational rotator has the parameters [36] (**Figure 14**): masses m_1 m_2, distance r_0, mass ratio $\mu = m_2/m_1 \le 1$, total mass $m = m_1 + m_2$, Schwarzschild radius $r_s = \frac{2Gm}{c^2}$, gravitational wave number $k = \frac{\sqrt{r_s}}{\sqrt{2r_0^3}}$ and is described by Kerr spacetime in first order approximation for $\alpha \ll 1$

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{1}{r}\right) & 0 & 0 & -\frac{\alpha \sin^2\theta}{r} \\ & \frac{1}{1-\frac{1}{r}} & 0 & 0 \\ & & r^2 & 0 \\ & & & r^2 \sin^2\theta \end{pmatrix}$$

$$\alpha = \frac{c_0}{r_0}, \text{ exactly: } \alpha = \frac{8\pi}{5r_0 F}\frac{\mu}{(1+\mu)^7 (3+8i\mu-4\mu^2)}$$

The celebrated Einstein's power formula for gravitational waves of the bgr is [9]:

$$P_{gr} = \frac{32}{5} m_1^2 m_1^2 m \frac{G^4}{r_0^5 c^5} \tag{39a}$$

The binary gravitational rotator, abbreviated bgr, (two masses rotating around

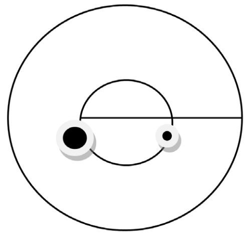


Figure 14. Binary gravitational rotator (bgr).

their center-of-mass in their own gravitational field) is the simplest source of gravitational waves, a single rotating mass (*i.e.* with axial symmetry) does not emit gravitational waves.

Bgr has an axial symmetry and can be described by a Kerr-spacetime with an appropriate Kerr-parameter α, which determines the power of the generated gravitational wave as shown in [36].

The exact formula derived there is

$$\alpha = \frac{8\pi}{5 r_0 F} \frac{\mu}{(1+\mu)^7 \left(3 + 8i\mu - 4\mu^2\right)},$$

where $F \approx 1$ is the relativistic velocity factor, $\mu = \frac{m_2}{m_1} \le 1$ is the mass ratio and r_0 is the average distance of the masses., masses $m_1 m_2$, total mass $m = m_1 + m_2$, Schwarzschild radius $r_s = \frac{2Gm}{c^2}$

Einstein's power formula for gravitational waves of the bgr is [9]:

$P_{gr} = \frac{32}{5} m_1{}^2 m_1{}^2 m \frac{G^4}{r_0{}^5 c^5}$ or formulated with α

$$P_\alpha = \frac{\Delta E_\alpha}{T} = \frac{(1+i\mu)\alpha}{r_0^5} \frac{F}{4\pi} \left(3 + 8i\mu - 4\mu^2\right) \tag{39b}$$

The gravitational waves of the bgr have the (dimensionless) wave number $k = \frac{1}{\sqrt{2 r_0^3}}$, with dimension $k = \frac{\sqrt{r_s}}{\sqrt{2 r_0^3}}$

In the following, we need only $\alpha = \frac{c_0}{r_0}$ with a constant c_0 and the bgr to be described by a Kerr-spacetime to be exact of order $O\left(\frac{\alpha^2}{r^2}\right)$.

We prove the validity of the Einstein power formula for the gravitational waves of the bgr in [34] chap. 4.7.1.

8. Numeric Solutions of Time-Independent Equations with Coupling Λ = 1, Λ = 0.01

Numeric solutions of static AK equations with different Λ-values give an impression of the behavior of the A-tensor and E-tensor, and show that there is a smooth transition of AK gravity to GR for $\Lambda \to 0$.

We consider the time-independent equations *eqtoiv* with strong and weak coupling ($\Lambda = 1$, $\Lambda = 0.01$).

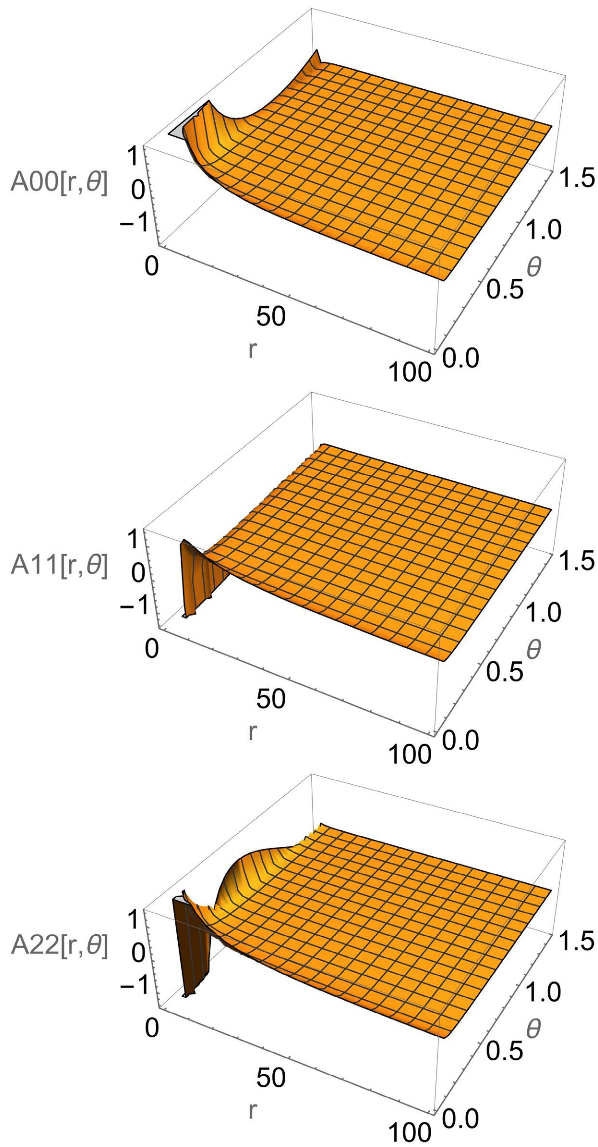
The calculation is carried out by Ritz-Galerkin method with trigonometric polynomials in $\theta\{\cos(\theta), \sin(\theta), \frac{1}{\sin(\theta)^{3/4}}\}$ and in r with polynomials of $\{\frac{1}{\sqrt{r-1}}, \sqrt{r-1}\}$, which can approximate the Schwarzschild-singularity at $r = 1$, in total 49 base functions.

The solution $\left(Aij(r,\theta), Eij(r,\theta)\right)$ generates the metric

$$g\left(Eij(r,\theta)\right) = \left(E\,\eta\,E^t\right)^{-1}\det(E)^{3/8}$$

The lattice is here a 30 × 12 $\{r, \theta\}$-lattice and the Ritz-Galerkin minimization runs in parallel with 8 processes on random sublattices with 100 points.

The resulting solution (for $\Lambda = 1$) $\{A00v(r, \theta), \ldots, A33v(r, \theta), r^{3/2}E00v(r, \theta), \ldots, r^{3/2}E33v(r, \theta)\}$ is shown below in **Figure 15** for some variables.



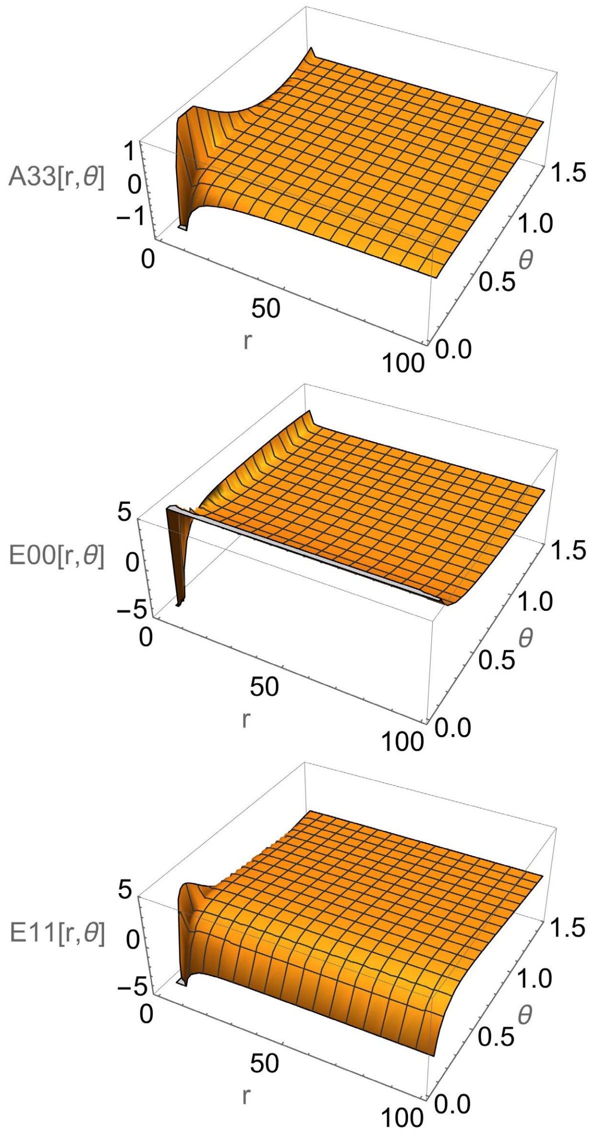


Figure 15. A-tensor and E-tensor for $\Lambda = 1$.

The Metric in AK-Gravity: No Horizon and No Singularity

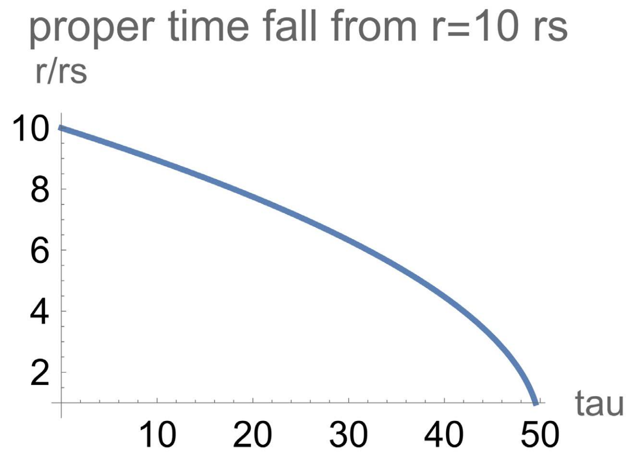
From the resulting solution $\{A00v(r,\theta), \ldots, A33v(r,\theta), E00v(r,\theta), \ldots, E33v(r,\theta)\}$ the generated metric $fgijv(r,\theta)$ is calculated.

Using this metric we can approximately calculate the velocity $v \approx \frac{\frac{1}{g_{00}}-1}{g_{11}} \le 1$ during the free fall to the horizon $r = 1$ (**Figure 16**).

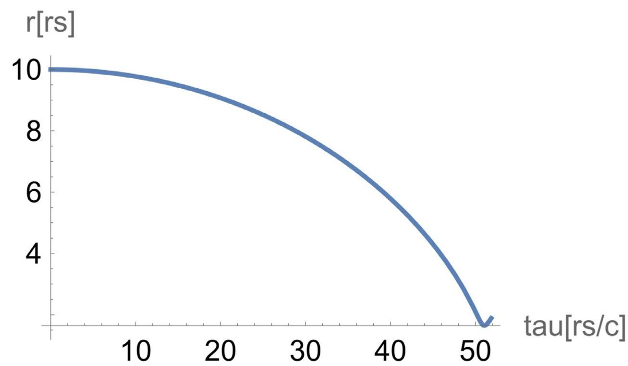
Now we calculate the Christoffel symbols $\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa}\left(\frac{\partial g_{\kappa\mu}}{\partial x^{\nu}} + \frac{\partial g_{\kappa\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\kappa}}\right)$ from the metric and solve the equations-of-motion for the free fall from $r_2 = 10$.

In GR, we have the following picture ([37] P3s1):

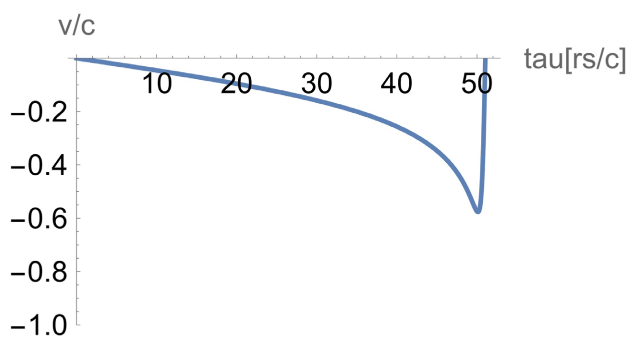
The proper time $\tau(r)$ of fall in dependence of radius r: the fall time is $\tau_f = \tau(r = 1) = 49.5$ and of course $v(r = 1) = 1$ and $\tau(r = 10) = 0$.


proper time fall from r=10 rs
r/rs
10
8
6
4
2
tau
10
20
30
40
50

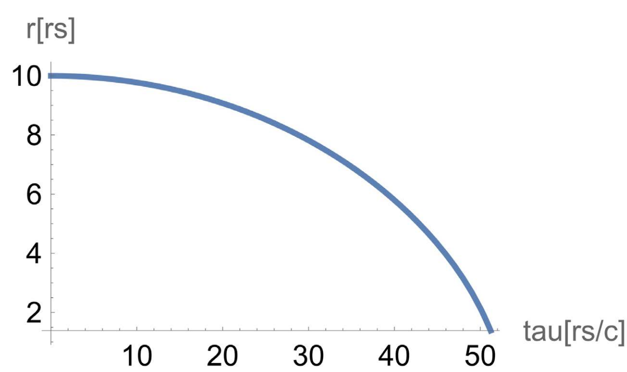
(a)


r[rs]
10
8
6
4
tau[rs/c]
10
20
30
40
50

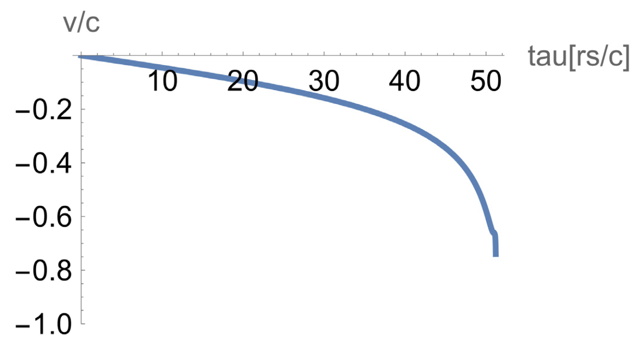
(b)


v/c
tau[rs/c]
10
20
30
40
50
−0.2
−0.4
−0.6
−0.8
−1.0

(c)


r[rs]
10
8
6
4
2
tau[rs/c]
10
20
30
40
50

(d)



(e)

Figure 16. (a) Distance $r(\tau)$ during direct fall in GR; (b) Distance $r(\tau)$ during direct fall in AK gravity with Λ = 1. (c) Velocity $v(\tau)/c$ during direct fall in AK gravity with Λ = 1. (d) Distancer (τ) during direct fall in AK gravity with Λ = 0.01. (e) Velocity $v(\tau)$/c during direct fall in AK gravity with Λ = 0.01.

