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Models and Monte Carlo Simulations of the Mean Sinuosity of Major Meandering Rivers

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Abstract

The purpose of this research is to investigate the sinuosity of major rivers in the United States and the world, and to compare them to that predicted by the existing theories. It is shown that the average sinuosity of meandering rivers deviates considerably from what has been reported previously as π. Calculations of the mean value of actual sinuosities of major rivers in the United States and in the World show that this average is very close to 2. Exact models as well as a Monte Carlo simulation for meandering rivers that is based on Gaussian probability distribution function are also presented, and the possibility of composite meandering is discussed.

Keywords

Meandering, River, Sinuosity, Simulation, Gaussian

1. Introduction

As a river or stream flows, as a result of various disturbances it normally does not flow in a straight path, but winds snakelike so that the curvilinear length (actual length along the curve) of the river L is longer than its Euclidean distance (straight end-to-end distance) D. This phenomenon is known as "meandering" [1].

The dynamics of meandering rivers have been a subject of interest to geologists and geographers alike. However, due to the complexity of the phenomenon and nonlinearity of the governing equations, understanding the process from theoretical and computational points of view has remained a challenge. Nevertheless, many computational models have been developed to simulate the dynamics of such rivers involving various assumptions and approximations [2]-[10].

Various aspects of meandering rivers have been subject of discussions for many years. For example, the fractal nature of these rivers has been suggested by Mandelbrot [11] and further studied by Snow [12], Montgomery [13], and Stolum [14]. Another characteristic of a river is its sinuosity s, defined as the ratio L/D,

$$s = \frac{L}{D} \tag{1}$$

where, clearly $s \geq 1$.

The actual profile of a river can have infinitely many shapes, and its sinuosity can have any value greater than or equal to unity. However, various models have been suggested to approximate the shape of a river. This includes circular and other types of smooth curves [15], as well as simulation models [16] [17] [18]. Based on the assumption of fractal geometry and computer simulations, it has been suggested that the mean sinuosity of rivers should have a value of π [18]. However, this result has not been verified by the actual sinuosities of rivers. In fact, a simple examination of the data reveals that, on the average, sinuosity of major rivers is substantially different from π.

In this article, we evaluate the actual sinuosities of major rivers in the United States and around the World, and calculate their average. We then suggest two Monte Carlo models, a parabolic and a zig-zag model, each involving a single-parameter Gaussian probability density function to calculate the average theoretical sinuosity of rivers. The parameter of the models is then adjusted to produce the observed mean sinuosity of the rivers in each case.

2. Observed Sinuosities of Rivers

Using the available data for the major US and World rivers, as well as the information obtained from the Google Maps, we calculate the sinuosity of each river, and then we find the average value of the sinuosities for each category.

2.1. Major US Rivers

Table 1 shows the curvilinear lengths, Euclidean distances, and the sinuosities of major rivers in the United States [19]. **Figure 1** shows sinuosity as a function of river number as listed in **Table 1**. Since curvilinear lengths of the rivers decrease with river number, we see from the figure that there is no correlation between curvilinear length and sinuosity of the rivers. The mean and the standard deviation of these sinuosities are $\langle s \rangle = 2.10 \pm 0.49$. This mean value is shown by the horizontal line in **Figure 1**.

2.2. Major World Rivers

Table 2 shows the curvilinear lengths, Euclidean distances, and the sinuosities of major rivers in the World, including some of those in the United States listed in **Table 1** [20]. The sinuosities of these rivers as a function of river number are shown in **Figure 2**. Since according to **Table 2**, curvilinear length of the rivers decrease with river number, **Figure 2** shows that again there is no correlation

Table 1. Major rivers of the United States and their lengths L, straight end-to-end distances D, and sinuosities L/D.

