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Equivalence of Paths under the Action of the Real Representation of *Sp*(*n*)

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Abstract

In this article, we will consider questions of G-equivalence of paths for the case when G was the group of the real representation of a symplectic transformation in an n-dimensional quaternion vector space. In determining the solution of this problem, we give an explicit description of differential generators of a differential field of differential rational functions that are invariant under the action of this group. Necessary and sufficient conditions for the G-equivalence of paths in a 4n-dimensional real space are obtained with the help of differential generators.

Keywords

Real Representation, Differential Invariant, Differential Generators, Path in a Finite-Dimensional Space

1. Introduction

Let V be a finite-dimensional linear space over the field K (in generally skew-field K), $GL(V)$ be the group of all invertible transformations of space V, and γ_1, γ_2 be two smooth curves in V.

Curves γ_1 and γ_2 in V are said to be G-equivalent if $\sigma(\gamma_1)=\gamma_2$ for some $\sigma\in G$, where G is a subgroup of the group $GL(V)$.

It is known that the problem on the G-equivalence of curves lying in V, *i.e.*, to fiend necessary and sufficient conditions that guarantee the G-equivalence of the curves γ_1 and γ_2, is an important problem in the differential geometry of curves.

One version of this problem was posed by E. Cartan at the beginning of the 20th century and it is now known as Cartan's problem. This problem consists of the search for all motions of the space that superpose the given curves γ_1 and

γ_2 (see [1]). A thorough study of this problem was carried out by E. Cartan himself by using the method of moving frame (see [1] [2]).

In the work of many other scientists, including Yu. Aminov [3], Blaschke [4], Pommaret [5], one can see a geometric approach to the solution of this problem. The solution of this problem using geometric methods will be clearly described by certain geometric notions (for example, curvature, torsion, an arc length, etc.). However, it is difficult to use geometric methods for the solution of the problem of G-equivalence of systems consisting of a large number of curves. This requires the use of methods of invariant theory. The methods of invariant theory are very useful, in particular, in solving the problem of G-equivalence of finite path systems (infinitely differentiable vector-valued functions). For this, it is necessary to establish the finite generators of the differential field of all G-invariant differential rational functions and find the explicit form of a rational basis of this field. This formulation of the problem was considered by D. Khadjiyev [6] [7], K.K. Muminov [8] [9], V.I. Chilin [10], R.G. Aripov [11], with respect to the action of various classical groups of transformations and was discussed in detail, in the monographs [6] [12]. In these articles, the authors obtained the effective criteria of G-equivalence paths with respect to the action of certain classical groups $G \subset GL(V)$, for example, orthogonal, symplectic and pseudo-orthogonal groups. Currently, the results are used solution for some problems of non-euclidean geometries, and solution of problems in computer vision and vision-based applications, (see [13] [14] [15] [16]).

In all the works listed above, the posed problem was studied for finite-dimensional real and complex spaces. It is known that in the classical theory of invariants, in addition to real and complex spaces, linear spaces over the skew field of quaternion numbers are also considered and invariants with respect to the action of subgroups invertible linear transformation in such spaces are studied. This is represented a special case of the theory of non-commutative invariants (see, for example, [17] [18] [19] [20]).

In this article, we will consider questions of G-equivalence of paths for the case when G was the group of the real representation of a symplectic transformation in an n-dimensional quaternion vector space; also we show its solution using a G-invariant matrix function and d-generators of the d-field of a G-invariant d-rational function. This article is organized as follows: In Section 2, the group of symplectic transformations in quaternion space, the group of their real representation, and the problem of G-equivalence of paths are introduced briefly. Also, the solution of this problem will be given by G-invariant matrix functions. In Section 3, the ring of G-invariant polynomial is studied, and the system of its generators is described. Using the results of Sections 2 and 3, the system of generators of a differential field of G-invariant differential rational functions is restored and expounded in detail in Section 4. Section 5 is the conclusion part, in which all the results obtained are summarized and necessary and sufficient conditions for G-equivalence of paths are given.

