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A Way to Quantum Gravity

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Abstract

We present a simple way to approach the hard problem of quantization of the gravitational field in four-dimensional space-time, due to non-linearity of the Einstein equations. The difficulty may be overcome when the cosmological constant is non-null. Treating the cosmological contribution as the energy-momentum of vacuum, and representing the metric tensor onto the tetrad of its eigenvectors, the corresponding energy-momentum and, consequently, the Hamiltonian are easily quantized assuming a correspondence rule according to which the eigenvectors are replaced by creation and annihilation operators for the gravitational field. So the geometric Einstein tensor, which is opposite in sign respect to the vacuum energy-momentum (plus the possible known matter one), is also quantized. Physical examples provided by Schwarzschild-De Sitter, Robertson-Walker-De Sitter and Kerr-De Sitter solutions are examined.

Keywords

General Relativity, Quantum Gravity, Cosmological Constant

1. Introduction

In the present paper we propose a straightforward way to quantization of the gravitational field when the cosmological constant is non-vanishing within the Einstein equations.

In Section 2 we propose an energy-momentum interpretation of the negative Einstein tensor, when it is shifted from the l.h.s. to the r.h.s., so that the total energy-momentum including matter and vacuum (or dark) contribution is identically null.

In Section 3 we represent the metric tensor onto the tetrad of its ortho-normalized eigenvectors and propose a correspondence rule according to which quantization is obtained replacing the eigenvectors with the creation and annihilation operators for the field. Some care is required in evaluating the product of the complex operators in order to provide a Hermitian result, the metric tensor being real. More, it is shown how the space-time interval expectation values are discretized.

In Section 4 we apply the proposed quantization method to physical solutions such as Schwarzschild-De Sitter, Robertson-Walker-De Sitter and Kerr-De Sitter metrics.

In Section 5 some conclusions and perspectives are suggested.

2. Energy-Momentum Interpretation of the Equations of the Gravitational Field

Let us consider the Einstein field equations of the gravitational field, including the cosmological constant [1] [2] in presence of observable matter and non-gravitational interaction fields:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}^{(mf)}, \tag{1}$$

the universal constant κ being the Einstein gravitational constant equal to:

$$\kappa = \frac{8\pi G}{c^4}, \tag{2}$$

with $G = 6.67430 \times 10^{-11}\ \mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{sec}^{-2}$, Newton gravitational constant.

As usual $\mathcal{g} \equiv \left(g_{\mu\nu}\right)$ is the metric tensor of signature $(+,-,-,-)$, $R \equiv \left(R_{\mu\nu}\right)$, is the Ricci tensor, where:

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\nu\alpha,\nu} - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\beta} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}, \tag{3}$$

of trace $R = g^{\mu\nu} R_{\mu\nu}$, $g^{\mu\nu}$ being the contravariant components of the metric tensor (inverse matrix):

$$g^{\mu\rho} g_{\rho\nu} = \delta^{\mu}_{\nu}, \tag{4}$$

and:

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta} \right), \tag{5}$$

the symmetric connection coefficients (Christoffel symbols) in a V^4 space-time torsionless manifold. Moreover $T^{(mf)} \equiv \left(T^{(mf)}_{\mu\nu}\right)$ is the energy-momentum tensor of the observable matter and non-gravitational fields, while Λ is the cosmological constant.

On introducing the notation $T^{(g)} \equiv \left(T^{(g)}_{\mu\nu}\right)$, where:

$$\kappa T^{(g)}_{\mu\nu} = -R_{\mu\nu} + \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu}, \tag{6}$$

which we may interpret as the energy-momentum tensor of the gravitational field, we can write the field Equation (1) as an energy-momentum balance equation.

We have:

$$T^{(g)}_{\mu\nu} + T^{(mf)}_{\mu\nu} = 0. \tag{7}$$

More, it results to be convenient to split the gravitational contribution in two parts, the former due to ordinary attractive gravitation (geometric), and the latter due to the repulsive gravitation (cosmological or dark) [3] [4] [5] [6]:

$$T_{\mu\nu}^{(g)} = T_{\mu\nu}^{(R)} + T_{\mu\nu}^{(\Lambda)}, \tag{8}$$

where:

$$\kappa T_{\mu\nu}^{(R)} = -R_{\mu\nu} + \frac{1}{2} R g_{\mu\nu}, \quad \kappa T_{\mu\nu}^{(\Lambda)} = \Lambda g_{\mu\nu}. \tag{9}$$

Of course, in absence of observable matter and non-gravitational fields the energy-momentum components of the full gravitational field $T_{\mu\nu}^{(g)}$ vanish.

