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Availability of the Boundary Condition for Vorticity

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Abstract

By applying a boundary condition for vorticity [1] in addition to that for velocity, a velocity distribution on a flat plate set in a parallel homogeneous flow has been numerically obtained through a one-way calculation from surface to infinity, without the "matching" procedure between an analysis from surface to infinity and that from infinity to surface. The numerical results obtained were in excellent agreement with those by Howarth [2]. The usage of the boundary condition for vorticity has raised the accuracy of velocity distribution near a plate's surface and made it possible to realize the one-way calculation from surface to infinity.

Keywords

Equation of Vorticity Transport, Blasius's Equation, Boundary Condition for Vorticity, One-Way Calculation from Surface to Infinity

1. Introduction

It is recognized in a world of the present fluid dynamics that the boundary conditions for the conventional vorticity, $\boldsymbol{\omega} \equiv \nabla \times \boldsymbol{u}$, at the interface between two phases, say phase-A and phase-B, do not provide new information because the difference between the vorticity flux carried into the surface of phase-A's side of the interface and that carried out of the surface of phase-B's side of the interface is a result of the baroclinic generation of vorticity in the interface where the value of density jumps in an infinitesimal distance, and the value of the baroclinic generation is *a poteriori* after the flow field has been determined otherwise [3]. Here, $\boldsymbol{\omega}$ and $\boldsymbol{u}$ respectively denote the conventional vorticity vector and a velocity vector of fluid, and the word "flux" means a transporting rate of a physical quantity through a unit area of the surface in unit time.

Recently, it was found for a newly defined vorticity, $\boldsymbol{\omega} \equiv \nabla \times (\rho \boldsymbol{u})$, as a natural consequence inherent in transporting phenomena of angular momentum in fluid, that the normal component of vorticity is completely transferred from one phase to the other due to the shear forces working at both sides of an interface, and that the tangential components of vorticity flux are perfectly shut off at an interface between two phases [1]. Here, $\boldsymbol{\varpi}$ and ρ respectively denote the newly defined vorticity vector and a density of the fluid. The latter finding for the tangential components of vorticity works as a boundary condition for vorticity.

The main purpose of this work is to evaluate the availability of the boundary condition for vorticity by numerically calculating a velocity field on a flat plate fixed in a parallel homogeneous flow, $\boldsymbol{U} = \boldsymbol{i}_X U_\infty$, where, $\boldsymbol{i}_X$ and U_∞ are a unit vector along the *X*-axis parallel to the homogeneous flow and an *X*-component of the homogeneous flow, respectively.

It is well known that the velocity distribution in a boundary layer on a flat plate set in a parallel homogeneous flow is numerically obtained as a solution of Blasius's equation [4] by using the boundary condition for the velocity at a surface of the plate and that in a region infinitely far from the surface. However, it took around 30 years to refine the calculation to be capable of reliable results [2] [4] [5] [6] [7] because it was not an easy task to combine a velocity profile far from the plate's surface and that in a vicinity of the surface. A long history concerning the improvement of the solution is briefly summarized by Schlichting [8].

The paper is organized as follows. In Section 2, the equation of vorticity transport is transcribed for a two-dimensional flow on the flat plate by using a stream function $\psi = \left(\nu x U_\infty\right)^{0.5} f\left(\xi,\eta\right)$, to a differential equation of *f* with two non-dimensional variables, $\xi = \dfrac{U_\infty}{x} t$ and $\eta = y\left(\dfrac{U_\infty}{\nu x}\right)^{0.5}$, and a non-dimensional parameter, $\alpha = \dfrac{\nu}{x U_\infty}$. Here, *t*, *x*, *y* and *ν*, respectively, are time, a tangential distance along the *X*-axis from the front edge of the plate, a normal distance from the plate's surface and a kinematic viscosity of fluid defined as a ratio of viscosity to density. The contents of the equation are composed of two parts; one is with *α*, and the other is without *α*. Admitting the boundary condition for vorticity, it is shown in subsection 2.1 that Blasius's equation, which is a basic equation for a flow in a boundary layer on a flat plate, is a mathematical consequence of the part without *α*.

Basic concepts for the numerical calculation are described in Section 3, focusing on boundary conditions for velocity and vorticity (subsection 3.1), a frame work for the discretization (subsection 3.2), values of *f* near the plate's surface (subsection 3.3) and a summary of the one-way calculation route from the plate's surface to infinity (subsection 3.4).

In Section 4, numerical results are obtained through the one-way calculation route. A solution to Blasius's equation is obtained in subsection 4.1, which excellently agrees with the previous results by Howarth [2]. It is also illustrated in the subsection that the boundary condition for vorticity raises the accuracy of the preliminarily given values of *f* near the plate's surface and makes the one-way calculation route capable of obtaining a reliable solution. A solution to the equation of vorticity transport is obtained in subsection 4.2, which agrees well with the solution of Blasius's equation. Here again, the one-way calculation provides reliable results. Velocity distributions near the front edge of the plate are first obtained in this subsection.

A present understanding of the one-way calculation is briefly outlined in Section 5, and the concluding remarks are summarized in Section 6.

Since the flat plate is assumed to be fixed, the shear force working at the surface of the plate is a matter of *a posteriori*, to be calculated after the velocity field has been obtained. A pressure field is not discussed here, but it can be obtained by introducing the velocity distribution into the Navier-Stokes equation, if necessary.

2. Equation of Vorticity Transport

For incompressible fluid with constant viscosity, the vorticity transport equation is given by Equation (1):

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \rho\left[\nabla\cdot\left\{(\nabla\times\boldsymbol{u})\boldsymbol{u}\right\} - \nabla\cdot\left\{\boldsymbol{u}(\nabla\times\boldsymbol{u})\right\}\right] + \mu\nabla\cdot\left[\nabla(\nabla\times\boldsymbol{u})\right]. \tag{1}$$

Here, $\boldsymbol{\omega}$ and μ are the newly defined vorticity and viscosity of fluid, respectively. Though the newly defined vorticity is used here, it is easily confirmed that (1) can be obtained from the conventional equation of vorticity transport by changing the definition of vorticity.

A velocity field on a flat plate set in a homogeneous parallel flow, $\boldsymbol{U} = \boldsymbol{i}_X U_\infty$, becomes two-dimensional with *X*- and *Y*-components of velocity as $\boldsymbol{u} = \boldsymbol{i}_X u + \boldsymbol{i}_Y v$. Here *Y*-axis is perpendicular to the plate's surface, and $\boldsymbol{i}_X$ and $\boldsymbol{i}_Y$ are unit vector along *X*-and *Y*-axes, respectively. Equation (1) can be simplified by using the stream function, $\psi = (\nu x U_\infty)^{0.5} f(\xi,\eta)$.

The values of u and v are given by (2) and (3), respectively.

$$u = \frac{\partial\psi}{\partial y} = (\nu x U_\infty)^{0.5}\frac{\partial f}{\partial\eta}\frac{\partial\eta}{\partial y} = U_\infty f^{(1)}. \tag{2}$$

$$v = -\frac{\partial\psi}{\partial x} = -f\frac{\partial}{\partial x}\left\{(\nu x U_\infty)^{0.5}\right\} - (\nu x U_\infty)^{0.5}\frac{\partial f}{\partial\eta}\frac{\partial\eta}{\partial x}$$
$$= \frac{1}{2}\left(\frac{\nu U_\infty}{x}\right)^{0.5}\left(\eta f^{(1)} - f\right). \tag{3}$$

Here, $f^{(1)} \equiv \frac{\partial f}{\partial\eta}$.

Equation (1) is transformed to (4) for incompressible fluid with constant viscosity.

$$\frac{\partial}{\partial\xi}\left[-\frac{1}{4}\alpha\left(\eta f^{(1)} - f\right) - \left(\frac{1}{4}\alpha\eta^2 + 1\right)f^{(2)}\right]$$

$$= \alpha^2\left(\frac{15}{16}f - \frac{15}{16}\eta f^{(1)} - \frac{45}{16}\eta^2 f^{(2)} - \frac{7}{8}\eta^3 f^{(3)} - \frac{1}{16}\eta^4 f^{(4)}\right) \tag{4}$$
$$+\alpha\left(\frac{3}{8}ff^{(1)} - \frac{3}{8}\eta ff^{(2)} - \frac{1}{8}\eta^2 ff^{(3)} - \frac{3}{8}\eta f^{(1)}f^{(1)} - \frac{3}{8}\eta^2 f^{(1)}f^{(2)}\right.$$
$$\left. - \frac{3}{2}f^{(2)} - \frac{5}{2}\eta f^{(3)} - \frac{1}{2}\eta^2 f^{(4)}\right) + \left(-\frac{1}{2}ff^{(3)} - \frac{1}{2}f^{(1)}f^{(2)} - f^{(4)}\right).$$

Here, $f^{(2)} \equiv \frac{\partial^2 f}{\partial \eta^2}$, $f^{(3)} \equiv \frac{\partial^3 f}{\partial \eta^3}$, $f^{(4)} \equiv \frac{\partial^4 f}{\partial \eta^4}$, $\alpha = \frac{\nu}{xU_\infty}$, $\eta = y\left(\frac{U_\infty}{\nu x}\right)^{0.5}$ and $\xi = \frac{U_\infty}{x}t$. A calculating process leading to (4) is shown in **Appendix**.

Equation (4) is a differential equation for a function *f*, the value of which depends on two variables, η and ξ, and a parameter α. Since the value of α depends solely on *x*, the value of *f* is obtained by integrating (4) with respect to two variables, η and ξ, keeping the value of α constant corresponding to a certain distance *x* from a front edge of the plate. The right-hand side of (4) is composed of three terms. The first and second terms are accompanied with parameter α. The third term does not contain α, and becomes dominant when $\alpha \ll 1$.

