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Gravitation, Density, Black Holes and Spatial Quantization

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Abstract

Making use of Newton's classical shell theorem, the Schwarzschild metric is modified. This removes the singularity at $r = 0$ for a standard object (not a black hole). It is demonstrated how general relativity evidently leads to quantization of space-time. Both classical and quantum mechanical limits on density give the same result. Based on Planck's length and the assumption that density must have an upper limit, we conclude that the lower limit of the classical gravitation theory by Einstein is related to the Planck length, which is a quantum phenomenon posed by dimensional analysis of the universal constants. The Ricci tensor is considered under extreme densities (where Kretschmann invariant = 0) and a solution is considered for both outside and inside the object. Therefore, classical relativity and the relationship between the universal constants lead to quantization of space. A gedanken experiment of light passing through an extremely dense object is considered, which will allow for evaluation of the theory.

Keywords

Newton's Shell Theorem, Schwarzschild Singularities, Photon Sphere, Planck's Units, Quantization of Space

1. Introduction

The Schwarzschild solution appears to have singularities at $r = 0$ and $r = r_s$, with $r_s = \frac{2GM}{c^2}$, which causes the metric to diverge at these radii. There is no problem as long as $R > r_s$.

The singularity at $r = r_s$ divides the Schwarzschild coordinates into two disconnected regions. The exterior Schwarzschild solution with $r > r_s$ relates to the gravitational fields of stars and planets. The interior Schwarzschild solution with $0 \le r < r_s$, including the singularity at $r = 0$, is separated from the outer Schwarzschild region by the so-called singularity shell at $r = r_s$.

The Schwarzschild coordinates give no physical connection between these two regions.

The singularity at $r = r_s$, however, is an illusion; it is the result of a bad choice of coordinates or coordinate conditions. When changing to a different coordinate system, the metric becomes regular at $r = r_s$ and can extend the external region to values of r larger than r_s. Using a different coordinate transformation, one can then relate the extended external region to the inner region.

The case $r = 0$ is different, however. If one asks that the solution be valid for all r, one runs into a true physical singularity, or gravitational singularity, at the origin.

At $r = 0$, the curvature can become infinite, if and only if, it represents a point particle of zero mass. One cannot compress a finite mass into an infinitesimal point (of radius $R = 0$). Any finite mass should have a finite radius $R > 0$. Therefore, when looking into the point $r = 0$, one needs to consider the effects of the outer shells of matter which surround the $r = 0$ point.

2. Spherically Symmetric, Non-Rotating, Uncharged Spherical Body

Consider a spherically symmetric, non-rotating, uncharged mass (a stellar object or an elementary particle).

Under spherical symmetry, and at a remote distance in empty space outside the object, the Schwarzschild metric [1] [2] [3] [4] is given by:

$$g_{\mu\nu} = \begin{pmatrix} 1-\frac{2GM}{rc^2} & & & \\ & \frac{-1}{1-\frac{2GM}{rc^2}} & & \\ & & -r^2 & \\ & & & -r^2\sin^2\theta \end{pmatrix} \tag{1}$$

This metric describes very well the gravitational effects of that mass on gravitational field curvature, and dynamics of moving particles, in the empty regions outside the object.

However, it suffers of two problems:

1) *At the Schwarzschild radius* $r_s = \frac{2GM}{c^2}$, *the metric has a singularity.*

2) *At* $r \to 0$, *the metric diverges.*

The first singularity is not a real one, as it can be removed by Painlevé-Gullstrand-Kruskal-Szekeres coordinate transformations [5]-[17].

The second singularity at $r \to 0$, cannot be removed by coordinate transformation. It is an inherent singularity in any inverse-square-law, central-force problem.

To solve this problem, we must investigate the behaviour of the field inside the mass.

If a mass source has a finite radius *R*, the solution to the potential as a function of distance must be modified, so that it includes the inner region where $r < R$.

In addition, even though the Einstein field equation relates to a non-empty space, to a very good approximation, the Einstein equation inside of even the heaviest stellar objects can be considered same as empty space.

This assumption will only be valid for a perfect fluid solution, where of the $T_{\mu\nu}$ tensor, only energy density is considered, while momentum density, shear stress, momentum flux and pressure are either null or irrelevant.

3. Impact of Shell Theorem

Let *R* be the radius of an uncharged, non-rotating, spherically symmetric mass, of given density $\rho(r)$, with *r* being the radial distance from center (at $r = 0$).

This object has a volume $v = \frac{4\pi r^3}{3}$ and radius *R*, its average density will be $\langle\rho\rangle = \frac{3M}{4\pi R^3}$ and since $M = \int_0^R 4\pi r^2 \rho(r)\mathrm{d}r$ it is straightforward to see that

$$M = \frac{4\pi}{3}\lim_{R\to 0} R^3 \rho(0)$$

This means that unless $\rho(R)$ goes to infinity as $R \to 0$, the mass must vanish ($M = 0$).

The only case of $M > 0$ is possible if and only if $\lim_{R\to 0}\rho(R) \to \infty$.

However, as will be shown later, density of any object must have an upper limit and cannot be infinitely large. one must accept, that no mass can be condensed to a singular point $r = 0$. A mass will always have some finite density and a finite radius (volume).

To resolve this contradiction, one must consider the effects of surrounding mass. Condensed as it might be, but still of finite density.

In classical mechanics, shell theorem, proven by Isaac Newton [18] [19] states that:

A spherically symmetric body affects external objects gravitationally as though all of its mass were concentrated at a point at its center.

If the body is a spherically symmetric shell (*i.e.*, a hollow ball), no net gravitational force is exerted by the shell on any object inside it, regardless of the object's location inside the shell.

A corollary is that inside a solid sphere of constant density, the gravitational *force* within the object varies linearly with distance from its center, becoming zero, by symmetry, at the center.

