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# The Barrier Binary Options

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## Abstract

We extend the binary options into barrier binary options and discuss the application of the optimal structure without a smooth-fit condition in the option pricing. We first review the existing work for the knock-in options and present the main results from the literature. Then we show that the price function of a knock-in American binary option can be expressed in terms of the price functions of simple barrier options and American options. For the knock-out binary options, the smooth-fit property does not hold when we apply the local time-space formula on curves. By the properties of Brownian motion and convergence theorems, we show how to calculate the expectation of the local time. In the financial analysis, we briefly compare the values of the American and European barrier binary options.

## Keywords

Binary Option, Barrier Option, Arbitrage-Free Price, Optimal Stopping, Geometric Brownian Motion, Parabolic Free Boundary Problem

## 1. Introduction

Barrier options on stocks have been traded in the OTC (Over-The-Counter) market for more than four decades. The inexpensive price of barrier options compared with other exotic options has contributed to their extensive use by investors in managing risks related to commodities, FX (Foreign Exchange) and interest rate exposures.

Barrier options have the ordinary call or put pay-offs but the pay-offs are contingent on a second event. Standard calls and puts have pay-offs that depend on one market level: the strike price. Barrier options depend on two market levels: the strike and the barrier. Barrier options come in two types: in options and out options. An in option or knock-in option only pays off when the option is in the money with the barrier crossed before the maturity. When the stock price crosses the barrier, the barrier option knocks in and becomes a regular option. If the stock price never passes the barrier, the option is worthless no matter it is in the money or not. An out barrier option or knock-out option pays off only if the option is in the money and the barrier is never being crossed in the time horizon. As long as the barrier is not being reached, the option remains a vanilla version. However, once the barrier is touched, the option becomes worthless immediately. More details about the barrier options are introduced in [1] and [2].

The use of barrier options, binary options, and other path-dependent options has increased dramatically in recent years especially by large financial institutions for the purpose of hedging, investment and risk management. The pricing of European knock-in options in closed-form formulae has been addressed in a range of literature (see [3] [4] [5] and reference therein). There are two types of the knock-in option: up-and-in and down-and-in. Any up-and-in call with strike above the barrier is equal to a standard call option since all stock movements leading to pay-offs are knock-in naturally. Similarly, any down-and-in put with strike below the barrier is worth the same as a standard put option. An investor would buy knock-in option if he believes the movements of the asset price are rather volatile. Rubinstein and Reiner [6] provided closed form formulas for a wide variety of single barrier options. Kunitomo and Ikeda [7] derived explicit probability formula for European double barrier options with curved boundaries as the sum of infinite series. Geman and Yor [8] applied a probabilistic approach to derive the Laplace transform of the double barrier option price. Haug [9] has presented analytic valuation formulas for American up-and-input and down-and-in call options in terms of standard American options. It was extended by Dai and Kwok [10] to more types of American knock-in options in terms of integral representations. Jun and Ku [11] derived a closed-form valuation formula for a digit barrier option with exponential random time and provided analytic valuation formulas of American partial barrier options in [12]. Hui [13] used the Black-Scholes environment and derived the analytical solution for knock-out binary option values. Gao, Huang and Subrahmanyam [14] proposed an early exercise premium presentation for the American knock-out calls and puts in terms of the optimal free boundary.

There are many different types of barrier binary options. It depends on: 1) in or out; 2) up or down; 3) call or put; 4) cash-or-nothing or asset-or-nothing. The European valuation was published by Rubinstein and Reiner [6]. However, the American version is not the combination of these options. This paper considers a wide variety of American barrier binary options and is organised as follows. In Section 2 we introduce and set the notation of the barrier binary problem. In Section 3 we formulate the knock-in binary options and briefly review the existing work on knock-in options. In Section 4 we formulate the knock-out binary option problem and give the value in the form of the early exercise premium representation with a local time term. We conduct a financial analysis in Section 5 and discuss the application of the barrier binary options in the current financial market.

## 2. Preliminaries

American feature entitles the option buyer the right to exercise early. Regardless of the pay-off structure (cash-or-nothing and asset-or-nothing), for a binary call option there are four basic types combined with barrier feature: up-in, up-out, down-in and down-out. Consider an American (also known as "One-touch") up-in binary call. The value is worth the same as a standard binary call if the barrier is below the strike since it naturally knocks-in to get the pay-off. On the other hand, if the barrier is above the strike, the valuation turns into the same form of the standard with the strike price replaced by the barrier since we cannot exercise if we just pass the strike and we will immediately stop if the option is knocked-in. Now let us consider an up-out call. Evidently, it is worthless for an up-out call if the barrier is below the strike. Meanwhile, if the barrier is higher than the strike the stock will not hit it since it stops once it reaches the strike. For these reasons, it is more mathematically interesting to discuss the down-in or down-out call and up-in or up-output.

Before introducing the American barrier binary options, we give a brief introduction of European barrier binary options and some settings for this new kind of option.

**Figure 1** and **Figure 2** show the value of eight kinds of European barrier binary options and the comparisons with corresponding binary option values. All of the European barrier binary option valuations are detailed in [6]. Note that the payment is binary, therefore it is not an ideal hedging instrument so we do not analyse the Greeks in this paper and more applications of such options in financial market will be addressed in Section 5. Since we will study the American-style options, we only consider the cases that barrier below the strike for the call and barrier above the strike for the put as reasons stated above. As we can see in **Figure 1** and **Figure 2**, the barrier-version options in the blue or red curves are always worth less than the corresponding vanilla option prices. For the binary call option in **Figure 1** when the asset price is below the in-barrier, the knock-in value is same as the standard price and the knock-out value is worthless. When the stock price goes very high, the effect of the barrier is intangible. The knock-intends to worth zero and the knock-out value converges to the knock-less value. On the other hand in Panel (a) of **Figure 2**, the value of the binary put decreases with an increasing stock price. As Panel (b) in **Figure 2** shows, the asset-or-nothing put option value first increases and then decreases as stock price going large. At a lower stock price, the effect of the barrier for the knock-out value is trifle and the knock-in value tends to be zero. When the stock price is above the barrier, the knock-out is worthless and the up-in value gets the peak at the barrier. The figures also indicate the relationship

$$\text{knock-out} + \text{knock-in} = \text{knock-less}. \tag{2.1}$$

Above all, barrier options create opportunities for investors with lower premiums than standard options with the same strike.