No.	Name	L (km)	D (km)	L/D
1	Missouri River	3768	1913	1.970
2	Mississippi River	3544	2041	1.736
3	Yukon River	3190	1681	1.898
4	Rio Grande	2830	1652	1.713
5	Colorado River	2330	1254	1.858
6	Arkansas River	2322	1490	1.558
7	Columbia River	2000	755	2.649
8	Red River	1811	864	2.096
9	Snake River	1674	726	2.306
10	Ohio River	1575	877	1.796
11	Colorado River of Texas	1560	710	2.197
12	Tennessee River	1504	441	3.410
13	Canadian River	1458	917	1.590
14	Brazos River	1390	660	2.106
15	Green River	1230	548	2.245
16	Pecos River	1175	794	1.480
17	White River	1159	312	3.715
18	James River	1140	550	2.073
19	Kuskokwim River	1130	532	2.124
20	Cimarron River	1123	606	1.853
21	Cumberland River	1120	452	2.478
22	Yellowstone River	1091	638	1.710
23	North Platte River	1070	483	2.215
24	Milk River	1005	504	1.994
25	Gila River	960	593	1.619
26	Sheyenne River	951	288	3.302
27	Tanana River	940	540	1.741
28	Smoky Hill River	927	497	1.865
29	Niobrara River	914	540	1.693
30	Little Missouri River	900	382	2.356
31	Sabine River	890	373	2.386
32	Red River of the North	890	464	1.918
33	Des Moines River	845	458	1.845
34	White River (Missouri River)	815	374	2.197
35	Trinity River	815	401	2.032
36	Wabash River	810	400	2.025

Table 2. Major rivers of the world and their curvilinear lengths L, Euclidean distances D, and sinuosities L/D.

No.	Name	L (km)	D (km)	L/D
1	Nile	6650	3619	1.838
2	Amazon	6400	3001	2.133
3	Yangtze	6300	2597	2.426
4	Yenisei	5539	2501	2.215
5	Yellow River	5464	2068	2.642
6	Congo-Chambeshi	4700	1500	3.133
7	Amur-Argun-Kherlen	4444	2285	1.945
8	Lena	4400	2235	1.969
9	Mekong	4350	2879	1.511
10	Mackenzie	4241	1225	3.462
11	Niger	4200	1178	3.565
12	Volga	3647	1669	2.185
13	Indus	3610	2136	1.690
14	Purus	3211	1384	2.320
15	Yukon	3185	1709	1.864
16	San Francisco	3180	821	3.873
17	Syr Darya-Naryn	3078	1051	2.929
18	Salween	3060	1920	1.594
19	Saint Lawrence	3058	1090	2.806
20	Rio Grande	3057	1658	1.844
21	Lower Tunguska	2989	837	3.571
22	Danube-Breg	2888	1677	1.722
23	Irrawddy River	2809	1455	1.931
24	Zambezi	2740	1526	1.796
25	Vilyuy	2720	1081	2.516
26	Ganges-Hooghly-Padma	2704	1448	1.867
27	Amu Darya-Panj	2620	1379	1.900
28	Japura	2615	2017	1.296
29	Paraguay	2549	1605	1.588
30	Kolyma	2513	999	2.516
31	Ishim	2450	723	3.389
32	Ural	2428	1582	1.535
33	Arkansas	2348	1484	1.582
34	Colorado (western US)	2333	1254	1.860
35	Olenyok	2292	806	2.844
36	Dnieper	2287	1056	2.166

Continued

37	Aldan	2273	824	2.758
38	Ubangi-Uele	2270	1292	1.757
39	Negro	2250	1263	1.781
40	Columbia	2250	755	2.980
41	Red (USA)	2188	866	2.527
42	Ohio-Allegheny	2102	1109	1.895
43	Orinoco	2101	715	2.938
44	Tarim	2100	967	2.172
45	Orange	2092	1167	1.793
46	Vitim	1978	637	3.105
47	Tigris	1950	1102	1.770
48	Don	1870	776	2.410
49	Pechora	1809	713	2.537
50	Limpopo	1800	731	2.462
51	Guapore	1749	721	2.426
52	Indigirka	1726	666	2.592
53	Snake	1670	731	2.285
54	Uruguay	1610	966	1.667
55	Churchill	1600	893	1.792
56	Tobol	1591	995	1.599
57	Alazeya	1590	909	1.749
58	Ica	1575	1139	1.383
59	Magdalena	1550	734	2.112
60	Han	1532	764	2.005
61	Kura	1515	581	2.608
62	Oka	1500	671	2.235
63	Guaviare	1497	1046	1.431
64	Pecos	1490	798	1.867
65	Murrumbidgee River	1485	501	2.964
66	Godavari	1465	930	1.575
67	Colorado (Texas)	1438	716	2.008
68	Upper Tocantins	1427	375	3.805
69	Belaya	1420	377	3.767
70	Dniester	1411	396	3.563
71	Benue	1400	1086	1.289
72	Fraser	1368	523	2.616
73	Lachlan River	1339	1088	1.231