It is seemingly necessary to integrate (4) for various values of α, but the value of α is very small in most practical cases. If we take $\nu = 1\times10^{-6}\ \mathrm{m}^2\cdot\mathrm{s}^{-1}$ and $\nu = 1\times10^{-5}\ \mathrm{m}^2\cdot\mathrm{s}^{-1}$ for liquid and gas, respectively, as their representative values, the value of α at $x = 1\times10^{-2}\ \mathrm{m}$ becomes 10^{-4} and 10^{-3} for liquid and gas, respectively, for $U_\infty = 1\ \mathrm{m}\cdot\mathrm{s}^{-1}$. The more become the values of U_∞ and *x*, the smaller becomes the value of α. Then, we need not care for a contribution of α, except when investigating very slow flow or flow near the front edge of the plate.

Blasius's Equation

Assuming $\alpha \ll 1$, (5) is obtained from (4) for a stationary flow.

$$0 = \frac{1}{2}ff^{(3)} + \frac{1}{2}f^{(1)}f^{(2)} + f^{(4)}. \tag{5}$$

Equation (6) is obtained by integrating (5) with respect to η.

$$ff^{(2)} + 2f^{(3)} = A. \tag{6}$$

Here, *A* is an integral constant.

The boundary condition for vorticity is given by (7) [1].

$$0 = \boldsymbol{n}\times\boldsymbol{F}_\omega\text{, on an interface.} \tag{7}$$

Here, $\boldsymbol{n}$ is a unit normal vector at an interface, and $\boldsymbol{F}_\omega$ is a total vorticity flux at an interface, given by (8).

$$\boldsymbol{F}_\omega = \boldsymbol{n}\cdot\left[\left\{(\nabla\times\boldsymbol{u})(\rho\boldsymbol{u})\right\} - \left\{(\rho\boldsymbol{u})(\nabla\times\boldsymbol{u})\right\} + \nabla\left\{\nabla\times(\mu\boldsymbol{u})\right\}\right]. \tag{8}$$

A vorticity flux is an amount of vorticity transported through a unit area in unit time. Equation (7) means that tangential components of vorticity do not go through an interface between two phases.

Let us transform (8) and (7) for a present target by using the stream function,

$$\psi = \left(\nu x U_\infty\right)^{0.5} f\left(\xi,\eta\right).$$

$$\begin{aligned}\boldsymbol{F}_\omega &= \boldsymbol{n}\cdot\left[\left\{(\nabla\times\boldsymbol{u})(\rho\boldsymbol{u})\right\}-\left\{(\rho\boldsymbol{u})(\nabla\times\boldsymbol{u})\right\}+\nabla\left\{\nabla\times(\mu\boldsymbol{u})\right\}\right] \\ &= \boldsymbol{i}_Y\cdot\left[\rho\left\{\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\left\{u\left(\boldsymbol{i}_Z\boldsymbol{i}_X-\boldsymbol{i}_X\boldsymbol{i}_Z\right)+v\left(\boldsymbol{i}_Z\boldsymbol{i}_Y-\boldsymbol{i}_Y\boldsymbol{i}_Z\right)\right\}\right\}\right. \\ &\quad \left.+\mu\left\{\left(\frac{\partial^2 v}{\partial x^2}-\frac{\partial^2 u}{\partial x\partial y}\right)\boldsymbol{i}_X\boldsymbol{i}_Z+\left(\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 u}{\partial y^2}\right)\boldsymbol{i}_Y\boldsymbol{i}_Z\right\}\right] \\ &= \left\{-\rho\left(v\frac{\partial v}{\partial x}-v\frac{\partial u}{\partial y}\right)+\mu\left(\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 u}{\partial y^2}\right)\right\}\boldsymbol{i}_Z. \end{aligned} \tag{9}$$

$$\begin{aligned} 0 &= \boldsymbol{n}\times\boldsymbol{F}_\omega \\ &= \boldsymbol{i}_Y\times\left[\left\{-\rho\left(v\frac{\partial v}{\partial x}-v\frac{\partial u}{\partial y}\right)+\mu\left(\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 u}{\partial y^2}\right)\right\}\boldsymbol{i}_Z\right] \\ &= \left\{-\rho\left(v\frac{\partial v}{\partial x}-v\frac{\partial u}{\partial y}\right)+\left(\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 u}{\partial y^2}\right)\right\}\boldsymbol{i}_X \text{ at } y=0. \end{aligned} \tag{10}$$

Following relations are obtained from (2) and (3).

$$\frac{\partial v}{\partial x}=\frac{1}{4}U_\infty\frac{y}{x^2}\left\{\frac{f}{\eta}-f^{(1)}-\eta f^{(2)}\right\}. \tag{11}$$

$$\frac{\partial u}{\partial y}=U_\infty\left(\frac{U_\infty}{\nu x}\right)^{0.5}f^{(2)}. \tag{12}$$

$$\frac{\partial^2 v}{\partial x\partial y}=\frac{1}{4}U_\infty\frac{1}{x^2}\left\{-3\eta f^{(2)}-\eta^2 f^{(3)}\right\}. \tag{13}$$

$$\frac{\partial^2 u}{\partial y^2}=U_\infty\frac{U_\infty}{\nu x}f^{(3)}. \tag{14}$$

Equation (15) is obtained by introducing (11), (12), (13) and (14) into (10).

$$0=-\frac{1}{8}\alpha f^2+\frac{1}{2}ff^{(2)}+f^{(3)}, \text{ at } \eta=0. \tag{15}$$

Equation (15) is the boundary condition for vorticity.

For a case where $\alpha \ll 1$, (16) is obtained.

$$0=\frac{1}{2}ff^{(2)}+f^{(3)}, \text{ at } \eta=0. \tag{16}$$

Then, the value of the integral constant in (6) is zero, and the Blasius's equation is obtained.

$$ff^{(2)}+2f^{(3)}=0. \tag{17}$$

3. Basic Concepts for Numerical Calculation

3.1. Boundary Conditions for Velocity and Vorticity

Equations (18) and (19) hold because the values of u and v are zero at a surface of the plate.

$$f = 0\text{, at } \eta = 0. \tag{18}$$

$$f^{(1)} = 0\text{, at } \eta = 0. \tag{19}$$

Equation (20) is obtained by introducing (18) into the boundary condition for vorticity, (15).

$$f^{(3)} = 0\text{, at } \eta = 0. \tag{20}$$

Another boundary condition is given by (21), because the flow field should coincide with the original flow, $\boldsymbol{U} = \boldsymbol{i}_X U_\infty$, in a region infinitely far from the plate.

$$f^{(1)} \to 1 \ (\eta \to \infty). \tag{21}$$

3.2. Discretization

Let us set lattice points along η axis denoting the value of f at $\eta = j\delta$ as f_j. Here, δ is a distance between two adjacent lattice points. Equations (22)-(26) are obtained as Taylor expansions of f around a point $\eta = j\delta$.

$$f_{j-2} = f_j - \frac{2\delta}{1!} f_j^{(1)} + \frac{(2\delta)^2}{2!} f_j^{(2)} - \frac{(2\delta)^3}{3!} f_j^{(3)} + \frac{(2\delta)^4}{4!} f_j^{(4)} - \frac{(2\delta)^5}{5!} f_j^{(5)} + O\delta^6. \tag{22}$$

$$f_{j-1} = f_j - \frac{\delta}{1!} f_j^{(1)} + \frac{\delta^2}{2!} f_j^{(2)} - \frac{\delta^3}{3!} f_j^{(3)} + \frac{\delta^4}{4!} f_j^{(4)} - \frac{\delta^5}{5!} f_j^{(5)} + O\delta^6. \tag{23}$$

$$f_{j+1} = f_j + \frac{\delta}{1!} f_j^{(1)} + \frac{\delta^2}{2!} f_j^{(2)} + \frac{\delta^3}{3!} f_j^{(3)} + \frac{\delta^4}{4!} f_j^{(4)} + \frac{\delta^5}{5!} f_j^{(5)} + O\delta^6. \tag{24}$$

$$f_{j+2} = f_j + \frac{2\delta}{1!} f_j^{(1)} + \frac{(2\delta)^2}{2!} f_j^{(2)} + \frac{(2\delta)^3}{3!} f_j^{(3)} + \frac{(2\delta)^4}{4!} f_j^{(4)} + \frac{(2\delta)^5}{5!} f_j^{(5)} + O\delta^6. \tag{25}$$

$$f_{j+3} = f_j + \frac{3\delta}{1!} f_j^{(1)} + \frac{(3\delta)^2}{2!} f_j^{(2)} + \frac{(3\delta)^3}{3!} f_j^{(3)} + \frac{(3\delta)^4}{4!} f_j^{(4)} + \frac{(3\delta)^5}{5!} f_j^{(5)} + O\delta^6. \tag{26}$$

Here, $f_j \equiv f\big|_{\eta=j\delta}$, $f_j^{(1)} \equiv \left.\frac{\mathrm{d}f}{\mathrm{d}\eta}\right|_{\eta=j\delta}$, $f_j^{(2)} \equiv \left.\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2}\right|_{\eta=j\delta}$, $f_j^{(3)} \equiv \left.\frac{\mathrm{d}^3 f}{\mathrm{d}\eta^3}\right|_{\eta=j\delta}$, $f_j^{(4)} \equiv \left.\frac{\mathrm{d}^4 f}{\mathrm{d}\eta^4}\right|_{\eta=j\delta}$ and $f_j^{(5)} \equiv \left.\frac{\mathrm{d}^5 f}{\mathrm{d}\eta^5}\right|_{\eta=j\delta}$, and $O\delta^6$ is a sum of residual terms the value of which is an order of δ^6. By using (22)-(26), the values of $f_j^{(1)}$, $f_j^{(2)}$, $f_j^{(3)}$ and $f_j^{(4)}$ can be approximated in terms of f_{j-2}, f_{j-1}, f_j, f_{j+1}, f_{j+2} and f_{j+3}.

$$f_j^{(1)} = \frac{1}{2\delta}\left(-f_{j-1} + f_{j+1}\right) + O\delta^2. \tag{27}$$

$$f_j^{(2)} = \frac{1}{\delta^2}\left(f_{j-1} - 2f_j + f_{j+1}\right) + O\delta^2. \tag{28}$$

$$f_j^{(3)} = \frac{1}{2\delta^3}\left(-f_{j-2} + 2f_{j-1} - 2f_{j+1} + f_{j+2}\right) + O\delta^2. \tag{29}$$

$$f_j^{(4)} = \frac{1}{\delta^4}\left(f_{j-2} - 4f_{j-1} + 6f_j - 4f_{j+1} + f_{j+2}\right) + O\delta^2. \tag{30}$$

Here, $O\delta^2$ is a sum of residual terms with an order of δ^2.