As we will see soon, the representation (7) of the field equations has the non-trivial advantage of facilitating quantization of the gravitational field.

We emphasize that, as a consequence of the *equivalence principle,* the description of a *physical gravitational field* within a pseudo-Euclidean (*i.e.,* Minkowskian) space-time is equivalent to the *geometry* of a suitable non-Euclidean (*i.e.,* Riemannian) space-time. Therefore the Einstein equations may be interpreted either *geometrically* as a set of partial differential equations governing metric, connection and curvature of space-time ($R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0$), or as *energetic* conditions ($T_{\mu\nu}^{(g)} = 0$) governing the energy-momentum of a *physical gravitational field* living within a flat space-time. Adopting the latter interpretation we have split the energy-momentum tensor of the free *physical gravitational field* (which is null because of the Einstein equations and then results to be non-localizable) into two non-vanishing localizable contributions ($T_{\mu\nu}^{(R)}$) and ($T_{\mu\nu}^{\Lambda}$). Quantization will be performed on the Hamiltonian of the cosmological part, so providing quantization also of the opposite Riemannian part. It is remarkable that even if the null Hamiltonian of the whole gravitational field cannot be quantized being non-localizable, each one of its half opposite contributions can be quantized separately.

3. Approaching the Quantization of the Gravitational Field

As a first step, we will be concerned with the gravitation in absence of external non-gravitational fields, *i.e.,* when:

$$T_{\mu\nu}^{(mf)} = 0. \tag{10}$$

Then Equation (7) becomes simply:

$$T_{\mu\nu}^{(R)} + T_{\mu\nu}^{(\Lambda)} = 0. \tag{11}$$

In order to open a way towards a method for gravity quantization resembling the well known procedures for Abelian (electromagnetic) and non-Abelian Yang-Mills (weak and strong interaction) fields, we represent the metric tensor on the tetrad of its ortho-normalized eigenvectors $\left\{ a_\mu^{(\sigma)}, \sigma = 0,1,2,3 \right\}$:

$$g_{\mu\nu} = \lambda^{[\sigma]} \eta_{(\sigma)(\tau)} a_\mu^{(\sigma)} a_\nu^{(\tau)}, \tag{12}$$

where:

$$\left(g_{\mu\nu}-\lambda^{[\sigma]}\eta_{\mu\nu}\right)a^{(\sigma)\nu}=0, \tag{13}$$

being:

$$\eta_{(\sigma)(\tau)}=\mathrm{diag}\left(+1,-1,-1,-1\right), \tag{14}$$

and:

$$g_{\mu\nu}\,a^{\mu}_{(\sigma)}\,a^{\nu}_{(\tau)}=\eta_{(\sigma)(\tau)}. \tag{15}$$

Then the components of the energy-momentum tensor $T^{(\Lambda)}$ result:

$$T^{(\Lambda)}_{\mu\nu}=\frac{\Lambda}{\kappa}\lambda^{[\sigma]}a_{(\sigma)\mu}\,a^{(\sigma)}_{\nu}, \tag{16}$$

where:

$$a_{(\sigma)\mu}=\eta_{(\sigma)(\tau)}\,a^{(\tau)}_{\mu}. \tag{17}$$

Let us now introduce the notation:

$$\omega^{[\sigma]}=\frac{\Lambda V}{\kappa\hbar}\lambda^{[\sigma]}, \tag{18}$$

where V is any constant volume in the physical space and $\hbar$ the reduced Planck constant. We emphasize that the eigenvalues $\lambda^{[\sigma]}$ are positive, the signature of the metric tensor being involved in $\eta_{\mu\nu}$.

It follows in (16):

$$T^{(\Lambda)}_{\mu\nu}=\frac{1}{V}\hbar\omega^{[\sigma]}a_{(\sigma)\mu}\,a^{(\sigma)}_{\nu}, \tag{19}$$