The proper fall-time from $r = r_2$ to r is the same as in Newtonian gravity

$$\tau(r_2, r) = \frac{r_2^{\ 2} - r^2}{2}$$

The inverse function radius in dependence on the fall time τ is $r1t0s(\tau)$ (in r_s units):

In AK-metric with Λ = 1 we have the following picture ([32] P2s1s2):

The radius in dependence on the fall time τ is $r1t0s(\tau)$:

and the velocity $vt0s(\tau)$

The fall-time is here τ_f = 51.1 reached at r_f = 1.65, the maximal velocity is v_{max} = 0.56, then there is a rebound.

In AK-metric with Λ = 0.01 we have the following picture ([32] P3s1s2):

The radius in dependence on the fall time τ is $r1t0s(\tau)$:

and the velocity $vt0s(\tau)$

The fall-time is here τ_f = 51.2 reached at r_f = 1.39, the maximal velocity is v_{max} = 0.74, then the orbit is finished: the solution of the equation-of-motion cannot be continued. The behavior is similar as in GR, but the escape velocity is $v_{max} < 1$, so there is no singularity.

So indeed, there is a covergent behavior for $\Lambda \to 0$ with $v_{\max} \to 1$, although GR emerges as a degenerate solution of the AK-equations (see chap. 6.2).

The overall behavior of the A-tensor and the E-tensor is as follows.

Some components (e.g. $A02$, $E11$, $E33$) diverge like $1/(\sin\theta)^{3/4}$ for $\theta \to 0$, as in the Gauss-Schwarzschild tetrad E_{GS}. But there is no apparent metric singularity for $r \to 1$, there are only some numerical artefacts near $r = 1$, because some of the Ritz-Galerkin base functions are divergent at $r = 1$.

9. Numeric Solutions of Time-Dependent Equations with Weak Coupling and Binary Gravitational Rotator

Numeric solutions of time-dependent AK equations in the case of a binary gravitational rotator (bgr = two masses orbiting their center-of-mass) show the interplay of the static background A-tensor $Ab_{\mu}^{\ \nu}(r,\theta)$ and E-tensor $Eb_{\mu}^{\ \nu}(r,\theta)$ with the gravitational wave $\Lambda\frac{As_{\mu}^{\ \nu}(r,\theta)}{r}\exp(-ik(r-t))$ emitted by bgr.

We consider the time-dependent equations *eqtoiev* with weak coupling (Λ = 0.001) and binary gravitational rotator (bgr) ([32] P4).

We start, as in chap. 7.1, with the Λ-scaled ansatz for the A-tensor

$$A_{\mu}^{\ \nu}(r,\theta,t)=Ab_{\mu}^{\ \nu}(r,\theta)+\Lambda\frac{As_{\mu}^{\ \nu}(r,\theta)}{r}\exp(-ik(r-t))$$

and correspondingly for the E-tensor

$$E^{\mu\nu}(r,\theta,t)=Eb^{\mu\nu}(r,\theta)+\frac{Es^{\mu\nu}(r,\theta)}{r}\exp(-ik(r-t))$$

We introduce the disturbance dAb and $Ab = A_{hab} + dAb$ from the bgr, as in chap. 7.6.

With this ansatz we derive from *eqtoiev* the static part $eqtoievnu3b(dAb, Eb)$ and the wave part $eqtoievnu3w(As, Es, dAb)$, but without the limit $\Lambda\to 0$, we set $\Lambda = \Lambda_0 = 0.001$ and the wave number $k=k_0=\frac{1}{\sqrt{2r_0^{\ 3}}}$ with $r_0 = 1$ average distance from the bgr.

At $r\to\infty\{dAb, Eb, As, Es\}$ take the values derived for the bgr in chap. 7.6.

$\{As, Es\}\to\{Asinfv, Esinfv\}$ =

$A_{s00}=\frac{c_0}{r_0}$, $A_{s02}=\frac{c_0}{r_0}$ (c_0, r_0 are bgr-parameters, see chap 7.6)

$E_{s2i}(r,\theta)=0\quad i=(0,1,2,3)$, $A_{s0i}(r,\theta)=A_{s00}(\theta)(1,1,-1,-1)\quad i=(0,1,2,3)$

$A_{s1i}(r,\theta)=A_{s00}(\theta)(1,1,-1,-1)\quad i=(0,1,2,3)$, $A_{s2i}(r,\theta)=0\quad i=(0,1,2,3)$

$A_{s3i}(r,\theta)=0\quad i=(0,1,2,3)$,

$$E_{s0i}(r,\theta)=(3A_{s00}(\theta)+A_{s00}'(\theta))(1,-1,1,-1)\quad i=(0,1,2,3)$$

$$E_{s1i}(r,\theta)=\frac{(3A_{s00}(\theta)+A_{s00}'(\theta))}{r}(1,-1,1,-1)\quad i=(0,1,2,3)$$

$$E_{s3i}(r,\theta)=\frac{3A_{s00}(\theta)}{r}(1,-1,1,-1)\quad i=(0,1,2,3)$$

$dAb\to dAb_{infv}$ =

$$dAb_{0i}=\frac{\alpha}{r^3}(1,1,1,1)\quad i=(0,1,2,3)$$

$$dAb_{1i}=0\quad i=(0,1,2,3)$$

$$dAb_{2i}=\frac{i}{sin\theta}(1,1,1,1)\quad i=(0,1,2,3)$$

$$dAb_{3i}=1\quad i=(0,1,2,3)$$

$Eb\to Eb_{infv} = E_{GK}$ the Gauss-Kerr-tetrad from chap. 7.6.

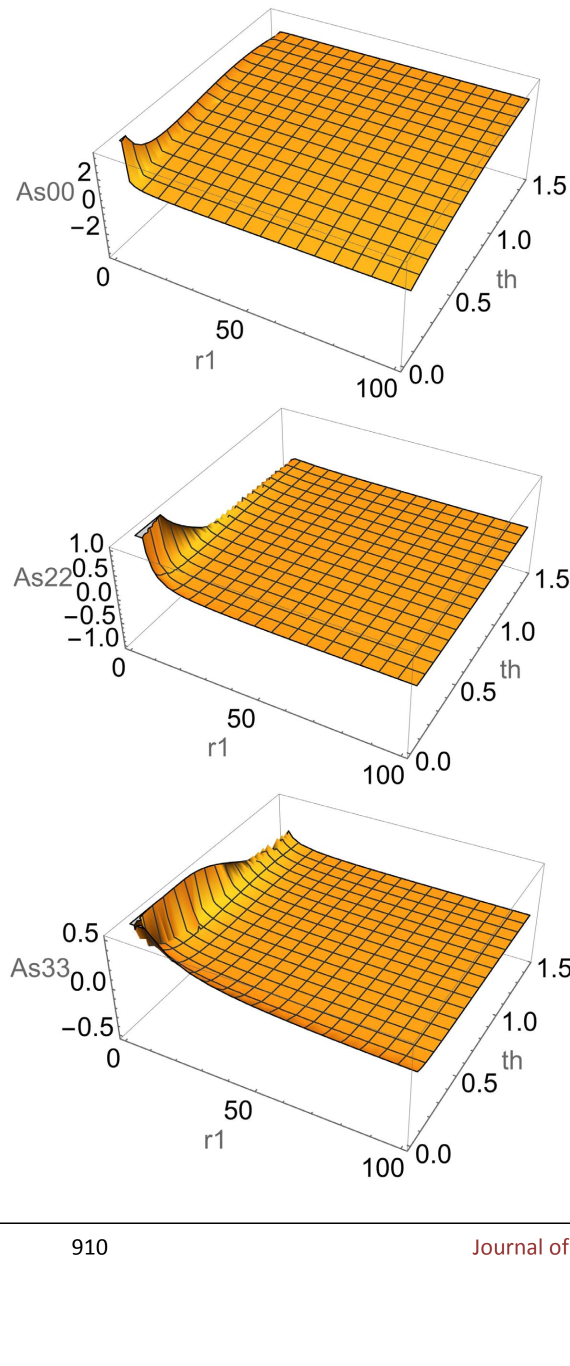
We apply the Ritz-Galerkin method with trigonometric polynomials in θ {cos(θ),

$\sin(\theta)$, $\frac{1}{\sin(\theta)^{3/4}}$ } and in r with polynomials of { $\frac{1}{r}$ }, 40 base functions.

The metric $g^{\mu\nu}(Eb)$ generated by the background Eb has a horizon at $r \approx 1.9$ for the free fall, which means that for weak coupling ($\Lambda = 0.001$) the singularity of GR still exists. So there is a Λ, ($0.001 < \Lambda < 1$), where the singularity disappears. The resulting solution is $\{dAb(r, \theta), Eb(r, \theta), As(r, \theta), Es(r, \theta)\}$, is shown for the wave with factor $Sin(\theta)^{3/4}$ in **Figure 17**.

We see that the wave component As00 shows a constant behavior, whereas As22 and As33 show linearly resp. exponentially decreasing behavior over r, i.e. they represent undamped or linearly resp. exponentially damped waves, as shown in chap. 7.2.

The behavior in θ is non-harmonic (non-linear in $(\sin\theta, \cos\theta)$), since the wave is quadrupole or higher order.



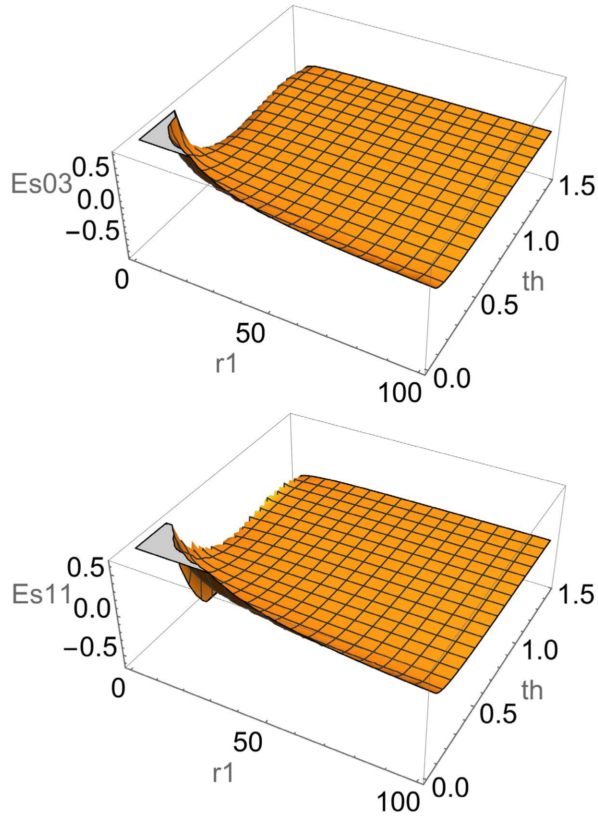


Figure 17. Wave components A-tensor (shown $\mathrm{Sin}(\theta)^{3/4}\, As_{ik}$) and E-tensor (shown $\mathrm{Sin}(\theta)^{3/4} Es_{ik}$) for bgr gravitational waves in AK gravity with weak coupling $\Lambda = 0.001$.

10. The Energy Tensor for the Gravitational Wave

Electromagnetic energy tensor

The electromagnetic energy tensor in cgs units $\varepsilon_0 = \frac{1}{4\pi}$ $\mu_0 = 4\pi$ is

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$ $[T]$ = energy/r^3 = endensity

$$T^{\mu\nu} = \begin{pmatrix} \left(E^2+B^2\right)/8\pi & S_x/c & S_y/c & S_z/c \\ S_x/c & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ S_y/c & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ S_z/c & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{pmatrix} \tag{40a}$$

where $\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$ Poynting vector, $[S]$ = energy/($r^2 * t$) = energy-flux, $[S/c]$ = energy/r^3 = endensity, $\sigma_{ij} = \varepsilon_0 E_i E_j + \frac{1}{\mu_0}B_i B_j - \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\delta_{ij}$ Maxwell stress tensor.

The conservation of momentum and energy yields

$$\partial_\nu T^{\mu\nu} + \eta^{\mu\rho} f_\rho = 0$$

where f_ρ is the (4D) Lorentz force density.

The electromagnetic energy density is

$$u_{em} = \frac{\varepsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2$$

and electromagnetic momentum density is

$$\vec{p}_{em} = \frac{\vec{S}}{c^2}$$

The energy tensor is symmetric, traceless and positive-definite

$$T^{\mu\nu} = T^{\nu\mu} \quad T^{\mu}{}_{\mu} = 0 \quad T^{00} \geq 0$$

Gravitational Ashtekar-Kodama energy

GR gravitational wave energy density (plane wave) is ([9] 34.23),

$$t_{\mu\nu} = \frac{\hbar c}{16\pi l_P{}^2} k_\mu k_\nu \left(e^{\lambda\kappa *} e_{\lambda\kappa} - \frac{1}{2}\left|e^{\lambda}{}_{\lambda}\right|^2 \right)$$

where dimension $[t_{\mu\nu}]$ = energy/r^3 = endensity ([9]), $e^{\lambda\kappa}$ is the polarization.