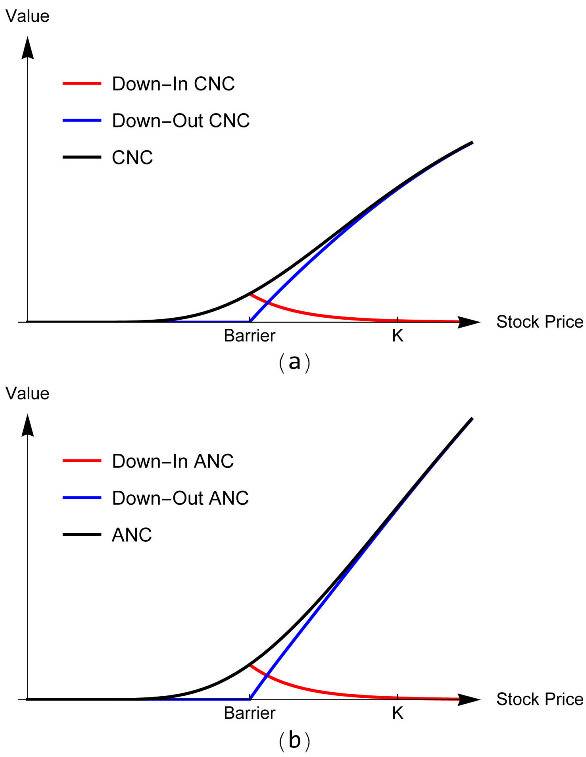


**Figure 1.** A computer comparison of the values of the European barrier cash-or-nothing call(CNC) and asset-or-nothing call(ANC) options for $t$ given and fixed.

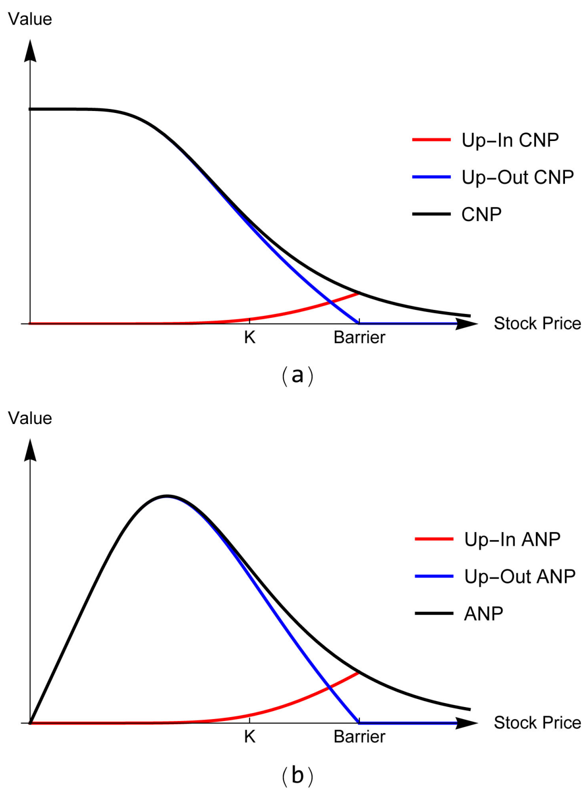


**Figure 2.** A computer comparison of the values of the European barrier cash-or-nothing put (CNP) and asset-or-nothing put (ANP) options for $t$ given and fixed.

## 3. The American Knock-In Binary Option

We start from the cash-or-nothing option. There are four types for the cash-or-nothing option: up-and-in call, down-and-in call, up-and-input and down-and-input. For the up-and-in call, if the barrier is below the strike the option is worth the same as the American cash-or-nothing call since it will cross the barrier simultaneously to get the pay-off. On the other hand, if the barrier is above the strike the value of the option turns into the American cash-or-nothing call with the strike replaced by the barrier level. Mathematically, the most interesting part of the cash-or-nothing call option is down-and-in call (also known as a down-and-up option). For the reason stated above, we only discuss up-and-input and down-and-in call in this section.

We assume that the up-in trigger clause entitles the option holder to receive a digital put option when the stock price crosses the barrier level.

1) Consider the stock price $X$ evolving as

$$\mathrm{d}X_t = rX_t\mathrm{d}t + \sigma X_t\mathrm{d}W_t \tag{3.1}$$

with $X_0 = x$ under P for any interest rate $r > 0$ and volatility $\sigma > 0$. Throughout $W = (W_t)_{t\geq 0}$ denotes the standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. The arbitrage-free price of the American cash-or-nothing knock-in put option at time $t \in [0,T]$ is given by

$$V(t,x) = \sup_{0\leq\tau\leq T-t} \mathrm{E}\left[\mathrm{e}^{-r\tau}\mathrm{I}(X_{t+\tau} \leq K)\mathrm{I}(M_{t+\tau} \geq L)\right], \tag{3.2}$$

where $K$ is the strike price, $L$ is the barrier level and $M_t = \max_{0\leq s\leq t} X_t$ is the maximum of the stock price process $X$. Recall that the unique strong solution for (3.1) is given by

$$X_{t+s} = x\exp\left(\sigma W_s + \left(r - \sigma^2/2\right)s\right) \tag{3.3}$$

under $\mathrm{P}_{t,x}$. The process $X$ is strong Markov with the infinitesimal generator given by

$$\mathbb{L}_X = rx\frac{\partial}{\partial x} + \frac{\sigma^2}{2}x^2\frac{\partial^2}{\partial x^2}. \tag{3.4}$$

We introduce a new process $X^L = \left(X_t^L\right)_{t\geq 0}$ which represents the process $X$ stopped once it hits the barrier level $L$. Define $X_t^L = \left(X_{t\wedge\tau_L}\right)_{t\geq 0}$, where $\tau_L$ is the first hitting time of the barrier $L$ as

$$\tau_L = \inf\{t \geq 0 : X_t \geq L\}. \tag{3.5}$$

It means that we do not need to monitor the maximum process $(M)_{t\geq 0} = \max_{0\leq t\leq T} X_t$ since the process $X_t^L$ behaves exactly the same as the process $X$ for any time $t < \tau_L$ and most of the properties of $X$ follow naturally for $X_t^L$.