In order to quantization we assume the following correspondence relation:

$$a_{(\sigma)\mu}\rightarrow \mathrm{a}_{(\sigma)\mu}, \tag{20}$$

where $\mathrm{a}_{(\sigma)\mu}$ are complex operators, and replace the non-quantized relation (19) with the correspondent quantum equation:

$$\mathrm{T}^{(\Lambda)}_{\mu\nu}=\frac{1}{2V}\hbar\omega^{[\sigma]}\left(\mathrm{a}^{+}_{(\sigma)\mu}\,\mathrm{a}^{(\sigma)}_{\nu}+\mathrm{a}_{(\sigma)\nu}\,\mathrm{a}^{(\sigma)+}_{\mu}\right), \tag{21}$$

symmetrization being required to preserve hermiticity of the energy-momentum operator $\mathrm{T}^{(\Lambda)}_{\mu\nu}$. The relation between the creation and annihilation operators with the co-ordinates and momenta operators:

$$\mathrm{Q}^{(\sigma)}_{\mu}=\frac{1}{\sqrt{2}}\left(\mathrm{a}^{(\sigma)}_{\mu}+\mathrm{a}^{(\sigma)+}_{\mu}\right),\quad \mathrm{P}^{(\sigma)}_{\mu}=\frac{1}{i\sqrt{2}}\left(\mathrm{a}^{(\sigma)}_{\mu}-\mathrm{a}^{(\sigma)+}_{\mu}\right), \tag{22}$$

provides the energy-momentum tensor for a set of harmonic oscillators:

$$\mathrm{T}^{(\Lambda)}_{\mu\nu}=\frac{1}{2V}\hbar\omega^{[\sigma]}\left(\mathrm{Q}^{(\sigma)}_{\mu}\mathrm{Q}_{(\sigma)\nu}+\mathrm{P}^{(\sigma)}_{\mu}\mathrm{P}_{(\sigma)\nu}\right). \tag{23}$$

Definitions (22) ensure, as usual in quantum field theory, the commutation relation:

$$\mathrm{a}^{(\sigma)}_{\nu}\mathrm{a}^{(\tau)+}_{\mu}-\mathrm{a}^{(\tau)+}_{\mu}\mathrm{a}^{(\sigma)}_{\nu}=\eta^{(\sigma)(\tau)}\eta_{\mu\nu}\mathrm{I}. \tag{24}$$

The the operators $a^{+}_{(\sigma)\mu}, a_{(\sigma)\mu}$ result to be the creation and respectively annihilation operators for the quantized gravitational field. The inidces $(\sigma),(\tau)$ are to be interpreted as labels of the oscillation modes. Finally we obtain:

$$T^{(\Lambda)}_{\mu\nu} = \frac{1}{V}\hbar\omega^{[\sigma]}\left(a^{+}_{(\sigma)\mu} a^{(\sigma)}_{\nu} + \frac{1}{2}\delta^{(\sigma)}_{(\sigma)}\eta_{\mu\nu} I\right), \tag{25}$$

The Hamiltonian operator follows by integration on the region of volume V of the *time-time* component T_{00}:

$$H = \int_V T_{00} d^3 x, \tag{26}$$

resulting:

$$H^{(\Lambda)} = \hbar\omega^{[\sigma]}\left(a^{+}_{(\sigma)0} a^{(\sigma)}_{0} + \frac{1}{2}\delta^{(\sigma)}_{(\sigma)} I\right), \tag{27}$$

Or expanding the sum:

$$\begin{aligned} H^{(\Lambda)} = {} & \hbar\omega^{[0]}\left(a^{+}_{(0)0} a^{(0)}_{0} + \frac{1}{2} I\right) + \hbar\omega^{[1]}\left(a^{+}_{(1)0} a^{(1)}_{0} + \frac{1}{2} I\right) \\ & + \hbar\omega^{[2]}\left(a^{+}_{(2)0} a^{(2)}_{0} + \frac{1}{2} I\right) + \hbar\omega^{[3]}\left(a^{+}_{(3)0} a^{(3)}_{0} + \frac{1}{2} I\right). \end{aligned} \tag{28}$$

The previous results provide soon, thanks to (11) also the information, holding in empty space-time:

$$T^{(R)}_{\mu\nu} = -T^{(\Lambda)}_{\mu\nu}, \tag{29}$$

and in presence of matter:

$$T^{(R)}_{\mu\nu} = -T^{(\Lambda)}_{\mu\nu} - T^{(mf)}, \tag{30}$$

the r.h.s. terms been known.

Eventually we point out that, after quantization, the metric tensor itself becomes an operator, resulting:

$$g_{\mu\nu} = \lambda^{[\sigma]}\eta_{(\sigma)(\tau)}\left(a^{(\sigma)+}_{\mu} a^{(\tau)}_{\nu} + \frac{1}{2}\eta^{(\sigma)(\tau)}\eta_{\mu\nu} I\right). \tag{31}$$

Therefore the interval becomes an operator too:

$$ds^2 = dx^{\mu} g_{\mu\nu} dx^{\nu}. \tag{32}$$

The expectation values of discretized interval are given by:

$$ds^2 = \lambda^{[\sigma]}\eta_{(\sigma)(\tau)} dx^{\mu}\left\langle \psi^{[\sigma]} \middle| a^{(\sigma)+}_{\mu} a^{(\tau)}_{\nu} \middle| \psi^{[\tau]} \right\rangle dx^{\nu}. \tag{33}$$

4. Applications to Physical Solutions

The previous results are easily applied to the known physical solutions to Einstein equations in presence of cosmological constant.