When the metric wave is spherical $h_{\mu\nu} = \frac{e_{\mu\nu}}{r}\exp\left(-ik_\mu x^\mu\right)$, we perform the transition from spherical wave A_r to plane wave A_p via energy condition: $4\pi r^2 \left|A_r\right|^2 = r_s{}^2 \left|A_p\right|^2$, and we obtain for the AK gravitational wave energy density $t_{\mu\nu} = D_\kappa A_\mu{}^\kappa D_\lambda A_\nu{}^\lambda \hbar c \left(\frac{1}{l_P{}^2 \Lambda^2 r_s{}^2} \right)$, with dimension $[t_{\mu\nu}]$ = energy/r^3 = endensity, where the dimensionless factor $\frac{r_{P\Lambda}{}^2}{r_s{}^2} = \frac{1}{l_P{}^2 \Lambda^2 r_s{}^2}$ is inserted for compatibility with GR and to account for the Λ-scaled wave ansatz), and $r_{P\Lambda} = \frac{1}{l_P \Lambda} = 5.64 \times 10^{86}$ m Planck-lambda scale.

The second term in gravitational stress energy: $t^e{}_{\mu\nu} = D_\kappa E_\mu{}^\kappa D_\lambda E_\nu{}^\lambda \Lambda \hbar c$ (Λ must be inserted for dimensional reasons), which is normally negligible.

For the standard spherical wave $k_\mu = \left(-k_0, k_0, 0, 0\right)$ x-y-polarization unit amplitude

GR energy density is $t_{\mu\nu} = k_0{}^2 \frac{e_{11}{}^2 r_s{}^2}{r^2} \frac{\hbar c}{4 l_P{}^2} \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

AK energy density for a standard scaled spherical wave with a single r-t-amplitude $A_\mu{}^\nu = \frac{\Lambda A s_{00}}{r} \begin{pmatrix} 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \exp\left(-ik_0\left(r-t\right)\right)$ is

$$t_{\mu\nu} = k_0{}^2 \frac{A s_{00}{}^2}{r^2 r_s{}^2} \hbar c \left(\frac{1}{l_P{}^2} \right) \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{40b}$$

which is identical to the GR expression apart from a dimensionless factor 1/4, which can be incorporated in As00.

The AK energy density has the form: $t_{\mu\nu} = s_\mu s_\nu \hbar c$, where $s_\mu = D_\kappa A_\mu{}^\kappa$ and dimension $[s_\mu] = 1/r^2$

The current is $j_\nu = c\dfrac{x^\kappa}{|x^\kappa|} t_{\kappa\nu}$, where $n^\kappa = \dfrac{x^\kappa}{|x^\kappa|}$ is a unit direction 4-vector, the energy flux in the direction n_i is then ([9] 41.11) $S = \sum ct_{0i} n_i$, dimension $[S]$ = energy/$r^2 t$

The total power of gravitational radiation for a quadrupole Q is in GR ([9] 42.21) $P = \dfrac{32G\omega^6 Q}{5c^5}$, in the special case of a binary gravitational rotator with masses m_1 and m_2 (total mass $m = m_1 + m_2$) and the average orbit radius r_0 we obtain $r_s = \dfrac{2mG}{c^2}$ $\quad k = \dfrac{\omega}{c} = \dfrac{\sqrt{r_s}}{\sqrt{2r_0^3}}$ $\quad P_{GR} = \dfrac{r_s^2 c^5}{2G} k^6 r_0^4 \left(\dfrac{m_1 m_2}{m^2}\right)^2 = P_0 \dfrac{r_s^5}{r_0^5}\left(\dfrac{m_1 m_2}{m^2}\right)^2$, where $P_0 = \dfrac{\hbar c^2}{2 l_P^2}$ is a constant with dimension of power.

In chap 7.6 we have shown that for binary gravitational rotator

$$As00(r,\theta,r_0) = \frac{As00n01(\theta)}{r_0},$$

we get

$$t_{00} = k_0^2 \frac{As_{00}^2}{r^2 r_s^2} \hbar c \left(\frac{1}{l_P^2}\right), \quad P_{KA} = t_{00} c 4\pi r^2 = k_0^2 As_{00}^2 4\pi \hbar c^2 \left(\frac{1}{l_P r_s}\right)^2$$

Setting $As_{00} = \dfrac{c_0}{r_0}$, with $k_0^2 = \dfrac{r_s}{2r_0^3}$ it follows from $P_{KA} = P_{GR}$,

$$c_0^2 = r_s^6 \frac{\left(\frac{m_1 m_2}{m^2}\right)^2}{32\pi}, \quad c_0 = r_s^3 \frac{\frac{m_1 m_2}{m^2}}{4\sqrt{2\pi}}$$

So the amplitude of the gravitational wave of the binary gravitational rotator becomes $As_{00} = \dfrac{\frac{m_1 m_2}{m^2}}{4\sqrt{2\pi}} \dfrac{r_s^3}{r_0}$, where $r_s = \dfrac{2Gm}{c^2}$ is the Schwarzschild radius of the total mass m, and $f_m = \dfrac{m_1 m_2}{m^2} = \dfrac{m_r}{m} = \dfrac{\mu}{(1+\mu)^2}$ is the ratio of the reduced mass to the total mass $\mu = \dfrac{m_1}{m_2} \le 1$.

This formula can be easily generalized to multiple masses rotating around their common center-of mass:

$As_{00} = \dfrac{f_m}{4\sqrt{2\pi}} \dfrac{r_s^3}{r_0}$ with $f_m = \dfrac{m_1 m_2 \ldots m_n}{m^n} = \dfrac{m_r}{m}$ and r_0 the average diameter of the rotator.

11. Quantum AK-Gravitation

We recall the Ashtekar-Kodama equations

spacetime curvature (field tensor) $F_{\mu\nu}{}^{\kappa} = \partial_{\mu} A_{\nu}{}^{\kappa} - \partial_{\nu} A_{\mu}{}^{\kappa} + \varepsilon^{\kappa}{}_{\kappa_1\kappa_2} A_{\mu}{}^{\kappa_1} A_{\nu}{}^{\kappa_2}$

4 gaussian constraints $G^{\mu} = \partial_{\nu} E^{\nu\mu} + \varepsilon^{\mu}{}_{\kappa\lambda} A_{\nu}{}^{\kappa} E^{\nu\lambda}$ (covariant derivative of $E^{\mu\nu}$ vanishes)

4 diffeomorphism constraints $I_{\mu} = E^{\kappa}{}_{\nu} F_{\mu\kappa}{}^{\nu}$

24 hamiltonian constraints $H_{(\mu,\nu)}{}^{\kappa} = F_{\mu\nu}{}^{\kappa} + \frac{\Lambda}{3} \varepsilon_{\mu\nu\rho} E^{\rho\kappa}$

In this section, we will find the Lagrangian, from which the AK equations can be derived.

11.1. Lagrangian of the Hamiltonian Equations

In electrodynamics, the Lagrangian of the fundamental Maxwell equations is

$$L_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Therefore we make at first the analogous ansatz for the AK-Lagrangian of the hamiltonian equations

$$L_F = F_{\mu\nu}{}^{\kappa} F^{\mu\nu}{}_{\kappa}$$

The formal expression for the variation of action for the variables $A_{\mu}{}^{\nu}$ is:

$$\frac{\delta L}{\delta A_{\rho}{}^{\sigma}} = \frac{\partial L}{\partial A_{\rho}{}^{\sigma}} - \partial_{\tau} \frac{\partial L}{\partial A_{\rho}{}^{\sigma}{}_{,\tau}}, \text{ where } A_{\rho}{}^{\sigma}{}_{,\tau} = \frac{\partial A_{\rho}{}^{\sigma}}{\partial x^{\tau}}$$

We have 4 intermediate results

$$\frac{\partial F_{\mu\nu}{}^{\kappa}}{\partial A_{\rho}{}^{\sigma}} = \delta_{\mu\rho} \varepsilon_{\sigma\kappa_2}{}^{\kappa} A_{\nu}{}^{\kappa_2} + \delta_{\nu\rho} \varepsilon_{\kappa_1\sigma}{}^{\kappa} A_{\mu}{}^{\kappa_1}$$

$$\partial_{\tau} \frac{\partial F_{\mu\nu}{}^{\kappa}}{\partial A_{\rho}{}^{\sigma}{}_{,\tau}} = \left(\delta_{\mu\tau} \delta_{\nu}{}^{\rho} - \delta_{\nu\tau} \delta_{\mu}{}^{\rho}\right) \delta^{\kappa}{}_{\sigma}$$

$$\frac{\partial L_F}{\partial A_{\rho}{}^{\sigma}} = 2\varepsilon_{\sigma\kappa_1}{}^{\kappa} \left(\delta_{\mu}{}^{\rho} A_{\nu}{}^{\kappa_2} - \delta_{\nu}{}^{\rho} A_{\mu}{}^{\kappa_1}\right) F^{\mu\nu}{}_{\kappa} = 4\varepsilon_{\sigma\lambda_1\kappa} F^{\rho\nu\kappa} A_{\nu}{}^{\lambda_1}$$

$$\partial_{\tau} \frac{\partial L_F}{\partial A_{\rho}{}^{\sigma}} = 2\partial_{\tau} \left(\delta_{\mu\tau} \delta_{\nu}{}^{\rho} - \delta_{\nu\tau} \delta_{\mu}{}^{\rho}\right) \delta_{\kappa\sigma} F^{\mu\nu\kappa} = 4\partial^{\tau} F_{\tau}{}^{\rho}{}_{\sigma}$$

and the result of the variation follows

$$\frac{\delta L_F}{\delta A_{\rho}{}^{\sigma}} = -4\partial^{\tau} F_{\tau}{}^{\rho}{}_{\sigma} + 4\varepsilon_{\sigma\lambda_1\kappa} F^{\rho\nu\kappa} A_{\nu}{}^{\lambda_1}$$

This is a derived equation $\tilde{H}^{\rho}_{\sigma} = 4\left(-\partial^{\tau} H_{\tau}{}^{\rho}{}_{\sigma} + \varepsilon_{\sigma\lambda_1\kappa} H^{\rho\nu\kappa} A_{\nu}{}^{\lambda_1}\right)$ from a 3-tensor $H = F$, which is the first term in the AK hamiltonian equations.

Now consider the following Lagrangian

$$L_{\Lambda} = \varepsilon^{\kappa}{}_{\mu\lambda} E^{\lambda\nu} \partial_{\kappa} A_{\mu\nu} + \varepsilon_{\mu_2\lambda_2\kappa_2} \varepsilon^{\mu_1\lambda_1}{}_{\kappa_1} E^{\kappa_1\kappa_2} A_{\lambda_1}{}^{\lambda_2} A_{\mu_1}{}^{\mu_2}$$

One can show easily that for $H_{\Lambda}(E)_{\rho\sigma}{}^{\tau} = \varepsilon_{\rho\sigma\lambda} E^{\lambda\tau}$

$$\frac{\partial L_F}{\partial A_{\rho}{}^{\sigma}} = -\partial^{\tau} H_{\Lambda}(E)_{\tau}{}^{\rho}{}_{\sigma} + \varepsilon_{\sigma\lambda_1\kappa} H_{\Lambda}(E)^{\rho\nu\kappa} A_{\nu}{}^{\lambda_1}$$

So the complete Lagrangian for the derived hamiltonian equations is

$$L_H = -\left(\frac{1}{4}L_F + \frac{\Lambda}{3}L_\Lambda\right)$$
$$= -\left(\frac{1}{4}F_{\mu\nu}{}^{\kappa}F^{\mu\nu}{}_{\kappa} + \frac{\Lambda}{3}\left(\varepsilon^{\kappa}{}_{\mu\lambda}E^{\lambda\nu}\partial_\kappa A_{\mu\nu} + \varepsilon_{\mu_2\lambda_2\kappa_2}\varepsilon^{\mu_1\lambda_1}{}_{\kappa_1}E^{\kappa_1\kappa_2}A_{\lambda_1}{}^{\lambda_2}A_{\mu_1}{}^{\mu_2}\right)\right)$$

The corresponding derived hamiltonian equations are

$$\frac{\delta L_H}{\delta A_\rho{}^\sigma} = -\partial^\tau H\left(A, \frac{\Lambda}{3}E\right)_{\tau\ \sigma}^{\ \rho} + \varepsilon_{\sigma\lambda_1\kappa}H\left(A, \frac{\Lambda}{3}E\right)^{\rho\nu\kappa}A_\nu{}^{\lambda_1},$$

where $H\left(A, \frac{\Lambda}{3}E\right)_{\mu\nu}^{\ \ \kappa} = F_{\mu\nu}{}^{\kappa} + \frac{\Lambda}{3}\varepsilon_{\mu\nu\rho}E^{\rho\kappa}$ are the AK hamiltonian equations.

Furthermore, we follow the ansatz of Smolin in [13] and let Λ be generated by a scalar field φ_Λ with the constraint $\bar{\varphi}_\Lambda\varphi_\Lambda = \Lambda$

$$L_H = -\hbar c\left(\frac{1}{4}F_{\mu\nu}{}^{\kappa}F^{\mu\nu}{}_{\kappa} + \frac{\bar{\varphi}_\Lambda\varphi_\Lambda}{3}\left(\varepsilon^{\kappa}{}_{\mu\lambda}E^{\lambda\nu}\partial_\kappa A_{\mu\nu} + \varepsilon_{\mu_2\lambda_2\kappa_2}\varepsilon^{\mu_1\lambda_1}{}_{\kappa_1}E^{\kappa_1\kappa_2}A_{\lambda_1}{}^{\lambda_2}A_{\mu_1}{}^{\mu_2}\right)\right)$$

which brings the action to the correct dimension $[L_H] = \frac{energy * r}{r^4}$, because $[\varphi_\Lambda] = \frac{1}{r}$ and $[\Lambda] = \frac{1}{r^2}$, therefore this action is formally renormalizable.

If we carry out the variation for $\varphi_{\mu\nu,}$ we get the following expression

$$\frac{\delta L_H}{\delta\varphi_{\rho\sigma}} = -\hbar c\left(\frac{\varphi_\Lambda}{3}\left(\varepsilon^{\kappa}{}_{\mu\lambda}E^{\lambda\nu}\partial_\kappa A_{\mu\nu} + \varepsilon_{\mu_2\lambda_2\kappa_2}\varepsilon^{\mu_1\lambda_1}{}_{\kappa_1}E^{\kappa_1\kappa_2}A_{\lambda_1}{}^{\lambda_2}A_{\mu_1}{}^{\mu_2}\right)\right),$$

which becomes the Λ-gauge condition for the AK equations in the form

$$G_\Lambda = \varepsilon^{\mu\nu}{}_{\lambda}E^{\lambda\kappa}\partial_\mu A_{\nu\kappa} + \varepsilon_{\mu_2\mu\kappa_2}\varepsilon^{\nu}{}_{\mu\kappa_1}E^{\kappa_1\kappa_2}A_\mu{}^{\lambda_2}A_\nu{}^{\mu_2}, \quad G_\Lambda \mathrel{!}= 0$$

We use the hamiltonian equations, and after some algebra we get the expression

$$G_\Lambda = -\frac{\Lambda}{3}\sum_{\lambda,\rho}\left(E \bullet \eta \bullet E^t\right)^{\lambda\rho} + \sum_{\kappa,\lambda}E^{\lambda\kappa}\sum_{(\mu,\nu)=C(\kappa,\lambda)}\left(A_{\mu\mu}A_{\nu\nu} - A_{\mu\nu}A_{\nu\mu}\right),$$

where $(\mu,\nu) = C(\kappa,\lambda)$ is the complementary index pair.