3.3. Velocity Distribution near Plate's Surface

Equations (31)-(33) are obtained from (24)-(26), by applying the boundary conditions (18)-(20).

$$f_1 = \frac{1}{2}\delta^2 f_0^{(2)} + O\delta^4. \tag{31}$$

$$f_2 = 2\delta^2 f_0^{(2)} + O\delta^4. \tag{32}$$

$$f_3 = \frac{9}{2}\delta^2 f_0^{(2)} + O\delta^4. \tag{33}$$

The values of f_1, f_2 and f_3 are respectively given by $\frac{1}{2}\delta^2 f_0^{(2)}$, $2\delta^2 f_0^{(2)}$ and $\frac{9}{2}\delta^2 f_0^{(2)}$, with a relative error $O\delta^2$, that is 1% for $\delta = 0.1$.

3.4. Calculating Route

We already know that $f_0 = 0$, $f_1 = \frac{1}{2}\delta^2 f_0^{(2)}$, $f_2 = 2\delta^2 f_0^{(2)}$ and $f_3 = \frac{9}{2}\delta^2 f_0^{(2)}$. Then, the value of f_j for $j \geq 4$ are to be obtained by using a discretized equation of vorticity transport. Our target has thus come down to iteratively determine the value of $f_0^{(2)}$ which leads to a numerical sequence, $f_0, f_1, f_2, \cdots, f_j, f_{j+1}, \cdots$, satisfying a relation, $\frac{1}{2\delta}\left(-f_{j-1} + f_{j+1}\right) = 1 + O\delta^2$, for sufficiently large *j*.

4. Numerical Results

4.1. Blasius's Equation

Let us obtain a numerical solution to Blasius's equation first. Equation (34) is obtained as Blasius's equation discretized at a lattice point $\eta = j\delta$.

$$0 = f_j f_j^{(2)} + 2 f_j^{(3)}. \tag{34}$$

By using (28) and (29), the right-hand side of (34) can be given in terms of f_{j-2}, f_{j-1}, f_j, f_{j+1} and f_{j+2}, and (35) is obtained.

$$f_{j+2} = f_{j-2} - 2 f_{j-1} + 2 f_{j+1} - \delta\left(f_{j-1} f_j - 2 f_j f_j + f_j f_{j+1}\right). \tag{35}$$

The values of f_0, f_1, f_2 and f_3 are given by (18), (31), (32) and (33). Then, the value of $f_j\left(j \geq 4\right)$ can be obtained by iteratively using (35) for arbitrary values of $f^{(2)}$ and δ.

In **Figure 1**, numerical results of $f^{(1)}$ for $f_0^{(2)} = 0.2$, 0.3 and 0.4 are shown. Obviously, the value of $f^{(1)}$ is approaching to a constant value corresponding to each value of $f_0^{(2)}$. If we suppose that $f_{j-2} = f_j - 2\beta$, $f_{j-1} = f_j - \beta$ and $f_{j+1} = f_j + \beta$, it is easily confirmed that (35) gives a relation $f_{j+2} = f_j + 2\beta$. Then, once f_{j-2}, f_{j-1}, f_j and f_{j+1} are on a straight line in a figure of *f* vs. *η*, the following value of $f_i\left(i > j+1\right)$ is also on the same straight line. Hence, the

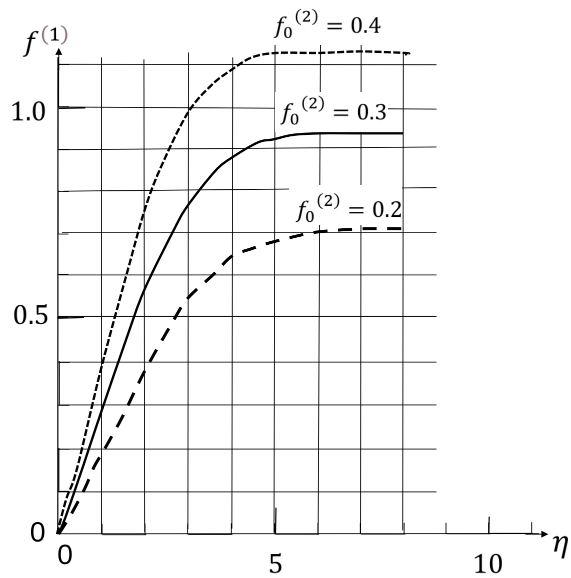

Figure 1. Dependence of numerical results on the value of $f_0^{(2)}$, for $\delta = 0.1$.

values of $f_j^{(1)}$ in the final stage of the calculation generally satisfy a relation, $f_j^{(1)} = \beta + O\delta^2$, for several consecutive lattice numbers. In this work, the value of β for each assumed value of $f_0^{(2)}$ was determined when the condition, $f_j^{(1)} = \beta + O\delta^2$, was satisfied for 5 or more consecutive lattice numbers. The exact value of $f^{(2)}$ was fixed by the try-and-error method to realize $\beta = 1$.

Numerical values are shown in **Table 1**, together with the previous results by Howarth cited in Boundary Layer Theory [8]. Obviously, the numerical results of this work and those by Howarth excellently agree with each other.

Blasius first presented Blasius's equation and obtained its solution by matching a series expansion around $\eta = 0$ and an asymptotic expansion for η very large [4]. It was not an easy task to obtain a solution of Blasius's equation and various researchers attempted to improve the analytical approach in following several decades, as briefly reviewed in Boundary Layer Theory [8], in which Howarth's solution [2] was cited as one of the most precise work among them.

Considering that the calculation route taken here is very simple, it is astonishing that the numerical results of this work excellently agree with those by Howarth. The calculation route in this work is straightforward. It starts from preliminarily given values $f_0 = 0$, $f_1 = \frac{1}{2}\delta^2 f_0^{(2)}$, $f_2 = 2\delta^2 f_0^{(2)}$ and $f_3 = \frac{9}{2}\delta^2 f_0^{(2)}$, and calculates the values of $f_i\,(i \ge 4)$ by iteratively using discretized Blasius's equation. The matching process between an analysis from the plate's surface to infinity and that from infinity to the surface is not necessary at all. The boundary condition for vorticity has been applied here in addition to the boundary conditions for velocity, and sufficiently accurate numerical values of f_0, f_1, f_2 and f_3 have been preliminarily given. This is the reason why such a simple and straightforward calculation route leads to the accurate numerical values. If the boundary

Table 1. Numerical results for $\delta = 0.1$ and $f_0^{(2)} = 0.332$.

η	f		$f^{(1)}$	
	Howarth	this work	Howarth	this work
0.4	0.02656	0.02656	0.13277	0.13278
0.8	0.10611	0.10611	0.26471	0.26470
1.2	0.23795	0.23796	0.39378	0.39377
1.6	0.42032	0.42037	0.51676	0.51676
2.0	0.65003	0.65012	0.62977	0.62977
2.4	0.92230	0.92245	0.72899	0.72898
2.8	1.23099	1.23121	0.81152	0.81149
3.2	1.56911	1.56939	0.87609	0.87604
3.6	1.92954	1.92986	0.92333	0.92326
4.0	2.30576	2.30611	0.95552	0.95543
4.4	2.69238	2.69273	0.97587	0.97577
4.8	3.08534	3.08567	0.98779	0.98770
5.2	3.48189	3.48221	0.99425	0.99417
5.6	3.88031	3.88061	0.99838	0.99742
6.0	4.27964	4.27993	0.99898	0.99894
6.4	4.67938	4.67964	0.99961	0.99961
6.8	5.07928	5.07951	0.99987	0.99993
7.2	5.47925	5.47940	0.99996	1.00021
7.6	5.87924	5.87918	0.99999	1.00086
8.0	6.27923	6.27846	1.00000	1.00314

condition for vorticity was known at the beginning of 20th century, the solution of the Blasius's equation could have been easily obtained.

4.2. Equation of Vorticity Transport

Equation (36) is the equation of vorticity transport for a stationary flow discretized at a lattice point $\eta = j\delta$.

$$\begin{aligned} 0 = {} & \alpha^2\left(\frac{15}{16} f_j - \frac{15}{16}\eta_j f_j^{(1)} - \frac{45}{16}\eta_j^2 f_j^{(2)} - \frac{7}{8}\eta_j^3 f_j^{(3)} - \frac{1}{16}\eta_j^4 f_j^{(4)}\right) \\ & + \alpha\left(\frac{3}{8} f_j f_j^{(1)} - \frac{3}{8}\eta_j f_j f_j^{(2)} - \frac{1}{8}\eta_j^2 f_j f_j^{(3)} - \frac{3}{8}\eta_j f_j^{(1)} f_j^{(1)} - \frac{3}{8}\eta_j^2 f_j^{(1)} f_j^{(2)} \right. \\ & \left. - \frac{3}{2} f_j^{(2)} - \frac{5}{2}\eta_j f_j^{(3)} - \frac{1}{2}\eta_j^2 f_j^{(4)}\right) + \left(-\frac{1}{2} f_j f_j^{(3)} - \frac{1}{2} f_j^{(1)} f_j^{(2)} - f_j^{(4)}\right). \end{aligned} \tag{36}$$

By using (27)-(30), the right-hand side of (36) can be given in terms of f_{j-2}, f_{j-1}, f_j, f_{j+1} and f_{j+2}, and (37) is obtained. Then, a calculation route for the equation of vorticity transport becomes similar with that for Blasius's equation.