Hence, the gravitational potential of a spherically symmetric object of mass *M* and radius *R*, as a function of distance *r*, from the object's center is given by:

$$\Phi(r) = -\frac{GM}{R^3}\begin{cases} \frac{1}{2}\left(3R^2 - r^2\right) & \text{for } r \le R \\ \frac{R^3}{r} & \text{for } r > R \end{cases} \tag{2}$$

The result for $r < R$ is obtained by summation of the potential at some point p inside the mass from its spherical shell between R and r, and the remaining mass inside sphere of radius r.

The potential at point p due to a shell is given by

$$\Phi_{shell}(p) = -2\pi\varrho G\left(R^2 - r^2\right) \tag{3}$$

The potential at same point p due to the remaining inner sphere is given by

$$\Phi_{inner}(p) = -4\pi\varrho G r^2 \tag{4}$$

The total potential at point p inside the sphere is a superposition of both $\Phi_{shell}(p)$, and $\Phi_{inner}(p)$.

Thus, for any point inside a spherical mass, at distance r from its center (for $r < R$):

$$\Phi_p(r) = \Phi_{shell}(r) + \Phi_{inner}(r) = -4\pi\rho G\left(\frac{R^2}{2} - \frac{r^2}{6}\right) \tag{5}$$

$$\Phi_p(r) = -\frac{3GM}{2R^3}\left(R^2 - \frac{1}{2}r^2\right) \tag{6}$$

Since $M = \frac{4\pi}{3}\lim_{R\to 0} R^3\rho(0)$, the $1/R$ term is not singular as $M/R \propto R^2$.

The second singularity at $r \to 0$, is now removed (**Figure 1** and **Figure 2**).

4. The Identification of Φ(*r*) in the Metric

In the Schwarzschild metric, g_{00} is a function of distance r and one may write:

$$g_{00}(r) = 1 + \frac{2\Phi(r)}{c^2} \tag{7}$$

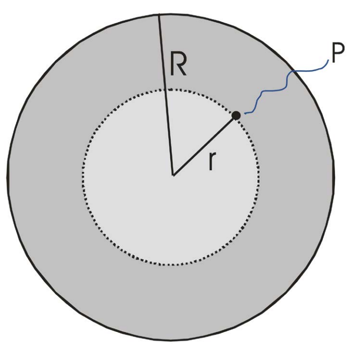

Figure 1. The gravitational potential of a spherically symmetric mass at point p, is depicted in the following figure, as a function of the distance r form the mass center. Notice, that unlike the case in a hollow sphere, the potential is not constant inside the sphere. Rather it falls of proportional with the square of the distance r. The minimum is obtained at $r = 0$, where the potential becomes $3GM/2R$.

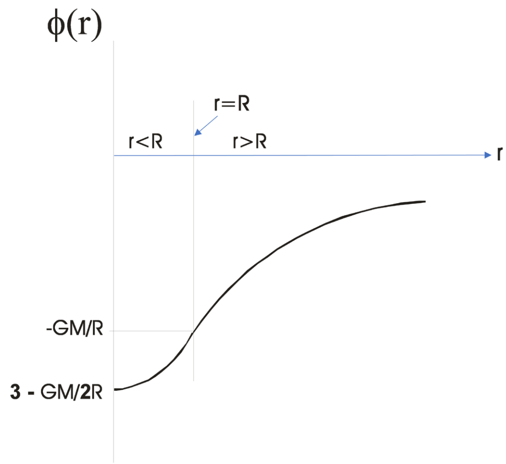

Figure 2. Classical gravitational field as a function of radial distance from a spherically symmetric source mass. At $r = 0$, the potential becomes $-3GM/2R$ with no singularity at $r \to 0$. The $1/R$ term is not singular since it cancels with the mass term ($M/R \propto R^2$).

First argument—dimensions

$\Phi(r)$ is a function of distance r, and must have the dimensions of c^2. It must obey the constraint of $\lim_{r\to\infty} \Phi(r) \to 0$ as should be the case at infinite distance in empty flat space.

Its units should be [G] [Kg]/[m] in order for $\frac{\Phi(r)}{c^2}$ to be dimensionless.

We therefore have a good reason to assume that $\Phi(r)$ is the gravitational potential due to a spherically symmetric object, independent of the size and density of that object. Exactly as described by Equation (2).

Second argument—Einstein equation under spherical symmetry

For a curved spacetime, based on the Riemann tensor, it can be shown that $\Phi(r)$ must have the form of a gravitational potential (see Appendix 1).

Following these arguments, it is justified to identify $\Phi(r)$, as the gravitational potential of the field created by a spherically symmetric, non-rotating, uncharged mass.

5. Inside the Sphere ($r < R$)

To see whether this is a true singularity one must look at quantities that are independent of the choice of coordinates. One such important quantity is the Kretschmann invariant [15] [16] [17], which is given by $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$.

For a Schwarzschild black hole of mass M and radius R, the Kretschmann invariant is $K_r = \frac{48G^2M^2}{c^4R^6} \approx \frac{842G^2}{\rho^2c^4}$. Obviously, it is independent of the radius R, and is always greater than 0 (provided we accept the physical assumption that density ρ, is never infinite).

General relativity predicts that any object collapsing beyond a certain point (for stars this is the Schwarzschild radius) would form a black hole, inside which a singularity (covered by an event horizon) would be formed.

As will be shown in the following, no such singularities exist.

In the case where $r < R$ (inside the mass), $\mathbb{K} \neq 0$ where

$$\mathbb{K} = \frac{16\pi G}{c^2}$$

$G = 6.6743 \times 10^{-11}$ and so $\mathbb{K} \approx 37 \times 10^{-27}$, so unless the internal density ρ is of the order of 10^{27} Kg/m^3, the Einstein field equation can be solved under the assumption of $\mathbb{K}\rho = 0$.

Even the heaviest neutron stars have overall densities of the order of 6×10^{17} Kg/m^3. For elementary particles, the neutron for instance, has density of approximately 3×10^{17} Kg/m^3. So, for all practical calculations one may assume $\mathbb{K} = 0$ inside any mass (that is, for $r < R$).