For the classical case with Λ ≈ 0 with the constant half-antisymmetric background Ahab and the Gauss-Schwarzschild tetrad EGS the first term in GΛ is negligible and the second vanishes for $A = A_{hab}$.

In the general case, GΛ is a single gauge condition, which fixes one free parameter of the AK-solution.

11.2. Lagrangian of the Remaining Equations

For the diffeomorphism equations $I_\mu = E^{\kappa}{}_{\nu}F_{\mu\kappa}{}^{\nu}$, we set simply the variable $C_\mu := E^{\kappa}{}_{\nu}F_{\mu\kappa}{}^{\nu}$ and take the Lagrangian $L_I = \hbar c C_\mu C^\mu = \hbar c E^{\kappa_1}{}_{\nu_1}F_{\mu\kappa_1}{}^{\nu_1}E^{\kappa_2}{}_{\nu_2}F^{\mu}{}_{\kappa_2}{}^{\nu_2}$ as the corresponding Lagrangian

As for the gaussian equations $G^\mu = \partial_\nu E^{\nu\mu} + \varepsilon^{\mu}{}_{\kappa\lambda}A_\nu{}^{\kappa}E^{\nu\lambda}$, they can be derived from the fact, that this is the covariant derivative for the tetrad *E*, so it must vanish.

With

$$L_H = -\hbar c\left(\frac{1}{4}F_{\mu\nu}{}^{\kappa}F^{\mu\nu}{}_{\kappa} + \frac{\overline{\varphi}_\Lambda\varphi_\Lambda}{3}\left(\varepsilon^{\kappa}{}_{\mu\lambda}E^{\lambda\nu}\partial_\kappa A_{\mu\nu} + \varepsilon_{\mu_2\lambda_2\kappa_2}\varepsilon^{\mu_1\lambda_1}{}_{\kappa_1}E^{\kappa_1\kappa_2}A_{\lambda_1}{}^{\lambda_2}A_{\mu_1}{}^{\mu_2}\right)\right)$$

the complete AK Lagrangian is then (41)

$$\begin{aligned} L_{gr} &= L_H + L_I \\ &= \hbar c\left(-\frac{1}{4}F_{\mu\nu}{}^{\kappa}F^{\mu\nu}{}_{\kappa} - \frac{\overline{\varphi}_\Lambda\varphi_\Lambda}{3}\left(\varepsilon^{\kappa}{}_{\mu\lambda}E^{\lambda\nu}\partial_\kappa A_{\mu\nu} + \varepsilon_{\mu_2\lambda_2\kappa_2}\varepsilon^{\mu_1\lambda_1}{}_{\kappa_1}E^{\kappa_1\kappa_2}A_{\lambda_1}{}^{\lambda_2}A_{\mu_1}{}^{\mu_2}\right)\right. \\ &\quad \left. + E^{\kappa_1}{}_{\nu_1}F_{\mu\kappa_1}{}^{\nu_1}E^{\kappa_2}{}_{\nu_2}F^{\mu}{}_{\kappa_2}{}^{\nu_2}\right) \end{aligned}$$

11.3. Dirac Lagrangian for the Graviton

The Dirac Lagrangian for the photon reads where $\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c}$ is the fine-structure-constant (in the following $\alpha = \frac{e^2}{4\pi}$ in natural units $\hbar = c = \varepsilon_0 = 1$ used in particle physics) $L_{Dem} = \overline{\psi}\left(-\hbar c i\gamma^\mu D_\mu - mc^2\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where $D_\mu = \partial_\mu + \frac{ie}{\sqrt{\hbar c}}A_\mu$ or $D_\mu = \partial_\mu + i\sqrt{4\pi\alpha}A_\mu$ in natural units is the covariant derivative of the photon (note the negative sign in the first term: we use here the metric $\eta = \mathrm{diag}(-1, 1, 1, 1)$)

This describes the interaction of the photon with a fermion and yields the corresponding Feynman diagrams and cross sections.

The Dirac Lagrangian for the graviton reads

$$L_{Dgr} = \overline{\psi}\left(-\hbar c i\gamma^\mu D_\mu - mc^2\right)\psi + L_{gr} \tag{42}$$

where $\left(D_\mu\right)^\lambda{}_\kappa = \partial_\mu + \left(\varepsilon_a\right)^\lambda{}_\kappa A_\mu{}^a$ is the covariant derivative of the graviton, and the generator matrix is $\left(\varepsilon_a\right)^\lambda{}_\kappa = \varepsilon^\lambda{}_{\kappa_1\kappa}$

The electron-graviton interaction term is

$$\delta_I L_{Dgr} = -\hbar c i\,\overline{\psi}\left(\gamma^\mu\left(\varepsilon_a\right)^\lambda{}_\kappa A_\mu{}^a\right)\psi\,,$$

where $A_\mu{}^\nu = \Lambda\frac{As_\mu{}^\nu}{r}\exp(2i\theta)\exp\left(-ik(r-t)\right)$ is the graviton quadrupole wave function, so background $Ab \approx 0$, so the term is linear in As, like in the electromagnetic case.

The presence of Λ makes the term very small.

Let us compare this to the GR-Dirac Lagrangian

$$L_{GRD} = -\frac{\sqrt{\det(-g)}}{2\kappa}(R - 2\Lambda) + \sqrt{-g}\,\overline{\psi}\left(i\hbar c\gamma^\mu(x)\nabla_\mu - mc^2\right)\psi$$

where $\nabla_\mu\psi = \left(\partial_\mu - \frac{i}{4}\omega^{ab}{}_\mu\sigma_{ab}\right)\psi$ and $\sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right]$ are the Dirac σ-matrices and ω the GR connection field in tetrad-expression

$$\omega^{ab}{}_\mu = \frac{1}{2}e^{a\nu}\left(\partial_\mu e^b{}_\nu - \partial_\nu e^b{}_\mu\right) + \frac{1}{4}e^{a\rho}e^{b\sigma}\left(\partial_\sigma e^c{}_\rho - \partial_\rho e^c{}_\sigma\right) - (a \leftrightarrow b)$$

with the tetrad $e^a{}_\mu e^a{}_\nu = g_{\mu\nu}$ *i.e.* $e \bullet \eta \bullet e = g$ compared to the metric condition for the inverse densitized background tetrad Eb

$$Eb \bullet \eta \bullet Eb^t = g^{-1} \Big/ \left(-\det(g)\right)^{3/4}, \text{ so } \; e = \left(Eb^{-1}\right)^t \Big/ \left(-\det(g)\right)^{3/8}$$

Here the interaction term is

$$\begin{aligned}\delta_I L_{GRD} &= -\frac{\hbar c}{4}\sqrt{\det(-g)}\,\bar{\psi}\left(\gamma^\mu \omega_\mu{}^{ab}\sigma_{ab}\right)\psi \\ &= -\frac{\hbar c}{4}\sqrt{\det(-g)}\,\bar{\psi}\left(\sum_\mu \gamma^\mu f^\mu\left(Eb^{-1}\right)\right)\psi\end{aligned}$$

where the middle term $\sum_\mu \gamma^\mu f^\mu\left(Eb^{-1}\right)$ is a sum of γ-matrices with coefficients, which are quadratic functions of Eb^{-1} so $\delta_I L_{GRD}$ is quite different from the AK-interaction term $\delta_I L_{Dgr}$.

11.4. The Graviton Wave Function and Cross-Sections

For the Compton effect, *i.e.* electron-photon scattering the Thompson cross-section for small energies is

$$\sigma_{th} = \alpha^2 \left(\frac{\hbar c}{mc^2}\right)^2 \frac{8\pi}{3} = 0.665 \times 10^{-24}\,\text{cm}^2,$$

where $m = m_e$ is the electron mass, and the reduced de-Broglie wavelength of the electron $\tilde{\lambda}_e = \frac{\hbar c}{m_e c^2} = 0.38 \times 10^{-12}$ m.

So the electron-photon Thompson cross-section is with these denominations $\sigma_{th} = \alpha^2 \frac{1}{\tilde{\lambda}_e^{\,2}} \frac{8\pi}{3}$

The photon wave function is here ([1] 7.53)

$$\left(A_e\right)^\mu = \frac{\varepsilon^\mu}{\sqrt{2kV}}\left(\exp(-ik \bullet x) + \exp(ik \bullet x)\right)$$

where ε^μ is unit-polarization vector, $k^\mu k_\mu = 0$ and $\varepsilon^\mu k_\mu = 0$.

A^μ is normalized to give the energy $E\left(A^\mu\right) = \hbar c \int \left(\nabla \times A\right)^2 d^3x = \hbar\omega = \hbar ck$

We use the results from chap. Solutions of the Gravitational Wave Equation

$$As30 = As30c\, i \exp\left(-\frac{4\sqrt{r}}{\sqrt{3}}\right)\frac{r^{17/12}}{6} \to 0$$

$$As10 = -\frac{As20c}{2}\exp(2i\theta)$$

$$As00 = -\frac{As20c}{2}\exp(2i\theta)$$

$$As20 = \frac{As20c}{r}\exp(2i\theta) \to 0$$

and write the graviton wave function as a plane wave analogous to the photon (the quadrupole characteristics disappear in the plane wave, therefore $\exp(2i\theta)$ is skipped) $\left(A_g\right)_\mu{}^\nu = \Omega_\mu{}^\nu \frac{1}{2}\Lambda f_m \frac{r_s{}^2\sqrt{\pi}}{2\sqrt{2}r_0}\frac{r_s{}^{3/2}}{\sqrt{2V}}\left(\exp(-ik \bullet x) + \exp(ik \bullet x)\right)$, with the polarization matrix according to the results from chap. 7.3 is a combination of the 4 columns of

$$\Omega_\mu^{\ \nu} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

According to chap. 10 we get now for the energy density

$$t_{00} = t_{11} = \left(2k\,As00\right)^2 \frac{\hbar c}{\Lambda^2 l_p^{\ 2} r_s^{\ 2}}$$

$$t_\mu^{\ \mu} = 8k^2 As00^2 \frac{\hbar c}{\Lambda^2 l_p^{\ 2} r_s^{\ 2}},$$

and as $\int_V \frac{1}{2V}\left(\exp(-ik \bullet x) + \exp(ik \bullet x)\right)^2 d^3x = 1$ and $k = \sqrt{\frac{r_s}{2r_0^3}}$, we get for the energy

$$E\left(A_g\right) = \int_V t_\mu^{\ \mu}\, d^3x = \frac{r_s \hbar c}{l_p^{\ 2}} \frac{\pi f_m^{\ 2}}{4^{2/3}} \left(kr_s\right)^{10/3},$$

now we demand that $E\left(A_g\right) = \hbar ck$, so the normalization factor is

$c_n = \dfrac{1}{\frac{r_s}{l_p} \frac{\sqrt{\pi} f_m}{4^{1/3}} \left(kr_s\right)^{7/6}}$ and the normalized wave function becomes

$$\left(A_{gn}\right)_\mu^{\ \nu} = \left(A_g\right)_\mu^{\ \nu} c_n = \Omega_\mu^{\ \nu} \sqrt{\alpha_{gr}} \frac{1}{\sqrt{kr_s}} \frac{r_s^{1/2}}{\sqrt{2V}} \left(\exp(-ik \bullet x) + \exp(ik \bullet x)\right),$$

where $r_s = r_{gr}$,

$$\left(A_g\right)_\mu^{\ \nu} = \Omega_\mu^{\ \nu} \frac{1}{2} \Lambda f_m \frac{r_s^{\ 2}\sqrt{\pi}}{2\sqrt{2r_0}} \frac{r_s^{\ 3/2}}{\sqrt{2V}} \left(\exp(-ik \bullet x) + \exp(ik \bullet x)\right) \tag{43a}$$

$\sqrt{\alpha_{gr}} = \frac{r_{gr}\Lambda l_p}{\sqrt{2}} = 0.55 \times 10^{-91}$ and α_{gr} is the gravitational fine structure constant and the photon-like wave function can be written

$$\left(A_{gn}\right)_\mu^{\ \nu} = \sqrt{\alpha_{gr}} \left(A_p\right)_\mu^{\ \nu} \tag{43b}$$

With running Λ we get a much higher value in the quantum regime $r < r_{gr}$

$$\sqrt{\alpha_{gr}} = g_{gr}\left(k = \frac{1}{r_{gr}}\right) = \frac{2}{M^2 r_{gr}^2 + 2w_*} = 1.75 \times 10^{-11} \quad \text{(see chap. 11.9).}$$

The covariant derivative is then

$$\left(D_\mu\right)^\lambda_{\ \kappa} = \partial_\mu + \left(\varepsilon_a\right)^\lambda_{\ \kappa} \sqrt{\alpha_{gr}} \left(A_p\right)_\mu^{\ a} \tag{44}$$

where A_p is completely analogous to the photon wave function A_e, and matrices $\left(\varepsilon_a\right)^\lambda_{\ \kappa} = \varepsilon^\lambda_{\ a\kappa}$ $a = 0,1,2,3$ in analogy to the Dirac gamma-matrices, and we have the correspondence $\alpha_{gr} \leftrightarrow 4\pi\alpha$ between the gravitational and the electromagnetic fine structure constants.

By analogy we can then assess the electron-graviton scattering cross-section $\sigma_{eg} \approx \left(\frac{\alpha_{gr}}{4\pi}\right)^2 \frac{1}{\tilde{\lambda}_e^{\,2}}$, ignoring the tensor form and the θ-dependence.