2) Standard Markovian arguments lead to the following free-boundary problem

$$V_t + \mathbb{L}_X = rV \quad \text{in } C \tag{3.6}$$

$$V(t,x) = \mathrm{I}(x \leq K) \quad \text{for } x = K \text{ or } t = T \tag{3.7}$$

$$V(t,x) > \mathrm{I}(x \le K) \quad \text{in } C \tag{3.8}$$

$$V(t,x) = \mathrm{I}(x \le K) \quad \text{in } D \tag{3.9}$$

$$V(t,x) = 0 \quad \text{for } x \in [L,\infty), \tag{3.10}$$

where the continuation set is expressed as

$$C = \{(t,x) \in [0,T) \times (0,\infty) \mid x > K\} \tag{3.11}$$

and the stopping set is given by

$$D = \{(t,x) \in [0,T) \times (0,\infty) \mid x \le K\} \cup \{(T,x) \mid x > K\} \tag{3.12}$$

and the optimal stopping time is given by

$$\tau_K = \inf\{t \in [0,T] : X_t^L \le K\}. \tag{3.13}$$

The proof is easy to attend by applying the definition of optimal stopping time.

3) Summarising the preceding facts, we can now apply the approach used in [10] and [15] to obtain a representation for the price of the American knock-in binary option as follows:

$$\begin{aligned} V(t,x) &= \int_0^{T-t} \mathrm{e}^{-rs} V_A(t+s,L)\,\mathbf{P}(\tau_L \in \mathrm{d}s \mid X_t = x) \\ &= \int_t^T \mathrm{e}^{-r(u-t)} V_A(u,L)\, f(u-t,x)\,\mathrm{d}u \end{aligned} \tag{3.14}$$

for $t \in [0,T]$ and $x \in (0,L)$, where $v \to f(v,x)$ is the probability density function of the first hitting time of the process (3.1) to the level $L$. The density function is given by (see e.g. [16])

$$f(v,x) = \log\left(\frac{L}{x}\right)\frac{1}{\sigma v^{3/2}}\phi\left(\frac{1}{\sigma\sqrt{v}}\left[\log\left(\frac{L}{x}\right) - \left(r - \frac{\sigma^2}{2}\right)v\right]\right) \tag{3.15}$$

for $v \in [0,T)$ and $x \in (0,L)$, where $\phi$ is the standard normal density function given by $\phi(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}$ for $x \in \mathbf{R}$. Therefore, the expression for the arbitrage-free price is given by (3.14) and can be solved by inserting the price of the American cash-or-nothing put option.

The value of the American cash-or-nothing put option is given by [6]

$$\begin{aligned} V_A(t,x) = {} & \left(\frac{x}{K}\right)\Phi\left(\frac{\log\frac{K}{x} - (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\ & + \left(\frac{K}{x}\right)^{2r/\sigma^2}\Phi\left(\frac{\log\frac{x}{K} + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right). \end{aligned} \tag{3.16}$$

The other three types of binary options: cash-or-nothing call, asset-or-nothing call and put follow the same pricing procedure and their American values can be referred in [6].

## 4. The American Knock-Out Binary Options

### 4.1. The American Knock-Out Cash-Or-Nothing Options

1) Consider the stock price $X$ evolving as

$$\mathrm{d}X_t = rX_t\mathrm{d}t + \sigma X_t\mathrm{d}W_t \tag{4.1}$$

with $X_0 = x$ under P for any interest rate $r > 0$ and volatility $\sigma > 0$. Throughout $W = (W_t)_{t\geq 0}$ denotes the standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. The arbitrage-free price of the American up-out cash-or-nothing put option at time $t \in [0,T]$ is given by

$$V(t,x) = \sup_{0\leq\tau\leq T-t} \mathrm{E}\left[\mathrm{e}^{-r\tau}\mathrm{I}(X_{t+\tau} \leq K)\mathrm{I}(M_{t+\tau} \leq L)\right], \tag{4.2}$$

where $K$ is the strike price, $L$ is the barrier level and $M_t = \max_{0\leq s\leq t} X_t$ is the maximum of the stock price process $X$. Recall that the unique strong solution for (4.1) is given by

$$X_{t+s} = x\exp\left(\sigma W_s + \left(r - \sigma^2/2\right)s\right) \tag{4.3}$$

under $\mathrm{P}_{t,x}$. The process $X$ is strong Markov with the infinitesimal generator given by

$$\mathbb{L}_X = rx\frac{\partial}{\partial x} + \frac{\sigma^2}{2}x^2\frac{\partial^2}{\partial x^2}. \tag{4.4}$$

We introduce a new process $X^L = (X_t^L)_{t\geq 0}$ which represents the process $X$ stopped once it hits the barrier level $L$. Define $X_t^L = (X_{t\wedge\tau_L})_{t\geq 0}$, where $\tau_L$ is the first hitting time of the barrier $L$:

$$\tau_L = \inf\{t \geq 0 : X_t \geq L\}. \tag{4.5}$$

It means that we do not need to monitor the maximum process $(M)_{t\geq 0} = \max_{0\leq t\leq T} X_t$ since the process $X_t^L$ behaves exactly the same as the process $X$ for any time $t < \tau_L$ and most of the properties of $X$ follow naturally for $X_t^L$.

2) Let us determine the structure of the optimal stopping problem (4.2). Standard Markovian arguments lead to the following free-boundary problem (see [17])

$$V_t + \mathbb{L}_X = rV \quad \text{in } C \tag{4.6}$$

$$V(t,x) = \mathrm{I}(x \leq K) \quad \text{for } x = K \text{ or } t = T \tag{4.7}$$

$$V(t,x) > \mathrm{I}(x \leq K) \quad \text{in } C \tag{4.8}$$

$$V(t,x) = \mathrm{I}(x \leq K) \quad \text{in } D \tag{4.9}$$

$$V(t,x) = 0 \quad \text{for } x \in [L,\infty), \tag{4.10}$$

where the continuation set is expressed as

$$C = \{(t,x) \in [0,T) \times (0,\infty) \mid x > K\}, \tag{4.11}$$

the stopping set is given by

$$D=\{(t,x)\in[0,T)\times(0,\infty)\mid x\le K\}\cup\{(T,x)\mid x>K\},\tag{4.12}$$

and the optimal stopping time is given by

$$\tau_K=\inf\{t\ge 0: X_t^L\le K\}\tag{4.13}$$

denoting the first time the stock price is equal to $K$ before the stock price is equal to $L$. We will prove that $K$ is the optimal boundary and $\tau_K$ is optimal for (4.2) below.