For the internal pressure $\boldsymbol{P}$, as long as the internal pressure is much less than 10^{24} Pa, and so $\mathbb{K}\boldsymbol{P} = 0$. For the Sun for instance, the internal core pressure is 10^{11} Pa.

Finding a solution to the homogeneous differential equation with $\mathbb{K}\rho = 0$ and $\mathbb{K}\boldsymbol{P} = 0$, will lead to the non-homogeneous solution with $\mathbb{K} = \text{constant}$. But we will concentrate on the homogeneous solution, since $\mathbb{K} \approx 0$.

In case of a constant $\mathbb{K}$, the non-homogeneous solution can be found.

This is the case for a perfect fluid.

Under the assumption of $\mathbb{K} \approx 0$, for the interior of the sphere, the proper time separation inside the sphere is given by $\mathrm{d}\tau^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$:

$$\mathrm{d}\tau^2 = \left(1 + \frac{2\Phi(r)}{c^2}\right)\mathrm{d}t^2 - \frac{1}{c^2}\left(h(r)r^2\mathrm{d}r - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\varphi^2\right) \tag{8}$$

6. A Modified Schwarzschild Metric and the Photon-Sphere

Inserting the expression for $\Phi(r)$:

$$\Phi(r) = -4\pi\rho G\left(\frac{R^2}{2} - \frac{r^2}{6}\right) \quad (\text{for } r \leq R)$$

$$\Phi(r) = -4\pi\rho G\left(\frac{R^3}{3r}\right) \quad (\text{for } r > R)$$

The Schwarzschild metric inside the sphere ($r \leq R$) becomes

$$\mathrm{d}\tau^2 = \left(1 - \frac{3}{2}\frac{r_s}{R}\left(1 - \frac{1}{3}\left(\frac{r}{R}\right)^2\right)\right)\mathrm{d}t^2 - \frac{1}{c^2}\left(h(r)r^2\mathrm{d}r - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\varphi^2\right) \tag{9}$$

while outside the sphere ($r > R$) it is

$$\mathrm{d}\tau^2 = \left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 - \frac{1}{c^2}\left(h(r)r^2\mathrm{d}r - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\varphi^2\right) \tag{10}$$

Defining the photon sphere radius $r_{ph} = \frac{3}{2}r_s$, there are now three different possibilities:

A. When both r_s and r_{ph} are inside the object's radius R. $r_s < r_{ph} < R$. This is a standard object.

B. When r_s is inside the object's radius R (a standard object) while r_{ph} is outside the object. $r_s < R < r_{ph}$.

C. When both r_s and r_{ph} are outside R. $R < r_s < r_{ph}$. (a Black hole). Therefore, B represents a possible situation where a photon-sphere light ring exists, surround a standard (not a black hole) body.

For said cases, the photon-sphere radius $r_{ph} = \frac{3}{2} r_s$ may fall inside or outside R (Table 1).

Case A is a standard object, with both r_s, and r_{ph} falling inside R.

Case B is a white object, with r_s falling inside R but r_{ph} falling outside R. Light is trapped in the photon sphere, causing the object to have a white glowing ring around it. There is no gravitational collapse.

Case C is a true black body, with both r_s and r_{ph} falling outside R.

For case A, both r_s and r_{ph} are inside R (holds for most stellar objects – but not for black holes). One obtains:

$$\mathrm{d}\tau^2 = \left(1 - \frac{3}{2}\frac{r_s}{R}\left(1 - \frac{1}{3}\left(\frac{r}{R}\right)^2\right)\right)\mathrm{d}t^2 - \frac{1}{c^2}\left(h(r)r^2\mathrm{d}r - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\varphi^2\right) \quad (11)$$

Picking the coordinate system so that the radius r is along the x axis, the θ and φ terms are zero

In other words, for $r > R$

$$\mathrm{d}\tau^2 = \mathrm{d}t^2\left(1 - \frac{r_s}{r}\right) \quad (12)$$

while for $r \le R$:

$$\mathrm{d}\tau^2 = \mathrm{d}t^2\left(1 - \frac{3}{2}\frac{r_s}{R}\left(1 - \frac{1}{3}\left(\frac{r}{R}\right)^2\right)\right) \quad (13)$$

Obviously, for $r = R$, $\mathrm{d}\tau^2 = 1 - \frac{r_s}{R}$, for both.

Investigating the condition

$$1 - \frac{r_{ph}}{R}\left(1 - \frac{1}{3}\left(\frac{r}{R}\right)^2\right) > 0$$

we see that in Case B, for all $r \in [0, R]$, as long as $r_s \le \frac{2}{3}R$ there is no singularity in the Schwarzschild metric.

Table 1. Three possible zones defining photon-sphere for different objects.

A	Standard object	r_s < R	r_{ph} < R
B	Standard object surrounded by white ring	r_s < R	R < r_{ph}
C	Black hole	r_s > R	r_{ph} > R

At $r_s = \frac{2}{3}R$ (and $r_{ph} = R$). $\mathrm{d}\tau^2$ becomes zero, and the object evaporates into photons ($\mathrm{d}\tau = 0$). Thus, $r_s > \frac{2}{3}R$ is the condition for a gravitational collapse.

As long as $r_s < \frac{2}{3}R$ the object is a standard object and does not undergo any gravitational collapse. This, in contrast to the current assertion (for $r > R$), is based on Equation (37), which states that $r_s > R$, is the condition for a gravitational collapse.

We see that as long as the photon sphere is inside the object ($r_{ph} \in [0, R]$), the Schwarzschild radius $r_s \in \left[0, \frac{2}{3}R\right]$. In other words, as long as the photon sphere does not show, the object is stable and does not suffer gravitational collapse. Once the photon sphere shows up ($r_{ph} > R$), the object undergoes gravitational collapse.