2. Preliminaries

This section is devoted to the main concepts of the paper, which describe the group of symplectic transformations and its group of real representations, as well as the problem *G*-equivalence of paths.

2.1. Symplectic Group

Let H^n be an *n*-dimensional linear space over the skew-field *H*, (multiplication of numbers is defined on the left), where H is a skew-field of quaternion numbers. Denote by $GL(H^n)$ the group of all invertible linear transformations of the space H^n. Let $\langle x, y\rangle$ be a mapping of the Cartesian product $H^n \times H^n$ onto H and satisfies the following conditions:

$$\begin{cases} \langle \alpha x+\beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle,\ \alpha,\beta\in H,\ x,y,z\in H^n; \\ \langle x, y\rangle = \overline{\langle y, x\rangle}; \\ \langle x, x\rangle > 0,\ \text{for every } x\in H^n,\ x\neq 0 \end{cases} \tag{1}$$

where $\overline{q}$ means the conjugate of a quaternion $q = a+bi+cj+dk$, *i.e.* $\overline{q} = a-bi-cj-dk$, $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$, $a,b,c,d \in R$.

We obtain the linear form

$$\langle x, y\rangle = x_1\overline{y}_1 + x_2\overline{y}_2 + \cdots + x_n\overline{y}_n \tag{2}$$

as a metric function $\langle x, y\rangle$. Then the symplectic group $Sp(n)$ with respect to the metric function is defined as a subgroup of $GL(H^n)$ as follows:

$$Sp(n) = \left\{\sigma\in GL(H^n) : \langle \sigma x, \sigma y\rangle = \langle x, y\rangle\right\} \tag{3}$$

It is plain that for $\forall x\in V$ and $\forall\sigma\in GL(H^n)$ the relation $\sigma x \leftrightarrow xg$ is true, where $g\in GL(n,H)$. In this case, the symplectic group $Sp(n)$ is defined as follows

$$Sp(n) = \left\{g\in GL(n,H) : g^*g = gg^* = E\right\} \tag{4}$$

where the matrix g^* is hermitian conjugate of the matrix *g*, *i.e.*, $g^* = \overline{g}^{\mathrm{T}}$, E is identity element of the group $GL(n,H)$, (see, [21]).

It is known that the space H^n can be considered as to a $4n$ dimensional real space using the following operation:

$$\begin{aligned} x &= (x_1, x_2, \cdots, x_n) \\ &= (x_{11}+x_{12}i+x_{13}j+x_{14}k, \cdots, x_{n1}+x_{n2}i+x_{n3}j+x_{n4}k) \\ &= (x_{11}+x_{12}i+x_{13}j+x_{14}k)e_1 + \cdots + (x_{n1}+x_{n2}i+x_{n3}j+x_{n4}k)e_n \\ &= x_{11}e_1 + x_{12}(ie_1) + x_{13}(je_1) + x_{14}(ke_1) + \cdots \\ &\quad + x_{n1}e_n + x_{n2}(ie_n) + x_{n3}(je_n) + x_{n4}(ke_n) \\ &\approx (x_{11}, x_{12}, x_{13}, x_{14}, \cdots, x_{n1}, x_{n2}, x_{n3}, x_{n4}) = \vec{x}, \end{aligned}$$

where $x_{lm}\in R$, $l = \overline{1,n}$, $m = \overline{1,4}$.

We conditionally call the ***realification*** of this operation and denoted by "≈", (see, [22]). We are denoted by V the space of the realification H^n.

It is obvious that as a result of applying the operation "≈", the sum of arbitrary vectors $x, y \in H^n$ turns into the sum of vectors $\vec{x}, \vec{y} \in V$, where $\vec{x}, \vec{y}$ are real vectors corresponding to the vectors x, y. However, this property does not hold for the operation of the multiplication (on the left) of an arbitrary vector $x \in H^n$ by a number $\lambda \in H$. Therefore, when realification the space, the concepts associated with the operation of scalar multiplication are defined with the help of certain conditions through their equivalent concepts. For example, ***linearly dependent, orthogonally and other***. Accordingly, we introduce the following definition.