11.5. The Graviton Propagator

As is well known, the photon propagator in QFT is [1]

$$D(q) = \frac{-1}{q^2 + i\varepsilon},$$

which follows from the Maxwell equations $\Box A_\mu(x) = J_\mu(x)$

The classical graviton propagator in GR is [3]

$$G_{\alpha\beta,\mu\nu} = \frac{g_{\alpha\mu} g_{\beta\nu} - g_{\beta\mu} g_{\alpha\nu} + g_{\mu\nu} g_{\alpha\beta}}{k^2 + i\varepsilon}$$

with non-zero elements for Minkowski metric

$$G_{\alpha\beta,\alpha\beta} = \frac{g_{\alpha\alpha} g_{\beta\beta}}{k^2 + i\varepsilon} \quad G_{\alpha\alpha,\beta\beta} = -\frac{g_{\alpha\alpha} g_{\beta\beta}}{k^2 + i\varepsilon}$$

We consider the wave equations eqgravlxA0, eqgravlxEn for spherical A-waves and spherical E-waves in the momentum representation, *i.e.* in the k-space.

Then the r-derivatives transform into k-powers under the Fourier transform

$$\partial_r^{\,n} f(r,k) = (ik)^n f(r,k)$$

At infinity for the basic E-wave $Es10(r)$:

$$\text{eqgravlxEninf} = -2k^2 Es10'(r) - 3ik\, Es10''(r) + Es10'''(r)$$

and after rescaling $r \to kr$

$$-2Es10'(r) - 3i\, Es10''(r) + Es10'''(r)$$

For the Fourier transform we have then the propagator

$$D(q) = \frac{1}{q^3 - 3iq^2 - 2q} = \frac{1}{q(q-i)(q-2i)}$$

and for the basic A-wave $As00(r)$:

$$\begin{aligned}
&\text{eqgravlxA0} \\
&= 3(+i\,lx)(lx + kr^2)\,As00(r) - r\Big(\big(-1 + lx^2 + 2lx(-i + kr)\big)Es10(r) \\
&\quad + 3(lx + kr)^2 As00'(r) + r\,(1 + 2i\,lx + 2i\,kr)\,Es10'(r) - rEs10''(r)\Big)
\end{aligned}$$

where lx is the spin, here $lx = 2$ for the simplest quadrupole wave.

And at infinity for the basic A-wave $As00(r)$:

$$\text{eqgravlxA0inf} = 3r^2 k^2 As00'(r) + 2i\,kr^2 Es10'(r)$$

it follows: $As00'(r) = \frac{2i}{k} Es10'(r)$, *i.e.* $As00(r) = c_1 Es10(r) + c_0$ with constants c_0, c_1 so for A-wave we get the same propagator as above.

The propagator $D(q)$ is finite-integrable in d^3q.

The propagator must have dimension 1/energy2, so it gets as factor an energy-constant of the first vertex, the only such constant is the mass m of the corresponding fermion.

The final form of the graviton propagator is then

$$D_g(q,m) = \frac{m}{q(q-im)(q-2im)} \tag{45}$$

where m = mass of source fermion.

11.6. The Gravitational Compton Cross Section

For the Compton effect, *i.e.* electron-photon scattering (**Figure 18**).

The total Klein-Nishina cross-section [3] is

$$\sigma = \left(\frac{8\pi\alpha^2}{4m^2}\right)\left(\frac{1+a}{a^3}\left(\frac{2a(1+a)}{1+2a} - \log(1+2a)\right) + \frac{\log(1+2a)}{2a} - \frac{1+3a}{(1+2a)^2}\right),$$

where $a = \frac{k}{m}$, for small energies it becomes the Thompson cross-section $\sigma_{th} = \frac{\alpha^2}{m^2}\frac{8\pi}{3}$.

The start formula for the calculation of the differential cross-section according to the Feynman rules is ([1] 7.7.2, [2] 4.218)

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{1}{2}\sum_{\pm s_i, \pm s_f}\frac{d\sigma}{d\Omega}$$
$$= \frac{\alpha^2}{2}\left(\frac{k'}{k}\right)^2 Tr\frac{\not{p}_f + m}{2m}\left(\frac{\not{\epsilon}'\not{\epsilon}\not{k}}{2k\cdot p_i} + \frac{\not{\epsilon}\not{\epsilon}'\not{k}'}{2k'\cdot p_i}\right)\frac{\not{p}_i + m}{2m}\left(\frac{\not{k}\not{\epsilon}'\not{\epsilon}}{2k\cdot p_i} + \frac{\not{k}'\not{\epsilon}'\not{\epsilon}}{2k'\cdot p_i}\right)$$

with the initial and final momenta p_i p_f of the electron, k k' momenta of the photon and polarizations ε ε' of the photon. The following conditions have to be satisfied:

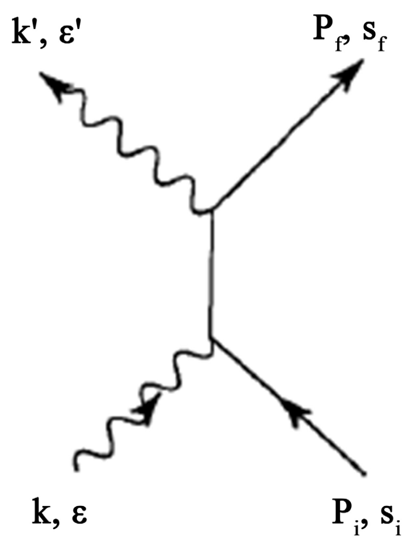


Figure 18. Gravitational Compton scattering.

$p_i \bullet p_i = m^2 \quad p_f \bullet p_f = m^2 \quad k \bullet k = k' \bullet k' = 0$ energy relations

$p_f + k' = p_i + k$ 4-momentum conservation

$k_0' = \frac{k_0}{1 + \frac{k_0}{m}(1 - \cos\theta)}$ Compton condition for the photon energy

There is 3 degrees of freedom in the choice of the polarization, the choice is made to simplify the expression above

$$\varepsilon \bullet \varepsilon = \varepsilon' \bullet \varepsilon' = -1 \quad \varepsilon \bullet k = \varepsilon' \bullet k' = 0 \quad \varepsilon \bullet p_i = \varepsilon' \bullet p_i = 0$$

After some manipulations using the conditions and commutation rules for Dirac matrices, the famous Klein-Nishina formula results [1] 7.74

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{\alpha^2}{4}\left(\frac{\hbar c}{mc^2}\right)^2 \left(\frac{k'}{k}\right)^2 \left(\frac{k'}{k} + \frac{k}{k'} + 4(\varepsilon'\varepsilon)^2 - 2\right),$$

where the scalar denomination $k\,k'$ is used for the energy $k_0\,k_0'$.

We get the total cross-section using the Compton condition and integrating over $z = \cos\theta$

$$\bar{\sigma} = \pi\alpha^2 \int_{-1}^{+1} dz \left(\frac{1}{\left(1 + a(1-z)\right)^3} + \frac{1}{\left(1 + a(1-z)\right)} - \frac{1 - z^2}{\left(1 + a(1-z)\right)^2}\right)$$

and averaging over polarizations ([2] 4.221)

$$\overline{(\varepsilon' \cdot \varepsilon)^2} = \sum_{ij} \frac{1}{2}\left(\delta_{ij} - \frac{k^i k^j}{k^2}\right)\left(\delta_{ij} - \frac{k'^i k'^j}{k'^2}\right)$$

for small energies $\frac{k}{m} \to 0$, the Thomson cross section arises

$$\sigma_{th} = \frac{\alpha^2}{m^2}\frac{8\pi}{3} = \alpha^2 \tilde{\lambda}_e^2 \frac{8\pi}{3}$$

For the graviton, we insert the photon-like (dimensionless, dropping the scale $r_s = r_{gr}$) wave function $\left(A_p\right)_\mu^{\ \nu} = \Omega_\mu^{\ \nu} \frac{1}{\sqrt{2Vk}}\left(\exp(-ik \bullet x) + \exp(ik \bullet x)\right)$

with the covariant derivative $\left(D_\mu\right)^\lambda_{\ \kappa} = \partial_\mu + \left(\varepsilon_a\right)^\lambda_{\ \kappa} \sqrt{\alpha_{gr}} \left(A_p\right)_\mu^{\ a}$

and the gravitational fine structure constant $\sqrt{\alpha_{gr}} = \frac{r_{gr}\Lambda l_p}{\sqrt{2}} = 0.55 \times 10^{-91}$

and again the starting formula above, where the only change is in the polarization terms $\not{\varepsilon} = \varepsilon_\mu \gamma^\mu$ and $\not{\varepsilon}' = \varepsilon'_\mu \gamma^\mu$, which, with the setting

$\Omega_\mu^{\ \nu} = \begin{pmatrix} e_0 & e_1 & e_2 & e_3 \\ e_0 & e_1 & e_2 & e_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and the new initial polarization $e_\mu = \left(e_0, e_1, e_2, e_3\right)$

and final polarization $e'_\mu = \left(e'_0, e'_1, e'_2, e'_3\right)$.

$\|e\| = \sqrt{e_0^{\ 2} + e_1^{\ 2} + e_2^{\ 2} + e_3^{\ 2}} = \|e'\| = 1$, and the totally antisymmetric matrices ε_a, become

$$\varepsilon_\alpha^{\ \lambda_1}{}_\kappa \Omega_\mu^{\ \alpha} \left(\gamma^\mu\right)^\kappa_{\ \lambda_2} = e^\alpha \left(\varepsilon_\alpha\right)\left(\gamma^0 + \gamma^1\right) = e^\alpha g_\alpha,$$

where $g_\alpha = \varepsilon_\alpha(\gamma^0 + \gamma^1)$ are the matrices analogous to the γ-matrices in the "Dirac-dagger" $\not{\varepsilon} = \varepsilon_\mu \gamma^\mu$ in the quantum-electrodynamics.

After going into the rest frame of the electron $p = (m, 0, 0, 0)$ and some manipulations we get

$$\frac{d\bar{\sigma}}{d\Omega} = \left(\frac{\alpha_{gr}}{4\pi}\right)^2 \frac{1}{32m^2}\left(\frac{k_0'}{k_0}\right)^2 \left(d_{s0}(e, e', \theta) + \frac{k_0}{m} d_{s1}(e, e', \theta) + O\left(\frac{k_0^{\ 2}}{m^2}\right)\right),$$

where the functions $d_{s0}(e, e', \theta)$ and $d_{s1}(e, e', \theta)$ are series-coefficients in the $\frac{k_0}{m}$-series.

Now perform the integration over θ and averaging over e_μ and $e'_\mu = e_\mu$ to get the total cross-section

$$\bar{\sigma} = \frac{\alpha_{gr}^{\ 2}}{2\pi}\tilde{\lambda}_e^{\ 2}\left(1.170 + \frac{k_0}{m}0.400 + \cdots\right) \approx 1.170\frac{\alpha_{gr}^{\ 2}}{2\pi}\tilde{\lambda}_e^{\ 2} \tag{46}$$

where the last expression is the gravitational low-energy Thomson cross-section.

The different form of the bracket expression in the differential cross-section as compared to the electromagnetic cross-section is due to the different nature of polarization: for the photon the polarization is transversal to the momentum, so the averaging depends on k and k', for the graviton the polarization is a free parameter independent of momentum.

11.7. The Gravitational Electron-Electron Cross Section

In QED, the electron-electron scattering via photon exchange is called Moeller-scattering.

The corresponding two Feynman diagrams are called t-channel and u-channel (**Figure 19**).

In the center-of mass system, the 4-momentums read [3] [38]

$$p_1 = (E, 0, 0, p) \quad p_2 = (E, 0, 0, -p)$$

$$p_3 = (E, p\sin\theta, 0, p\cos\theta), \quad p_4 = (E, -p\sin\theta, 0, -p\cos\theta)$$

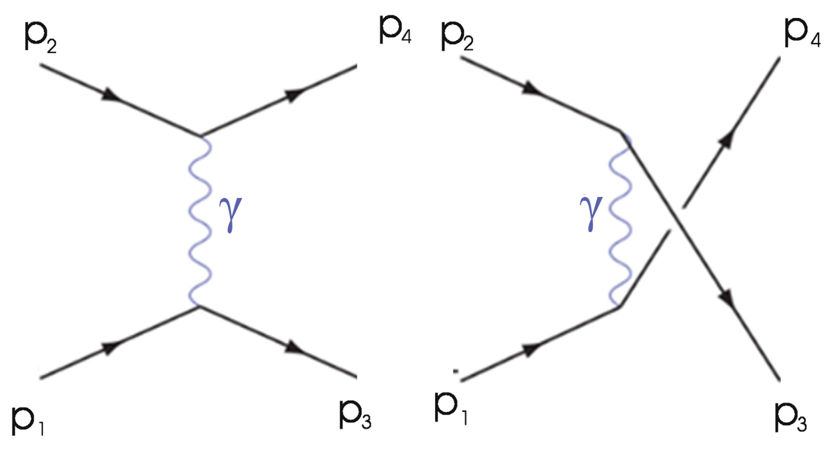


Figure 19. Gravitational electron-electron interaction, t-channel left, u-channel right.