Anywhere in the region where $r_s \in \left[\frac{2}{3}R, R\right]$ the object undergoes gravitational collapse, and since $r_{ph} = 3/2\,R$, the photon sphere is outside the object, creating a bright light ring outside R, at $r = r_{ph}$.

Near the sphere's center, where $r \to 0$ and $r < R$, one has

$$\mathrm{d}\tau^2 = \mathrm{d}t^2\left(1 - \frac{r_{ph}}{R}\right) \tag{14}$$

And, as long as, $r_{ph} < R$, $\mathrm{d}\tau > 0$. It may be re-written as:

$$\mathrm{d}\tau = \mathrm{d}t\sqrt{1 - \frac{4\pi\rho G}{c^2}R^2} \tag{15}$$

In the case of infinitesimally small r, the time component becomes independent of distance r from origin. In any case, the divergence at $r \to 0$ is removed.

The proper time $\mathrm{d}\tau$, must remain well defined. Therefore, an upper limit on the density must exist for a given radius R, otherwise the expression for $\mathrm{d}\tau$ becomes undefined.

Notice that this assertion leads to the constraint

$$\rho \le \frac{c^2}{4\pi G R^2} \tag{16}$$

In other words, there is an upper limit on the density ρ, of any object with radius R. When $R \to 0$, the density may increase indefinitely, but as will be shown next, there is an upper limit on the density

7. Density Must Have an Upper Limit

Simple dimensional arguments show that the physical phenomena where quantum gravitational effects become relevant are those characterized by the Planck length $\ell_p = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35}\ \mathrm{m}$. Here $\hbar$ is the Planck constant that governs the scale of the quantum effects, G is the Newton constant that governs the strength of the gravitational force, and c is the speed of light, that governs the scale of the relativistic effects. The Planck length is many times smaller than what current technology is capable of observing. Physical effects at scales are so small. Because of this, we have no direct experimental guidance for building a quantum theory of gravity [14].

Suppose there exists a quantum minimum for distance. We call it the Planck length and denote it by ℓ_p.

It is given by

$$\ell_p = \sqrt{\frac{\hbar G}{c}} = 1.616 \times 10^{-35}\ \mathrm{m}.$$

Define in addition the Planck mass
Planck's mass

$$m_p = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8}\ \mathrm{kg}.$$

If this assumption is true, then the minimal spherical volume possible is

$$V = \frac{4\pi \ell_p^3}{3}$$

Let m' denote the mass of this volume, so its density will be given by

$$\rho_P = \frac{3m'}{4\pi \ell_p^3}$$

Since by assumption ℓ_p is the minimal length possible in nature, then for any mass m' the density ρ_P is the maximum possible.

It can be showed, that for a classical spherically symmetric object of mass m' and radius R, the general relativistic limit gives

$$\mathrm{d}\tau = \mathrm{d}t \sqrt{1 - \frac{4\pi \rho G}{c^2} R^2} \tag{17}$$

Since the expression in brackets must be real, we arrive at the restriction:

$$1 - \frac{4\pi \rho G}{c^2} R^2 \geq 0 \tag{18}$$

and therefore

$$\rho \leq \frac{c^2}{4\pi G R^2} \tag{19}$$

For an object of any given mass m' and radius R we have

$$\rho = \frac{m'}{V} = \frac{3m'}{4\pi R^3} \tag{20}$$

Since for any mass m' of radius R one must have, by Equation (16) above,

$$\rho = \frac{3m'}{4\pi R^3} \le \frac{c^2}{4\pi G R^2} \tag{21}$$

The result is that for any mass m'

$$m'(R) \le \frac{Rc^2}{G} \tag{22}$$

Obviously, the smaller the radius R, the smaller the allowed mass m'.

For the minimal possible length (according to Planck) $R = \ell_p$ one obtains:

$$m' \le \frac{\ell_p c^2}{G} \tag{23}$$

Recall now, by Planck's dimensionality analysis) that $\ell_p = \sqrt{\frac{\hbar G}{c}}$ and $m_p = \sqrt{\frac{\hbar c}{G}}$ so

$$\frac{m_p}{\ell_p} = \frac{c^2}{G} \tag{24}$$

Therefore

$$m' \le \frac{\ell_p c^2}{G} = m_p \tag{25}$$

Therefore, for any mass m'(with radius ℓ_p)

$$m' \le m_p \tag{26}$$

And since

$$\rho = \frac{m'}{V} \le \frac{m_p}{V} = \rho_P \tag{27}$$

We have the result that ρ_P is the maximal possible density, namely Planck density.

In other words, for any given radius R, the mass $m'(R)$ becomes smaller and smaller with R, but when one reaches the smallest possible radius ℓ_p, the mass must be smaller than the Planck mass m_p, and so, the density will always be smaller than the Planck density ρ_P.

One can reduce the radius R, but the density will never exceed the Planck density.

$$\lim_{R \to \ell_p} \rho(R) = \left(\frac{\ell_p}{R}\right)^3 \rho_P \tag{28}$$

Assume next, that the sphere has density $\rho(r)$, which varies with distance r from its center.

Assume the sphere is of minimal possible radius ℓ_p.

We need to calculate the radius R of a quantized particle by its average normalized density so that we obtain a reduced average radius.

By comparing the integrated variable density over the Planck radius, to a volume, with constant density ρ_0 one obtains:

$$4\pi\int_0^{\ell_p} r^2\rho(r)\,\mathrm{d}r = \frac{4\pi\rho_0\ell_p^3}{3} \tag{29}$$

By definition, the average classical distance $\langle r^2\rangle$ is given by the integral over the normalized density:

$$\langle r^2\rangle \overset{\text{def}}{=} \frac{1}{\ell_p}\int_0^{\ell_p} r^2\,\rho(r)/\rho_0\,\mathrm{d}r \tag{30}$$

Thus

$$4\pi\ell_p\rho_0\langle r^2\rangle = \frac{4\pi\rho_0\ell_p^3}{3} \tag{31}$$

and so

$$\langle r\rangle = \sqrt{\langle r^2\rangle} = \frac{1}{\sqrt{3}}\ell_p \tag{32}$$

Hence, the actual measured classical radius R is given by

$$R = \langle r\rangle = \frac{1}{\sqrt{3}}\ell_p \tag{33}$$

The above result shows how the lower limit of the classical gravitation theory by Einstein, is related to the Planck length, which is a quantum phenomenon posed by dimensional analysis of the universe constants.

