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Mean-Variance Portfolio Choice with Uncertain Variance-Covariance Matrix

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Abstract

Expected returns, variances, and co-variances are key inputs of mean-variance portfolio selection problems. In traditional mean-variance portfolio models, the model uncertainty is excluded a priori. But in practice, these parameters are not known a priori and are usually estimated with error. Current researches incorporate the model uncertainty into the mean-variance framework but mainly focus on the uncertain means. The aim of this dissertation is to incorporate uncertain variance-covariance into mean-variance portfolio model via the concept of ambiguity and ambiguity aversion. The approaches developed in this study numerically compare the impact from return ambiguity and variance ambiguity. In particular, re-examine if uncertain variance-covariance can lead to "No-Participation in Stock Market" and/or "Home Bias" via stock indexes data.

Keywords

Mean-Variance Portfolio, Uncertain Variance-Covariance

1. Introduction

The mean-variance capital asset pricing model (CAPM) delivers one key result is that all market participants will hold some fraction of the market portfolio. In the case that the investor has no particular preference would be advised to hold the market portfolio combined with borrowing or lending. More realistically, the investor should hold an index with a combination of market portfolio assets. The concept of portfolio diversification is generated from the CAPM model and included as the key concept of Harry Markowitz's portfolio selection theory. In his research, Markowitz argues that putting a large portion of investment in a few stocks is insufficient and risky, investors should instead diversify across many stocks summarized as the following by Markowitz (1952):

Diversification is both observed with personal opinion and sensible to market movement, a rule of investment strategy which does not imply diversification must be rejected either as a hypothesis or as a maxim.

Beside Markowitz, there is another major school of thought generated by John Keynes (1983) concerning the portfolio selection process. The key point of Keynes's theory is that the investor should allocate investment in a few stocks with a strong feeling of preference. While Markowitz's portfolio diversification theory is widely accepted by modern financial economic, supported by Keynes' theory, lots of empirical evidence showed that the investor should rather put a heavy weight of investment in a few assets. Assets expected returns, variance-covariance matrixes are key inputs of the classical Markowitz mean-variance portfolio optimization model. In traditional mean-variance portfolio models, the model uncertainty is not generated with a priori as the investor is assumed to have information accurate enough to process; accordingly, variables in the model are estimated with infinite precision. However, in practice, such model variables are usually estimated with error. Without applying estimation error, such a model could generate extreme weight and fluctuation over time.

During investors' decision making process in the stock market, one crucial determinant is uncertainty about the probability distribution of expected asset return, which is also known as model uncertainty. Ambiguity and ambiguity aversion is demonstrated to compose an investor's level of familiarity toward different assets. Higher ambiguity level to an asset means the investor is less familiar with the assets return distribution, and the estimated return comes with larger error. Under such circumstance, it is insufficient to weight a large portion of portfolio in such asset, the portfolio should be more diversified under Markowitz's portfolio theory.

This study mainly focuses on using empirical data to test the relation between portfolio diversification and invest in assets of better familiarity with. Since the model developed by Garlappi, Uppal and Wang (2007) does not use realistic data for testing purpose, this study will generate some basic example to test their model. Secondly, this study also tests the impact of specified familiarity levels of each asset to the corresponding estimated returns and return variances. Furthermore, how the modified estimated returns and variance would impact the optimal portfolio weight and return. In the end, risk aversion and ambiguity aversion are compared to demonstrate their influence on portfolio weight and return.

The main contribution of this paper is to demonstrate how to apply the ambiguity aversion model to real index data and to the portfolio selection problem, while sample variance-covariance is estimated with error. The max-min objective function presented later is consistent with the multi-prior approach by Gilboa and Schmeidler (1989) and further developed by several other researchers.

The aim of this dissertation is to incorporate uncertain variance-covariance into mean-variance portfolio model. In particular, re-examine if uncertain variance-covariance can lead to "No-Participation in Stock Market" and/or "Home Bias". To better demonstrate this modified model and compare with Boyle, Garlappi, Uppal and Wang (2012) method, two applications are implied to test the impact of investors' ambiguity aversion level on the return of optimal portfolio. The first application follows the idea of applying a confidence interval onto assets estimated returns. The second application applies the confidence interval onto variances of estimated returns based on the statistic distribution of sample variance. Furthermore, to incorporate assets' Sharpe ratios in the process of calculating expected return via modified return variance, test the impact of Sharpe ratio estimation.

2. Literature Review

Individual investor's attitude towards uncertain investment has been modelled by many papers basing on the concept of ambiguity and ambiguity aversion. Such a concept has been explored to imply on asset pricing, portfolio choice and many other implications.

Study based on the *Panel Study of Income Dynamics* data (PSID) (Mankiw & Zeldes, 1991) shows that a majority portion of US families do not put investment in the US security market. Resent survey shows that more than 70% of the secondary stock market investment comes from institutional investors in the US stock market. On the contrary, more than 60% of the secondary stock market investment comes from personal investors in China's stock market. Evidence of lacking diversification, which obeys Keynes's theory mentioned earlier, is provided as early as 1975 by Blume and Friend (1975). By testing the income-tax return data, they conclude that a larger portion of investors prefers to hold less than three stocks in their investment portfolio. Two of the researches using *Survey of Consumer Finances* data have similar finding about US households' investment: Kelly (1995) discovered that most personal stock holder only carries one single stock traded in the market and often is the company where he/she works for; Polkovnichenko (2005) finds that the median number of stocks holding by US household was no more than 3 during the last two decades, a significant part of stock is invested in employer stock. Holding a few assets that investors are familiar with is quite typical in the personal investment routine.

Many researchers have been working on the former phenomenon of US stock market and trying to find possible explanations. Their objects of research include transaction cost, liquidity condition (Williamson, 1994; Allen & Gale, 1994), entry cost (Vissing-Jorgensen, 1999; Yaron & Zhang, 2000), risk aversion (Haliassos & Bertaut, 1995) and so on. Although most of these factors do create limited market participation, none of them could provide enough evidence to fully explain the investor's investment decision. The difference between ambiguity aversion and risk aversion is crucial and complicated. Ambiguity aversion could be applied when the probabilities of outcomes are unknown. Risk aversion applies when each possible outcome situation could be assigned to a probability (Epstein, 1999). Ambiguity aversion arises from the idea of risk aversion and the two theories encompass each other. Haliassos and Bertaut (1995) explore if limited market participation could be potentially caused by risk aversion and find that it is unable to explain the phenomenon either.

Empirical studies show that in cases when people feel that the possibility of other people assessing a similar probability distribution is small, investor's ambiguity aversion is particularly strong. After the financial crisis in the first decade of 20th Century, especially the sub-prime crisis triggered in 2007, people have raised their attention to ambiguity and ambiguity aversion again recently. The model adopted to build the portfolio selection framework is developed by Garlappi, Uppal and Wang (2007). The model they developed is simply based on and a mild modification of the Markowitz portfolio optimization model. In their model, they showed that when investors are equally familiar with all assets, then the Markowitz's diversification theory will be used to generate the optimized portfolio. On the other hand, if the investors have different degrees of ambiguity across assets, the weight of different assets in the optimal portfolio will be influenced by the relative degree of ambiguity, and the standard deviation of the estimated return. If the standard deviation is low and ambiguity levels of different assets are similar with each other, the optimal portfolio would be a mix of both familiar and unfamiliar assets. On the contrary, if the standard deviation of the asset's expected return is high and the difference of ambiguity levels of assets in the portfolio is large, the optimal portfolio contains only the asset(s) with lower level of ambiguity. In the extreme case in which both the ambiguity levels and the volatility of the expected returns are large, optimally the investor should not hold any asset except the risk free ones.

The model mentioned earlier is closely related to a few papers on portfolio optimization, which are newly developed to model level of ambiguity or ambiguity aversion. In the paper by Boyle, Garlappi, Uppal and Wang (2012), the uncertainty is modelled by "confidence interval" around estimated returns generated by the sample mean. They also generated a closed form solution for the composition of optimal portfolio, which also known as the portfolio weight, so that to provide economic implication to the effect of ambiguity.