Continued

74	Olyokma	1320	766	1.723
75	Krishna	1300	821	1.583
76	Narmada	1289	1004	1.284
77	Ottawa	1271	449	2.831
78	Rhine	1233	789	1.563
79	Athabasca	1231	880	1.399
80	Canadian	1223	732	1.671
81	Vaal	1210	566	2.138
82	Shire	1200	595	2.017
83	Ogooué (or Ogowe)	1200	586	2.048
84	Nen	1190	338	3.521
85	Green	1175	568	2.069
86	White	1162	340	3.418
87	Wu	1150	393	2.926
88	Red (Asia)	1149	1112	1.033
89	James (Dakotas)	1143	727	1.572
90	Kapuas	1143	476	2.401
91	Madre De Dios	1130	745	1.517
92	Tiete	1130	668	1.692
93	Sepik	1126	338	3.331
94	Cimarron	1123	605	1.856
95	Anadyr	1120	346	3.237
96	Liard	1115	554	2.013
97	Cumberland	1105	454	2.434
98	Gambia	1094	526	2.080
99	Chenab	1086	719	1.510
100	Yellowstone	1080	639	1.690
101	Ghaghara	1080	632	1.709
102	Aras	1072	579	1.851
103	Chu River	1067	715	1.492
104	Seversky Donets	1053	475	2.217
105	Fly	1050	472	2.225
106	Kuskokwim	1050	528	1.989
107	Tennessee	1049	439	2.390
108	Oder-Warta	1045	502	2.082
109	Aruwimi	1030	826	1.247
110	Daugava	1020	571	1.786

Continued

111	Gila	1015	599	1.694
112	Loire	1012	566	1.788
113	Essequibo	1010	586	1.724
114	Tagus	1006	658	1.529
115	Flinders River	1004	505	1.988

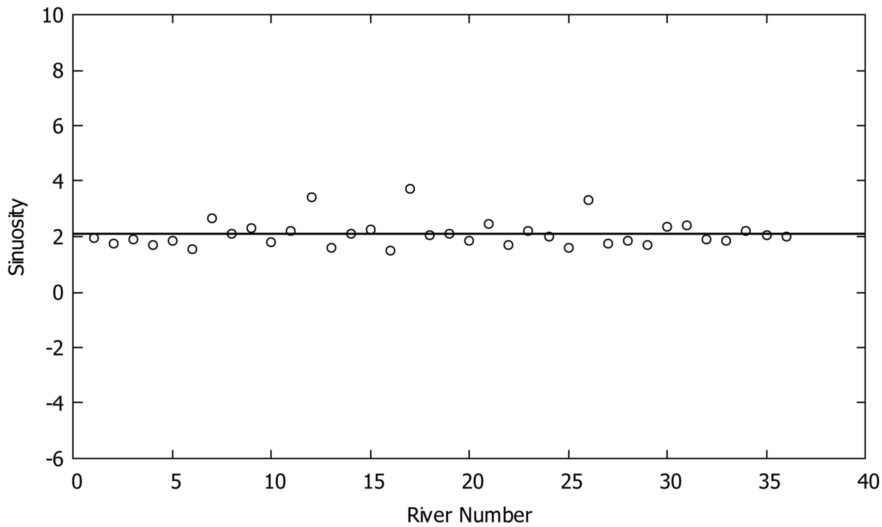

Figure 1. Sinuosities of major rivers in the United States as a function of river number (circles) according to **Table 1**. The horizontal line represents the mean value of the sinuosities.