Using the Mandelstamm variables

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$

$$t = (p_1 - p_3)^2 = (p_4 - p_2)^2$$

$$u = (p_1 - p_4)^2 = (p_3 - p_2)^2$$

we get the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{8E^2}\overline{|M|^2},$$

where $|M|^2$ is the usual Feynman matrix element

$$\begin{aligned} |M|^2 = &\frac{1}{t^2}\left(\bar{u}(p_3)\gamma^\mu u(p_1)\right)\left(\bar{u}(p_1)\gamma^\nu u(p_3)\right)\left(\bar{u}(p_4)\gamma_\mu u(p_2)\right)\left(\bar{u}(p_2)\gamma_\nu u(p_4)\right) \\ &\frac{1}{u^2}\left(\bar{u}(p_3)\gamma^\mu u(p_2)\right)\left(\bar{u}(p_2)\gamma^\nu u(p_3)\right)\left(\bar{u}(p_4)\gamma_\mu u(p_1)\right)\left(\bar{u}(p_1)\gamma_\nu u(p_4)\right) \\ &-\frac{1}{tu}\left(\bar{u}(p_3)\gamma^\mu u(p_1)\right)\left(\bar{u}(p_2)\gamma^\nu u(p_3)\right)\left(\bar{u}(p_4)\gamma_\mu u(p_2)\right)\left(\bar{u}(p_1)\gamma_\nu u(p_4)\right) \\ &-\frac{1}{tu}\left(\bar{u}(p_3)\gamma^\mu u(p_2)\right)\left(\bar{u}(p_1)\gamma^\nu u(p_3)\right)\left(\bar{u}(p_4)\gamma_\mu u(p_1)\right)\left(\bar{u}(p_2)\gamma_\nu u(p_4)\right) \end{aligned}$$

which is averaged over spins to get the general unpolarized result

$$\begin{aligned} |M|^2 = &\frac{1}{t^2}Tr\left(\gamma^\mu(\not{p}_1 + m)\gamma^\nu(\not{p}_3 + m)\right)Tr\left(\gamma_\mu(\not{p}_2 + m)\gamma_\nu(\not{p}_4 + m)\right) \\ &\frac{1}{u^2}Tr\left(\gamma^\mu(\not{p}_2 + m)\gamma^\nu(\not{p}_3 + m)\right)Tr\left(\gamma_\mu(\not{p}_1 + m)\gamma_\nu(\not{p}_4 + m)\right) \\ &-\frac{2}{tu}Tr\left((\not{p}_3 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu\right) \end{aligned}$$

The resulting Moeller differential cross-section reads [1] [3] [38]

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2p^4\sin^4\theta}\left(m^4 + 4m^2p^2 + 9p^4 + 3\left(m^4 + 4m^2p^2 + 2p^4\right)\cos^2\theta + p^4\cos^4\theta\right)$$

The high-energy limit $p \gg m$ becomes

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2\sin^4\theta}\left(3 + \cos^2\theta\right)^2$$

The low-energy limit $p \ll m$ becomes

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2m^4}{4E^2\sin^4\theta}\left(1 + 3\cos^2\theta\right)$$

In the gravitational Feynman diagram with graviton exchange we have to replace the propagators $\left(\frac{1}{q^2}\right)^2$ in the matrix element, which here appear as $\frac{1}{t^2}$, $\frac{1}{u^2}$, $\frac{1}{tu}$, expressions in Mandelstamm variables, by the corresponding squared real gravitational propagator $|D(q,m)|^2 = \frac{m}{q^2(q^2+1)(q^2+4)}$.

The modified propagator-factors in the matrix element become then

$\frac{1}{t^2}=\frac{m}{q^2\left(q^2+m^2\right)\left(q^2+4m^2\right)}$, where q^2 is replaced $q^2\to\left(p_1-p_3\right)^2$ by the corresponding Mandelstamm variable $\frac{1}{u^2}=\frac{m}{q^2\left(q^2+m^2\right)\left(q^2+4m^2\right)}$, where q^2 is replaced $q^2\to\left(p_1-p_4\right)^2$ $\frac{1}{tu}=\frac{m}{q^2\left(q^2+m^2\right)\left(q^2+4m^2\right)}$, where q^2 is replaced $q^2\to\left|p_1-p_3\right|\left|p_1-p_4\right|$.

With these adaptations we get the gravitational differential cross section with graviton exchange (47a)

$$\frac{d\sigma_{gr}}{d\Omega}=\frac{{\alpha_{gr}}^2}{8E^2}\left(-\frac{m^2\left(24m^4+16\left(m^2+p^2\right)^2-8m^2\left(4\left(m^2+p^2\right)2m^2+2p^2\left(-1+\cos\theta\right)\right)\right)+4p^4\left(-1+\cos\theta\right)^2}{\left(2m^2-2\left(m^2+p^2\left(1+\cos\theta\right)\right)\right)\left(3m^2-2\left(m^2+p^2\left(1+\cos\theta\right)\right)\right)\left(6m^2-2\left(m^2+p^2\left(1+\cos\theta\right)\right)\right)}\right.$$
$$+\frac{m^2\left(24m^4+16\left(m^2+p^2\right)^2-8m^2\left(4\left(m^2+p^2\right)2m^2-2p^2\left(-1+\cos\theta\right)\right)\right)+4p^4\left(-1+\cos\theta\right)^2}{\left(2m^2-2\left(m^2-p^2\left(1+\cos\theta\right)\right)\right)\left(3m^2-2\left(m^2-p^2\left(1+\cos\theta\right)\right)\right)\left(6m^2-2\left(m^2+p^2\left(1-\cos\theta\right)\right)\right)}$$
$$\left.+\frac{\left(12m^4-32\left(m^2+p^2\right)+16\left(m^2+p^2\right)^2\right)}{2\sin\theta p^2\left(2m^2+5p^2\sin\theta+2p^4\sin^2\theta/m^2\right)}\right)$$

where α_{gr} is the gravitational fine structure constant (see chap. 11.9)

$$\sqrt{\alpha_{gr}}=g_{gr}\left(k=\frac{1}{r_{gr}}\right)=\frac{2}{M^2r_{gr}^2+2w_*}=1.75\times10^{-11}$$

The high-energy limit $p\gg m$ becomes

$$\frac{d\sigma_{gr}}{d\Omega}=\frac{{\alpha_{gr}}^2m^2}{64E^2p^2\sin^6\theta}\left(143+108\cos2\theta+5\cos4\theta-24\sin\theta+8\sin3\theta\right)\quad(47b)$$

The low-energy limit $p\ll m$ becomes

$$\frac{d\sigma_{gr}}{d\Omega}=\frac{{\alpha_{gr}}^2m^6}{8E^2p^6\sin^3\theta}\quad(47c)$$

11.8. The Role of Gravity in the Objective Collapse Theory

The objective collapse theory put forward by Penrose [39], links the spontaneous collapse of the wave function to quantum gravitation, the limit being one graviton. In the formulation of Ghirardi-Rimini-Weber (GRW) the wave function collapse is characterized by the wave function width rc and by the decay rate λ.

In the recent test of collapse models carried out by Bassi *et al.* [40], possible values of these parameters are measured:

As shown above in **Figure 20**, $\lambda(r_c)$ has a minimum at $r_c=10^{-5}$ m, which is a good candidate for the limit of the quantum regime, and there is $\lambda(r_c)=10^{-11}\mathrm{s}^{-1}$.

This is in good agreement with the quantum limit

$r_{gr}=\sqrt{l_P\sqrt{\frac{1}{\Lambda}}}=3.9\times10^{-5}\ \mathrm{m}=39\ \mu\mathrm{m}$ of the AK-gravitation.

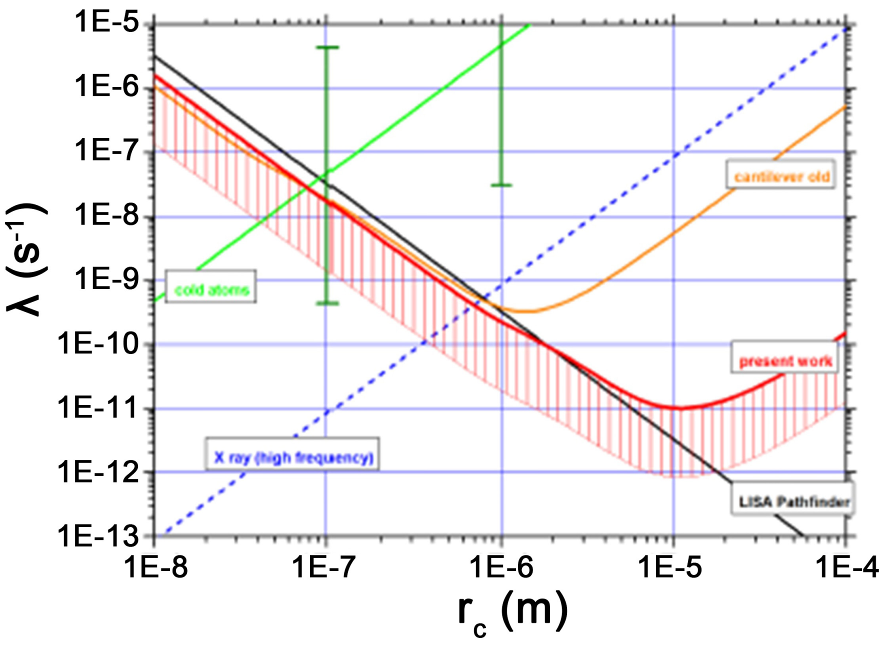


Figure 20. Objective wave-function collapse measurement [40].

The decay rate can be assessed from $\lambda = \frac{E_{gr}}{\hbar} = 0.19 \times 10^{-11}\ \mathrm{s}^{-1}$, where the gravitational energy $E_{gr} = \frac{G m_e^2}{r_e} = 1.22 \times 10^{-27}\ \mathrm{eV}$, where m_e is the electron mass and $r_e = 2.8 \times 10^{-15}\ \mathrm{m}$ is the classical electron radius.

11.9. The Running Cosmological Constant

The gravitational coupling constant $g_{gr} = \sqrt{\alpha_{gr}}$, where $\sqrt{\alpha_{gr}} = \frac{r_{gr} \Lambda l_p}{\sqrt{2}} = 0.69 \times 10^{-91}$ and $r_{gr} = \sqrt{l_P \sqrt{\frac{1}{\Lambda}}} = 39\ \mu\mathrm{m}$, is expected to depend on the momentum *k*, it is a "running coupling", like its counterparts in the QFT.

- lambda flow equation

In [11] Christof Wetterich derives a "flow equation" for $w(k) = g_{gr}^{-1}(k)$, which is analogous to the Callan-Symanzik equation in QCD.

The flow equation is

$$\frac{\partial w(k)}{\partial \ln k} = -2w(k) + 2c_M,$$

it has the same form as the flow equation for the mass generating Higgs potential $\lambda(k)$

$$\frac{\partial \lambda(k)}{\partial \ln k} = A\lambda(k)$$

The solution of the gravitational flow equation is $w(k) = \frac{M^2}{2k^2} + w_*$ with a dimensionless constant w_* and a mass *M*, so we get for g_{gr}

$$g_{gr}(k)=\frac{r_{gr}\Lambda l_p}{\sqrt{2}}=w^{-1}(k)=\frac{2k^2}{M^2+w_*2k^2}\text{, so }\frac{\Lambda^{3/4}l_p^{\,3/2}}{\sqrt{2}}=\frac{2k^2}{M^2+w_*2k^2}$$

we determine M from the "macroscopic" cosmological constant $\Lambda=\Lambda_0=1.1\times10^{-52}\,\mathrm{m}^{-2}$ for the smallest possible k-value $k=\sqrt{\Lambda_0}$, neglecting the second term in the denominator: $\frac{\Lambda_0^{\,3/4}l_p^{\,3/2}}{\sqrt{2}}=\frac{2\Lambda_0}{M^2}$ $M^2=0.45\times10^{40}\,\mathrm{m}^{-2}$ (we use the $\hbar=c=1$ units, the only unit is meter m).

The limit of the expression for $g_{gr}(k)$ is $g_{gr}(k\to\infty)=\frac{1}{w_*}$ and the turning point k_{cr} before the limit is reached, when the 2 terms in the denominator become equal $M^2=w_*2k_{cr}^{\,2}$, we set the turning point to the energy scale of the classical electron radius $r_{ce}=2.5\times10^{-15}\,\mathrm{m}$, $k_{cr}=\frac{1}{r_{ce}}$ and get

$$w_*=\frac{M^2}{2k_{cr}^{\,2}}=3.45\times10^{10}.$$

Now we can calculate the running $\Lambda=\left(\frac{2\sqrt{2}k^2}{M^2+w_*2k^2}\right)^{4/3}l_p^{\,-2}$ with the limit

$$\Lambda_{inf}=\Lambda(k\to\infty)=\left(\frac{\sqrt{2}}{w_*}\right)^{4/3}l_p^{\,-2}=0.34\times10^{56}\,\mathrm{m}^2 \tag{48a}$$

which is one order larger than $\Lambda_{GUT}=0.14\times10^{55}\frac{1}{\mathrm{m}^2}$ (see below), and for the dimensionless interaction constant we get

$$\sqrt{\alpha_{gr}}=g_{gr}\left(k=\frac{1}{r_{gr}}\right)=\frac{2}{M^2r_{gr}^2+2w_*}=1.75\times10^{-11} \tag{48b}$$

In the quantum Ashtekar-Kodama equations we get

$$\Lambda_{dl}=\Lambda_{\inf}l_P^{\,2}=\left(\frac{\sqrt{2}}{w_*}\right)^{4/3}=1.4\times10^{-14} \tag{48c}$$

so the coupling of $E^{\nu\mu}$ to $A_\nu^{\ \kappa}$ is still weak, and after imposing the Minkowski boundary metric condition we get the Minkowski metric in the quantum regime (see chap. 6.1).

- quantum vacuum energy and cosmic inflation

In quantum field theory, one can assess the quantum vacuum energy, based on the collapse of the original unified interaction (GUT) [3], the result is about 100 orders of magnitude larger than the cosmological constant Λ, which constitutes the Dark-Energy, the cosmological vacuum energy density. This well-known discrepancy is the so called "vacuum energy density problem".

The breakdown of GUT happens at the energy $E_{GUT}\approx10^{15}\,\mathrm{GeV}$ and the corresponding curvature is [3]

$\Lambda_{GUT}=\frac{1}{3}\kappa E_{GUT}^{\ \ 4}/(\hbar c)^3$, where κ is the Einstein constant with dimensionality

$\kappa = \frac{8\pi G_N}{c^4} = 2.07\times10^{-43}\,\frac{\text{m}}{\text{J}}$ and $\Lambda_{GUT} = 0.14\times10^{55}\,\frac{1}{\text{m}^2}$.

So the ratio of Λ_{GUT} and the cosmological constant $\Lambda_0 = 1.1\times10^{-52}\,\text{m}^{-2}$ is

$$\frac{\Lambda_{GUT}}{\Lambda_0} = \frac{\frac{1}{3}\kappa E_{GUT}{}^4/(\hbar c)^3}{\Lambda_0} = \frac{0.14\times10^{55}\,\text{m}^{-2}}{1.1\times10^{-52}\,\text{m}^{-2}} = 0.13\times10^{107}$$

The blow-up of the cosmological constant in the quantum regime causes the cosmological inflation with expansion factor $f_{\inf} = \exp\left(r_{\inf}\sqrt{\frac{\Lambda}{3}}\right) \ge 10^{30}$ ($f_{\inf} \ge 10^{30}$ is required to produce the degree of homogeneity and isotropy measured in the cosmic microwave background CMB [41]), between $r = l_p = 1.61\times10^{-35}\,\text{m}$ and limit of the quantum regime $r_{gr} = 3.1\times10^{-5}\,\text{m}$ by the factor $f_{\inf} = \frac{r_{gr}}{l_p} = \frac{3.1\times10^{-5}}{1.61\times10^{-35}} = 1.9\times10^{30}$.