Boyle, Garlappi, Uppal and Wang (2012) approach to model uncertainty follows the Knightian approach and is different from the commonly used Bayesian approach. Bayesian approach assumes that the investors are neutral to model uncertainty while they can be risk averse. Their decision making process is based on the expected assets return with respect to certain statistic distribution. On the other hand, Knightian approach assumes that investors no longer use single prior to model uncertainty but adopt a framework via multi-priors applied on the expected assets return. The investment strategy is evaluated under the worst case scenario generated from the expected probability distribution such that the investor is averse to both risk and ambiguity. In recent papers, Chen, Peng, Zhang and Rosyida (2016) proposed two types of mean-semivariance diversified models for uncertain portfolio selection, based on the concept of semivariance of uncertain variables, a hybrid intelligent algorithm which is based on 99-method and genetic algorithm is designed to solve the models. Qin, Kar and Zheng (2014) propose uncertain mean-semiabsolute deviation adjusting models for portfolio optimization problem in the trade-off between risk and return on investment, and use examples with synthetic uncertain returns to demonstrate the effectiveness of the proposed models and the influence of transaction cost in portfolio selection.

With the development of derivatives and other financial products, except studies on the impact of equity premium relating issues to limit market participation, the implication of other asset-pricing issues are now being explored. Cao, Wang and Zhang (2005) show that due to the heterogeneity of investors' uncertainty about stock returns, diversification discount may also be related to limited participation of the stock market.

3. Methodology

This section introduces the portfolio optimization model that incorporates investors' ambiguity about assets' returns and variances. The approach derived by Boyle, Garlappi, Uppal and Wang (2012) concerning the expected return distribution is described first. A further modified model concerning the return variance is derived to test the impact when assuming expected variance is also estimated with error. Being treated as random variables, variance estimation errors are and generated from a predictive distribution of sample variances.

The model adopted by Boyle, Garlappi, Uppal and Wang (2012) that dealt with estimation error is based on a Bayesian approach. However, such an investor is assumed to have a prior which also means the investor is neutral to ambiguity. Empirical evidences shows that investors are not neutral to ambiguity and also due to the connection between the portfolio weights and the chosen prior, a multi-prior approach is adopted to meet the requirement for a model with a number of possibilities.

Garlappi and Uppal's model allows for multi-priors to examine implication parameter and model uncertainty. Assuming that assets' expected returns are estimated with error, their paper mainly focus on demonstrating how to use the model in practical problem for portfolio optimization, furthermore, comparing the portfolio weights generated from this approach with the traditional mean-variance portfolio theory.

The optimal portfolio generated from this model mainly depends on three variables: 1) the ambiguity level of each asset, 2) the estimated returns, and 3) the variances of estimated returns. The expected returns and variances are firstly estimated using historical data.

If the ambiguity interval of the expected variance of a certain asset is large, which represents large estimation error, then the portfolio selection model would reduce the weight of portfolio invested in this asset, which represents the investor relies less on the estimation data. On the contrary, when the interval is relatively small, the portfolio will generate asset weights close to a model without considering estimation error. This model will generate the optimal weight same as the classical mean-variance model when the confidence interval is set to zero.

3.1. Return Ambiguity

The starting point of this study is the static Markowitz (1952) mean variance portfolio model. Based on the classical Markowitz model, the solution of the following optimization will generate the optimal portfolio including N risky assets:

$$\max_{\pi} \pi^{T}\mu - \frac{\gamma}{2}\pi^{T}\Sigma\pi \tag{1}$$

μ represents the vector of true returns, γ is risk aversion vector, Σ is assets variance covariance matrix, portfolio weights π sum to one.

In the standard mean-variance model formulated in **(1)**, there is a precondition that assuming the true returns and variance-covariance matrix is known as a prior by investors. However, such quantities have to be estimated in practice. For simplicity, they assume that the investor has perfect knowledge of standard deviation of asset returns, σ and the assets correlation ρ, but is uncertain about the error generated by estimated returns. Taken into account that expected returns are estimated with error, two new factors are introduced into the standard portfolio optimization process. Based on classical statistics, they model the level of ambiguity by adding a confidence interval onto the estimated returns. First, they imposed a confidence interval on the mean-variance optimization as an additional constraint. Second, the investor's ambiguity level is represented by minimizing the expected returns of uncertain assets according to the confidence interval **in (2)**.

$$\alpha_n\text{-confidence interval} = \left\{ \mu_n : \frac{(\mu_n - \hat{\mu}_n)^2}{\sigma^2_{\hat{\mu}_n}} \le \alpha_n^2 \right\} \tag{2}$$

ε_n is asset *n*'s level of ambiguity, it is considered as *the product of ambiguity aversion and ambiguity*. The extended model is modified as the following form:

$$\max_{\pi} \min_{\mu} \pi^{T}\mu - \frac{\gamma}{2}\pi^{T}\Sigma\pi \tag{3}$$

$\varepsilon_n = 0$ means the investors have confidence that there is no error with respect to the estimated assets return, the estimated mean return $\hat{\mu}_n$ equals to the expected return μ_n. The optimal portfolio in (3) will be equal to the mean variance portfolio in (1).

The confidence interval of expected returns is modified in this study to be consistent with that of variance returns is. Following the naming convention from Garlappi and Uppal's, the max-min Equation is (3) while taking the constraint in (2) into account equals to the following **maximization problem:**

$$\max_{\pi}\left\{\pi^{T}\left(\hat{\mu}-\mu^{adj}\right)-\frac{\gamma}{2}\pi^{T}\Sigma\pi\right\} \tag{4}$$

μ-adj is a N-vector adjustment made to the estimated expected return:

$$\mu^{adj}=\left\{sign(w_1)\frac{\sigma_1}{\sqrt{T}}\sqrt{\varepsilon_1},\cdots,sign(w_N)\frac{\sigma_N}{\sqrt{T}}\sqrt{\varepsilon_N}\right\},$$

where

$$\varepsilon_n=F^{-1}_{N\left(0,\sigma_n^2\right)}\left(\frac{1+\alpha}{2}\right) \tag{5}$$

α, value from 0 to 1 in (5), is the final variable represents the investor's **ambiguity level** about the corresponding asset.

Furthermore, for a portfolio with only two risky assets and $\varepsilon_1\le\varepsilon_2$, it will generate the following portfolio weight,

Case 1. If $\varepsilon_1<\frac{\mu}{\sigma}<\varepsilon_1+\frac{1}{1-\rho}(\varepsilon_2-\varepsilon_1)$:

the portfolio weight will be:

$$\begin{bmatrix}\pi_1\\\pi_2\end{bmatrix}=\frac{1}{\gamma\sigma^2}\begin{bmatrix}\mu-\sigma\varepsilon_1\\0\end{bmatrix} \tag{6}$$

Case 2. If $\varepsilon_1>\frac{\mu}{\sigma}$

the portfolio weight will be:

$$\begin{bmatrix}\pi_1\\\pi_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \tag{7}$$

3.2. Variance Ambiguity

Based on the model in (3), this study developed a new model which applies the investor's uncertainty/ambiguity about assets onto the variance of estimated returns. An intuitive way of modelling this ambiguity is also to rely on results from classical statistics as the return ambiguity model. To quantitatively compare the impact of estimation error between expected returns and variance, the confidence interval for the variance is set up similarly as that of the expected returns via Bayesian statistics. The sample variance could be considered as a random variable. If the sample assets return follows a normal distribution, the return variance follows a scaled chi-squared distribution as following:

$$(n-1)\frac{\hat{\sigma}^2}{\sigma^2}\sim\chi^2 \tag{8}$$

σ-hat square is the sample variance, σ^2 is the true variance and n is the sample size.

By knowing the distribution of sample variance, the confidence interval of expected variance is derived as the following:

$$\delta_1\le(n-1)\frac{\hat{\sigma}^2}{\sigma^2}\le\delta_2 \tag{9}$$

As the chi-squared distribution is not symmetrical as the normal distribution, the absolute value of the upper boundary δ_1 and the lower boundary δ_2 of the confidence interval in (9) unequal, comparing with the return confidence interval set up in (2). Based on the confidence interval for expected variance in (9), the interval for expected variance with estimated error is derived as the following:

$$\frac{\hat{\sigma}^2(n-1)}{\delta_2} \le \sigma^2 \le \frac{\hat{\sigma}^2(n-1)}{\delta_1} \rightarrow \delta_1 = F^{-1}_{\chi^2_{n-1}}\left(\frac{1-\alpha}{2}\right), \ \delta_2 = F^{-1}_{\chi^2_{n-1}}\left(\frac{1+\alpha}{2}\right) \quad (10)$$

α is the confidence interval with the same value range as in (5), valued from 0 to 1.