3) We will show that (4.13) is optimal for (4.2). The fact that the value function (4.2) is a discounted price indicates that the larger $\tau$ is, the less value we will get. As to the payoff, it is either £1 or nothing. Therefore, the optimal stopping time is just the very first time that the stock price hits $K$, which is (4.13). To prove this, we define $\tau$ as any stopping time. We need to show that

$$\mathbb{E}_{t,x}\left(e^{-r\tau_K}I\left(X_{t+\tau_K}\le K\right)I\left(M_{t+\tau_K}\le L\right)\right)\ge\mathbb{E}_{t,x}\left(e^{-r\tau}I\left(X_{t+\tau}\le K\right)I\left(M_{t+\tau}\le L\right)\right).\tag{4.14}$$

Actually,

$$\begin{aligned}&\mathbb{E}_{t,x}\left(e^{-r\tau_K}I\left(X_{t+\tau_K}\le K\right)I\left(M_{t+\tau_K}\le L\right)\right)\\&=\mathbb{E}_{t,x}\left(e^{-r\tau_K}I\left(X_{t+\tau_*}\le K\right)I\left(M_{t+\tau_K}\le L\right)I\left(t+\tau_K<T\right)\right.\\&\quad\left.+e^{-r\tau_K}I\left(X_{t+\tau_K}\le K\right)I\left(M_{t+\tau_K}\le L\right)I\left(t+\tau_K\ge T\right)\right)\\&=\mathbb{E}_{t,x}\left(e^{-r\tau_K}I\left(X_{t+\tau_K}\le K\right)I\left(M_{t+\tau_K}\le L\right)I\left(t+\tau_K<T\right)+e^{-r\tau_K}I(\varnothing)\right)\\&=\mathbb{E}_{t,x}\left(e^{-r\tau_K}I\left(t+\tau_K<T\right)\right).\end{aligned}\tag{4.15}$$

On the other hand,

$$\begin{aligned}&\mathbb{E}_{t,x}\left(e^{-r\tau}I\left(X_{t+\tau}\le K\right)I\left(M_{t+\tau}\le L\right)\right)\\&=\mathbb{E}_{t,x}\left(e^{-r\tau}I\left(X_{t+\tau}\le K\right)I\left(M_{t+\tau}\le L\right)I\left(\tau<\tau_K\right)\right.\\&\quad\left.+e^{-r\tau}I\left(X_{t+\tau}\le K\right)I\left(M_{t+\tau}\le L\right)I\left(\tau\ge\tau_K\right)\right)\\&=\mathbb{E}_{t,x}\left(e^{-r\tau}I(\varnothing)+e^{-r\tau}I\left(X_{t+\tau}\le K\right)I\left(M_{t+\tau}\le L\right)I\left(\tau\ge\tau_K\right)\right)\end{aligned}$$

$$\begin{aligned}&=\mathbb{E}_{t,x}\left(e^{-r\tau}I\left(X_{t+\tau}\le K\right)I\left(M_{t+\tau}\le L\right)I\left(\tau\ge\tau_K\right)I\left(t+\tau_K<T\right)\right.\\&\quad\left.+e^{-r\tau}I\left(X_{t+\tau}\le K\right)I\left(M_{t+\tau}\le L\right)I\left(\tau\ge\tau_K\right)I\left(t+\tau_K\ge T\right)\right)\\&=\mathbb{E}_{t,x}\left(e^{-r\tau}I\left(X_{t+\tau}\le K\right)I\left(M_{t+\tau}\le L\right)I\left(\tau\ge\tau_K\right)I\left(t+\tau_K<T\right)+e^{-r\tau}I(\varnothing)\right)\\&\le\mathbb{E}_{t,x}\left(e^{-r\tau_K}I\left(t+\tau_K<T\right)\right).\end{aligned}\tag{4.16}$$

Hence we conclude that $\tau_K$ is optimal in (4.2).

4) Based on the optimal stopping time (4.13), a direct solution for (4.2) can be expressed as

$$\begin{aligned}V(t,x)&=E_{t,x}\left[e^{-r\tau_K}I\left(X_{t+\tau_K}\le K,M_{t+\tau_K}\le L,\tau_K\le T-t\right)\right]\\&=E_{t,x}\left[e^{-r\tau_K}I\left(M_{t+\tau_K}\le L,\tau_K\le T-t\right)\right].\end{aligned}\tag{4.17}$$

For the geometric Brownian motion the density $P_x\left(M_{\tau_K}\le L,\tau_K\in dt\right)$ is known in closed form (cf. ([16], Page 622):

$$P_x\left(M_{\tau_K} \le L, \tau_K \in \mathrm{d}t\right) = \left(\frac{K}{x}\right)^{r/\sigma^2 - 1/2} \mathrm{e}^{-\frac{\left(r-\sigma^2/2\right)^2 t}{2\sigma^2}} ss_t\left(\frac{\log(L/x)}{\sigma}, \frac{\log(L/K)}{\sigma}\right)\mathrm{d}t \quad (4.18)$$

for $K \le x \le L$, where $ss_t(x, y)$ is given by (cf. [16])

$$ss_t(x,y) = \sum_{k=-\infty}^{\infty} \frac{y - x + 2ky}{\sqrt{2\pi t^3}} \mathrm{e}^{-\frac{(y-x+2ky)^2}{2t}} \quad (4.19)$$

for $x < y$. The result is straightforward

$$V(t,x) = \left(\frac{K}{x}\right)^{r/\sigma^2 - 1/2} \int_0^{T-t} \mathrm{e}^{-\frac{\left(r+\sigma^2/2\right)^2 s}{2\sigma^2}} \sum_{j=-\infty}^{\infty} \frac{\log(x/K) + 2j\log(l/K)}{\sigma\sqrt{2\pi s^3}} \times \mathrm{e}^{-\frac{\left(\log(x/K)+2j\log(L/K)\right)^2}{2\sigma^2 s}} \mathrm{d}s \quad (4.20)$$

for $K \le x \le L$. The value function concerns with the convergence due to the sum of an infinite series. More precisely we will apply the optimal stopping theory to value (4.2) and get a better result. However, the result from (4.20) indicates some properties of the pricing (4.2). It is easy to verify that local time-space formula is applicable to our problem (4.2).