Therefore, classical relativity and the relationship between the universal constants leads to quantization of space.

8. Quantization by Planck's Units and Maximal Density Limit

In the limit where $R \to 0$, one needs to consider quantum limits and the uncertainty principle.

Planck's mass $m_p = \sqrt{\frac{\hbar c}{G}} = 2.176\times10^{-8}\ \text{kg}$.

Planck's length $\ell_p = \sqrt{\frac{\hbar G}{c}} = 1.616\times10^{-35}\ \text{m}$.

However, dimensional analysis can only determine the Planck's units up to a dimensionless multiplicative factor. For instance, use h instead of $\hbar$.

Based on Planck's units one obtains Planck's maximal density ρ_P as Planck's mass/Planck's volume (assuming ℓ_p is the lowest possible physical distance, leads to the maximal possible physical density assumption).

$$\rho_P = \frac{m_p}{4\pi\ell_p^3/3} = 1.23074\times10^{+96} \tag{34}$$

(If one uses the Planck constant instead of the reduced Planck's constant, the Planck density will be modified by a factor of 2π).

Since one must assume that Planck's density is the maximum possible density allowed, we now ask, what should the minimum classical radius R be, in order for $\rho \le \rho_P$. (Any classical density cannot exceed the Planck density).

Therefore:

$$\rho = \frac{c^2}{4\pi G R^2} \le \rho_P = \frac{3m_p}{4\pi \ell_p^3} \tag{35}$$

Doing the calculation, we find that one must have

$R \ge \frac{1}{\sqrt{3}} \ell_p = 0.93325 \times 10^{-35}\ \text{m}$.

Compared to Planck's length $\ell_p = 1.61625 \times 10^{-35}\ \text{m}$, the classically derived radius R is smaller than the allowed Planck's length ℓ_p, by an unexplained factor of 0.5774.

These calculations were based on the assumptions of negligible $\mathbb{K} = \frac{16\pi G}{c^2}\rho$, and $R \to 0$ $(r < R)$.

It seems now, that the classical solution to the metric of a spherically symmetric object, puts a lower limit on the radius R of an object. It looks like there is a sort of quantization limit to small scale objects in spacetime.

Instead of having $R \ge \ell_p$, we find $R \ge 0.5774 \ell_p$.

The reason for this 0.5774 factor is due to the fact that we have assumed a constant density inside the sphere. If instead we assume variation in density with radius (highest density at $r \to 0$ and then it falls off to zero at ℓ_p). We need to calculate the radius R of a quantized particle by its average density so that we obtain a reduced average radius.

By comparing the integrated variable density over the Planck radius, to the same volume, but with a constant average density ρ_0 one obtains (see paragraph VI on density upper limit)):

$$R = \langle r \rangle = \frac{1}{\sqrt{3}} \ell_p \tag{36}$$

Since $0.5774 = \frac{1}{\sqrt{3}}$, it is now clear where the reduction factor came from.

The above result shows how the lower limit of the classical gravitation theory by Einstein, (Schwarzschild radius), and the dimensional constraints between the constants of the universe, result in a quantized material space.

9. A Tunnelled Sphere "Thought Experiment"

Consider a sphere with a thin penetrating tunnel through center of sphere between opposite surfaces.

The sphere of radius R and of mass M will exert a gravitational attractive force on a remote particle. The force will be directed towards the center of the sphere, as if all of its mass is concentrated at its center (where $r = 0$).

The attracted particle will be accelerated until it reaches the surface, but then will continue its path through the hole. It will continue with no acceleration inside the shell and then come out through the opposite hole.

On its journey, the particle will pass twice the Schwarzschild radius $r_s = \frac{GM}{c^2}$

on its way towards the center and on its way out.

The metric of this shell will be similar to that of a spherical mass. Only that this time, even though r_s is inside the body, one can reach the Schwarzschild radius independently of whether it is a black hole or a standard mass.

One way of doing the experiment would be to send a photon through the hole in the mass and compare its time of arrival at the other end, with a simultaneously emitted photon traveling outside the mass. One would then be able to compare differences in the time of arrival for the two photons.

This gedanken experiment can reveal whether the approach to r_s will last forever in the eyes of the external observer. If it does, then the particle will never be able to come out through the opposite hole.

10. Dynamics near Center

It is interesting to see what are the equations of motion near center (where $r \ll R$).

Assuming spherical symmetry [20] [21] [22], it is allowed to investigate it along the radial distance r, and assume it is the x axis.

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = -\Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \tag{37}$$

where

$$\Gamma^\lambda_{\gamma\mu} = \frac{1}{2} g^{\lambda r} \left(\partial_\mu g_{r\gamma} + \partial_\gamma g_{r\mu} - \partial_r g_{\gamma\mu} \right) \tag{38}$$

The total time of flight in the classical case will be given by (see Appendix 2):

$$2T = \frac{2}{\sqrt{Q_0}} \cos^{-1}\left[2\left(1 - \frac{Q_0 R^3}{GM} \right) \right]$$

Whereas in the general relativistic case it will be:

$$2T = \frac{2}{\sqrt{Q}} \cos^{-1}\left[2\left(1 - \frac{Q R^3}{GM} \right) \right]$$

Comparing the two results, $T_{Gen.Rel.} > T_{class}$.