The modified portfolio optimization process based on (3) is as the following:

$$\max_{\pi} \min_{\sigma} \pi^T \mu - \frac{\gamma}{2} \pi^T \Sigma \pi \quad (11)$$

The major difference between the optimized portfolio in (3) and the portfolio in this modified model in (11) is the inner minimization parameter. In Garlappi and Uppal's model, the inner minimization is based on the investor's ambiguity and ambiguity aversion about assets return μ, on the other hand, the inner minimization in this modified model is based on the investor's ambiguity level on the assets variance σ^2.

Since the inner minimization is based on the sample variance, which is included in the variance-covariance matrix Σ, this minimization process could be as simplified as to maximize the later part of (11). For a portfolio contains two assets, the derived function is as following:

$$\pi^T \Sigma \pi = w^2 \sigma_1^2 + 2w(1-w)\sigma_{12} + (1-w)^2 \sigma_2^2 \quad (12)$$

σ_{12} is the covariance between the two assets in the portfolio. To achieve the maximum value of (12), the upper bound of σ_1 and σ_2 in the confidence interval in (10) is taken as the ambiguity adjusted volatility. The max-min problem in (11) where the constraint in (10) takes from (12) is equivalent to the following maximization problem:

$$\max_{\pi} \left\{ \pi^T \mu - \frac{\gamma}{2} \left(\frac{\omega^2 \hat{\sigma}^2 (n-1)}{\delta_1} + 2\omega(1-\omega)\sigma_{12} + \frac{(1-\omega)^2 \hat{\sigma}^2 (n-1)}{\delta_1} \right) \right\}$$
$$\text{where } \delta_1 = F^{-1}_{\chi^2_{n-1}}\left(\frac{1-\alpha}{2}\right) \quad (13)$$

the maximum value of σ_1 and σ_2 in (10) would be used in this maximization process.

Since the distribution of covariance σ_{12} between assets is unknown, this study adopts the sample covariance σ_{12} and set it as constant inside the variance-covariance matrix. Copula function could be used to further modify the distribution of sample correlation, due to the aims of this study, this part is not included in this paper.

3.3. Sharpe Ratio Modified Asset Return

After testing the impact of return ambiguity and variance ambiguity on the optimal portfolio, Sharpe ratio is adopted in the variance ambiguity model to test how portfolio will change with the change of return and ambiguity modified variance. The Sharpe ratio computed in this study uses historic data but justified on the basis of predicted relationships. This further modified optimization approach in introduced by imposing another constraint as following:

$$\mu_n' = r_0 + \lambda_n \sigma_n \quad (14)$$

λ is asset Sharpe ratio, μ' is modified asset return, σ is return standard deviation with ambiguity level adjustment.

Due to the relation between expected return and variance shows in (14), simply use the sample return for the inner minimization process shown in (11) is no longer suitable for this approach. The ordinary derivative of the minimization function in (11) subject to (14) shows that it only contains one extreme value point, the secondary derivative regarding return variance further verifies that the extreme value point in the inner minimization problem is a maxim point, so that the minimum value for the inner minimization should be found on one of the boundary points of the variance interval.

Another important determinant to the minimization process in (11) subject to (14) is the portfolio weight. The implication to quantify the impact of portfolio weight to portfolio return is to numerically calculate the minimum portfolio return in each weight level and choose the largest return with its corresponding weight level as the result of the max-min optimization process.

Sharpe ratio is introduced to test whether the change of portfolio's returns is due to investors' ambiguity aversion or a result of excess risk. This measurement is quite significant as even if an investor's portfolio could achieve higher returns comparing to others', it is only a wise investment if too much additional risk does not come with it. A portfolio's Sharpe ratio is positively correlated with its risk-adjusted performance.

To calculate the Sharpe ratio of asset return, the risk free rate is a required factor. In this study, the market price of three month US Treasury bill is taken as the risk free rate.

4. Data

The optimal portfolios calculated in this paper contain 2 risky assets**: asset1** and **asset2**. The ambiguity level is represented by **alpha1** and **alpha2**, alpha1 and alpha2 represent either return ambiguity level or variance ambiguity level through separate approach. The weight of the two assets in the portfolio is represented by **weight1** and **weight2**. The weights of the two assets add up to 1. To test the impact of multiple ambiguity level on the optimal portfolio return, this study does not divide the assets into a familiar group and an unfamiliar group, but simply test multiple combinations of ambiguity levels. Both return and variance ambiguity levels are changed from 0.1 to 0.9 with an increasing step of 0.1 for both assets.

All historical assets prices and indexes value are downloaded from the Bloomberg System. The data processing and model calculation are programmed in Matlab version R2011b.

As a starting point, the Standard and Pool 500 Index and the Dow Jones Industrial Average Index are firstly adopted to test the ambiguity aversion approaches. S&P 500 index is asset1, Dow Jones index is asset2. The market price of three-month US T-bill is adopted as the risk free rate to calculate the Sharpe Ratio of the index returns. The yearly index values are used to calculate the sample return. For yearly index return, the data used is from 1934 to 2011. The values of yearly sample returns, variances, co-variances and Sharpe ratios are listed in Table 1.

In order to apply the confidence interval on sample variance, the index return data is firstly test by a Lilliefors test, the default null hypothesis is that the sample comes from a distribution in the normal family. To calculate Sharpe ratio for monthly data, the yearly risk free rate is converted to monthly base.

To test the hypothesis of home bias and no participation in stock markets, this paper also adopted monthly return data from the Shanghai Composite Index (SHCOMP) to compare with that from the US stock market indexes. The monthly data of Shanghai Composite Index is dated since January 1992 to July 2012, the starting date is soon after the starting date of the new Shanghai Stock Exchange re-established. Table 2 lists the index mean return and return variance of SHCOMP and S&P500. In this part of the study, SHCOMP index is asset1, S&P 500 index is asset2. During the sample period, SHCOMP index has much higher return and variance comparing to S&P500. The co-variance between these two indexes is much smaller than the covariance of S&P500 and Dow Jones. How would the ambiguity level affect the portfolio weight and return constituting these two indexes is inspected in this study.

Table 1. Sample return and variance data of S&P 500 and Dow Jones index.

	Yearly Data 1934-2011			
	Return	Variance	Co-variance	Sharpe Ratio
S&P 500	0.081	0.032	0.029	0.2424
Dow Jones	0.077	0.028		0.2356

Table 2. Sample returns and variance of SHCOMP and S&P 500 index.

	Monthly Data 1992.1-2012.7		
	Return	Variance	Co-variance
S&P 500	0.0059	0.0019	7.8E-4
SHCOMP	0.0197	0.0356	

5. Result and Discussion

5.1. US Stock Market Multi-Index Test

The following test will first test a portfolio contains two assets: the S&P 500 index and Dow Jones index. The major input variables in this study are ambiguity level of estimated returns, ambiguity level of variance of estimated returns and risk aversion factor γ. The level of ambiguity for both assets is changed from 0.1 to 0.9 with a growth rate of 0.1. The risk aversion factor γ is firstly tests from 1 to 5 with a growth rate of 1.

5.1.1. Original Portfolio Weights

The key results of this study are portfolio weights and returns, the optimal weights and returns generated by the Markowitz mean variance optimization are listed in Table 3. The portfolio return decrease with the increase of risk aversion level, so as the weight of S&P 500 index. The major reason of this situation is that S&P 500 index has a larger sample variance and sample return comparing to Dow Jones. For investors with low level of risk aversion (smaller than 2), the optimal portfolio would be dominated by the asset with higher estimated return, so that the S&P 500 index takes most of the portfolio weight. On the other hand, investors require higher level of risk aversion would automatically choose the asset with lower estimated return, which in this case it is the Dow Jones Index. With higher risk aversion level, the expected portfolio return would be reduced by the risk aversion factor as well.