5) To get the solution to the optimal stopping problem (4.2), apply Ito's formula to $\mathrm{e}^{-rs}V\left(t+s, X_{t+s}^L\right)$ and get

$$\begin{aligned} \mathrm{e}^{-rs}V\left(t+s, X_{t+s}^L\right) &= V(t,x) + \int_0^s \mathrm{e}^{-ru} H\left(t+u, X_{t+u}^L\right)\mathrm{d}u + M_s \\ &\quad + \frac{1}{2}\int_0^s \left(V_x\left(u, X_{u^+}\right) - V_x\left(u, X_{u^-}\right)\right) \mathrm{I}\left(X_u = K\right)\mathrm{d}\ell_u^K(X), \end{aligned} \quad (4.21)$$

where the function $H = H(t,x)$ is defined by

$$H = V_t^{\mu_c} + rxV_x + \frac{\sigma^2}{2}x^2 V_{xx} - rV, \quad (4.22)$$

$\ell_u^K$ is given by

$$\ell_u^K = \mathbb{P} - \lim_{\xi \downarrow 0} \frac{1}{2\xi}\int_0^u \mathrm{I}\left(K - \xi < X_r < K + \xi\right)\mathrm{d}\langle X, X\rangle_r \quad (4.23)$$

and $\mathrm{d}\ell_u^K$ refers to integration with respect to the continuous increasing function $u \to \ell_u^K$, and $M_s = \sigma\int_0^s \mathrm{e}^{-ru} X_u^L V_x\left(t+u, X_{t+u}^L\right)\mathrm{d}W_u$ is a continuous local martingale for $s \in [0, T-t]$ with $t \in [0, T)$.

The martingale term vanishes when taking E on both sides. From the optional sampling theorem we get

$$\begin{aligned} \mathrm{E}\left[\mathrm{e}^{-rs}V\left(t+s, X_{t+s}^L\right)\right] &= V(t,x) + \mathrm{E}\left[\int_0^s \mathrm{e}^{-ru} H\left(t+u, xX_u^L\right)\mathrm{d}u\right] \\ &\quad + \frac{1}{2}\mathrm{E}\int_0^s \mathrm{e}^{-ru} V_x\left(u, K^+\right)\mathrm{d}\ell_u^K(X) \end{aligned} \quad (4.24)$$

for all stopping times $\tau$ of $X$ with values in $[0, T-t]$ with $t \in [0, T)$ and $x \in (0, \infty)$ given and fixed. Replacing $s$ by $T-t$ in (4.24), we get

$$\begin{aligned} V(t,x) &= \mathrm{e}^{-r(T-t)}\mathrm{E}_{t,x} G\left(X_T^L\right) - \int_0^s \mathrm{e}^{-ru}\mathrm{E}_{t,x}\left[H\left(t+u, xX_u^L\right)\mathrm{I}\left(X_{t+u}^L \le K\right)\right]\mathrm{d}u \\ &\quad - \frac{1}{2}\mathrm{E}\int_0^s \mathrm{e}^{-ru} V_x\left(u, K^+\right)\mathrm{d}\ell_u^K(X) \end{aligned} \quad (4.25)$$

for all $(t,x) \in [0,T] \times \mathbb{R}_+$, where $G(x) = I(x \le K)$ and $H = -r$ for $x \le K$. We obtain the following early exercise premium representation of the value function

$$\begin{aligned} V(t,x) = & \mathrm{e}^{-r(T-t)}\mathrm{E}_{t,x}G\left(X_T^L\right) + r\int_0^{T-t}\mathrm{e}^{-ru}\mathrm{P}_{t,x}\left(X_{t+u}^L \le K\right)\mathrm{d}u \\ & -\frac{1}{2}\mathrm{E}\int_0^{T-t}\mathrm{e}^{-ru}V_x\left(t+u,K^+\right)\mathrm{d}\ell_u^K\left(X\right). \end{aligned} \tag{4.26}$$

The first term on the RHS is the arbitrage-free price of the European knock-out cash-or-nothing put option $V_E$ at the point $(t,x)$ and can be written explicitly as (see [6])

$$\begin{aligned} V_E(t,x) = & \mathrm{e}^{-r(T-t)}\Phi\left(\frac{\log(K/x)-\left(r-\sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\ & -\mathrm{e}^{-r(T-t)}\left(\frac{L}{x}\right)^{\frac{2r}{\sigma^2}-1}\Phi\left(\frac{\log\left(Kx/L^2\right)-\left(r-\sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}\right). \end{aligned} \tag{4.27}$$

We write

$$\mathrm{P}_{t,x}\left(X_{t+u}^L \le K\right) = \mathrm{P}\left(X_{t+u} \le K, \max_{0\le s\le u} X_{t+s} < L\right). \tag{4.28}$$

Recall that the joint density function of geometric Brownian motion and its maximum $(X_t, M_t)$ under P with $x_0 = M_0 = 1$ is given by (see [16])

$$f(t,x,m) = \frac{2}{\sigma^3\sqrt{2\pi t^3}}\frac{\log\left(m^2/x\right)}{xm}\exp\left(-\frac{\log^2\left(m^2/x\right)}{2\sigma^2 t}+\frac{\beta}{\sigma}\log x-\frac{\beta^2}{2}t\right) \tag{4.29}$$

for $0 < x \le m$ with $\beta = r/\sigma - \sigma/2$.