$$f_{j+2} = \frac{1}{\alpha^2\left(\frac{7}{16}\eta_j^3\delta+\frac{1}{16}\eta_j^4\right)+\alpha\left(\frac{5}{4}\eta_j\delta+\frac{1}{2}\eta_j^2\right)+1+\left(\alpha\frac{1}{16}\eta_j^2+\frac{1}{4}\right)\delta f_j} \tag{37}$$

$$\times\left[\left\{\alpha^2\left(\frac{7}{16}\eta_j^3\delta-\frac{1}{16}\eta_j^4\right)+\alpha\left(\frac{5}{4}\eta_j\delta-\frac{1}{2}\eta_j^2\right)-1\right\}f_{j-2}\right.$$

$$+\left\{\alpha^2\left(\frac{15}{32}\eta_j\delta^3-\frac{45}{16}\eta_j^2\delta^2-\frac{7}{8}\eta_j^3\delta+\frac{1}{4}\eta_j^4\right)\right.$$

$$\left.+\alpha\left(-\frac{3}{2}\delta^2-\frac{5}{2}\eta_j\delta+2\eta_j^2\right)+4\right\}f_{j-1}$$

$$+\left\{\alpha^2\left(\frac{15}{16}\delta^4+\frac{45}{8}\eta_j^2\delta^2-\frac{3}{8}\eta_j^4\right)+\alpha\left(3\delta^2-3\eta_j^2\right)-6\right\}f_j$$

$$+\left\{\alpha^2\left(-\frac{15}{32}\eta_j\delta^3-\frac{45}{16}\eta_j^2\delta^2+\frac{7}{8}\eta_j^3\delta+\frac{1}{4}\eta_j^4\right)\right.$$

$$\left.+\alpha\left(-\frac{3}{2}\delta^2+\frac{5}{2}\eta_j\delta+2\eta_j^2\right)+4\right\}f_{j+1}+\left(\alpha\frac{1}{16}\eta_j^2\delta+\frac{1}{4}\delta\right)f_{j-2}f_j$$

$$+\left\{\alpha\left(-\frac{3}{32}\eta_j\delta+\frac{3}{16}\eta_j^2\right)+\frac{1}{4}\right\}\delta f_{j-1}f_{j-1}$$

$$+\left\{\alpha\left(-\frac{3}{16}\delta^2-\frac{3}{8}\eta_j\delta-\frac{1}{2}\eta_j^2\right)-1\right\}\delta f_{j-1}f_j+\alpha\frac{3}{16}\eta_j\delta^2 f_{j-1}f_{j+1}$$

$$+\alpha\frac{3}{4}\eta_j\delta^2 f_j f_j+\left\{\alpha\left(\frac{3}{16}\delta^2-\frac{3}{8}\eta_j\delta+\frac{1}{2}\eta_j^2\right)+1\right\}\delta f_j f_{j+1}$$

$$\left.+\left\{\alpha\left(-\frac{3}{32}\eta_j\delta-\frac{3}{16}\eta_j^2\right)-\frac{1}{4}\right\}\delta f_{j+1}f_{j+1}\right].$$

The value of f_{j+2} given by (37) is with $O\delta^2$ error, because the residual terms, $O\delta^2$, in (27)-(30) have been neglected. We already have the values of f_0, f_1, f_2 and f_3 as 0, $\frac{1}{2}\delta^2 f_0^{(2)}$, $2\delta^2 f_0^{(2)}$ and $\frac{9}{2}\delta^2 f_0^{(2)}$, respectively. Then, the value of $f_j\,(j\ge 4)$ can be obtained by iteratively using (37) for arbitrary values of $f^{(2)}$ and δ.

As for the judgement of the final stage of the numerical calculation, we need to separately treat two cases; $\alpha=0$ and $\alpha\ne 0$.

when $\alpha = 0$,

Equation (37) is simplified to (38).

$$f_{j+2} = \frac{1}{1+\frac{1}{4}\delta f_j}\times\left\{-f_{j-2}+4f_{j-1}-6f_j+4f_{j+1}\right. \tag{38}$$

$$\left.+\delta\left(\frac{1}{4}f_{j-2}f_j+\frac{1}{4}f_{j-1}f_{j-1}-f_{j-1}f_j+f_j f_{j+1}-\frac{1}{4}f_{j+1}f_{j+1}\right)\right\}.$$

If we suppose that $f_{j-2}=f_j-2\beta$, $f_{j-1}=f_j-\beta$ and $f_{j+1}=f_j+\beta$, it is easily confirmed that (38) gives a relation $f_{j+2}=f_j+2\beta$. This means that f_{j+2} is also on the same straight line in a figure of f vs. η. Then, the value of $f_0^{(2)}$ can be determined so as to realize a condition, $f_j^{(1)}=1+O\delta^2$, just like the numerical calculation for Blasius's equation. Numerical results agree with those for Blasius's equation as shown in Table 2.

Numerical results for the equation of vorticity transport are rather scattered but agree well with those for Blasius's equation. The preliminarily given values of f_0, f_1, f_2 and f_3 are thus shown to be sufficiently accurate to hold the one-way calculating route for the equation of vorticity transport.

when $\alpha \neq 0$,

The condition, $f^{(1)} \to 1(\eta \to \infty)$, means that the values of f_{j-2}, f_{j-1}, f_j and f_{j+1} can be given by $f_j - 2\delta$, $f_j - \delta$, f_j and $f_j + \delta$, respectively in the final stage of the calculation. Let us examine whether (37) fits to the condition, $f^{(1)} \to 1(\eta \to \infty)$.

Equation (39) is obtained by introducing $f_{j-2} = f_j - 2\delta$, $f_{j-1} = f_j - \delta$, $f_{j+1} = f_j + \delta$, and $f_{j+2} = f_j + 2\beta$ into (37).

$$\beta = \delta + \frac{\left(f_j - \eta_j\right)\left(\frac{15}{16}\alpha + \frac{3}{8}\right)\alpha}{f_j\left(\frac{1}{8}\alpha\eta_j^2\delta + \frac{1}{2}\delta\right) + \alpha^2\left(\frac{1}{8}\eta_j^4 + \frac{7}{8}\eta_j^3\delta\right) + \alpha\left(\frac{5}{2}\eta_j\delta + \eta_j^2\right) + 2}\delta^4$$
$$= \delta\left(1 + O\delta^3\right). \tag{39}$$

Table 2. Numerical data for $\alpha = 0$ ($\delta = 0.1$).

η	$\alpha = 0$			
	Equation (38) ($f_0^{(2)}$ = 0.332)		Equation (35) ($f_0^{(2)}$ = 0.332)	
	f	$f^{(1)}$	f	$f^{(1)}$
0.4	0.02656	0.13277	0.02656	0.13278
0.8	0.10613	0.26481	0.10611	0.26470
1.2	0.23807	0.39407	0.23796	0.39377
1.6	0.42064	0.51732	0.42037	0.51676
2.0	0.65068	0.63065	0.65012	0.62977
2.4	0.92343	0.73019	0.92245	0.72898
2.8	1.23274	0.81302	1.23121	0.81149
3.2	1.57159	0.87788	1.56939	0.87604
3.6	1.93290	0.92546	1.92986	0.92326
4.0	2.31008	0.95806	2.30611	0.95543
4.4	2.69785	0.97890	2.69273	0.97577
4.8	3.09215	0.99132	3.08567	0.98770
5.2	3.49023	0.99834	3.48221	0.99417
5.6	3.89045	1.00232	3.88061	0.99742
6.0	4.29186	1.00451	4.27993	0.99894
6.4	4.69395	1.00584	4.67964	0.99961
6.8	5.09649	1.00677	5.07951	0.99993
7.2	5.49932	1.00736	5.47940	1.00021
7.6	5.90233	1.00760	5.87918	1.00086
8.0	6.30534	1.00739	6.27846	1.00314

Here, $O\delta^3$ on the right-hand side is an error term, which is sufficiently small to assure that once f_{j-2}, f_{j-1}, f_j and f_{j+1} are on a straight line with gradient unity in a figure of *f* vs. *η*, the values of $f_i(i > j+1)$ are on the same straight line. Then, the value of $f_0^{(2)}$ can be determined so as to realize a condition, $f_j^{(1)} = 1 + O\delta^2$, for several consecutive lattice points.

Numerical results for $\alpha = 0.1$ and $\alpha = 1.0$ are shown in **Figure 2** together with those for $\alpha = 0$. Since $f_0^{(2)}$ is closely related to a shear force at the plate's surface as $\tau_s = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu U_\infty \left(\frac{U_\infty}{\nu x}\right)^{0.5} f_0^{(2)}$, it is easily seen in **Figure 2** that the shear force becomes greater for a larger value of *α*, that is, in the vicinity of the front edge of the plate. Numerical data are tabulated in **Table 3**. These are the first obtained velocity distributions in the vicinity of the front edge of the plate. Velocity distributions for an arbitrary value of *α* can be easily obtained in the same way, if necessary.

Table 3. Numerical results for $\alpha = 0.1$, $\alpha = 1.0$ ($\delta = 0.1$).