The starting point of the following tests will be numerically comparing the return ambiguity and variance ambiguity modified optimal portfolio with the portfolio list in Table 3. The tests try to answer the following questions: 1) At what ambiguity level would the ambiguity aversion portfolio achieve the same weight as listed in Table 1, 2) how would the portfolio return change with the change of assets' ambiguity level, 3) at the same level, which ambiguity aversion, return ambiguity or variance ambiguity, would cause larger movement of the optimal portfolio, while all the other conditions stay the same, 4) what is the inner connection between the optimal portfolio and separate asset's return and variance, 5) is the ambiguity be able to cause "no participation in stock market".

Table 3. Optimal portfolio return and weight of assets in the portfolio at different risk aversion level.

Risk Aversion Factor	Portfolio Return	Weight of S&P 500	Weight of Dow Jones
1	0.065	1	0
2	0.050	0.49	0.51
3	0.036	0.17	0.83
4	0.022	0	1
5	0.008	0	1

After evaluating the impact of ambiguity to the optimal portfolio, the following topics are also studied: 6) the connection between risk aversion and ambiguity aversion would be studied, 7) estimated Sharpe ratio is introduced to further test the variance ambiguity model, 8) the concept of home bias is introduced in this model to test if home bias could be caused by ambiguity.

5.1.2. Compare the Ambiguity Adjusted Portfolio with the Mean-Variance Portfolio

To test the impact of ambiguity to optimal portfolio, this study first test if the ambiguity adjusted portfolio could generate the same optimal portfolio as in Table 1 at each risk aversion level. Since the portfolio return will always be smaller with ambiguity applied, the test only focuses on if the ambiguity adjustment could cause the portfolio weight reach the values in Table 1. By increasing the ambiguity level of both assets, the first time that portfolio weight equals to Table 1 are listed in Table 4. At all risk aversion level, portfolio with return ambiguity adjusted meet the portfolio in Table 3, return ambiguity also causes the portfolio return decreases faster. For variance ambiguity, its reduction power to the portfolio return is weaker while it would cause the weight of portfolio moves away from the optimal portfolio listed in Table 3, so that at all ambiguity level, the optimal portfolio would not generate weight equal to Table 3 when risk aversion factor larger than 3.

5.1.3. The Portfolios Returns Change with Ambiguity Levels

Set $\gamma = 3$ and alpha2 from 0.1 to 0.9, the portfolio returns are calculated while alpha1 is set at 0.1, 0.5 and 0.9. For each alpha1, 9 portfolios are generated. Table 5 shows the mean portfolio return for each alpha1 while alpha2 changes.

Table 4. Portfolio returns and assets ambiguity levels while the portfolio weights equals to values in Table 1.

	Return Ambiguity			Variance Ambiguity		
Gama	Return	Alpha1	Alpha2	Return	Alpha1	Alpha2
1	0.062	0.10	0.10	0.065	0.10	0.86
2	0.047	0.13	0.17	0.050	0.10	0.12
3	0.032	0.12	0.16	0.035	0.53	0.10
4	0.019	0.10	0.10		N/A	
5	0.005	0.10	0.10		N/A	

Table 5. Mean of portfolio return with 1SD for alpha2 move from 0.1 to 0.9.

Ambiguity level of asset1	Return Ambiguity	Variance Ambiguity
	Mean portfolio Return	Mean portfolio Return
0.1	0.0318 ± 0.00077	0.0339 ± 0.00059
0.5	0.0295 ± 0.0019	0.0327 ± 0.0012
0.9	0.0292 ± 0.0022	0.0315 ± 0.0020

It is easy to tell from Table 5 that for the mean portfolio return with same level of ambiguity, the return ambiguity causes relatively larger decrease to the portfolio return comparing to variance ambiguity as the return ambiguity are directly applied on the assets returns in (4) while the variance ambiguity are applied on the variance-covariance matrix in (13) in the mean-variance optimization process.

Figure 1 displays clear trend of portfolio return move with the increase of alpha2. The variance ambiguity causes steeper down sloop with the increase of alpha2 in Figure 1(b). The reason is as following, compare the ambiguity level α set up for return and variance in function (5) and (10), with the same ambiguity level α applied, the estimated variances change at a linear level with the α while the returns change at a level of the square root of α, so that the level of reduction of portfolio return increase with the increase of variance ambiguity level in Figure 1(a) and Figure 1(b). With the increase of ambiguity level of asset2, the portfolio return decrease due to the inner minimization of optimization function, higher ambiguity level cause lower expected returns of each asset and that of the optimized portfolio.

Figure 2 represent the change of assets returns and variances due to the application of ambiguity. The assets returns are presented in Figure 2(a), with the increase of the return ambiguity level, both return1 and return2 decrease at a nearly constant rate. In Figure 2(b), the increase of the variance ambiguity levels cause the variance increase at a nearly constant rate with smaller r-squared values comparing to Figure 2(a). The slope of the variance increases with the ambiguity level further supports that the variance ambiguity causes the variance change at a quadratic level compare to the return change due to return ambiguity in Figure 2(b). The shape change of the return and variance with increasing ambiguity level in Figure 2 further explained the portfolio change shows in Figure 1.

5.1.4. Assets with Equal Ambiguity Level

Figure 3 shows the weight of asset1 and asset 2 plus the optimized portfolio return with the same level of return ambiguity and variance ambiguity. Compare with the optimal portfolio in Table 1 when γ = 3, return ambiguity causes weight 1 decrease with increasing ambiguity level, on the contrary, the increase of variance ambiguity changes the portfolio weight dramatically and causes weight1 increase with it.

5.1.5. Return Ambiguity vs Variance Ambiguity

Set variance of Dow Jones in the optimization process increase with the increase of its ambiguity level, while asset1's variance remains the same. In Figure 4(a), the weight of asset2 has a much faster decrease speed comparing to Figure 4(b) and drop to 0 when its ambiguity level reach 0.8. In Figure 3(b), the weight of asset2 decrease slower due to its smaller variance, even applied with the highest ambiguity, its relative variance with asset1 still cause smaller reduction to its weight comparing to the return variance in Figure 4(a). Compare the portfolio returns in Figure 4(a) and 4b, the portfolio return decrease faster with the increase of asset2's ambiguity level since the weight of Dow Jones's is consistently higher while its estimated return is smaller than S&P 500. Compare to variance ambiguity, the return ambiguity causes more significant change to the portfolio weight and smaller change to the portfolio return especially when the ambiguity level is high. The return ambiguity and variance ambiguity modify the corresponding return and variance toward different direction, with the increase of return ambiguity, the modified asset return become smaller, as the estimated return2 is smaller than return1, the difference between return1 and return2 increases, on the contrary, the estimated variance of asset2 is smaller than asset1's, with the increase of variance ambiguity, the modified variance of asset2 increase so that the difference between variance1 and variance2 move toward the opposite direction compare to the return difference. And the weight of asset2 caused by variance ambiguity applied shows moderated change.

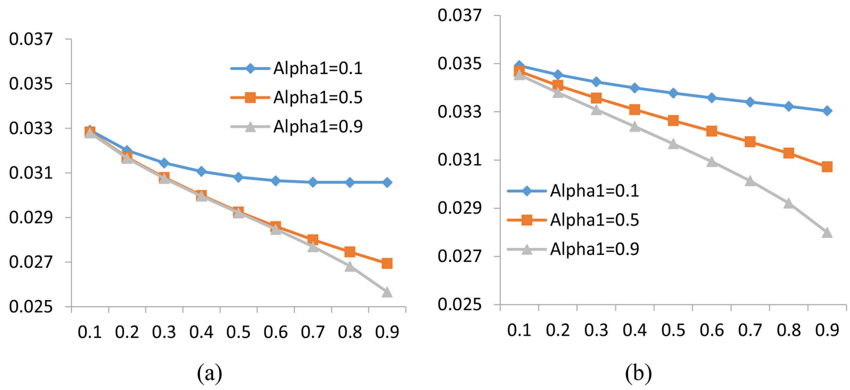


Figure 1. Portfolio Return variations with Alpha2 change from 0.1 to 0.9, x-axis is the value of alpha2; (a) shows the portfolio return generated with return ambiguity applied, (b) shows the portfolio return with variance ambiguity applied.