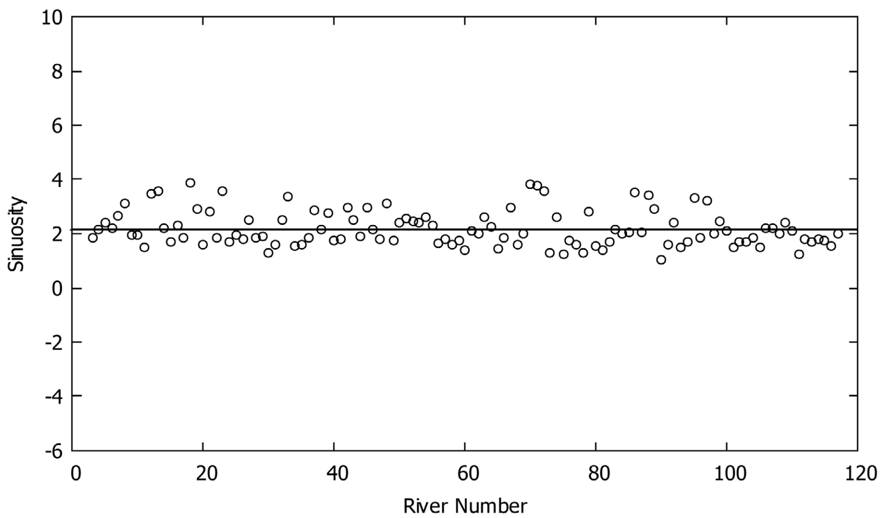

Figure 2. Sinuosities of major World rivers as a function of river number (circles) according to **Table 2**. The horizontal line represents the mean value of the sinuosities.

between the curvilinear length and the sinuosity of these rivers. The mean and the standard deviation of these sinuosities are $\langle s \rangle = 2.17 \pm 0.65$. This mean value is shown by the horizontal line in **Figure 2**.

3. Stochastic Models and Monte Carlo Simulations

3.1. Zig-Zag Paths

The zig-zag path of a meandering river is simulated by a Monte Carlo method [21], using random numbers drawn from a Gaussian (or normal) probability density function [22] [23],

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right] \tag{2}$$

with $\mu = 0$, and adjustable standard deviation σ.

We choose a river with straight end-to-end distance of 2000 km, and a unit length of 1 km. We draw random numbers according to the Gaussian probability density function, which can be done by either the Box-Muller method or the acceptance-rejection method [24]. These random numbers are taken to be the heights h in **Figure 3**. Then the length of each section of the zig-zag, and hence the total length of the river L is calculated. Finally, if D is the straight-line distance between the two ends of the river, the sinuosity of the river is calculated from

$$s = \frac{L}{D} \tag{3}$$

We repeat this Monte Carlo experiment 1000 times and calculate the average value of the sinuosity $\langle s \rangle$, and adjust the value of σ to obtain the observed value of the sinuosity. The results are shown in **Table 3** for the United States and World rivers along with the corresponding adjusted value of σ in each case.

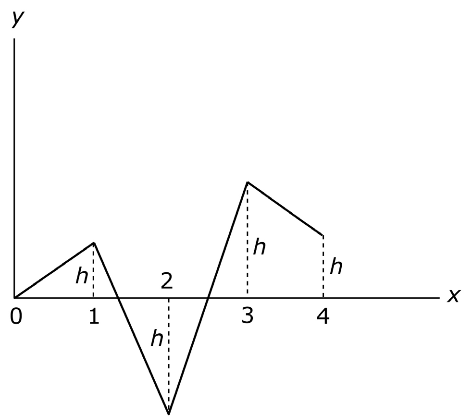

Figure 3. A zig-zag path model for meandering rivers.

Table 3. The observed and simulated (MC) values of sinuosities of the United States and World rivers. The simulated values are based on the zig-zag model with the Gaussian standard deviations σ shown.