η	$\alpha = 0$ $\left(f_0^{(2)} = 0.332\right)$		$\alpha = 0.1$ $\left(f_0^{(2)} = 0.48\right)$		$\alpha = 1.0$ $\left(f_0^{(2)} = 1.01\right)$	
	f	$f^{(1)}$	f	$f^{(1)}$	f	$f^{(1)}$
0.4	0.02656	0.13277	0.03839	0.19176	0.08063	0.39980
0.8	0.10613	0.26481	0.15287	0.37911	0.31168	0.73474
1.2	0.23807	0.39407	0.34009	0.55371	0.65135	0.93650
1.6	0.42064	0.51732	0.59308	0.70631	1.04457	1.00984
2.0	0.65068	0.63065	0.90140	0.82913	1.44976	1.00707
2.4	0.92343	0.73019	1.25213	0.91812		
2.8	1.23274	0.81302	1.63173	0.97401		
3.2	1.57159	0.87788	2.02780	1.00177		
3.6	1.93290	0.92546	2.43054	1.00864		
4.0	2.31008	0.95806	2.83300	1.00172		
4.4	2.69785	0.97890				
4.8	3.09215	0.99132				
5.2	3.49023	0.99834				
5.6	3.89045	1.00232				
6.0	4.29186	1.00451				
6.4	4.69395	1.00584				
6.8	5.09649	1.00677				
7.2	5.49932	1.00736				
7.6	5.90233	1.00760				
8.0	6.30534	1.00739				

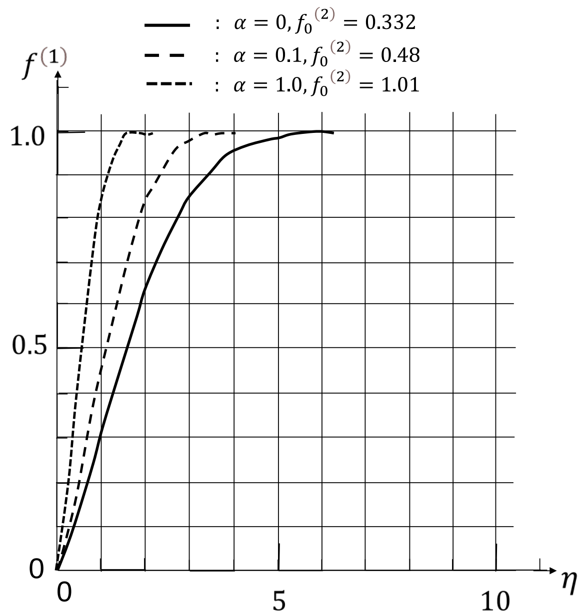

Figure 2. Velocity distributions for $\alpha = 0$, $\alpha = 0.1$ and $\alpha = 1.0$.

5. One-Way Calculation Route from Surface to Infinity

The matching procedure between an analysis from surface to infinity and that from infinity to the surface has been believed to be a theoretical core in studying force acting on the surface or a velocity distribution near the surface. For instance, the Saffman force acting on a spherical particle set in a shear flow was obtained by matching the inner expansion and the outer expansion of a flow field [9], and the matching procedure was inevitable for previous works on the numerical solution of the Blasius's equation briefly illustrated in Boundary Layer Theory [8]. However, a reliable numerical solution of Blasius's equation, which excellently coincides with Howarth's solution, has been obtained in this work following a one-way calculation route from surface to infinity without the matching process.

The one-way calculation taken here is composed of four steps. The first step is to arbitrarily set the values of δ and $f_0^{(2)}$. Since the numerical values treated here are involving $O\delta^2$ relative error, the value of δ should be sufficiently smaller than unity. The second step is to give the values of f_0, f_1, f_2 and f_3 as 0, $\frac{1}{2}\delta^2 f_0^{(2)}$, $2\delta^2 f_0^{(2)}$ and $\frac{9}{2}\delta^2 f_0^{(2)}$, respectively, on the basis of boundary conditions for velocity and vorticity. The third step is to obtain the value of f_j for $j \geq 4$ by iteratively using the discretized basic equation, and the final step is to fix the value of $f_0^{(2)}$ so as to satisfy the remaining boundary condition, $f^{(1)} \to 1(\eta \to \infty)$.

Considering that the value of $f_0^{(2)}$ is directly related to the shear force on the plate's surface, these aspects of the one-way calculation seem to reflect a given nature of a flow field, that is, the velocity distribution in a vicinity of a surface is exactly corresponding to a flow field far from the surface through the basic equation governing the flow field. We may become capable of obtaining the velocity distribution on the basis of the equation of vorticity transport and the boundary conditions for velocity and vorticity without the help of the matching procedure.

It may be meaningful to point out here that the same values, $\frac{1}{2}\delta^2 f_0^{(2)}$, $2\delta^2 f_0^{(2)}$ and $\frac{9}{2}\delta^2 f_0^{(2)}$ for f_1, f_2 and f_3, respectively, can also be obtained as approximated values with relative error $O\delta$, by applying (24), (25) and (26) for $j = 0$. Then, it is mathematically possible without a help of the boundary condition for vorticity to take the same simple calculation route as that taken in this work. The reason why no one has tried this before is that the values of f_1, f_2 and f_3 thus given are involving $O\delta$ relative error, which does not fit the following more precise calculation with $O\delta^2$ relative error. The boundary condition for vorticity made us believe the values, $\frac{1}{2}\delta^2 f_0^{(2)}$, $2\delta^2 f_0^{(2)}$ and $\frac{9}{2}\delta^2 f_0^{(2)}$ respectively for f_1, f_2 and f_3 are with $O\delta^2$ relative error, and encouraged us to take a simple and straightforward calculation route from surface to infinity, with $O\delta^2$ relative error.

6. Concluding Remarks

By admitting the boundary condition for vorticity, the following results have been obtained.

1) Blasius's equation, which is a basic equation for a flow in a boundary layer on a flat plate, is a mathematical consequence of the equation of vorticity transport for a case where $\alpha = \frac{\nu}{xU_\infty} \ll 1$.

2) A velocity field in a boundary layer can be easily obtained by one-way calculation from the surface of the plate to infinity, without the "matching" procedure between an analysis from surface to infinity and that from infinity to surface.

3) Numerical results for Blasius's equation excellently agreed with Howarth's results [2] cited in Boundary Layer Theory [8].

4) When $\alpha = 0$, the numerical solution of the equation of vorticity transport well coincides with those for Blasius's equation.

5) Velocity profiles for $\alpha = 0.1$ and $\alpha = 1.0$, which are velocity profiles in the vicinity of the front edge of the plate, were obtained.

It has been confirmed that the boundary condition for vorticity raises the precision of a preliminarily given velocity profile near an interface and considerably improves numerical calculations as shown in this work for Blasius's equation and the equation of vorticity transport. As for the analytical approach, the boundary condition for vorticity played a conclusive role in determining the values of integral constants in a general solution representing a flow field surrounding a fluid particle set in a simple shear flow [1].

The boundary condition for vorticity is expected to help us in investigating flow fields under the influence of surfaces of the container, conduit, rotating blade, dispersed particles and so on. It should also play an important role in pursuing a shape of deforming interface between two fluids. Furthermore, a new approach may be developed in the near future, where a velocity distribution is obtained first as a solution to the equation of vorticity transport by using boundary conditions for velocity and vorticity and a pressure field next by introducing the velocity distribution into the Navier-Stokes equation.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix

Basic relations

$$\eta = y\left(\frac{U_\infty}{\nu x}\right)^{0.5} \tag{A1}$$

$$\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial x}\left\{y\left(\frac{U_\infty}{\nu x}\right)^{0.5}\right\} = -\frac{1}{2}\frac{1}{x}\eta \tag{A2}$$

$$\frac{\partial \eta}{\partial y} = \frac{\partial}{\partial y}\left\{y\left(\frac{U_\infty}{\nu x}\right)^{0.5}\right\} = \left(\frac{U_\infty}{\nu x}\right)^{0.5} \tag{A3}$$

(2)

$$\begin{aligned} u(x,y) &= \frac{\partial}{\partial y}\psi = \frac{\partial}{\partial y}\left\{(\nu x U_\infty)^{0.5} f(t,\eta)\right\} = (\nu x U_\infty)^{0.5}\frac{\mathrm{d}f}{\mathrm{d}\eta}\frac{\partial \eta}{\partial y} \\ &= (\nu x U_\infty)^{0.5} f^{(1)}\frac{\partial}{\partial y}\left\{y\left(\frac{U_\infty}{\nu x}\right)^{0.5}\right\} = (\nu x U_\infty)^{0.5}\left(\frac{U_\infty}{\nu x}\right)^{0.5} f^{(1)} = U_\infty f^{(1)} \end{aligned} \tag{2}$$

(3)

$$\begin{aligned} v(x,y) &= -\frac{\partial}{\partial x}\psi = -\frac{\partial}{\partial x}\left\{(\nu x U_\infty)^{0.5} f(t,\eta)\right\} \\ &= -f\frac{\partial}{\partial x}(\nu x U_\infty)^{0.5} - (\nu x U_\infty)^{0.5}\frac{\mathrm{d}f}{\mathrm{d}\eta}\frac{\partial \eta}{\partial x} \\ &= -f(\nu U_\infty)^{0.5}\frac{1}{2}x^{-0.5} - (\nu x U_\infty)^{0.5} f^{(1)}\frac{\partial}{\partial x}\left\{y\left(\frac{U_\infty}{\nu x}\right)^{0.5}\right\} \\ &= -f(\nu U_\infty)^{0.5}\frac{1}{2}x^{-0.5} - (\nu x U_\infty)^{0.5} y\left(\frac{U_\infty}{\nu}\right)^{0.5} f^{(1)}\left(-\frac{1}{2}x^{-1.5}\right) \\ &= -f(\nu U_\infty)^{0.5}\frac{1}{2}x^{-0.5} + (\nu U_\infty)^{0.5}\eta f^{(1)}\frac{1}{2}x^{-0.5} \\ &= -f\left(\frac{\nu U_\infty}{x}\right)^{0.5}\frac{1}{2} + \left(\frac{\nu U_\infty}{x}\right)^{0.5}\eta f^{(1)}\frac{1}{2} = \frac{1}{2}\left(\frac{\nu U_\infty}{x}\right)^{0.5}\left(\eta f^{(1)} - f\right) \end{aligned} \tag{3}$$

(4)

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \rho\left[\nabla\cdot\left\{(\nabla\times\boldsymbol{u})\boldsymbol{u}\right\} - \nabla\cdot\left\{\boldsymbol{u}(\nabla\times\boldsymbol{u})\right\}\right] + \mu\nabla\cdot\left[\nabla(\nabla\times\boldsymbol{u})\right] \tag{1}$$

$$\begin{aligned} \boldsymbol{\omega} = \nabla\times(\rho\boldsymbol{u}) &= \left(\boldsymbol{i}_X\frac{\partial}{\partial x} + \boldsymbol{i}_Y\frac{\partial}{\partial y}\right)\times\left(\boldsymbol{i}_X\rho u + \boldsymbol{i}_Y\rho v\right) \\ &= \boldsymbol{i}_X\frac{\partial}{\partial x}\times(\boldsymbol{i}_Y\rho v) + \boldsymbol{i}_Y\frac{\partial}{\partial y}\times(\boldsymbol{i}_X\rho u) \\ &= \boldsymbol{i}_Z\frac{\partial}{\partial x}(\rho v) - \boldsymbol{i}_Z\frac{\partial}{\partial y}(\rho u) \\ &= \boldsymbol{i}_Z\left\{\frac{\partial}{\partial x}(\rho v) - \frac{\partial}{\partial y}(\rho u)\right\} \end{aligned}$$