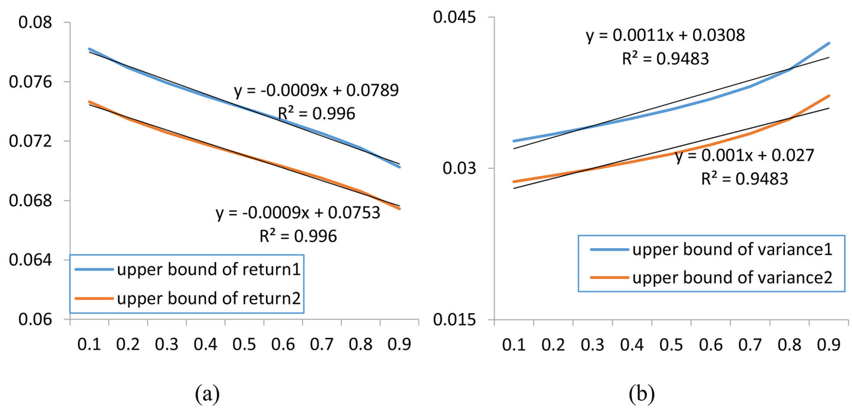


Figure 2. (a) Upper bound of assets return confidence interval under different return ambiguity level, x-axis is variance ambiguity level from 0.1 to 0.9, y-axis is asset return. (b) Upper bound of assets variance confidence interval under different variance ambiguity level. X-axis is variance ambiguity level, y-axis is asset variance.

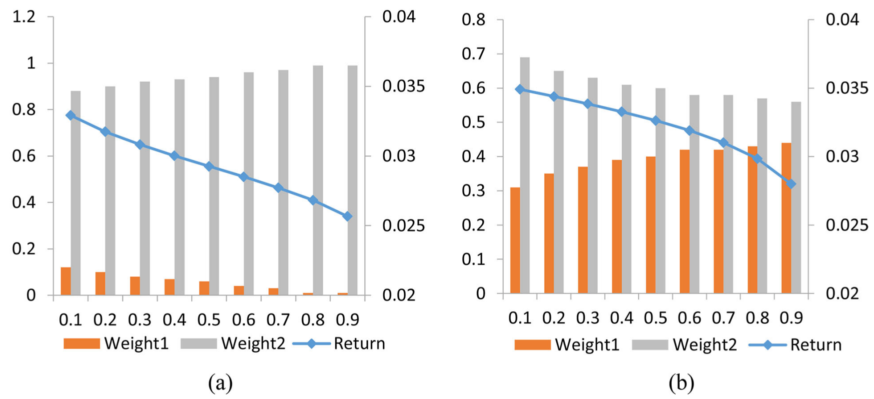


Figure 3. Weights of asset 1 and 2 and optimized portfolio returns. (a) is return ambiguity, x-axis shows the ambiguity level for both 1 and 2, primary y-axis shows the portfolio weight of assets, secondary y-axis shows the estimated portfolio return; (b) is variance ambiguity.

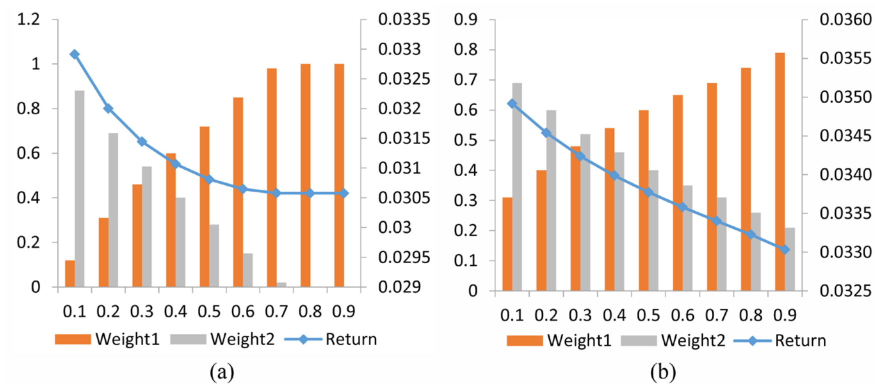


Figure 4. Weights of asset1 and 2 and optimized portfolio returns. Set ambiguity level of asset1 equal to 0.1; (a) is return ambiguity, x-axis shows the ambiguity level of asset2, primary y-axis shows the portfolio weight of assets, secondary y-axis shows the estimated portfolio return; (b) is variance ambiguity.

Set alpha1 = 0.9, Figure 5 shows the weight of Asset2, while its ambiguity level increases from 0.1 to 0.9. Portfolio weight is much heavier changed by the variance ambiguity. Since the variance of asset1 is larger, its weight in the optimal portfolio consistently stays less than the weight of asset2.

5.1.6. "No Participation in Stock Market" Test

To test if ambiguity could causes "No participation in stock market", a portfolio contains only risk free asset and one risky asset is modified. The variance of risk free asset and the covariance between two assets are set to zero. Figure 6 shows the weight of S&P 500 index as the risky asset in the portfolio mentioned earlier, the x-axis represent value of risk free rate γ change from 1 to 20, Figure 6(a) is return ambiguity, Figure 6(b) is variance ambiguity. The optimal portfolio keeps containing more than 5% of asset1, even with the largest ambiguity level alpha1 = 0.9. Except those investors who have extremely high risk aversion and highest level of ambiguity toward the risky asset, the optimal portfolio would not

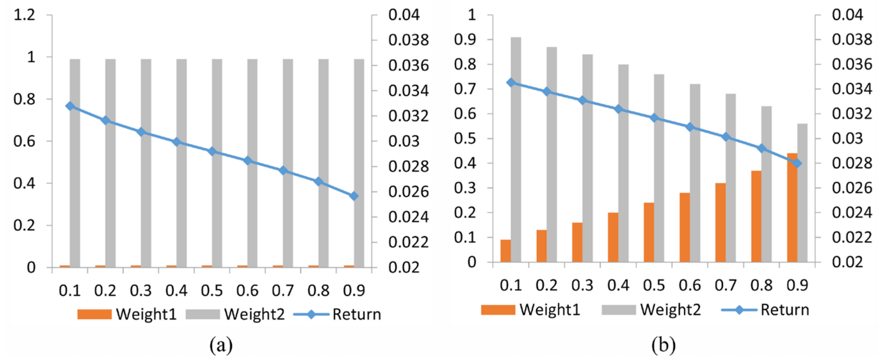


Figure 5. Weights of asset1 and 2 and optimized portfolio returns. Set ambiguity level of asset1 equal to 0.9; (a) is return ambiguity, x-axis is the ambiguity level of asset2, primary y-axis is the portfolio weight of assets, secondary y-axis shows the estimated portfolio return; (b) is variance ambiguity.

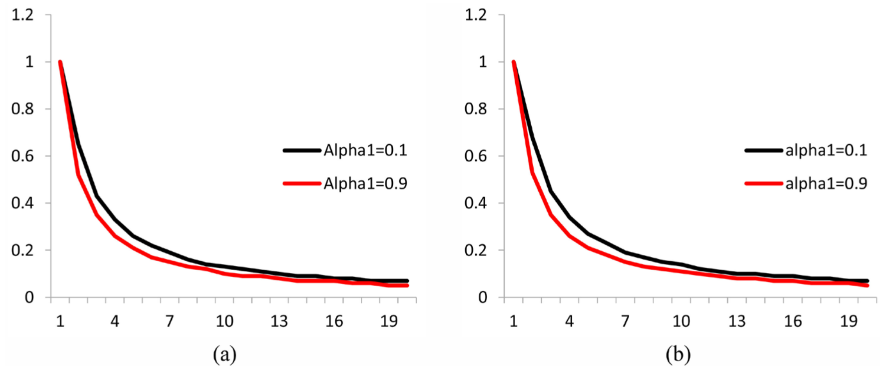


Figure 6. Weight of asset1 in optimized portfolio with different ambiguity, x-axis is γ-value from 1 to 20, (a) return ambiguity (b) variance ambiguity.

only take the risk free asset as the only asset in this case. "No participation of stack market" is not showing in this case even for assets with unrealistically high ambiguity.