6) We will discuss the calculation about the local-time term $\ell_u^K$ (see [18] and reference therein). Note that

$$\begin{aligned} & \mathrm{E}_{t,x}\int_0^{T-t}\mathrm{e}^{-ru}V_x\left(t+u,K^+\right)\mathrm{d}\ell_u^K(X) \\ & = \int_0^{T-t}\mathrm{e}^{-ru}V_x\left(t+u,K^+\right)\mathrm{d}\mathrm{E}_{t,x}\ell_u^K(X). \end{aligned} \tag{4.30}$$

From the definition of local time $\ell_u^K(X) = \mathbb{P} - \lim_{\xi\downarrow 0}\frac{1}{2\xi}\int_0^u \mathrm{I}\left(K-\xi < X_{t+s} < K+\xi\right)\mathrm{d}\left\langle X,X\right\rangle_s$, there exists a sequence $\xi_n$ such that $\lim_{n\to\infty}\xi_n = 0$ and $\lim_{n\to\infty}\frac{1}{2\xi_n}\int_0^u \mathrm{I}\left(K-\xi_n < X_{t+s} < K+\xi_n\right)\mathrm{d}\left\langle X,X\right\rangle_s$ - $P_{a.s}$. Using Dominated Convergence Theorem, we get

$$\begin{aligned} \mathrm{E}_{t,x}\ell_u^K(X) & = \mathrm{E}_{t,x}\int_0^u \lim_{n\to\infty}\frac{1}{2\xi_n}\mathrm{I}\left(K-\xi_n < X_{t+s} < K+\xi_n\right)\mathrm{d}\left\langle X,X\right\rangle_s \\ & = \int_0^u \lim_{n\to\infty}\frac{1}{2\xi_n}\mathrm{E}\left[\mathrm{I}\left(K-\xi_n < X_{t+s} < K+\xi_n\right)\sigma^2 X_{t+s}^2\right]\mathrm{d}s \\ & = \int_0^u \lim_{n\to\infty}\frac{1}{2\xi_n}\int_{K-\xi_n}^{K+\xi_n}\sigma^2 y^2 f_{X_s}(y)\mathrm{d}y\mathrm{d}s. \end{aligned} \tag{4.31}$$

The second step is attained by Fubini's Theorem and Dominated Convergence Theorem. By the definition of derivative, the last step in (4.31) equals

$$\mathrm{E}_{t,x}\ell_u^K(X)=\int_0^u \sigma^2K^2 f_{X_s}(K)\,\mathrm{d}s. \tag{4.32}$$

The density function $f_{X_s}(K)$ is given by

$$f_{X_s}(K)=\frac{1}{K\sigma\sqrt{s}}\phi\left(\frac{\log\frac{K}{x}-\left(r-\sigma^2/2\right)s}{\sigma\sqrt{s}}\right), \tag{4.33}$$

where $\phi(x)=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}$ is the density function for standard normal distribution. Therefore, (4.30) can be expressed as

$$\begin{aligned}&\mathrm{E}_{t,x}\int_0^{T-t}\mathrm{e}^{-ru}V_x\left(t+u,K^+\right)\mathrm{d}\ell_u^K(X)\\&=\sigma K\int_0^{T-t}\frac{\mathrm{e}^{-ru}}{\sqrt{u}}V_x\left(t+u,K^+\right)\phi\left(\frac{\log\frac{K}{x}-\left(r-\sigma^2/2\right)u}{\sigma\sqrt{u}}\right)\mathrm{d}u.\end{aligned} \tag{4.34}$$

Substituting the result (4.34) into (4.26), we get the early exercise premium (EEP) representation for the American knock-out cash-or-nothing put option

$$\begin{aligned}V(t,x)&=\mathrm{e}^{-r(T-t)}\mathrm{E}_{t,x}G\left(X_T^L\right)+r\int_0^{T-t}\mathrm{e}^{-ru}\mathrm{P}_{t,x}\left(X_{t+u}^L\le K\right)\mathrm{d}u\\&\quad-\frac{1}{2}\sigma K\int_0^{T-t}\frac{\mathrm{e}^{-ru}}{\sqrt{u}}V_x\left(t+u,K^+\right)\phi\left(\frac{\log\frac{K}{x}-\left(r-\sigma^2/2\right)u}{\sigma\sqrt{u}}\right)\mathrm{d}u,\end{aligned} \tag{4.35}$$

where the first and second terms are defined in (4.27) and (4.28).

The main result of the present subsection may now be stated as follows. Below, we will make use of the following function

$$\begin{aligned}J(t,x,v,z)&=\Phi\left(\frac{\log(z/x)-\left(r-\sigma^2/2\right)(v-t)}{\sigma\sqrt{v-t}}\right)\\&\quad-\left(\frac{L}{x}\right)^{\frac{2r}{\sigma^2}-1}\Phi\left(\frac{\log\left(zx/L^2\right)-\left(r-\sigma^2/2\right)(v-t)}{\sigma\sqrt{v-t}}\right)\end{aligned} \tag{4.36}$$

for all $t\in[0,T), x>0, z>0$ and $v\in(t,T)$.

**Theorem 1.** *The arbitrage-free price of the American knock-out cash-or-nothing put option follows the early-exercise premium representation*

$$\begin{aligned}V(t,x)&=\mathrm{e}^{-r(T-t)}\mathrm{E}_{t,x}G\left(X_T^L\right)+r\int_t^T\mathrm{e}^{-r(v-t)}J(t,x,v,K)\,\mathrm{d}v\\&\quad-\frac{1}{2}\sigma K\int_t^T\frac{\mathrm{e}^{-r(u-t)}}{\sqrt{u-t}}V_x\left(u,K^+\right)\phi\left(\frac{\log\frac{K}{x}-\left(r-\sigma^2/2\right)(u-t)}{\sigma\sqrt{u-t}}\right)\mathrm{d}u\end{aligned} \tag{4.37}$$

for all $(t,x)\in[0,T]\times(0,L)$, where the first term is the arbitrage-free price of the European knock-out cash-or-nothing put option and the second and third terms are the early-exercise premium.

The proof is straightforward following the points 4, 5 and 6 stated above. Note that our problem is based on the stopped process $X_L$ instead of the original process $X$ and that the value of $V_x(u,K^+)$ in (4.37) needs to be estimated by finite difference method otherwise we can not get the value $V(t,x)$.