The condition for the expansion factor

$$f_{\inf} = \exp\left(r_{\inf}\sqrt{\frac{\Lambda_{\inf}}{3}}\right) = 1.9\times10^{30} = \exp(69.7),$$

yields $r_{\inf} = 2\times10^{-26}\,\text{m}$ for the end of cosmic inflation with the inflation energy

$$E_{\inf} = \frac{\hbar c}{r_{\inf}} = \frac{1.96\times10^{-16}\,\text{GeV}}{2\times10^{-26}\,\text{m}} = 0.98\times10^{10}\,\text{GeV}$$

The resulting values $r_{\inf}$ and $E_{\inf}$ fit well to the assessed and calculated values in the Lambda-CDM model of cosmic evolution [41].

11.10. The Quasi-Classical Approximation in the Quantum Regime

In quantum electrodynamics (QED), we can use the Schrödinger equation with classical electrostatic self-energy $E_{el}(r) = \frac{e_0{}^2}{4\pi\varepsilon_0 r} = m_e c^2 \frac{r_{ce}}{r}$ of an electron for $r \gg r_{ce} = \frac{e_0{}^2}{4\pi\varepsilon_0 m_e c^2} = 2.8\times10^{-15}\,\text{m}$, where r_{ce} is the classical electron radius (quasi-classical approximation). Below r_{ce}, we have to use the full Dirac equation coupled to the Maxwell equation for the electron wave function ψel and the photon wave function Aμ.

In AK-gravity in the quantum regime $r < r_{gr}$, we have the Minkowski metric with approximately Gauss-Minkowski tetrad $E_{GM}(r,\theta)$ and lambda correction $l_0 \approx 64\Lambda r_s^3$, where the scale is mass-independent $r_s = l_P$, so $l_0 \approx 64\Lambda r_s^3$ with potential energy $E_{gr} = -mc^2\frac{l_0}{r}$. Inserting $\Lambda = \Lambda_{GUT}$ we obtain $E_{gr} = -mc^2\frac{r_\Lambda}{r}$,

$$\text{where}\quad r_\Lambda = 64\Lambda_{GUT} l_P{}^3 = 3.7\times10^{-49}\,\text{m} \tag{49a}$$

is the quasi-classical gravitational interaction radius, in analogy to r_{ce}.

The equation-of-motion: Dirac equation in weak-field approximation

$\left(i\hbar\gamma^{\mu}\partial_{\mu} - mc\frac{r_{\Lambda}}{r} - mc\right)\Psi = 0$, with the self-interaction potential energy

$V_N(r) = -mc^2\frac{r_{\Lambda}}{r}$

So in the **quasi-classical approximation in AK-quantum gravity** reads

$$E_{gr}(r) = \frac{r_{\Lambda}}{r}mc^2 \quad \text{for} \quad r \ll r_{gr} \tag{49b}$$

The gravitational interaction potential of two masses m_1, m_2 is the self-interaction of the reduced mass

$$V_N(r) = -\frac{m_1 m_2}{m_1 + m_2}c^2\frac{r_{\Lambda}}{r} \tag{49c}$$

If we consider in the quantum regime a medium with gas/fluid/solid with particle distance r_p and mass m_p, then there will be a characteristic time $t_p = \frac{r_p}{c}$, dissipative energy $E_{gr}(r_p)$, dissipative power loss (=gravitational quantum noise) of

$$P_p = \frac{E_{gr}(r_p)}{t_p} = \frac{r_{\Lambda}}{r_p}m_p c^2\frac{c}{r_p} \tag{50a}$$

Example: setting $r_p = 1\,\text{nm}$ $\quad m_p = m_{pr} = 0.938\,\text{GeV}/c^2$

$$\text{we get and} \quad m_p c^2 = 0.938 \times 10^9\,\text{eV} \quad P_p = 0.88 \times 10^{-22}\,\text{eV/s} \tag{50b}$$

We get as result the temperature-independent **gravitational quantum noise** of $P_p = \frac{r_{\Lambda}}{r_p}m_p c^2\frac{c}{r_p}$.

Compare this to the Nyquist thermal noise at $T = 300\,\text{K}$:

$$P_{th} = k_B T\frac{v_{th}}{\lambda_0} = k_B T\frac{\sqrt{3k_B T/m_p}}{\lambda_0},$$

where $v_{th} = \sqrt{3k_B T/m_p} = 260\,\text{m/s}$ $\quad k_B T = 25\,\text{meV}$

$$t_{th} = \frac{\lambda_0}{v_{th}} = 1.38 \times 10^{-11}\,\text{s} \quad \text{and} \quad P_{th} = \frac{k_B T}{t_{th}} = 1.8 \times 10^9\,\text{eV/s} = 2.89 \times 10^{-10}\,\text{W}$$

11.11. The Transition from Classical to Quantum Regime

In [42], the decoherence time of semi-macroscopic objects was measured for the first time. Two membranes sizes of (20 μm, 20 μm, 0.1 μm) with a mass of m = 70pg, were brought into an entangled state with a decoherence time of $t_{dc} \approx 1$ ms. This gives the characteristic length of the membrane as

$l_0 = \sqrt[3]{20 \times 20 \times 0.1} = 3.42\,\mu\text{m}$.

The corresponding decoherence energy is

$$E_{dc} = \frac{\hbar}{t_{dc}} = \frac{0.64 \times 10^{-15}\,\text{eV} \cdot \text{s}}{1 \times 10^{-3}\,\text{s}} = 0.64 \times 10^{-12}\,\text{eV}$$

The decoherence in absence of thermal effects happens by gravitational self-interaction, near the gravitational quantum boundary $r_{gr} = 3.9 \times 10^{-5}\,\text{m}$ and above we can use the Newtonian approximation

$$E_{gr} = \frac{Gm^2}{l_0} = \frac{6.67\times10^{-11}\,\frac{\text{Jm}}{\text{kg}^2}\left(7\times10^{-14}\,\text{kg}\right)^2}{3.42\times10^{-6}\,\text{m}} = 0.955\times10^{-31}\,\text{J},$$

and with $1\,\text{J} = 6.24\times10^{18}\,\text{eV}$ we get

$$E_{gr} = 0.955\times10^{-31}\times6.24\times10^{18}\,\text{eV} = 0.595\times10^{-12}\,\text{eV}$$

so we get the corresponding decoherence time $t_{gdc} = t_{dc}\frac{E_{dc}}{E_{gr}} = 1.08\,\text{ms}$, which a very good approximation for the measured decoherence time.

Based on this formula, we get in the classical and near-classical regime for the decoherence time $t_{dc} = \frac{\hbar}{E_{gr}} = \frac{\hbar l_0}{Gm^2} \sim \frac{l_0}{l_0^{\,6}} = \frac{1}{l_0^{\,5}}$, under the classical assumption that mass scales approximately with volume.

We can assess the decoherence time for $l_0 = 1\,\text{mm}$

$$t_{dc}\left(l_0 = 1\,\text{mm}\right) = t_{dc}\left(l_0 = 3.42\,\mu\text{m}\right)\left(\frac{1000\,\mu\text{m}}{3.42\,\mu\text{m}}\right)^{-5} = 4.67\times10^{-16}\,\text{s}$$

which is below the current limit of time-measurement.

At the scale of $l_0 = 100\,\text{m}$ the decoherence time reaches the Planck time $t_p = 5.39\times10^{-44}\,\text{s}$, *i.e.* it becomes zero in physical sense, and **quantum effects do not exist anymore.**

For molecular beam interference experiments the interference criterion is $t_{dc}\left(l_0\right) > t_0$ where l_0 is the size of the molecule, and $t_0 = \frac{L}{v_{th}} = \frac{L}{\sqrt{\frac{8k_BT}{m\pi}}}$ is its flight time with thermal velocity v_{th} in the apparatus of length L and at temperature T.

Consider a quantum object with size = AK-limit $t_{dc}\left(l_0 = 39\,\mu\text{m}\right) \approx 3.1\times10^{-8}\,\text{s}$ with thermal speed of oxygen at room temperature T = 300K $v_{th} = 310\frac{\text{m}}{\text{s}}$ in an apparatus of length $L = 1\,\text{m}$, then we get the flight time $t_0 = \frac{L}{v_{th}} = 3.2\times10^{-3}\,\text{s}$, so here the interference criterion $t_{dc}\left(l_0\right) > t_0$ is violated **and there is no quantum behavior anymore.**

11.12. Thermal Gravitational Radiation

The celebrated Planck formula for the spectral energy density of thermal black-body radiation is

$$u_e(\omega) = \frac{d\rho_e}{d\omega} = \frac{\hbar}{\pi^2}\left(\frac{\omega}{c}\right)^3\frac{1}{\exp\left(\frac{\hbar\omega}{k_BT}\right)-1} \tag{51a}$$

its dimension is action-density, *i.e.* $\left[u_e(\omega)\right] = \frac{E\,t}{V}$ = energy * time/volume.

It can be derived from the electric dipole radiation power (see Becker [43] chap. 84).

The electric dipole radiation power for an electric dipole with the dipole moment $p_0 = ql$ (q = charge, l = length) and angular frequency $\omega = kc$

$$P_e = \frac{1}{3}ck^4 p_0{}^2 = \frac{1}{3}\frac{\omega^4}{c^3} p_0{}^2 = \frac{1}{3}(kl)^4 P_{0e},$$

where $P_{0e} = \frac{cp_0{}^2}{l^4} = \frac{q^2}{l}\frac{c}{l}$ is the basic dipole radiation power.

The corresponding gravitational quadrupole radiation power with the quadrupole moment tensor Q_{ij}, $P_g = \frac{2}{15}Gck^6\left(Q_{11}{}^2 + Q_{22}{}^2\right) = \frac{1}{15}(kl)^6 P_{0g}$ (see Fliessbach [32] chap. 41) where for the symmetric quadrupole with mass m length l, we have $Q_{11} = Q_{22} = \frac{l^2 m}{2}$ and where $P_{0g} = \frac{Gm^2}{l}\frac{c}{l}$ is the basic quadrupole radiation power.

The ratio of the two basic radiation powers is very large, for the proton

$$f_{eg}\left(m = m_p, q = e\right) = \frac{P_{0e}}{P_{0g}} = 1.24 \times 10^{36}.$$

Now, the spectral energy density is derived from the dipole power as follows.

The "naive" spectral energy density derived from the dipole radiation power becomes

$$u_{e0}(\omega) = \frac{dP_e}{d\omega} t_\omega \frac{1}{V_l} = \frac{1}{3}(kl)^4 P_{0e} \frac{l}{c}\frac{1}{(2\pi l)^3} = \frac{4}{3}\left(\frac{\omega}{c}\right)^3 S_{0e}$$

where $t_\omega = \frac{l}{c}$ is the time period and $V_l = (2\pi l)^3$ is the elementary volume, and $S_{0e} = P_{0e}\left(\frac{l}{c}\right)^2 = \frac{p_0{}^2}{cl^2}$ is the basic action.

In the phase space, the basic action is replaced by the average one-dimensional phase space volume with Einstein-Bose statistics, multiplied by the spatial dimension $d_0 = 3$

$$S_{0e} \to d_0 \frac{h}{\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1}$$

and we get the correct expression for the thermal spectral energy density

$$u_e(\omega) = \frac{4}{3}\left(\frac{\omega}{c}\right)^3 \frac{1}{(2\pi)^3} \frac{3h}{\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1} = \frac{\hbar}{\pi^2}\left(\frac{\omega}{c}\right)^3 \frac{1}{\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1}$$

In the same way, we can derive the formula for the gravitational radiation thermal spectral energy density, when we take into account, that here the basic action is replaced by the average one-dimensional phase space volume reduced by the factor $\frac{1}{f_{eg}}$

$$S_{0g} = P_{0g} \left(\frac{l}{c} \right)^2 = \frac{Gm^2}{l} \frac{l}{c}, \quad S_{0g} \to \frac{1}{f_{eg}} d_0 \frac{h}{\exp\left(\frac{\hbar\omega}{k_B T} \right) - 1}$$

$$u_g(\omega) = \frac{1}{f_{eg}} \frac{3\hbar}{10\pi^2} \left(\frac{\omega}{c} l \right)^2 \left(\frac{\omega}{c} \right)^3 \frac{1}{\exp\left(\frac{\hbar\omega}{k_B T} \right) - 1} \tag{51b}$$

We see that the gravitational spectral energy density is in magnitude 36 powers smaller than the Planck formula and has a frequency dependence $u_g(\omega) \sim \omega^5$ instead of $u_e(\omega) \sim \omega^3$.

12. Conclusions

The starting point of the AK gravity is the 3-dimensional AK constraints. They can be derived from the Ashtekar version of the ADM-theory plus Kodama ansatz (chap. 3) or from the Plebanski action of the BF-theory, which is a generalized form of GR. Essential for the solvability and non-degeneracy of the AK-constraints is the existence of the positive cosmological constant Λ. It guarantees that the operator of the hamiltonian constraint (also known as the Wheeler-DeWitt-equation) is non-singular and invertible.

The 3-dimensional AK constraints can be generalized to 4 dimensions including time in a mathematically consistent and unique way, simply by generalization of the 3-dimensional antimetric tensor ε_{ijk} in the spatial indices (1, 2, 3) to the 4-dimensional tensor $\varepsilon_{\mu\nu\kappa}$ in the temporal-spatial-indices (0, 1, 2, 3), *i.e.* in the coordinates (t, r, θ, φ), using spherical spatial coordinates.

The 4-dimensional AK equations are 32 partial differential equations for the 16 variables $E^{\mu\nu}$ (E-tensor, inverse densitized tetrad of the metric $g_{\mu\nu}$) and 16 variables $A_\mu{}^\nu$ (A-tensor, gravitational wave tensor). We impose the boundary condition: $E^{\mu\kappa} E^\nu{}_\kappa = g^{\mu\nu} \big/ \left(-\det(g) \right)^{3/4}$ for $r \to \infty$ $g(E^{\mu\nu}) \to g_{\mu\nu}$ *i.e.* in the classical limit of large r the Kodama state generates the given asymptotic spacetime (normally Schwarzschild-spacetime).