Compare to the artificial risk free rate asset with no variance and correlation with other asset, the two indexes portfolio in earlier tests shows extreme cases, in which only one asset takes the whole portfolio, for certain risk aversion level. The reason of such circumstance could be found in (6) and (7). By the equation in (5), when $\alpha_1 = 0.9$, the confidence interval $\varepsilon_1 = 0.2931$, while $\mu_1/\sigma_1 = 0.4559$. In this case, even for the asset with ambiguity level of 0.9, the relation $\varepsilon > \mu/\sigma$ in (7) would not be established. On the other hand, with certain ambiguity level, the precondition generates Case2 shown in (6) would cause the optimal portfolio put all the weight in one index.

5.1.7. Impact of Risk Aversion Factor γ

To test the impact of risk aversion to the portfolio optimization, the risk aversion factor γ is modified from 1 to 5. Figure 7 shows the weight of asset1 when ambiguity level of asset2 changes from 0.1 to 0.9 for multiple γ values. Figure 7(a) shows weight1 with return ambiguity, Figure 7(b) shows weight1 with variance ambiguity. Figure 7 presents that 1) with the increase of γ, the weight of asset1 become smaller, 2) asset1 weights more in the optimized portfolio with higher ambiguity level of asset 2 for all γ. Figure 7(b) generated by variance ambiguity shows that when γ is larger than 1, the weight of asset1 shows a trend of convergence with the increase of alpha2. When γ equals one, due to its higher rate of return, asset1 dominates the optimal portfolio and has a weight of 1 no matter the value of alpha2. When γ grows higher, due to its smaller variance comparing to asset1, asset2 dominates the optimized portfolio when its ambiguity level is small. *The larger relative variance of asset*1 *causes weight*1 *decrease with increasing risk aversion factor in the optimization process*.

To better understand the relation between movement of γ and different ambiguity levels, the next test sets the range of γ from 0.1 to 5 with a growth rate of 0.1. The results are presented in Figure 8 and Figure 9. In Figure 8, how portfolio changes with ambiguity level and γ is tested. For return ambiguity, Figure 8(a) shows 1) the portfolio return decrease with the increase of γ; 2) when alpha1 is 0.9 and alpha 2 changes from 0.1 to 0.9, the slope of portfolio return slightly increase, so that the return with smaller γ increase while the return with larger γ decrease; 3) when alpha2 is low and alpha 1 changes from low to high, the slope of portfolio return moves in the opposite direction as in 2). *The reason behind this situation is that when* γ *is small*, *the impact from risk aversion factor to portfolio return is limited*, *the change of portfolio return is mainly caused by the assets*' *return ambiguity*. Expected return of asset1 is larger, when its ambiguity level is smaller, the portfolio weights more on asset1, and the optimal portfolio achieves larger return when γ is smaller than 2. With the increase of γ, the risk aversion factor shows stronger influence on the portfolio return, as asset 1 has higher variance, its weight in the optimized portfolio decreases, the optimal portfolio return decrease with it.

For variance ambiguity, Figure 8(b) shows similar variation of portfolio return as return ambiguity. As the variance of asset 2 is relatively smaller, its impact to the reduction of portfolio return is quite limited even with high ambiguity level. With the increase of γ, asset1 with a smaller variance ambiguity level stays dominate in the portfolio by weight, as its expected variance is also higher, its impact on portfolio return reduction is stronger, so that the red line is under the green line in Figure 8(b).

Figure 9 shows the portfolio return generated from three different approaches: the standard mean variance portfolio theory, variance ambiguity adjusted portfolio and return ambiguity adjusted portfolio. For small γ, although the two assets' return ambiguities equal to their variance ambiguities, the impact of return variance dominates, asset 1's higher estimated return and higher ambiguity level cause higher reduction of portfolio return while γ is small, therefore the red line in Figure 9(b) is lower than the blue line. On the other hand, when γ turns larger, its impact to the portfolio return become stronger, so as the impact from variance of assets, weight1 decrease due to its larger expected variance in Figure 9(a), even though its ambiguity level is smaller than asset2's.

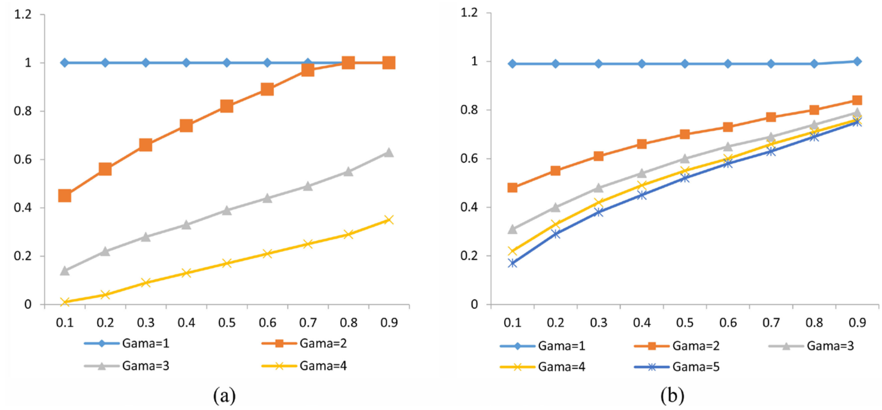


Figure 7. Weights of asset1 and 2 and optimized portfolio returns. Set ambiguity level of asset1 equal to 0.1; (a) is return ambiguity, x-axis is the ambiguity level of asset2, y-axis is the portfolio weight of asset 1; (b) is variance ambiguity.

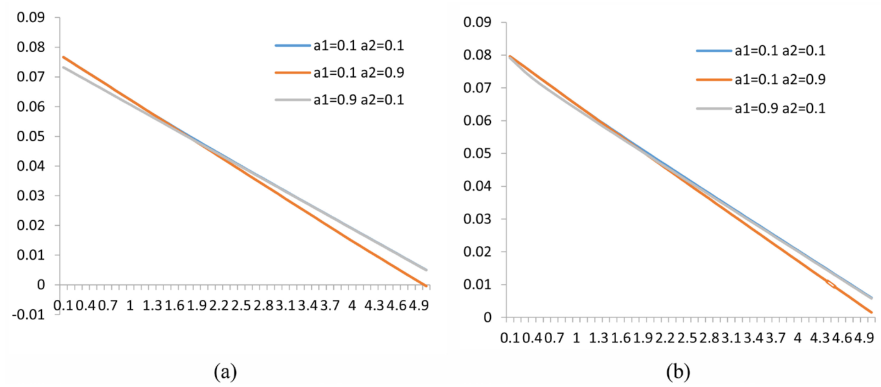


Figure 8. Portfolio return with multiple alpha1 and alpha2 levels, x-axis is γ-value from 0.1 to 5.0, (a) return ambiguity (b) variance ambiguity.

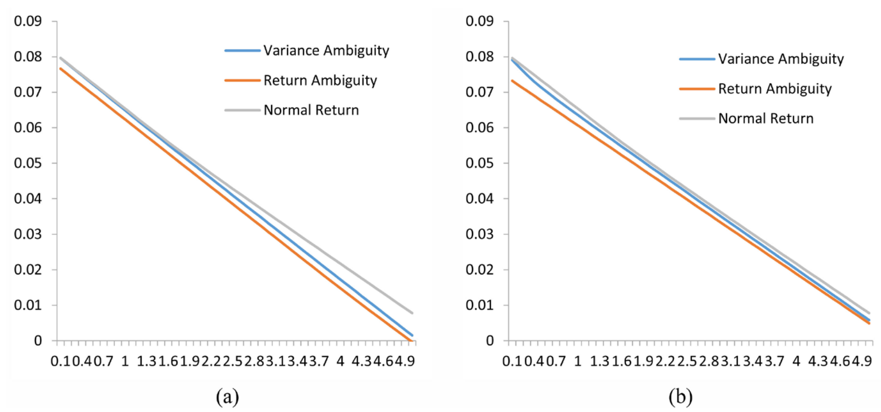


Figure 9. Portfolio returns generated with different approaches: variance ambiguity, return ambiguity and normal return w/o ambiguity adjustment, x-axis is γ-value from 0.1 to 5.0, (a) Set alpha1 = 0.1; alpha2 = 0.9, (b) alpha1 = 0.9; alpha2 = 0.1.