Rivers	s (observed)	s (simulation)	σ (zig-zag)	σ (parabolic)
United States	2.10 ± 0.49	2.10	1.86	1.03
World	2.17 ± 0.65	2.17	1.92	1.08

3.2. Parabolic Paths

Consider a parabolic curve shown in Figure 4, whose equation is

$$y = ax(x-1) \tag{4}$$

where a is a constant. But in terms of the height of the parabola h, the equation of this parabolic curve can be written as

$$y = -4hx(x-1) \tag{5}$$

The length of the parabolic curve described above is given by [25]

$$l = \int_0^1 \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_0^1 \sqrt{1+16h^2(2x-1)^2}\,\mathrm{d}x \tag{6}$$

Evaluation of this integral gives

$$l = \frac{1}{2}\sqrt{1+16h^2} - \frac{1}{8h}\ln\left(-4h+\sqrt{1+16h^2}\right) \tag{7}$$

We choose a river of length 2000 km, and take the unit of length to be 1 km. We assume that the river meanders on a parabolic curve of height h in each unit of distance along the straight line from the beginning to the end of the river, as shown in Figure 4. The height of the parabolic curve in each step h is randomly chosen from the Gaussian distribution function,

$$f(h) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{h^2}{2\sigma^2}\right) \tag{8}$$

The sinuosity of the river is then calculated by adding the length of the parabolic curves in each step and dividing it by the straight end-to-end distance of the river. This Monte Carlo experiment is then repeated 1000 times and the average sinuosity of the river is calculated which, of course, a function of the parameter σ of the Gaussian probability density function (8). We then adjust the value of σ to obtain the observed value of the sinuosity. The results are shown in Table 3 for the United States and the World rivers along with the corresponding adjusted value of σ in each case.

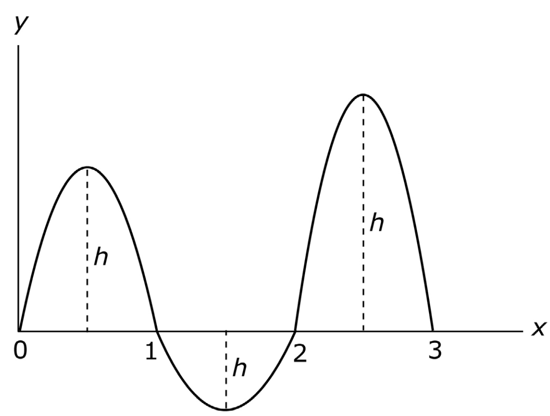

Figure 4. A parabolic curve with equation $y = -4hx(x-1)$.

4. Exact Models

A meandering river can also be modeled by a deterministic curve that repeats itself. Consider a function $y = f(x)$ defined on the interval $[0, d]$ as shown in Figure 5(a). The length of this curve is given by

$$l = \int_0^d \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x \tag{9}$$

If this curve (called the *basis*) repeats itself, alternating on two sides of a straight line, it generates the profile of a meandering river, as shown in Figure 5(b). If the river consists of n basis, its sinuosity is given by

$$s = \frac{nl}{nd} = \frac{l}{d} = \frac{1}{d}\int_0^d \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x \tag{10}$$

Analytical evaluation of the integral in Equation (10) is not always possible. Nevertheless, it can always be evaluated numerically. However, in simple cases, the sinuosity can be obtained in closed form [15]. For example, consider a basis consisting of a circular arc as shown in Figure 6(a). The length of this arc is $l = r\theta$, where r is the radius of the circular arc, and its Euclidean end-to-end distance is obtained from the cosine law,

$$d = r\sqrt{2(1-\cos\theta)} \tag{11}$$

Therefore, the sinuosity of a river obtained from this basis, shown in Figure 6(b), is given by

$$s = \frac{l}{d} = \frac{\theta}{\sqrt{2(1-\cos\theta)}} \tag{12}$$

which is independent of the radius of the circular arc. If the sinuosity of a river is