Hence; $$\frac{\partial \boldsymbol{\omega}}{\partial t} = \rho\boldsymbol{i}_Z\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{A4}$$

$$\frac{\partial v}{\partial x}=\frac{\partial}{\partial x}\left\{\frac{1}{2}\left(\frac{\nu U_\infty}{x}\right)^{0.5}\left(\eta f^{(1)}-f\right)\right\}$$

$$=-\frac{1}{4}\left(\nu U_\infty\right)^{0.5}x^{-1.5}\left(\eta f^{(1)}-f\right)-\frac{1}{4}\left(\frac{U_\infty}{\nu x}\right)^{0.5}U_\infty\frac{\nu}{xU_\infty}\eta^2 f^{(2)} \quad \text{(A5)}$$

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left\{U_\infty f^{(1)}\right\}=U_\infty\left(\frac{U_\infty}{\nu x}\right)^{0.5}f^{(2)} \quad \text{(A6)}$$

Hence;

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\left(\frac{U_\infty}{\nu x}\right)^{0.5}U_\infty\left\{-\frac{1}{4}\frac{\nu}{xU_\infty}\left(\eta f^{(1)}-f\right)-\left(\frac{1}{4}\frac{\nu}{xU_\infty}\eta^2+1\right)f^{(2)}\right\} \quad \text{(A7)}$$

By introducing Equation (A7) into Equation (A4);

$$\frac{\partial\omega}{\partial t}=\rho\left(\frac{U_\infty}{\nu x}\right)^{0.5}U_\infty\boldsymbol{i}_Z\frac{\partial}{\partial t}\left[\left\{-\frac{1}{4}\alpha\left(\eta f^{(1)}-f\right)-\left(\frac{1}{4}\alpha\eta^2+1\right)f^{(2)}\right\}\right] \quad \text{(A8)}$$

Here, $\alpha=\frac{\nu}{xU_\infty}$.

$$\nabla\times\boldsymbol{u}=\boldsymbol{i}_Z\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

$$\left(\nabla\times\boldsymbol{u}\right)\boldsymbol{u}=\left\{\boldsymbol{i}_Z\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right\}\left(\boldsymbol{i}_X u+\boldsymbol{i}_Y v\right)=\boldsymbol{i}_Z\boldsymbol{i}_X u\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\boldsymbol{i}_Z\boldsymbol{i}_Y v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

$$\boldsymbol{u}\left(\nabla\times\boldsymbol{u}\right)=\boldsymbol{i}_X\boldsymbol{i}_Z u\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\boldsymbol{i}_Y\boldsymbol{i}_Z v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

Hence;

$$\left(\nabla\times\boldsymbol{u}\right)\boldsymbol{u}-\boldsymbol{u}\left(\nabla\times\boldsymbol{u}\right)=\left(\boldsymbol{i}_Z\boldsymbol{i}_X-\boldsymbol{i}_X\boldsymbol{i}_Z\right)u\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\left(\boldsymbol{i}_Z\boldsymbol{i}_Y-\boldsymbol{i}_Y\boldsymbol{i}_Z\right)v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

Then;

$$\nabla\cdot\left\{\left(\nabla\times\boldsymbol{u}\right)\boldsymbol{u}-\boldsymbol{u}\left(\nabla\times\boldsymbol{u}\right)\right\}$$

$$=\left(\boldsymbol{i}_X\frac{\partial}{\partial x}+\boldsymbol{i}_Y\frac{\partial}{\partial y}\right)\cdot\left\{\left(\boldsymbol{i}_Z\boldsymbol{i}_X-\boldsymbol{i}_X\boldsymbol{i}_Z\right)u\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\left(\boldsymbol{i}_Z\boldsymbol{i}_Y-\boldsymbol{i}_Y\boldsymbol{i}_Z\right)v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right\}$$

$$=-\boldsymbol{i}_Z\left[u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right] \quad \text{(A9)}$$

By introducing (2), (3) and (A7) into (A9);

$$\nabla\cdot\left\{\left(\nabla\times\boldsymbol{u}\right)\boldsymbol{u}-\boldsymbol{u}\left(\nabla\times\boldsymbol{u}\right)\right\}$$

$$=-\boldsymbol{i}_Z\left[U_\infty f^{(1)}\frac{\partial}{\partial x}\left[\left(\frac{U_\infty}{\nu x}\right)^{0.5}U_\infty\left\{-\frac{1}{4}\frac{\nu}{xU_\infty}\left(\eta f^{(1)}-f\right)-\left(\frac{1}{4}\frac{\nu}{xU_\infty}\eta^2+1\right)f^{(2)}\right\}\right]\right. \quad \text{(A10)}$$

$$\left.+\frac{1}{2}\left(\frac{\nu U_\infty}{x}\right)^{0.5}\left(\eta f^{(1)}-f\right)\frac{\partial}{\partial y}\left[\left(\frac{U_\infty}{\nu x}\right)^{0.5}U_\infty\left\{-\frac{1}{4}\frac{\nu}{xU_\infty}\left(\eta f^{(1)}-f\right)\right.\right.\right.$$

$$\left.\left.\left.-\left(\frac{1}{4}\frac{\nu}{xU_\infty}\eta^2+1\right)f^{(2)}\right\}\right]\right]$$

$$\frac{\partial}{\partial x}\left[\left(\frac{U_\infty}{\nu x}\right)^{0.5}U_\infty\left\{-\frac{1}{4}\frac{\nu}{xU_\infty}\left(\eta f^{(1)}-f\right)-\left(\frac{1}{4}\frac{\nu}{xU_\infty}\eta^2+1\right)f^{(2)}\right\}\right]$$

$$=\frac{\partial}{\partial x}\left[-\frac{1}{4}\frac{\nu}{xU_\infty}\left(\frac{U_\infty}{\nu x}\right)^{0.5}U_\infty\left(\eta f^{(1)}-f\right)-\frac{1}{4}\frac{\nu}{x}\left(\frac{U_\infty}{\nu x}\right)^{0.5}\eta^2 f^{(2)}\right.$$

$$\left.-\left(\frac{U_\infty}{\nu x}\right)^{0.5}U_\infty f^{(2)}\right]$$

$$=\left\{-\frac{1}{4}\frac{\partial}{\partial x}\left\{\frac{\nu}{x}\left(\frac{U_\infty}{\nu x}\right)^{0.5}\right\}\left(\eta f^{(1)}-f+\eta^2 f^{(2)}\right)\right.$$

$$-\frac{1}{4}\frac{\nu}{x}\left(\frac{U_\infty}{\nu x}\right)^{0.5}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta f^{(1)}-f+\eta^2 f^{(2)}\right)\frac{\partial\eta}{\partial x}$$

$$\left.-\frac{\partial}{\partial x}\left(\frac{U_\infty}{\nu x}\right)^{0.5}U_\infty f^{(2)}-\left(\frac{U_\infty}{\nu x}\right)^{0.5}U_\infty\frac{\mathrm{d}}{\mathrm{d}\eta}\left(f^{(2)}\right)\frac{\partial\eta}{\partial x}\right\}$$

$$=\frac{1}{8}\left(\frac{\nu U_\infty}{x}\right)^{0.5}\frac{1}{x^2}\left(-3f+3\eta f^{(1)}+6\eta^2 f^{(2)}+\eta^3 f^{(3)}\right) \tag{A11}$$

$$+\frac{1}{2}\left(\frac{U_\infty}{\nu x}\right)^{0.5}\frac{1}{x}U_\infty\left(f^{(2)}+\eta f^{(3)}\right)$$

$$\frac{\partial}{\partial y}\left[\left(\frac{U_\infty}{\nu x}\right)^{0.5}U_\infty\left\{-\frac{1}{4}\alpha\left(\eta f^{(1)}-f\right)-\left(\frac{1}{4}\alpha\eta^2+1\right)f^{(2)}\right\}\right]$$

$$=\frac{1}{4}\frac{\partial}{\partial y}\left\{\frac{\nu}{x}\left(\frac{U_\infty}{\nu x}\right)^{0.5}\right\}\left(-\eta f^{(1)}+f-\eta^2 f^{(2)}\right)$$

$$+\frac{1}{4}\frac{\nu}{x}\left(\frac{U_\infty}{\nu x}\right)^{0.5}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(-\eta f^{(1)}+f-\eta^2 f^{(2)}\right)\frac{\partial\eta}{\partial y}$$

$$-\frac{\partial}{\partial y}\left\{\left(\frac{U_\infty}{\nu x}\right)^{0.5}\right\}U_\infty f^{(2)}-\left(\frac{U_\infty}{\nu x}\right)^{0.5}U_\infty\frac{\partial}{\partial\eta}\left(f^{(2)}\right)\frac{\partial\eta}{\partial y}$$

$$=-\frac{U_\infty}{\nu x}U_\infty\left\{\frac{3}{4}\alpha\eta f^{(2)}+\left(\frac{1}{4}\alpha\eta^2+1\right)f^{(3)}\right\} \tag{A12}$$

Here, (A2) and (A3) have been used.