To further understand the relation between risk aversion and ambiguity aversion and how the portfolio composition would be affected, the weight of asset1 is also evaluated. The results are shown in Figure 10. For asset1 and 2 with equal return ambiguity level, Figure 10(a) shows that the weight1 slightly shifts to the left when the assets ambiguity change from 0.1 to 0.9. Within a certain range of risk aversion, one asset's return ambiguity change causes a dramatic change of portfolio weight, if the ambiguity level of two assets equals, large change of the ambiguity level would only cause quite limited difference in the portfolio weights. Taking the blue curve as central line and keep one of the alphas constant, Weight1 shifts to left when alpha2 increasing, and shifts to the right when alpha1 increasing. The investor will only invest in asset1 for within a certain risk aversion interval while the two assets have a same return ambiguity (less than 1.3 in this case). This risk aversion interval for only holding the asset with higher estimated return moves to right with the increase of alpha2 and moves to left with the increase of alpha1. For the return ambiguity, portfolio composition is co-decided by ambiguity aversion and risk aversion.

Figure 10(b) shows particularly interesting results differs to Figure 10(a). At the same variance ambiguity level, asset1's weight would fall suddenly at a certain risk aversion level. The decrease magnitude becomes moderated gradually. Higher the ambiguity level of asset2, larger the constant weight of asset1 will be. When alpha1 is small, the investor will only invest in asset1 due to its higher expected return, the risk aversion range for asset1 takes the whole portfolio gets shorter with the increase of asset1's variance ambiguity level. For a certain level of ambiguity of asset1, the power of portfolio weight modification by risk aversion is limited into a certain range. The impact is limited after γ pass 2 in this case. After this certain region of risk aversion level, the major cause of portfolio weight change will be the relative ambiguity level of each asset. Assets which investors are more familiar with will dominate the optimal portfolio.

5.1.8. Sharpe Ratio Modified Asset Return

Sharpe ratio of S&P 500 index and Dow Jones index is first calculated via sample return data and risk free rate of return. The risk free rate of return is presented by 3 month T-Bill rate. The estimated return is calculated by sample Sharpe ratio and return standard deviation applied with multiple ambiguity level.

Figure 11 shows the optimal portfolio return and assets weights w/o and with Sharpe ratio adjustment. Figure 11(a) is variance ambiguity w/o Sharpe ratio adjustment, with the increase of alpha2, the weight of asset1 exceeds 50% when alpha2 increase from 0.3 to 0.4 and grows up to 80%, the optimal portfolio return decrease at a nearly stable speed from 0.035 to 0.033. In Figure 11(b), the weight of asset1 increases with a slower speed comparing to in 11a, weight1 grows to 50% when alpha2 increase from 0.4 to 0.5 and stays slightly larger than 50%. The portfolio return drops less than 0.001 which is much smaller comparing to 11a as well.

Compare portfolio return with and without Sharpe ratio adjusted. In Figure 12(a), set γ to 2, the optimal portfolio return w/o Sharpe adjustment shows a decreasing trend with the increase of ambiguity level. On the other hand, the

Sharpe adjusted portfolio return moves on the opposite direction and increase with higher ambiguity level. The gap between these two portfolio returns grows wider with higher level of ambiguity.

When set γ to 3 as shown in Figure 12(b), both portfolios returns decrease with higher variance ambiguity. The portfolio w/o Sharpe adjustment shows an even faster decreasing rate compare to γ = 2, the portfolio return with Sharpe adjustment is more stable and shows slightly decrease with the increase of ambiguity level. Furthermore, the gap between the two portfolio returns still grows wider as what it shows in Figure 12(a).

To understand the reason of the portfolio return movement with the change of ambiguity level, we need to first understand how the portfolio return calculated in the optimization process. Due to the increase of variance ambiguity level, the standard deviation of estimated returns moves toward its confidence interval boundary, so that to impact on the return of the assets. Although the inner minimization process restrict the return move too far away from the sample return, the minimum return are always chosen on the boundary of the variance interval. Such process will partially compromise the minimization process and increase the portfolio return to some extent.

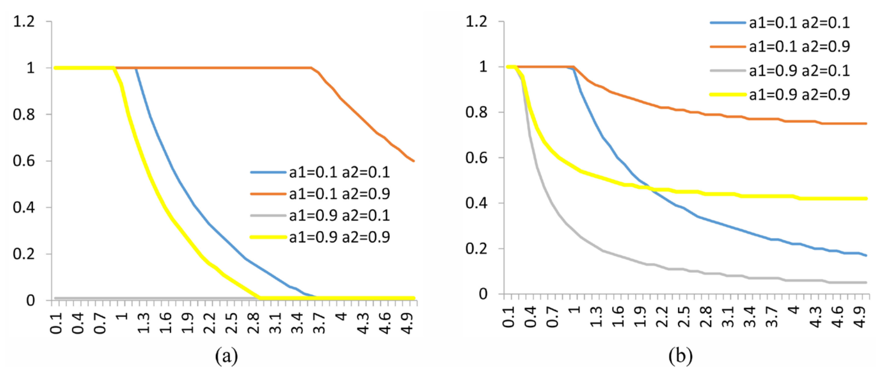


Figure 10. Weight of asset1 in optimized portfolio for different ambiguity combinations, x-axis is γ-value from 0.1 to 5.0, (a) return ambiguity (b) variance ambiguity.

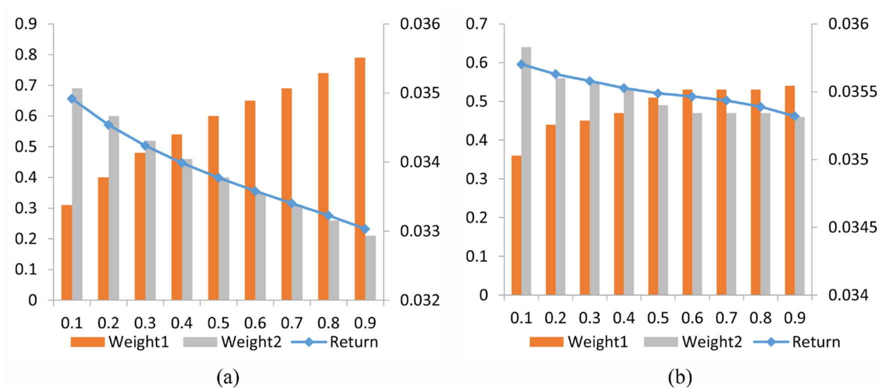


Figure 11. Weights of asset 1 and 2 and optimized portfolio returns. Set ambiguity level of asset1 equal to 0.1; (a) is variance ambiguity, x-axis is the ambiguity level of asset2, y-axis is the portfolio weight of asset 1; (b) is variance ambiguity with Sharpe ratio adjustment on asset return.

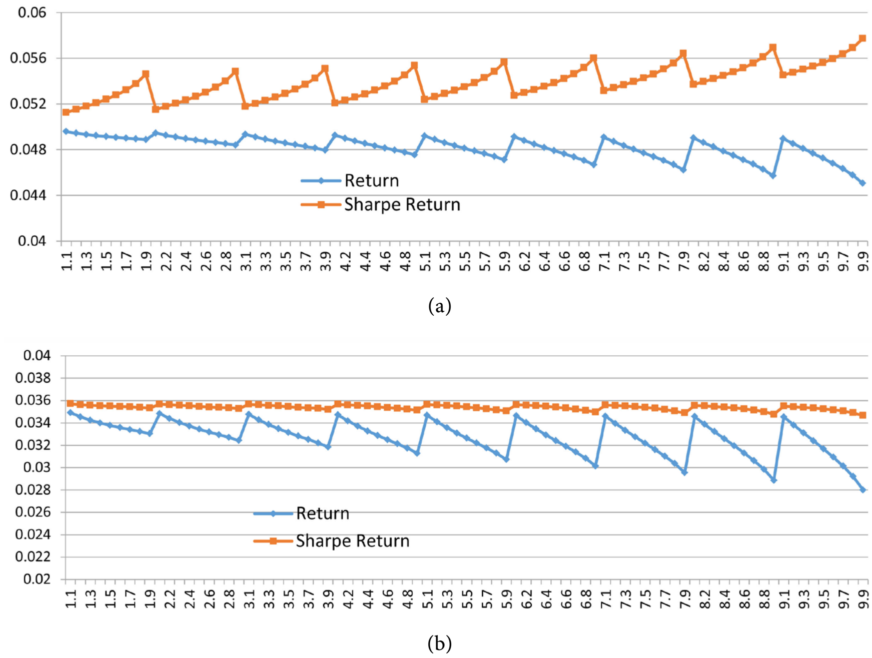


Figure 12. x-axis is the variance ambiguity level of asset1 and 2 (e.g. 1.2 means the ambiguity level of asset 1 is 0.1 and the ambiguity level of asset2 is 0.2); y-axis is the optimal portfolio return.