1) *Schwarzschild-De Sitter metric* (*spherical co-ordinates* t, r, θ, φ)

In correspondence to the diagonal Schwarzschild-De Sitter solution for which it results:

$$g_{00} = 1 - \frac{2GM}{c^2 r} - \frac{\Lambda}{3} r^2, \quad g_{11} = -\frac{1}{1 - \frac{2GM}{c^2 r} - \frac{\Lambda}{3} r^2},$$
$$g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2\theta, \tag{34}$$

the characteristic frequencies provided by (18) become:

$$\omega^{[0]} = \frac{\Lambda V}{\kappa\hbar}\left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda}{3} r^2\right), \quad \omega^{[1]} = \frac{\Lambda V}{\kappa\hbar} \frac{1}{1 - \frac{2GM}{c^2 r} - \frac{\Lambda}{3} r^2},$$
$$\omega^{[2]} = \frac{\Lambda V}{\kappa\hbar} r^2, \quad \omega^{[3]} = \frac{\Lambda V}{\kappa\hbar} r^2 \sin^2\theta, \tag{35}$$

Then the proper frequencies of the 4 characteristic modes depend on the mass M of the oscillating body, the volume V into which it is confined and on the cosmological constant.

2) *Robertson-Walker-De Sitter metric* (*spherical coordinates* t, r, θ, φ)

In correspondence to the diagonal Robertson-Walker-De Sitter solution we have:

$$g_{00} = 1, \quad g_{11} = -\frac{a(t)^2}{1 - Kr}, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2\theta, \tag{36}$$

and the characteristic frequencies provided by (18) become:

$$\omega^{[0]} = \frac{\Lambda V}{\kappa\hbar}, \quad \omega^{[1]} = \frac{\Lambda V}{\kappa\hbar} \frac{a(t)^2}{1 - Kr},$$
$$\omega^{[2]} = \frac{\Lambda V}{\kappa\hbar} r^2, \quad \omega^{[3]} = \frac{\Lambda V}{\kappa\hbar} r^2 \sin^2\theta. \tag{37}$$

Here, the proper frequencies of oscillation of the observed region of the universe depend on the evolution function $a(t)$, the volume of the region and, of course on the cosmological constant.

3) *Kerr-De Sitter metric* (*untwisted co-ordinates* T, y, θ, φ)

In untwisted co-ordinates [7] [8] the Kerr metric reduces to a Schwarzschild-De Sitter like metric and the results are the formally similar.

We have:

$$g_{00} = c^2\left(1 - \frac{\Lambda}{3} y^2\right), \quad g_{11} = -\frac{1}{1 - \frac{\Lambda}{3} y^2},$$
$$g_{22} = -y^2, \quad g_{33} = -y^2 \sin^2\theta. \tag{38}$$

The frequencies are:

$$\omega^{[0]} = \frac{\Lambda V}{\kappa\hbar}\left(1 - \frac{\Lambda}{3} y^2\right), \quad \omega^{[1]} = \frac{\Lambda V}{\kappa\hbar} \frac{1}{1 - \frac{\Lambda}{3} y^2},$$
$$\omega^{[2]} = \frac{\Lambda V}{\kappa\hbar} y^2, \quad \omega^{[3]} = \frac{\Lambda V}{\kappa\hbar} y^2 \sin^2\theta. \tag{39}$$

Now, the proper frequencies of oscillation of the observed region of the universe depend on the volume V of the region and on the cosmological constant, the mass being hidden within the resized ray coordinate y.

The proposed approach could offer a suggestion also for quantization of multidimensional gravity in $V^n(n>4)$ space-time manifolds [9] [10], and $f(R)$ [11] or higher power R^n theories of gravitation [12].

5. Conclusions

We presented a very simple method to approach the hard problem of quantization of the gravitational field, suggesting a way based on the presence of a non-vanishing cosmological constant, a condition which is today universally recognized. Quantization of the cosmological contribution owed to vacuum energy-momentum provides immediately also the quantization of the geometric gravity which is opposite in sign respect to the vacuum term, possibly incremented by other sources of matter and non-gravitational fields.

Our approach could be applied, in further investigations, even to generalized theories of gravitation.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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