As an example, for the Sun, with $R = 6.963 \times 10^8\ \mathrm{m}$ and density $\rho_{\square} = 150 \times 10^3\ \mathrm{kg/m^3}$ the predicted general relativistic cross time will be 6.6 mSec longer than the classical calculated cross time. This result is based on average Q over r from 0 to R.

For an average Neutron star of radius $R = 11{,}000$ m and density $\rho(r) = 4 \times 10^{17}\ \mathrm{kg/m^3}$, the passage time under general relativity prediction, will last 0.13 mSec longer than under classical calculation.

11. Conclusions

Using Newton's classical shell theorem, the Schwarzschild metric was modified. This removed the singularity at $r = 0$ for a standard object (not a black hole).

For all practical matters, $r < R$ can be treated as an empty space even for the densest known stellar objects (neutron stars) and also for elementary particles (neutrons for instance).

It was demonstrated how general relativity evidently leads to quantization of space-time. Both classical and quantum mechanical limits on density give the same result.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix 1

For a curved spacetime we define the Riemann tensor:

$$R^{\mu}_{\alpha\beta\gamma} = \partial_{\beta}\Gamma^{\mu}_{\alpha\gamma} - \partial_{\gamma}\Gamma^{\mu}_{\alpha\beta} + \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\alpha\gamma} - \Gamma^{\mu}_{\sigma\gamma}\Gamma^{\sigma}_{\alpha\beta}$$

where $\Gamma^{\mu}_{\alpha\beta} = g^{\mu\sigma}\Gamma_{\sigma\alpha\beta}$ are the Christoffel symbols:

$$\Gamma_{ikl} = \frac{1}{2}\left(\partial_l g_{ik} + \partial_k g_{il} - \partial_i g_{kl}\right) \tag{39}$$

The Ricci tensor $R_{\alpha\beta}$ is defined by:

$$R_{\alpha\beta} = g^{\sigma\varrho} R_{\alpha\varrho\sigma\beta} \tag{40}$$

And the curvature scalar R is defined by:

$$\mathcal{R} = g^{\alpha\beta} R_{\alpha\beta} = g^{\alpha\beta} g^{\sigma\varrho} R_{\alpha\beta\sigma\varrho} \tag{41}$$

For the Einstein equation one has:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{42}$$

In empty space $T_{\mu\nu} = 0$ everywhere and therefore $R_{\mu\nu} = 0$ everywhere.

Due to spherical symmetry, the metric must be of the form

$$g_{\mu\nu} = \begin{pmatrix} g(r) & & & \\ & h(r) & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix} \tag{43}$$

where $g(r)$ and $h(r)$ are functions of the distance r from the coordinate center (located at the center of the mass.

So, the infinitesimal proper time interval $\mathrm{d}\tau$ between two events along a time-like path is given by

$$\mathrm{d}\tau^2 = g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = g(r)\mathrm{d}t - \frac{1}{c^2}\left(h(r) r^2 \mathrm{d}r - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\varphi^2\right) \tag{44}$$

With the flat space metric $g_{\mu\nu} = (+---)$.

In order to solve Einstein's equation everywhere, one can no longer assume that the curvature $\mathcal{R}$ is null anywhere. This is true only outside the mass and so $\mathcal{R} = 0$ only for $r > R$.

Assuming the mass density of the object being homogeneous and at total freeze of motion, $T_{\mu\nu} = 0$, except for $T_{00} = \rho c^2$, with ρ being the mass density.

$$g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R}\right) = \frac{8\pi G}{c^4} g^{\mu\nu} T_{\mu\nu} \tag{45}$$

Under perfect fluid conditions, $T_{\mu\nu} = \begin{pmatrix} \rho c^2 & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$, where p is the pressure.

Define $\mathbb{K} = \frac{16\pi G}{c^2}$, and so,

$$R_{00} = \frac{16\pi G}{c^2}\rho = \mathbb{K}\rho \tag{46}$$

For all other terms, $R_{11} = R_{22} = R_{33}$:

$$R_{ii} = \frac{16\pi G}{c^2}P = \mathbb{K}\boldsymbol{P} \tag{47}$$

From the definitions of the Riemann tensor and the Ricci tensor we obtain

$$R_{\mu\gamma} = \partial_\delta \Gamma^\delta_{\mu\gamma} - \partial_\gamma \Gamma^\delta_{\mu\delta} + \Gamma^\delta_{\delta\lambda}\Gamma^\lambda_{\gamma\mu} - \Gamma^\delta_{\gamma\lambda}\Gamma^\lambda_{\delta\mu} \tag{48}$$

$$R_{00} = \partial_\delta \Gamma^\delta_{00} - \partial_0 \Gamma^\delta_{0\delta} + \Gamma^\delta_{\delta\lambda}\Gamma^\lambda_{00} - \Gamma^\delta_{0\lambda}\Gamma^\lambda_{\delta 0} \tag{49}$$

Following some tedious mathematical work, one arrives at [4]:

$$\partial_r \ln\left(h(r)g(r)\right) = -\frac{16\pi G}{c^2}\rho\frac{rh(r)}{g(r)} \tag{50}$$

And so

$$R_{22} = \frac{1}{h(r)} - 1 - \frac{r}{2h(r)}\left(\frac{h'(r)}{h(r)} - \frac{g'(r)}{g(r)}\right) = \frac{1}{h(r)} - 1 - \frac{r}{2h(r)}\partial_r \ln\left(\frac{h(r)}{g(r)}\right) \tag{51}$$

We have then two equations:

$$\partial_r \ln\left(h(r)g(r)\right) = -\mathbb{K}\frac{rh(r)}{g(r)} \tag{52}$$

$$R_{22} = \frac{1}{h(r)} - 1 - \frac{r}{2h(r)}\partial_r \ln\left(\frac{h(r)}{g(r)}\right) \tag{53}$$

<u>Outside the mass ($r > R$)</u>

$\mathbb{K} = 0$ and so:

$h(r)g(r) = \alpha$ (some constant).