Definition 1. Vectors $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n \in V$ are called a ***strongly linearly independent*** if the vectors $x_1, x_2, \cdots, x_n \in H^n$ corresponding them by realification, are linearly independent in H^n.

Note. Any set of the strongly linearly independent vectors in V is of course linearly independent, but the converse is not always true.

The Group of Real Representations of the Symplectic Group *Sp*(*n*)

Let V be a space that the realification of the space H^n. Then every element $\vartheta \in GL(H^n)$ defines a linear transformation $\vartheta' \in GL(V)$, and $GL(H^n)$ can be regarded as a subgroup in $GL(V)$ using the isomorphism in $\vartheta \to \vartheta'$. Then $Sp(n)$ can be regarded also a subgroup in $GL(V)$. This subgroup is called ***the real representation*** $Sp(n)$, (see, [20]). Now let's define the definition of the real representation $Sp(n)$.

Let $\Omega_1(\vec{x}, \vec{y}), \Omega_i(\vec{x}, \vec{y}), \Omega_j(\vec{x}, \vec{y}), \Omega_k(\vec{x}, \vec{y})$ coefficients $1, i, j, k$ in "metric form" $\langle x, y \rangle$ respectively:

$$\langle x, y \rangle = \Omega_1(\vec{x}, \vec{y}) - \Omega_i(\vec{x}, \vec{y}) i - \Omega_j(\vec{x}, \vec{y}) j - \Omega_k(\vec{x}, \vec{y}) k, \tag{5}$$

where $x, y \in H^n$, $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$, $x_l = x_{l1} + x_{l2} i + x_{l3} j + x_{l4} k$, $y_l = y_{l1} + y_{l2} i + y_{l3} j + y_{l4} k$, $x_{lm}, y_{lm} \in R$, $m = \overline{1,4}$, $l = \overline{1,n}$, i, j, k —imaginary units of quaternion numbers.

Then, as is easily seen, Ω_1 (respectively Ω_i, Ω_j, Ω_k) is a symmetric (respectively skew-symmetric) real-valued bilinear form on the real vector space V. Moreover, Ω_1 is positive defined and the bilinear forms Ω_1, Ω_i, Ω_j, Ω_k are defined as follows:

$$\Omega_1(\vec{x}, \vec{y}) = \sum_{l=1}^{n} (x_{l1} y_{l1} + x_{l2} y_{l2} + x_{l3} y_{l3} + x_{l4} y_{l4});$$

$$\Omega_i(\vec{x}, \vec{y}) = \sum_{l=1}^{n} (x_{l1} y_{l2} - x_{l2} y_{l1} + x_{l3} y_{l4} - x_{l4} y_{l3});$$

$$\Omega_j(\vec{x}, \vec{y}) = \sum_{l=1}^{n} (x_{l1} y_{l3} - x_{l3} y_{l1} + x_{l4} y_{l2} - x_{l2} y_{l4});$$

$$\Omega_k(\vec{x}, \vec{y}) = \sum_{l=1}^{n} (x_{l1} y_{l4} - x_{l4} y_{l1} + x_{l2} y_{l3} - x_{l3} y_{l2}).$$

Obviously, a symplectic transformation is a transformation that leaves invariant the bilinear form $\langle x, y \rangle$. Then the corresponding real transformation to it be a transformation that leaves invariant those bilinear forms Ω_1, Ω_i, Ω_j, Ω_k. From this property, we will have in following definition.

Definition 2. The group of linear transformations $\vartheta' \in GL(V)$ is called ***a group of the real representation*** $Sp(n)$ if it satisfies the following conditions:

$$\begin{cases} \vartheta' \in GL(V): \Omega_1(\vartheta'\vec{x}, \vartheta'\vec{y}) = \Omega_1(\vec{x}, \vec{y}), \Omega_i(\vartheta'\vec{x}, \vartheta'\vec{y}) = \Omega_i(\vec{x}, \vec{y}), \\ \Omega_j(\vartheta'\vec{x}, \vartheta'\vec{y}) = \Omega_j(\vec{x}, \vec{y}), \Omega_k(\vartheta'\vec{x}, \vartheta'\vec{y}) = \Omega_k(\vec{x}, \vec{y}) \end{cases}$$