5.2. Multi-Markets Test

Key feature of the multi-market model applied in this study is that the objective stock indexes are from two countries, which hold significant different financial market history, fiscal and monetary policies. Due to the different stock market history and development scheme, the correlation between SHCOPM and S&P 500 is minimized. Exclude all other possible causes to home bias. The following test will only be focusing on two possible reasons that would change the optimal portfolio: risk aversion and ambiguity aversion. The portfolio return and weights of the two index generated from the mean-variance portfolio optimization are listed in Table 6.

With ambiguity applied, Figure 13 shows the weight of asset1 in the optimal portfolio while ambiguity levels are increasing from 0.1 to 0.9 for each asset. Comparing to Figure 2 and Figure 3, it is easy to tell that the gap between weight1 from return ambiguity adjustment and variance ambiguity adjustment is quite small and varies slightly with variation of alpha2. The impact of ambiguity aversion is much weaker comparing to the US stock market case.

To test the implication of risk aversion on multi country markets, the test from section 5 is adopted in this section as well. Similar as Figure 10, three approaches are implied on the SHCOMP and S&P 500 data. Figure 14(a) shows how the portfolio return would change with increasing γ, when alpha1 equals to alpha2 for both variance and return ambiguity level. It clearly shows that while γ is small, the variance ambiguity has weaker influence to the portfolio return comparing to the return ambiguity, so that blue line is closer to the green line while γ is less than 3. The portfolio reduction caused by variance ambiguity becomes stronger with the increase of γ, therefore the later part of blue line in Figure 14(a) drops faster than the red line. The figure represents the weight of asset1 in (b) shows similar shape for each approach. As both assets have the same high ambiguity level, the weight is mainly decide by the γ value and variance of assets, asset1 has larger variance compare to asset2 so that its weight is reduced with the increase of γ during the optimization process. Figure 14(b) further demonstrates for this case, the composition of optimal portfolio is mainly decided by the γ value when it is adjusted within a lower range. When γ passes 1, its influence to the portfolio weight is reduced and limited, so as its impact on the portfolio returns. Furthermore, not like the US stock market case, ambiguity aversion in this case has quite limited power to influence the optimal portfolio. Although the ambiguity levels of assets tested in Figure 14 are the highest level in this study, the figures represent normal return and variance/return ambiguity only slightly shifted. It is easy to conclude that if the returns and variances of assets in the portfolio are quite different as the SHCOMP index and S&P 500, the main reason cause portfolio change is the risk aversion level, ambiguity level corresponding to either return or variance would not cause large difference comparing to the standard mean-variance portfolio.

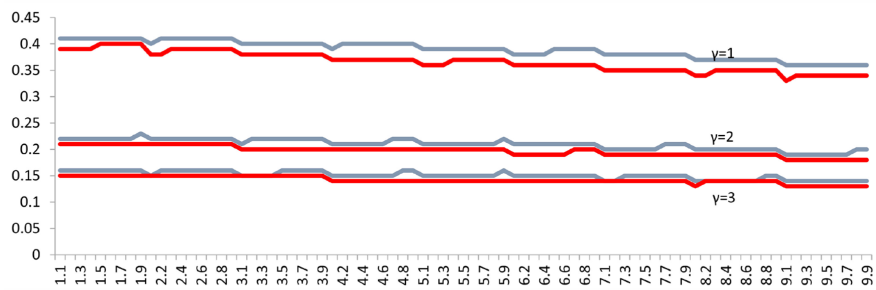
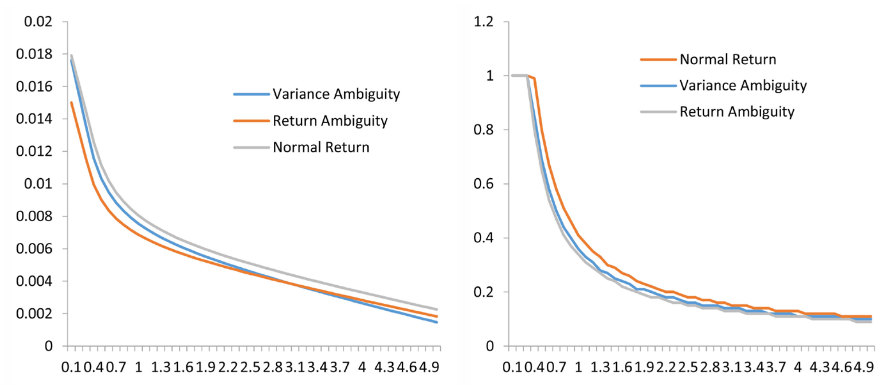


Figure 13. Weight of SHCOMP in the optimal portfolio for alpha1 and alpha2 vary from 0.1 to 0.9. γ = 1, 2, 3. In each of the two figures, the blue one is return ambiguity, red one is variance ambiguity.



(a) (b)

Figure 14. Set alpha1 = alpha2 = 0.9; (a) Portfolio return from variance ambiguity and return ambiguity with change of risk aversion factor γ, x-axis is γ. (b) Portfolio weight of asset1.

Table 6. Optimal portfolio return and weight of assets in the portfolio at different risk aversion level.

Risk Aversion Factor	Portfolio Return	Weight of S&P 500	Weight of SHCOMP
1	0.0081	0.59	0.41
2	0.0058	0.78	0.22
3	0.0044	0.84	0.16
4	0.0033	0.87	0.13
5	0.0023	0.89	0.11

6. Conclusion

The standard mean variance portfolio theory assumes that asset returns and variances are estimated without error. However, such an estimation dose applied with error and such error follows certain distribution. This study incorporates the estimation error generated from return and volatility into the mean variance portfolio optimization, tests and compares the impact from risk aversion and ambiguity aversion. Variance ambiguity is not only implied into the variance-covariance matrix, but also implied into the estimated return through expected Sharpe ratio.

The models in this study incorporate the view of ambiguity and ambiguity aversion about the true distribution of asset returns and volatilities. These models allow investors to distinguish their ambiguities toward risky assets and quantitatively apply such ambiguity into the Markowitz mean variance portfolio. The empirical data used in this model is quite typical as the asset with a larger return also has a larger variance so that the return ambiguity and variance ambiguity could cause the portfolio weight move in the opposite direction.

When **assets have similar expected returns and variance, furthermore, high correlation with each other**, the results show that numerically, the model has following implications for a portfolio containing two risky assets, 1) both return and variance ambiguity could cause dramatic change to the optimal portfolio weight; 2) return variance would cause a larger change to the return of optimal portfolio comparing to variance ambiguity; 3) the portfolio return changes linearly with return ambiguity, while higher variance ambiguity will increase the reduction speed of portfolio return as presented in (5) and (10); 4) when the assets are at the same ambiguity level, the weight of each asset is decided by its estimated variance and variance ambiguity level, the higher the ambiguity level, the lower the effect caused by its estimated variance; 5) relative return ambiguity cross assets decide the portfolio weight; 6) within certain risk aversion level, the absolute variance ambiguity of each asset decides the portfolio composition, the asset with a low ambiguity level will be largely hold by the investor.

Furthermore, when **assets have quite different returns and variance, especially low correlation** like assets from different stock markets, the ambiguity would not cause much difference but the risk aversion dominate the decision making process. Neither return ambiguity nor variance ambiguity could cause and explain the phenomenon like “No participation of stock market” or “Home bias”.

Limitations of this paper are listed below. First of all, the result of this study are mostly numerical results, whether such conclusions could be fit to other cases need to be proved theoretically in the next step. Secondly, some of the parameters in this study are set constant while they could follow certain distribution, like the correlation between assets. In the end, a model fit to portfolios with more than 2 assets should be developed to meet realistic needs.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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