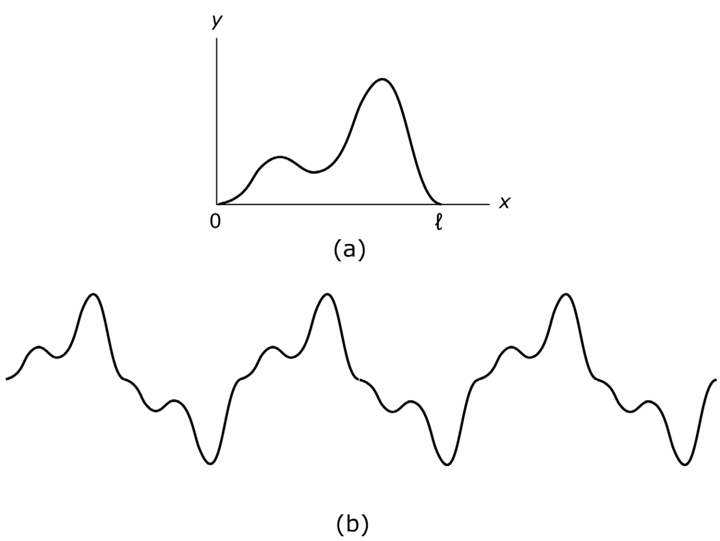

Figure 5. An exact model for meandering rivers. The profile of the river (b) is generated by alternating the basis curve (a) on two sides of a straight line.

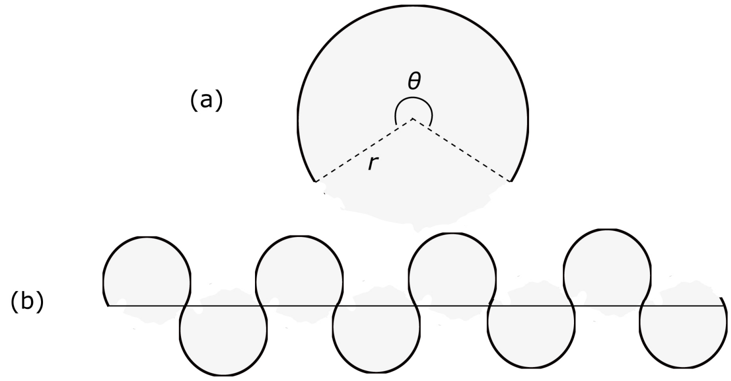

Figure 6. Circular model for meandering rivers.

known, this equation can be solved numerically for θ. For the major rivers of the United States and of the World with a mean sinuosity of about 2.0, we find $\theta = 3.79\ \text{rad} = 217^\circ$.

5. Discussion and Conclusions

The analysis of the data for major rivers in the United States and in the World shows that the mean sinuosity of rivers is not π as suggested previously [18]. Instead, the data for both classes of rivers show that the mean sinuosity is closer to 2. This is further evidenced by the fact that π does not fall within one standard deviation from the mean sinuosities of the major United States and World rivers.

Monte Carlo simulations using random numbers from a Gaussian probability distribution, with fairly small standard deviations, generate the observed mean sinuosity of the rivers with either a zig-zag model or a parabolic model as examples. Exact curves can also be used to model meandering rivers, as we have shown by simple circular curves.

In the calculation of sinuosities, there were a couple of rivers with sinuosities greater than 4. We ignored those rivers in our calculations due to the fact that large sinuosities are caused by higher composite meandering effects. To see this, consider a river shown in **Figure 7**, where a small-scale meandering is superimposed on a large-scale one. The total sinuosity is given by

$$s = \frac{L}{D} = \frac{L}{d}\frac{d}{D} \tag{13}$$

where d is the length of the curve shown by the dotted line in the **Figure 7**. But $L/d = s_1$ where s_1 is the sinuosity of the small-scale meandering, and $d/D = s_2$ where s_2 is the sinuosity of the large-scale meandering. Therefore, we have

$$s = s_1 s_2 \tag{14}$$

Examples of high composite meandering rivers are Kama River [26] and Kizilirmak River [27], with composite sinuosity of 6.78 and 4.66, respectively. It is, of course, possible for a river to have even higher composite sinuosity.

Figure 7. Composite meandering.

In conclusion, the observed sinuosities of major rivers in the United States and in the World have a mean value of about 2 (or more accurately 2.1 which is very close to $2\pi/3$), and not π that has been suggested based on the assumption of fractal geometry and idealizing the river geometry as a perfectly symmetrical sequence of bends.

These meandering rivers can be modeled by either stochastic simulations or by exact curves. In each case, a parameter in the governing equation needs to be adjusted to reproduce the observed mean sinuosity.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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