The cash-or-nothing call option can be handled in a similar way. The different part is the European value function in (4.27). The arbitrage-free price of the European down-out cash-or-nothing call option $V_E^{CNC}$ at the point $(t,x)$ is given by (see [6])

$$\begin{aligned} V_E^{CNC}(t,x) = {} & \mathrm{e}^{-r(T-t)}\Phi\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\ & -\mathrm{e}^{-r(T-t)}\left(\frac{L}{x}\right)^{\frac{2r}{\sigma^2}-1}\Phi\left(\frac{\log(L^2/xK)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right). \end{aligned} \tag{4.38}$$

## 4.2. The American Knock-Out Asset-Or-Nothing Options

The arbitrage-free price of the European knock-out asset-or-nothing option $V_E$ at the point $(t,x)$ can be written explicitly as (see [6])

$$\begin{aligned} V_E^{ANC}(t,x) = {} & x\Phi\left(\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\ & -\left(\frac{L}{x}\right)^{\frac{2r}{\sigma^2}+1}\Phi\left(\frac{\log(L^2/xK)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \end{aligned} \tag{4.39}$$

$$\begin{aligned} V_E^{ANP}(t,x) = {} & x\Phi\left(\frac{\log(K/x)-(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\ & -\left(\frac{L}{x}\right)^{\frac{2r}{\sigma^2}+1}\Phi\left(\frac{\log(xK/L^2)-(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right), \end{aligned} \tag{4.40}$$

where $V_E^{ANC}(t,x)$ represents the value for the European down-out asset-or-nothing call (ANC) option and $V_E^{ANP}(t,x)$ for the up-out put.

**Theorem 2.** *The arbitrage-free price of the American knock-out asset-or-nothing option follows the early-exercise premium representation*

$$V^{ANC}(t,x) = \mathrm{e}^{-r(T-t)}\mathrm{E}_{t,x}G(X_T^L) - \frac{1}{2}\mathrm{E}\int_0^{T-t}\mathrm{e}^{-ru}\left(1-V_x(t+u,K^-)\right)\mathrm{d}\ell_u^K(X) \tag{4.41}$$

for all $(t,x)\in[0,T]\times(L,K)$, and

$$V^{ANP}(t,x) = \mathrm{e}^{-r(T-t)}\mathrm{E}_{t,x}G(X_T^L) - \frac{1}{2}\mathrm{E}\int_0^{T-t}\mathrm{e}^{-ru}\left(V_x(t+u,K^+)-1\right)\mathrm{d}\ell_u^K(X) \tag{4.42}$$

for all $(t,x)\in[0,T]\times(K,L)$, where the first term is the arbitrage-free price of the European knock-out asset-or-nothing option and the second term is the early-exercise premium.

*Proof.* The proof is analogous to that of Theorem 1. Back to (4.22), it is easy to verify that the value of $H$ vanishes since $V=G$ in the stopping set. There are only two terms in (4.26).

## 5. Financial Analysis of the American Barrier Binary Options

The payment of the American barrier binary options is binary, so they are not ideal hedging instruments. Instead, they are ideal investment products. It is popular to use structured accrual range notes in the financial markets. Such notes are related to foreign exchanges, equities or commodities. For instance, in a daily accrual USD-BRP exchange rate range note, it pays a fixed daily accrual interest if the exchange rate remains within a certain range.

Generally, an investor buying a barrier option is seeking for more risk than that of a vanilla option since the barrier options can be stopped or "knocked-out" at any time prior to maturity or never start or "knock-in" due to not hitting the barrier. Basic reasons to purchase barrier options rather than standard options include a better expectation of the future behaviour of the market, hedging needs and lower premiums. In the liquid market, traders value options by calculating the expected value of the pay-offs based on all stock scenarios. It means to some extent we pay for the volatility around the forward price. However, barrier options eliminate paying for the impossible scenarios from our point of view. On the other hand, we can improve our return by selling a barrier option that pays off based on scenarios we think of little probability. Let us imagine that the 1-year forward price of the stock is 110 and the spot price is 100. We believe that the market is very likely to rise and if it drops below 95, it will decline further. We can buy a down-and-out call option with strike price 110 and the barrier level 95. At any time, if the stock falls below 95, the option is knocked-out. In this way, we do not pay for the scenario that the stock price drops firstly and then goes up again. This reduces the premium. For the hedgers, barrier options meet their needs more closely. Suppose we own a stock with spot 100 and decide to sell it at 105. We also want to get protected if the stock price falls below 95. We can buy a put option struck at 95 to hedge it but it is more inexpensive to buy an up-an-out put with a strike price 95 and barrier 105. Once the stock price rises to 105 when we can sell it and this put disappears simultaneously.

The relationship between knock-in option, knock-out option and knock-less option (standard option) of the same type (call or put) with the same expiration date, strike and barrier level can be expressed as

$$\text{knock-out} + \text{knock-in} = \text{knock-less}. \tag{5.1}$$

This relationship only holds for the European barrier options. It has not been obtained for the American version when we get the American values from the sections above.

We plot the value of the American barrier binary options using the free-boundary structure in the above sections. Note that the value of $V_x\left(u,K^+\right)$ in Equations (4.37), (4.41) and (4.42) separately is estimated by finite difference method (see [19]).

The American value curves in **Figure 3** and **Figure 4** are simulated from (15) by inserting different American binary option values. **Figure 3** shows that the value of the American down-in cash-or-nothing call options (asset-or-nothing call option follows a similar curve) increases with stock price $X_t$ before the in barrier and then decreases due to the uncertainty of knock-in. Figure 4 shows the value of the American up-in cash-or-nothing put option (asset-or-nothing put is similar ). As we can see before the barrier, the option value is increasing and gets its peak at the barrier. Then the value goes down as the stock price continues to go up after the barrier level. Generally, the price of the American version options is larger than the European version.

Figures 5-8 show the values for the knock-out binary options. Figure 5 illustrates that the value of the up-out cash-or-nothing put option is a decreasing function of the stock price below the barrier. However, in Figure 6 the up-out asset-or-nothing put first goes up and then down to the barrier. We can see the value of the down-out cash-or-nothing call option in Figure 7 is strictly increasing as the asset price above the barrier. The asset-or-nothing call value in Figure 8 is also in the similar situation but with different amount of payoff size. All of the out figures show that the smooth-fit condition is not satisfied at the stopping boundary *K*.