By introducing (A11) and (A12) into (A10);

$$\nabla\cdot\left\{(\nabla\times\boldsymbol{u})\boldsymbol{u}-\boldsymbol{u}(\nabla\times\boldsymbol{u})\right\}$$

$$=-\boldsymbol{i}_Z\left[\!\left[U_\infty f^{(1)}\frac{\partial}{\partial x}\left[\left(\frac{U_\infty}{\nu x}\right)^{0.5}U_\infty\left\{-\frac{1}{4}\frac{\nu}{xU_\infty}\left(\eta f^{(1)}-f\right)-\left(\frac{1}{4}\frac{\nu}{xU_\infty}\eta^2+1\right)f^{(2)}\right\}\right]\right.\right.$$

$$+\frac{1}{2}\left(\frac{\nu U_\infty}{x}\right)^{0.5}\left(\eta f^{(1)}-f\right)\frac{\partial}{\partial y}\left[\left(\frac{U_\infty}{\nu x}\right)^{0.5}U_\infty\left\{-\frac{1}{4}\frac{\nu}{xU_\infty}\left(\eta f^{(1)}-f\right)\right.\right.$$

$$\left.\left.\left.\left.-\left(\frac{1}{4}\frac{\nu}{xU_\infty}\eta^2+1\right)f^{(2)}\right\}\right]\right]\!\right]$$

$$=-\boldsymbol{i}_Z\left[\!\left[U_\infty f^{(1)}\left[\frac{1}{8}\left(\frac{\nu U_\infty}{x}\right)^{0.5}\frac{1}{x^2}\left(-3f+3\eta f^{(1)}+6\eta^2 f^{(2)}+\eta^3 f^{(3)}\right)\right.\right.\right.$$

$$+\frac{1}{2}\left(\frac{U_\infty}{\nu x}\right)^{0.5}\frac{1}{x}U_\infty\left(f^{(2)}+\eta f^{(3)}\right)\Bigg]$$

$$+\frac{1}{2}\left(\frac{\nu U_\infty}{x}\right)^{0.5}\left(\eta f^{(1)}-f\right)\left[-\frac{U_\infty}{\nu x}U_\infty\left\{\frac{3}{4}\alpha\eta f^{(2)}+\left(\frac{1}{4}\alpha\eta^2+1\right)f^{(3)}\right\}\right]\Bigg]$$

$$=\boldsymbol{i}_Z\left(\frac{U_\infty}{\nu x}\right)^{0.5}\frac{U_\infty^2}{x}\left\{\frac{3}{8}\alpha ff^{(1)}-\frac{3}{8}\alpha\eta ff^{(2)}-\frac{1}{2}\left(\frac{1}{4}\alpha\eta^2+1\right)ff^{(3)}\right.$$
$$\left.-\frac{3}{8}\alpha\eta f^{(1)}f^{(1)}+\left(-\frac{3}{8}\alpha\eta^2-\frac{1}{2}\right)f^{(1)}f^{(2)}\right\} \tag{A13}$$

$$\nabla(\nabla\times\boldsymbol{u})=\left(\boldsymbol{i}_X\frac{\partial}{\partial x}+\boldsymbol{i}_Y\frac{\partial}{\partial y}\right)\boldsymbol{i}_Z\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$
$$=\boldsymbol{i}_X\boldsymbol{i}_Z\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\boldsymbol{i}_Y\boldsymbol{i}_Z\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) \tag{A14}$$

From (A7);

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$
$$=\frac{\partial}{\partial x}\left\{\left(\frac{U_\infty}{\nu x}\right)^{0.5}U_\infty\left\{-\frac{1}{4}\frac{\nu}{xU_\infty}\left(\eta f^{(1)}-f\right)-\left(\frac{1}{4}\frac{\nu}{xU_\infty}\eta^2+1\right)f^{(2)}\right\}\right\}$$
$$=\left[-\frac{1}{4}\frac{\partial}{\partial x}\left\{\frac{\nu}{xU_\infty}\left(\frac{U_\infty}{\nu x}\right)^{0.5}\right\}U_\infty\left(\eta f^{(1)}-f\right)\right.$$
$$-\frac{1}{4}\frac{\nu}{xU_\infty}\left(\frac{U_\infty}{\nu x}\right)^{0.5}U_\infty\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta f^{(1)}-f\right)\frac{\partial\eta}{\partial x}$$
$$-\frac{1}{4}\frac{\partial}{\partial x}\left\{\frac{\nu}{xU_\infty}\left(\frac{U_\infty}{\nu x}\right)^{0.5}\right\}U_\infty\eta^2 f^{(2)}$$
$$-\frac{1}{4}\frac{\nu}{xU_\infty}\left(\frac{U_\infty}{\nu x}\right)^{0.5}U_\infty\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta^2 f^{(2)}\right)\frac{\partial\eta}{\partial x}$$
$$\left.-\frac{\partial}{\partial x}\left\{\left(\frac{U_\infty}{\nu x}\right)^{0.5}\right\}U_\infty f^{(2)}-\left(\frac{U_\infty}{\nu x}\right)^{0.5}U_\infty\frac{\mathrm{d}}{\mathrm{d}\eta}\left(f^{(2)}\right)\frac{\partial\eta}{\partial x}\right]$$
$$=-\frac{3}{8}\left(\frac{\nu}{U_\infty x}\right)^{0.5}\frac{1}{x^2}U_\infty f+\frac{3}{8}\left(\frac{\nu}{U_\infty x}\right)^{0.5}\frac{1}{x^2}U_\infty\eta f^{(1)} \tag{A15}$$
$$+\left\{\frac{3}{4}\left(\frac{\nu U_\infty}{x}\right)^{0.5}\frac{1}{x^2}\eta^2+\frac{1}{2}\left(\frac{U_\infty}{\nu x}\right)^{0.5}\frac{1}{x}U_\infty\right\}f^{(2)}$$
$$+\left\{\frac{1}{8}\left(\frac{\nu U_\infty}{x}\right)^{0.5}\frac{1}{x^2}\eta^3+\frac{1}{2}\left(\frac{U_\infty}{\nu x}\right)^{0.5}\frac{1}{x}U_\infty\eta\right\}f^{(3)}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$
$$=\frac{\partial}{\partial y}\left\{\left(\frac{U_\infty}{\nu x}\right)^{0.5}U_\infty\left\{-\frac{1}{4}\frac{\nu}{xU_\infty}\left(\eta f^{(1)}-f\right)-\left(\frac{1}{4}\frac{\nu}{xU_\infty}\eta^2+1\right)f^{(2)}\right\}\right\}$$

$$=\left\{-\frac{1}{4}\frac{\nu}{xU_\infty}\left(\frac{U_\infty}{\nu x}\right)^{0.5}U_\infty\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta f^{(1)}-f\right)\frac{\partial\eta}{\partial y}\right.$$

$$\left.-\frac{1}{4}\frac{\nu}{xU_\infty}\left(\frac{U_\infty}{\nu x}\right)^{0.5}U_\infty\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta^2 f^{(2)}\right)\frac{\partial\eta}{\partial y}-\left(\frac{U_\infty}{\nu x}\right)^{0.5}U_\infty\frac{\mathrm{d}}{\mathrm{d}\eta}\left(f^{(2)}\right)\frac{\partial\eta}{\partial y}\right\}$$

$$=\frac{U_\infty}{\nu x}U_\infty\left\{-\frac{3}{4}\alpha\eta f^{(2)}-\left(\frac{1}{4}\alpha\eta^2+1\right)f^{(3)}\right\} \tag{A16}$$

Here, (A2) and (A3) have been used.

By introducing (A15) and (A16) into (A14);

$$\nabla(\nabla\times\boldsymbol{u})$$

$$=\left(\boldsymbol{i}_X\frac{\partial}{\partial x}+\boldsymbol{i}_Y\frac{\partial}{\partial y}\right)\boldsymbol{i}_Z\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=\boldsymbol{i}_X\boldsymbol{i}_Z\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\boldsymbol{i}_Y\boldsymbol{i}_Z\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

$$=\boldsymbol{i}_X\boldsymbol{i}_Z\left[-\frac{3}{8}\left(\frac{\nu}{U_\infty x}\right)^{0.5}\frac{1}{x^2}U_\infty f+\frac{3}{8}\left(\frac{\nu}{U_\infty x}\right)^{0.5}\frac{1}{x^2}U_\infty\eta f^{(1)}\right. \tag{A17}$$

$$+\left\{\frac{3}{4}\left(\frac{\nu U_\infty}{x}\right)^{0.5}\frac{1}{x^2}\eta^2+\frac{1}{2}\left(\frac{U_\infty}{\nu x}\right)^{0.5}\frac{1}{x}U_\infty\right\}f^{(2)}$$

$$\left.+\left\{\frac{1}{8}\left(\frac{\nu U_\infty}{x}\right)^{0.5}\frac{1}{x^2}\eta^3+\frac{1}{2}\left(\frac{U_\infty}{\nu x}\right)^{0.5}\frac{1}{x}U_\infty\eta\right\}f^{(3)}\right]$$

$$+\boldsymbol{i}_Y\boldsymbol{i}_Z\left[\frac{U_\infty}{\nu x}U_\infty\left\{-\frac{3}{4}\alpha\eta f^{(2)}-\left(\frac{1}{4}\alpha\eta^2+1\right)f^{(3)}\right\}\right]$$