Substitute in Equation (2) to obtain:

$$R_{22} = \frac{g(r)}{\alpha} - 1 - \frac{rg(r)}{2\alpha}\partial_r \ln\left(\frac{\alpha}{g^2(r)}\right) = 0 \tag{54}$$

And after few steps:

$$\alpha = \partial_r\left(rg(r)\right) \tag{55}$$

So finally:

$$g(r) = \alpha r + \beta \tag{56}$$

$$h(r) = \frac{\alpha}{\alpha r + \beta} \tag{57}$$

The metric becomes:

$$g_{\mu\nu} = \begin{pmatrix} \alpha r + \beta & & & \\ & \frac{\alpha}{\alpha r + \beta} & & \\ & & r^2 & \\ & & & r^2 \sin^2\theta \end{pmatrix} \tag{58}$$

Comparing this metric with the asymptotic *weak-field* metric ($r \gg R$):

$$g_{\mu\nu} = \begin{pmatrix} 1-\frac{2GM}{rc^2} & & & \\ & \frac{-1}{1-\frac{2GM}{rc^2}} & & \\ & & -r^2 & \\ & & & -r^2\sin^2\theta \end{pmatrix} \tag{59}$$

It is suggested that

$$\alpha r + \beta = 1 + \frac{2\Phi(r)}{c^2} \tag{60}$$

Therefore,

$$\beta = 1 \tag{61}$$

$$\alpha = \frac{2\Phi(r)}{rc^2} \tag{62}$$

This shows us that the structure of the Schwarzschild metric in Equation (7), is not only a result of dimensional considerations discussed above, but also a result of spherical symmetry considerations of the Einstein Equation, Equation (17).

This result was obtained based on symmetry arguments only, and it is independent of the weak-field approximation.

Inside the mass ($r < R$)

$\mathbb{K} > 0$:

The only approximation made was the assumption on empty space solution. For non-empty space, it will be valid for a perfect fluid of zero pressure.

However, one may use a model where the pressure $\boldsymbol{P}$ is proportional o the density [16], and so $\boldsymbol{P} = \omega\rho$ where ω is a proportionality factor.

Appendix 2

Since by coordinates choice and symmetry, only Γ^{λ}_{00} and Γ^{λ}_{11} are non-zero (assuming constant gravitational field $\partial_t g_{r\mu} = 0$).

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} = -\Gamma^1_{00}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \Gamma^1_{11}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 \tag{63}$$

$$\begin{aligned} \Gamma^1_{00} &= \frac{1}{2} g^{1r}\left(\partial_0 g_{r0} + \partial_0 g_{r0} - \partial_r g_{00}\right) \\ &= \frac{1}{2} g^{11}\left(\partial_0 g_{10} + \partial_0 g_{10} - \partial_1 g_{00}\right) \\ &= -\frac{1}{2} g^{11}\partial_1 g_{00} \end{aligned} \tag{64}$$

and likewise

$$\Gamma^1_{11} = \frac{1}{2} g^{11}\left(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}\right) = \frac{1}{2} g^{11}\partial_1 g_{11} \tag{65}$$

The equation of motion in the x direction becomes

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2}=\left(\frac{1}{2}g^{11}\partial_1 g_{00}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 c^2-\left(\frac{1}{2}g^{11}\partial_1 g_{11}\right)\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 \tag{66}$$

Since the choice of coordinates is such that x is represented by r, one can write

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2}=\frac{1}{2}g^{11}\partial_r g_{00}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 c^2-\frac{1}{2}g^{11}\partial_r g_{11}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \tag{67}$$

From the proper-time result (which holds inside the sphere, where $r < R$)

$$\mathrm{d}\tau^2=\mathrm{d}t^2\left(1-\frac{4\pi G\rho R^2}{c^2}\left(1-\frac{1}{3}\left(\frac{r}{R}\right)^2\right)\right)=-g_{00}\mathrm{d}t^2 \tag{68}$$

Define for convenience

$$g_{00}=A^2(r)=1-\frac{4\pi G\rho R^2}{c^2}\left(1-\frac{1}{3}\left(\frac{r}{R}\right)^2\right) \tag{69}$$

And so

$$\frac{\mathrm{d}\tau}{\mathrm{d}t}=A \tag{70}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\tau}=\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau}=\frac{1}{A}\frac{\mathrm{d}r}{\mathrm{d}t} \tag{71}$$

And

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2}=\frac{1}{A^2}\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} \tag{72}$$

From the metric definition:

$$\mathrm{d}\tau^2=g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu=g_{00}\mathrm{d}x^0\mathrm{d}x^0+g_{11}\mathrm{d}x^1\mathrm{d}x^1 \tag{73}$$

and since

$$g_{00}=A^2 \tag{74}$$

$$g_{11}=-\frac{1}{g_{00}}=-\frac{1}{A^2} \tag{75}$$

One obtains:

$$\partial_r g_{11}=\frac{1}{g_{00}^2}\partial_r g_{00} \tag{76}$$

Inserting into Equation (49) one obtains:

$$\frac{1}{A^2}\frac{\mathrm{d}^2 r}{\mathrm{d}t^2}=\frac{1}{2}g^{11}\partial_r g_{00}\left[\frac{1}{A^2}c^2-\frac{1}{g_{00}^2}\frac{1}{A^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\right] \tag{77}$$

And so

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2}=-\frac{1}{2}\frac{1}{g_{00}}\partial_r g_{00}\left[c^2-\frac{1}{g_{00}^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\right] \tag{78}$$

For $\dfrac{r}{R}\ll 1$

$$g_{00} = \frac{4\pi G\rho R^2}{c^2} - 1 \tag{79}$$

And so $\partial_r g_{00} = 0$, which means that $\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = 0$.

The acceleration near the center is zero.