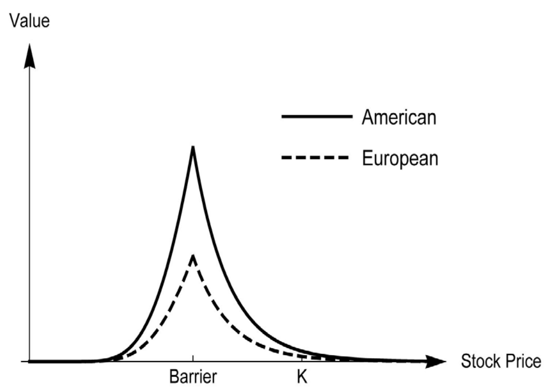


Figure 3. A computer comparison for the values of the European and the American down-in cash-or-nothing call options with parameters $r = 0.1, \sigma = 0.4, K = 10, T = 1, Barrier = 6$ and $t = 0$.

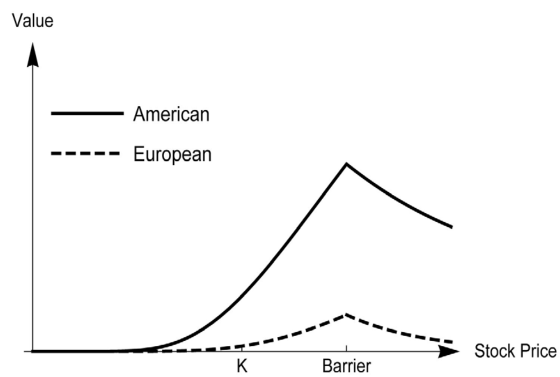


Figure 4. A computer comparison for the values of the European and the American up-in cash-or-nothing put options with parameters $r = 0.1, \sigma = 0.4, K = 10, T = 1, Barrier = 15$ and $t = 0$.

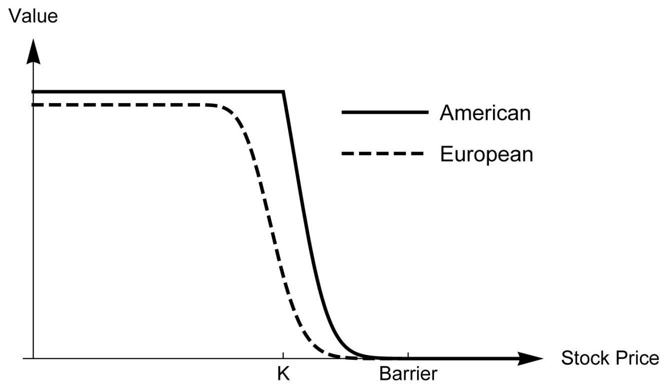


Figure 5. A computer comparison for the values of the European and the American up-out cash-or-nothing put options with parameters $r = 0.05, \sigma = 0.1, K = 10, T = 1, L = 15$ and $t = 0$.

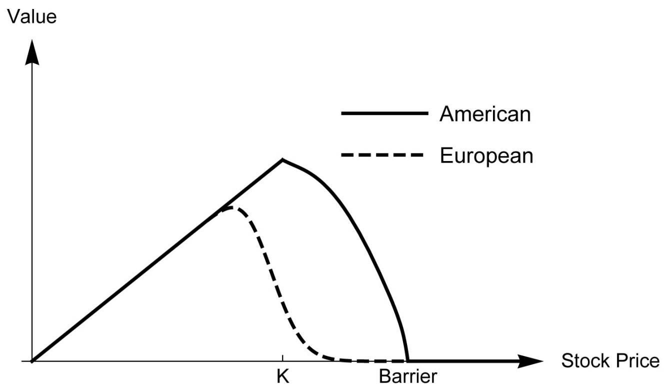


Figure 6. A computer comparison for the values of the European and the American up-out asset-or-nothing put options with parameters $r = 0.05, \sigma = 0.1, K = 10, T = 1, L = 15$ and $t = 0$.

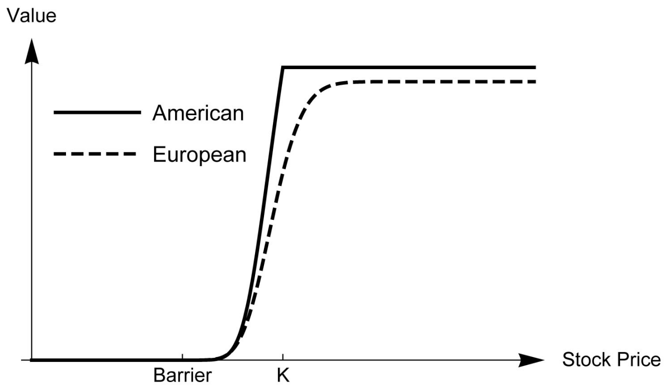


Figure 7. A computer comparison for the values of the European and the American down-out cash-or-nothing call options with parameters $r = 0.05, \sigma = 0.1, K = 10, T = 1, L = 6$ and $t = 0$.

The results of this paper also hold for an underlying asset with dividend structure. With minor modifications, the formulas developed here can be applied to handle those problems.

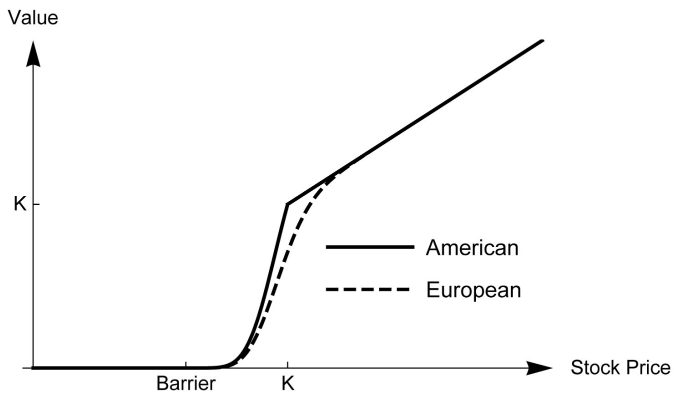


Figure 8. A computer comparison for the values of the European and the American down-out asset-or-nothing call options with parameters $r = 0.05, \sigma = 0.1, K = 10, T = 1, L = 6$ and $t = 0$.

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## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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