Then;

$$\nabla\cdot\left[\nabla(\nabla\times\boldsymbol{u})\right]$$

$$=\left(\boldsymbol{i}_X\frac{\partial}{\partial x}+\boldsymbol{i}_Y\frac{\partial}{\partial y}\right)\cdot\left[\!\left[\boldsymbol{i}_X\boldsymbol{i}_Z\left[-\frac{3}{8}\left(\frac{\nu}{U_\infty x}\right)^{0.5}\frac{1}{x^2}U_\infty f+\frac{3}{8}\left(\frac{\nu}{U_\infty x}\right)^{0.5}\frac{1}{x^2}U_\infty\eta f^{(1)}\right.\right.\right.$$

$$+\left\{\frac{3}{4}\left(\frac{\nu U_\infty}{x}\right)^{0.5}\frac{1}{x^2}\eta^2+\frac{1}{2}\left(\frac{U_\infty}{\nu x}\right)^{0.5}\frac{1}{x}U_\infty\right\}f^{(2)}$$

$$\left.+\left\{\frac{1}{8}\left(\frac{\nu U_\infty}{x}\right)^{0.5}\frac{1}{x^2}\eta^3+\frac{1}{2}\left(\frac{U_\infty}{\nu x}\right)^{0.5}\frac{1}{x}U_\infty\eta\right\}f^{(3)}\right]$$

$$\left.\left.+\boldsymbol{i}_Y\boldsymbol{i}_Z\left[\frac{U_\infty}{\nu x}U_\infty\left\{-\frac{3}{4}\alpha\eta f^{(2)}-\left(\frac{1}{4}\alpha\eta^2+1\right)f^{(3)}\right\}\right]\right]\!\right]$$

$$=\boldsymbol{i}_Z\frac{\partial}{\partial x}\left[-\frac{3}{8}\left(\frac{\nu}{U_\infty x}\right)^{0.5}\frac{1}{x^2}U_\infty f+\frac{3}{8}\left(\frac{\nu}{U_\infty x}\right)^{0.5}\frac{1}{x^2}U_\infty\eta f^{(1)}\right.$$

$$+\left\{\frac{3}{4}\left(\frac{\nu U_\infty}{x}\right)^{0.5}\frac{1}{x^2}\eta^2+\frac{1}{2}\left(\frac{U_\infty}{\nu x}\right)^{0.5}\frac{1}{x}U_\infty\right\}f^{(2)}$$

$$\left.+\left\{\frac{1}{8}\left(\frac{\nu U_\infty}{x}\right)^{0.5}\frac{1}{x^2}\eta^3+\frac{1}{2}\left(\frac{U_\infty}{\nu x}\right)^{0.5}\frac{1}{x}U_\infty\eta\right\}f^{(3)}\right]$$

$$+\boldsymbol{i}_Z\frac{\partial}{\partial y}\left[\frac{U_\infty}{\nu x}U_\infty\left\{-\frac{3}{4}\alpha\eta f^{(2)}-\left(\frac{1}{4}\alpha\eta^2+1\right)f^{(3)}\right\}\right]$$

$$= \boldsymbol{i}_Z \left[-\frac{3}{8}\left\{ -\frac{5}{2}\left(\frac{\nu}{U_\infty x}\right)^{0.5} \frac{1}{x^3} \right\} U_\infty f - \left(\frac{\nu}{U_\infty x}\right)^{0.5} \frac{1}{x^2} U_\infty \frac{\mathrm{d}}{\mathrm{d}\eta}(f) \frac{\partial \eta}{\partial x} \right.$$

$$+ \frac{3}{8}\left\{ -\frac{5}{2}\left(\frac{\nu}{U_\infty x}\right)^{0.5} \frac{1}{x^3} \right\} U_\infty \eta f^{(1)} + \frac{3}{8}\left(\frac{\nu}{U_\infty x}\right)^{0.5} \frac{1}{x^2} U_\infty \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta f^{(1)}\right) \frac{\partial \eta}{\partial x}$$

$$+ \frac{3}{4}\left\{ -\frac{5}{2}\left(\frac{\nu U_\infty}{x}\right)^{0.5} \frac{1}{x^3} \right\} \eta^2 f^{(2)} + \frac{3}{4}\left(\frac{\nu U_\infty}{x}\right)^{0.5} \frac{1}{x^2} \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta^2 f^{(2)}\right) \frac{\partial \eta}{\partial x}$$

$$+ \frac{1}{2}\left\{ -\frac{3}{2}\left(\frac{U_\infty}{\nu x}\right)^{0.5} \frac{1}{x^2} \right\} U_\infty f^{(2)} + \frac{1}{2}\left(\frac{U_\infty}{\nu x}\right)^{0.5} \frac{1}{x} U_\infty \frac{\mathrm{d}}{\mathrm{d}\eta}\left(f^{(2)}\right) \frac{\partial \eta}{\partial x}$$

$$+ \frac{1}{8}\left\{ -\frac{5}{2}\left(\frac{\nu U_\infty}{x}\right)^{0.5} \frac{1}{x^3} \right\} \eta^3 f^{(3)} + \frac{1}{8}\left(\frac{\nu U_\infty}{x}\right)^{0.5} \frac{1}{x^2} \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta^3 f^{(3)}\right) \frac{\partial \eta}{\partial x}$$

$$\left. + \frac{1}{2}\frac{\partial}{\partial x}\left\{ \left(\frac{U_\infty}{\nu x}\right)^{0.5} \frac{1}{x} \right\} U_\infty \eta f^{(3)} + \frac{1}{2}\left(\frac{U_\infty}{\nu x}\right)^{0.5} \frac{1}{x} U_\infty \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta f^{(3)}\right) \frac{\partial \eta}{\partial x} \right]$$

$$+ \frac{U_\infty}{\nu x} U_\infty \left\{ -\frac{3}{4}\alpha \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta f^{(2)}\right) \frac{\partial \eta}{\partial y} - \frac{1}{4}\alpha \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta^2 f^{(3)}\right) \frac{\partial \eta}{\partial y} \right.$$

$$\left. \left. - \left(\frac{1}{4}\alpha\eta^2 + 1\right) \frac{\mathrm{d}}{\mathrm{d}\eta}\left(f^{(3)}\right) \frac{\partial \eta}{\partial y} \right\} \right]$$

$$= i_Z \frac{1}{\nu}\left(\frac{U_\infty}{\nu x}\right)^{0.5} \frac{U_\infty^2}{x} \left\{ \alpha^2 \left(\frac{15}{16} f - \frac{15}{16}\eta f^{(1)} - \frac{45}{16}\eta^2 f^{(2)} - \frac{7}{8}\eta^3 f^{(3)} \right. \right. \quad \text{(A18)}$$

$$\left. \left. - \frac{1}{16}\eta^4 f^{(4)} \right) + \alpha\left(-\frac{3}{2} f^{(2)} - \frac{5}{2}\eta f^{(3)} - \frac{1}{2}\eta^2 f^{(4)} \right) - f^{(4)} \right\}$$

By introducing (A8), (A13) and (A18) into (1);

$$\rho\left(\frac{U_\infty}{\nu x}\right)^{0.5} U_\infty \boldsymbol{i}_Z \frac{\partial}{\partial t}\left[-\frac{1}{4}\alpha\left(\eta f^{(1)} - f\right) - \left(\frac{1}{4}\alpha\eta^2 + 1\right) f^{(2)} \right]$$

$$= \boldsymbol{i}_Z \rho\left(\frac{U_\infty}{\nu x}\right)^{0.5} \frac{U_\infty^2}{x} \left[\!\left[\frac{3}{8}\alpha f f^{(1)} - \frac{3}{8}\alpha\eta f f^{(2)} - \frac{1}{2}\left(\frac{1}{4}\alpha\eta^2 + 1\right) f f^{(3)} \right.\right.$$

$$\left.\left. - \frac{3}{8}\alpha\eta f^{(1)} f^{(1)} - \frac{3}{8}\alpha\eta^2 f^{(1)} f^{(2)} - \frac{1}{2} f^{(1)} f^{(2)} \right]\!\right]$$

$$+ \boldsymbol{i}_Z \frac{\mu}{\nu}\left(\frac{U_\infty}{\nu x}\right)^{0.5} \frac{U_\infty^2}{x} \left[\!\left[\alpha^2 \left(\frac{15}{16} f - \frac{15}{16}\eta f^{(1)} - \frac{45}{16}\eta^2 f^{(2)} - \frac{7}{8}\eta^3 f^{(3)} \right.\right.\right.$$

$$\left.\left.\left. - \frac{1}{16}\eta^4 f^{(4)} \right) + \alpha\left(-\frac{3}{2} f^{(2)} - \frac{5}{2}\eta f^{(3)} - \frac{1}{2}\eta^2 f^{(4)} \right) - f^{(4)} \right]\!\right]$$

Then;

$$\frac{x}{U_\infty}\frac{\partial}{\partial t}\left[-\frac{1}{4}\alpha\left(\eta f^{(1)} - f\right) - \left(\frac{1}{4}\alpha\eta^2 + 1\right) f^{(2)} \right]$$

$$= \alpha^2 \left(\frac{15}{16} f - \frac{15}{16}\eta f^{(1)} - \frac{45}{16}\eta^2 f^{(2)} - \frac{7}{8}\eta^3 f^{(3)} - \frac{1}{16}\eta^4 f^{(4)} \right)$$

$$+ \alpha\left(\frac{3}{8} f f^{(1)} - \frac{3}{8}\eta f f^{(2)} - \frac{1}{8}\eta^2 f f^{(3)} - \frac{3}{8}\eta f^{(1)} f^{(1)} - \frac{3}{8}\eta^2 f^{(1)} f^{(2)} \right.$$

$$\left. - \frac{3}{2} f^{(2)} - \frac{5}{2}\eta f^{(3)} - \frac{1}{2}\eta^2 f^{(4)} \right) + \left(-\frac{1}{2} f f^{(3)} - \frac{1}{2} f^{(1)} f^{(2)} - f^{(4)} \right)$$

Hence, (4) is obtained.

$$\begin{aligned}&\frac{\partial}{\partial \xi}\left[-\frac{1}{4}\alpha\left(\eta f^{(1)}-f\right)-\left(\frac{1}{4}\alpha\eta^{2}+1\right)f^{(2)}\right]\\&=\alpha^{2}\left(\frac{15}{16}f-\frac{15}{16}\eta f^{(1)}-\frac{45}{16}\eta^{2}f^{(2)}-\frac{7}{8}\eta^{3}f^{(3)}-\frac{1}{16}\eta^{4}f^{(4)}\right)\\&\quad+\alpha\left(\frac{3}{8}ff^{(1)}-\frac{3}{8}\eta ff^{(2)}-\frac{1}{8}\eta^{2}ff^{(3)}-\frac{3}{8}\eta f^{(1)}f^{(1)}-\frac{3}{8}\eta^{2}f^{(1)}f^{(2)}\right.\\&\quad\left.-\frac{3}{2}f^{(2)}-\frac{5}{2}\eta f^{(3)}-\frac{1}{2}\eta^{2}f^{(4)}\right)+\left(-\frac{1}{2}ff^{(3)}-\frac{1}{2}f^{(1)}f^{(2)}-f^{(4)}\right)\end{aligned}\tag{4}$$

Here, $\alpha=\frac{\nu}{xU_{\infty}}$ and $\xi=\frac{U_{\infty}}{x}t$.