However, if one moves away from the center:

$$g_{00} = \frac{4\pi G\rho R^2}{c^2}\left(1 - \frac{1}{3}\left(\frac{r}{R}\right)^2\right) - 1 \tag{80}$$

And so

$$\partial_r g_{00} = \frac{4\pi G\rho R^2}{c^2}\left(-\frac{2r}{3R^2}\right) = -\frac{8\pi G\rho}{3c^2} r \tag{81}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{1}{2}\frac{1}{g_{00}}\partial_r g_{00}\left[c^2 - \frac{1}{g_{00}^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\right] \tag{82}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{1}{2}\frac{1}{A^2}\left(-\frac{8\pi G\rho}{3c^2} r\right)\left[c^2 + \frac{1}{A^4}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\right] \tag{83}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{4\pi G\rho}{3c^2 A^2} r\left[c^2 + \frac{1}{A^4}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\right] \tag{84}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{4\pi G\rho}{3A^2} r\left[1 + \frac{1}{c^2 A^4}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\right] \tag{85}$$

Near center (where $r \to 0$), $A^2 = 1 - \frac{4\pi G\rho R^2}{c^2}$ depends on r only through the density $\rho(r)$.

We may write

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = Q(r) r\left(1 + W(r)\left(\frac{v(r)}{c}\right)^2\right) \tag{86}$$

Here, $v(r)$ is the approach velocity. Q and W depend on r and are defined by:

$$Q(r) \stackrel{\text{def}}{=} \frac{4\pi G\rho(r)}{3A^2(r)} \tag{87}$$

$$W(r) \stackrel{\text{def}}{=} \frac{1}{A^4(r)} \tag{88}$$

The result shows how on approach to center ($r \to 0$), where $\rho(r)$ is assumed constant, the acceleration decreases in proportion with r on one hand, but depends also on the square of the approach velocity.

For small approach-to-center velocity ($v/c \ll 1$)

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -Qr \tag{89}$$

One may approximate though for a slow approach velocity ($v/c \ll 1$):

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -Q(r)r \tag{90}$$

where

$$Q(r) = \frac{4\pi G\rho(r)}{3\left[1 - \frac{4\pi G\rho(r)R^2}{c^2}\left(1 - \frac{1}{3}\left(\frac{r}{R}\right)^2\right)\right]} \tag{91}$$

(Recall that by the assumption on the maximal density (Equation (38)), $\frac{4\pi G\rho(r)R^2}{c^2} < 1$, and so, $Q(r) \geq 0$. Also, note that $[Q] = 1/\sec^2$]).

One may solve this under certain approximation:

$\rho(r) = \rho_0$ constant density inside $r < R$.

In this case $Q(r)$ is nearly a constant independent of r, and the equation becomes

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -Qr \tag{92}$$

Which general solution is

$$r(t) = B + C\cos\left(\sqrt{Q}t\right) + D\sin\left(\sqrt{Q}t\right) \tag{93}$$

Even though $Q(r)$ is not constant, the term $\frac{4\pi G\rho(r)R^2}{c^2}$ in the denominator of $Q(r)$ is of the order of, or less than, 9.2×10^{-9}. $Q(r)$ may be approximated as:

$$Q(r) \cong \frac{4\pi G\rho(r)}{3\left[1 - 9.2\times 10^{-9}\left(1 - \frac{1}{3}\left(\frac{r}{R}\right)^2\right)\right]} \cong \frac{4\pi G\rho(r)}{3} \tag{94}$$

$$\dot{r}(t) = -C\sqrt{Q}\sin\left(\sqrt{Q}t\right) + D\sqrt{Q}\cos\left(\sqrt{Q}t\right) \tag{95}$$

Assuming zero start velocity, $\dot{r}(0) = -\sqrt{Q}D$ one must have $D = 0$.

Hence

$$r(t) = B - C\cos\left(\sqrt{Q}t\right) \tag{96}$$

Also, $r(0) = R$.

Therefore

$$R = B - C \tag{97}$$

$$r(t) = B - (B - R)\cos\left(\sqrt{Q}t\right) \tag{98}$$

Assume the acceleration $\ddot{r}(0) = -\frac{GM}{R^2}$,

$$\ddot{r}(t) = B - C\cos\left(\sqrt{Q}t\right) \tag{99}$$

Using these assumptions, one obtains:

$$r(t) = R + \frac{GM}{QR^2}\left(\cos\left(\sqrt{Q}t\right) - 1\right) \tag{100}$$

Furthermore, if the object arrives at the center ($r = 0$) at some later time T, one has:

$$r(T) = 0 = R + \frac{GM}{QR^2}\left(\cos\left(\sqrt{Q}T\right) - 1\right) \tag{101}$$

The time to cross the diameter $2R$ (from surface to surface through the center) is thus

$$2T = \frac{2}{\sqrt{Q}}\cos^{-1}\left[1 - \frac{QR^3}{GM}\right] \tag{102}$$

where

$$Q(r) = \frac{4\pi G\rho(r)}{3\left[1 - \frac{4\pi G\rho(r)R^2}{c^2}\left(1 - \frac{1}{3}\left(\frac{r}{R}\right)^2\right)\right]}$$

In this case

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{4\pi G}{3} r\rho(r) \tag{103}$$

Classically, a falling object of mass m, starting at $r < R$, will be accelerated towards center according to the potential:

$$\Phi(r) = -\frac{GM}{2R^3}\left(3R^2 - r^2\right) \tag{104}$$

And the acceleration will be

$$F = m\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -m\frac{\partial\Phi(r)}{\partial r} = -m\frac{GM}{2R^3}2r \tag{105}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{GM}{R^3}r = -\frac{G4\pi\rho_0 R^3}{3R^3}r = -\frac{4\pi G\rho_0}{3}r \tag{106}$$

Classically, the acceleration will be

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{4\pi G}{3}\rho_0 r = -Q_0 r \tag{107}$$

with $Q_0 = \frac{4\pi G}{3}\rho_0$

This, in comparison to the general relativistic solution:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -Q(r)r \tag{108}$$