The static equations (time-independent, *i.e.* without gravitational waves) in the limit $\Lambda \to 0$ degenerate in the 24 hamiltonian equations: for the A-tensor we get the trivial solution $A_\mu{}^\nu$ = *constant half-antisymmetric*, the E-tensor solutions of the remaining 4 gaussian equations (the last 4 vanish identically) are the *Gauss-Schwarzschild tetrad* (or *the Kerr-Schwarzschild tetrad*), which satisfies the Einstein equations *everywhere in r*, so GR is valid.

If we set the constant A-tensor $A_\mu{}^\nu = \frac{1}{l_P}$, the *modified Einstein-Hilbert action* $S = \frac{\hbar c}{\pi} \int \left(A_\mu{}^\nu A_\nu{}^\mu \right) R \sqrt{-g}\, d^4x$ becomes *dimensionally renormalizable* (chap. 4.2).

At the horizon in the limit $r \to 1$ the E-tensor becomes very large and the term $\Lambda\ E^{\mu\nu}$ not negligible any more, the singularity is removed and becomes a peak $|g_{11,c}| = \frac{1}{|g_{00,c}|} \approx \frac{r_s}{l_0}$, or with the AK correction length $l_0 = r_s \left(r_s{}^2 \Lambda \right)$ we get the corrected outer metric $\left|g_{11,c}\right| = \frac{1}{\left|g_{00,c}\right|} = \frac{1}{1 - \frac{r_s - l_0}{r}}$ with the sub-luminal escape velocity $v_c(r) = c\sqrt{1 - \frac{l_0}{r_s}} \approx c\left(1 - \frac{l_0}{2r_s}\right) < c$, e.g. for central Black Hole of the Milky Way $v_c(r) = c\left(1 - 0.8 \times 10^{-32}\right)$.

The full AK correction length is $l_0 \approx 64\Lambda r_s^3$.

From this results a limit for the *quantum gravitational scale*

$r_{gr} = \sqrt{l_P \sqrt{\frac{1}{\Lambda}}} = 3.1 \times 10^{-5}\,\text{m} = 31\,\mu\text{m}$ (chap. 6.2).

This quantum gravitational scale emerges in the objective wave collapse theory of Ghirardi-Rimini-Weber as the critical wave function width r_c (chap. 11.8), and the second critical parameter is the critical decay rate

$\lambda(r_c) = \frac{G m_e^2}{\hbar r_e} = 0.19 \times 10^{-11}\,\text{s}^{-1}$, where m_e is the electron mass and

$r_e = 2.8 \times 10^{-15}\,\text{m}$ is the classical electron radius. This means that the quantum gravitational scale marks the limit of the quantum coherence length, in other words, *it is the border between quantum and classical regime*.

Numerical calculation for strong coupling $\Lambda = 1$ (chap. The Metric in AK-Gravity: No Horizon and No Singularity) shows that in free fall from the distance $r_0 = 10\ r_s$ from the horizon the maximum velocity is $v_{\max} = 0.56c$ at $r = 1.65$, and then there is a rebound.

For weak coupling $\Lambda = 0.01$ (chap. The Metric in AK-Gravity: No Horizon and No Singularity) the maximum velocity is $v_{\max} = 0.78c$ at $r = 1.38$, and then the orbit ends.

It means that indeed there is a convergent behavior for $\Lambda \to 0$ *with* $v_{\max} \to 1$.

For the time-dependent AK equations, we make the Λ*-scaled wave ansatz*

$$A_\mu{}^\nu = Ab_\mu{}^\nu + \Lambda \frac{As_\mu{}^\nu}{r} \exp\left(-i\,k\left(r - t\right)\right)$$

$$E^{\mu\nu} = Eb^{\mu\nu} + \frac{Es^{\mu\nu}}{r} \exp\left(-i\,k\left(r - t\right)\right)$$

and solve the static part *eqtoievnu*3*b* for *Ab*, *Eb* and the time-dependent part *eqtoievnu*3*w* with the factor $\exp\left(-i\,k\left(r-t\right)\right)$ for *As*, *Es*.

With the multipole ansatz *Es*(*r*, *θ*) = *Es*(*r*)*exp*(*i* * *lx* * *θ*), *As*(*r*, *θ*) = *As*(*r*)*exp*(*I* * *lx* * *θ*), *eqtoievnu*3*w* boils downafter variable eliminations to the gravitational wave equation for the variable *Es*10 (and identical for variables *Es*11, *Es*12, *Es*13)

eqgravlxEninf =

$$2ik^2 lx\, fs(r) - 2k^2 r\, fs'(r) - 3ikr\, fs''(r) + r\, fs'''(r)$$

which is a differential equation of degree 3 in $r \equiv r1$ with the parameters *k* (wave number) and *lx* (angular momentum).

This equation has feasible solutions only for $lx \geq 2$ (*at least quadrupole wave*), as required in GR.

The overall solution is (chap. Solutions of the Gravitational Wave Equation):

- The *E-tensor is exponentially damped* with $\exp\left(-\frac{4\sqrt{r}}{\sqrt{3}}\right)$.

- The A-tensor components *As*0 and *As*1 are *pure quadrupole waves*, *As*2 is a linearly damped quadrupole wave, *As*3 is exponentially damped with $\exp\left(-\frac{4\sqrt{r}}{\sqrt{3}}\right)$.

This means that a classical wave source generates gravitational waves *As* via the metric, the *energy is carried away by the As-tensor* and, when the wave is absorbed, it dissipates energy and generates again a (locally damped) metric oscillation *Es*.

We examine spherical and planar gravitational waves, derive their *wave component relations*

$$Es_1 = 0 = Es_0,\ Es_2 = 3i\,k\left(As_0 - As_1 + As_2\right),\ Es_3 = -3i\,k\,As_2$$

$$As_3 = \left(As_0 - As_1 + As_2\right),$$

and calculate the reflection and absorption ratios $\frac{\delta A_r}{A} = \frac{\delta A_a}{A} = \frac{\delta k}{k} = \frac{r_s(M)}{2r_s}$, where $r_s(M)$ = Schwarzschild radius of the interacting matter *M*.

We demonstrate this solution procedure at the example of the binary gravitational rotator (bgr = binary black hole). The metric of bgr is a Kerr-metric with Kerr-parameter *a*, the corresponding (*Eb*, *Ab*)-solution is

Eb-tensor = the Gauss-Kerr tetrad E_{GK}:

$E_{GK} = E_{GS}$ except $\left(E_{GK}\right)_{03} = \frac{\alpha}{r^{9/2}\sin^{3/4}\theta}$, *Ab*-tensor $Ab = A_{hab} + dAb$ perturbed *half-antisymmetric background*

We develop in a series in *r* and r_0 and get in lowest order

$As00\left(r,\theta,r_0\right) = \frac{As00n01(\theta)}{r_0}$, the θ-functions are calculated numerically.

In chap. 11 we derive the quantum version of AK-gravity.

AK-quantum-gravity is a *renormalizable field gauge theory* with gauge Lie group = gen(SU(2)e), with four generators τ^a, $\left[\tau^a,\tau^b\right] = i\varepsilon_c{}^{ab}\tau^c$.

AK-quantum-gravity is concisely described as follows:

$r_{gr} = \sqrt{l_P\sqrt{\frac{1}{\Lambda}}} = 39\ \mu\text{m}$, E-scale $E_{gr} = \frac{\hbar c}{r_{gr}} = 5.04\times10^{-3}\ \text{eV}$

covariant derivative $\left(D_\mu\right)^\lambda{}_\kappa = \partial_\mu - i\sqrt{\alpha_{gr}}\,\tau_a A_\mu{}^a$, $\tau^a = i\varepsilon_\nu{}^a{}_\lambda$

gauge Lie group = gen(SU(2)e), $\left[\tau^a,\tau^b\right] = i\varepsilon_c{}^{ab}\tau^c$

Dirac eq. $\left(i\hbar\gamma^\mu D_\mu - mc\right)\Psi = 0$

field eq. = $AK\left(A_\mu{}^\nu, E^{\mu\nu}, \Lambda\right)$

boundary cond. $r\to\infty$ $E\,\eta\,E^t = \eta$

quantum metric $g = g_{Schw}\left(r_s = r_\Lambda\right)$ $\left|g_{00}\right| = 1 - \frac{r_\Lambda}{r}$ $\left|g_{11}\right| = \frac{1}{1-\frac{r_\Lambda}{r}}$

quasi-classical quantum grav. potential $E_{gr}(r) = \frac{r_\Lambda}{r}mc^2$, $r_\Lambda = 3.7\times10^{-49}\ \text{m}$

grav. quantum noise $P_p = \frac{r_\Lambda}{r_p} m_p c^2 \frac{c}{r_p} \sim 10^{-22}\ \text{eV/s}$

We derive the energy-momentum density tensor of the AK gravity in the form $t_{\mu\nu} = D_\kappa A_\mu{}^\kappa D_\lambda A_\nu{}^\lambda \hbar c\left(\frac{1}{l_P{}^2 \Lambda^2 r_s{}^2}\right)$, which is identical to the corresponding expression in GR (chap. 9) and is consistent (has the correct r_0-dependence) with the Einstein power formula for the gravitational waves of the binary gravitational rotator

$$P_{GR} = \frac{\hbar c^2}{2 l_P{}^2} \frac{r_s{}^5}{r_0{}^5} \left(\frac{m_1 m_2}{m^2}\right)^2 .$$

The Lagrangian of the AK gravity is (chap. 11.2)

$$\begin{aligned} L_{gr} &= L_H + L_I \\ &= \hbar c \Bigg(-\frac{1}{4} F_{\mu\nu}{}^\kappa F^{\mu\nu}{}_\kappa - \frac{\bar{\varphi}_\Lambda \varphi_\Lambda}{3} \left(\varepsilon^\kappa{}_{\mu\lambda} E^{\lambda\nu} \partial_\kappa A_{\mu\nu} + \varepsilon_{\mu_2 \lambda_2 \kappa_2} \varepsilon^{\mu_1 \lambda_1}{}_{\kappa_1} E^{\kappa_1 \kappa_2} A_{\lambda_1}{}^{\lambda_2} A_{\mu_1}{}^{\mu_2} \right) \\ &\quad + E^{\kappa_1}{}_{\nu_1} F_{\mu\kappa_1}{}^{\nu_1} E^{\kappa_2}{}_{\nu_2} F^\mu{}_{\kappa_2}{}^{\nu_2} \Bigg) \end{aligned}$$

with the spacetime curvature (field tensor) $F_{\mu\nu}{}^\kappa = \partial_\mu A_\nu{}^\kappa - \partial_\nu A_\mu{}^\kappa + \varepsilon^\kappa{}_{\kappa_1\kappa_2} A_\mu{}^{\kappa_1} A_\nu{}^{\kappa_2}$ and Λ is generated by a scalar field φ_Λ with the constraint $\bar{\varphi}_\Lambda \varphi_\Lambda = \Lambda$. This Lagrangian is dimensionally renormalizable.

Λ is expected to depend on the momentum *k*, it is a “running coupling”, like its counterparts in the QFT.

We get for the dimensionless quantum-Λ in the limit $k \to \infty$:

$\Lambda_{dl} = \sqrt{\Lambda_\infty}\, l_p = \left(\frac{\sqrt{2}}{w_*}\right)^{4/2} = 0.031$ and we have a *non-neglectable A-E coupling* in the quantum AK-equations.

In the quantum region of AK gravity $r \le r_{gr} = \sqrt{l_P \sqrt{\frac{1}{\Lambda}}} = 31\,\mu\text{m}$ we get for the *energy normalized graviton wave function* A_{gn} (we demand that $E\left(A_{gn}\right) = \hbar c k$):

$$\left(A_{gn}\right)_\mu{}^\nu = \Omega_\mu{}^\nu \sqrt{\alpha_{gr}} \frac{1}{\sqrt{k r_{gr}}} \frac{r_{gr}{}^{1/2}}{\sqrt{2V}} \left(\exp(-ik \bullet x) + \exp(ik \bullet x)\right),$$

where $\sqrt{\alpha_{gr}} = \frac{r_{gr} \Lambda l_p}{\sqrt{2}}$ and α_{gr} is the *gravitational fine structure constant* and the photon-like wave function can be written

$$\left(A_{gn}\right)_\mu{}^\nu = \sqrt{\alpha_{gr}} \left(A_p\right)_\mu{}^\nu \quad \text{(chap. 11.4).}$$

The covariant derivative of the AK gravity is then

$$\left(D_\mu\right)^\lambda{}_\kappa = \partial_\mu + \left(\varepsilon_a\right)^\lambda{}_\kappa \sqrt{\alpha_{gr}} \left(A_p\right)_\mu{}^a$$

where A_p is completely analogous to the photon wave function A_e, and matrices $\left(\varepsilon_a\right)^\lambda{}_\kappa = \varepsilon^\lambda{}_{a\kappa}$ $a = 0,1,2,3$ and where the generators $\tilde{\tau}^a = i\,\varepsilon^a$ satisfy the *ex-*

tended SU(2) Lie-algebra $\left[\tilde{\tau}^a,\tilde{\tau}^b\right]=i\,\varepsilon^{abc}\tilde{\tau}^c$. So the quantum AK gravity is a *full-fledged quantum gauge theory with the extended SU(2) (ε-tensor group) as the corresponding Lie-algebra* (chap. 11.4).

The propagator of the AK-gravity is the momentum-transform of the gravitational wave equation:

$$D_g\left(q,m\right)=\frac{m}{q\left(q-i\,m\right)\left(q-2i\,m\right)}\quad \text{where } m=\text{mass of source fermion}$$

Based on these results, *we can use for AK quantum gravity the full formalism of Feynman-diagrams* of quantum field theory.

We demonstrate this for the graviton-electron scattering cross-section in analogy to the Compton scattering (photon-electron scattering).

We get the result (chap. 11.6)

$$\bar{\sigma}=\frac{\alpha_{gr}^{\;2}}{2\pi}\tilde{\lambda}_e^{\;2}\left(1.170+\frac{k_0}{m}0.400+\cdots\right)\approx 1.170\frac{\alpha_{gr}^{\;2}}{2\pi}\tilde{\lambda}_e^{\;2},$$

as compared to the photon-electron Thompson cross-section $\sigma_{th}=\alpha^2\tilde{\lambda}_e^{\;2}\frac{8\pi}{3}$ with the *reduced de-Broglie wavelength* of the electron

$$\tilde{\lambda}_e=\frac{\hbar c}{m_e c^2}=0.38\times10^{-12}\ \text{m}.$$

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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