It is known that each transformation $\vartheta' \in GL(V)$ can be uniquely represented by the matrix $g \in GL(4n, R)$. This allows us to define the group of real representations $Sp(n)$ using matrices $g \in GL(4n, R)$. To do this, we use from *Definition* 2 and the following equalities

$$\begin{cases} \Omega_1(\vec{x}, \vec{y}) = \vec{x}(\vec{y})^{\mathrm{T}}; \Omega_i(\vec{x}, \vec{y}) = \vec{x}I(\vec{y})^{\mathrm{T}}; \\ \Omega_j(\vec{x}, \vec{y}) = \vec{x}J(\vec{y})^{\mathrm{T}}; \Omega_j(\vec{x}, \vec{y}) = \vec{x}K(\vec{y})^{\mathrm{T}} \end{cases} \tag{6}$$

where

$$I = \begin{bmatrix} I_1 & \theta & \cdots & \theta \\ \theta & I_1 & \cdots & \theta \\ \vdots & \vdots & \ddots & \vdots \\ \theta & \theta & \cdots & I_1 \end{bmatrix}, \ J = \begin{bmatrix} J_1 & \theta & \cdots & \theta \\ \theta & J_1 & \cdots & \theta \\ \vdots & \vdots & \ddots & \vdots \\ \theta & \theta & \cdots & J_1 \end{bmatrix}, \ K = \begin{bmatrix} K_1 & \theta & \cdots & \theta \\ \theta & K_1 & \cdots & \theta \\ \vdots & \vdots & \ddots & \vdots \\ \theta & \theta & \cdots & K_1 \end{bmatrix};$$

here θ is 4th ordered zero matrix, also

$$I_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \ J_1 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \ K_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

As a result, we have the following definition of the group of real representation $Sp(n)$ given by the matrices $g \in GL(4n, R)$:

Definition 3. The group of matrix $g \in GL(4n, R)$ is said to be group of the real representation $Sp(n)$, if it satisfies the following conditions:

$$\left\{ g \in GL(4n, R): gg^{\mathrm{T}} = E, gIg^{\mathrm{T}} = I, gJg^{\mathrm{T}} = J, gKg^{\mathrm{T}} = K, \det g = 1 \right\} \tag{7}$$

where E is $4n$th ordered unit matrix.

In what follows, we consider only the group of real representation of $Sp(n)$, and denote it by $\mathfrak{Sp}(4n)$.

2.2. Equivalence of Paths under the Action of the Group $\mathfrak{Sp}(4n)$

Let R be the field of real numbers and $T = (a, b)$ be an open interval in R.

Definition 4. A vector-valued function $\vec{x}(t) = \{x_l(t)\}_{l=1}^{4n}: T \to R^{4n}$ is called ***a path*** in R^{4n} if all of its coordinate functions $x_l(t): T \to R$ are infinitely differentiable (see, [6] [12]).

The k-th derivative of a path $\vec{x}(t) = \{x_l(t)\}_{l=1}^{4n}$ is the vector-valued function $\vec{x}^{(k)}(t) = \{x_l^{(k)}(t)\}_{l=1}^{4n}$, where $x_l^{(k)}(t)$ is the k-th derivative of the coordinate function $x_l(t)$, $t \in T$, $l = \overline{1, 4n}$, $k \in N$. The vector-valued function $\vec{x}^{(k)}(t)$

is also path for all $k = 1, 2, \cdots$.

Definition 5. A path $\vec{x}(t)$ in R^{4n} will be called ***regular*** if first derivative of the path $\vec{x}(t)$ is non-zero for all $t \in T$ (see, [6] [12]).

For any path $\vec{x}(t) = \{x_l(t)\}_{l=1}^{4n}$, we denote by $M(\vec{x}(t))$ the matrix $\left(x_l^{(m-1)}(t)\right)_{l,m=1}^{4n}$.

Definition 6. A path $\vec{x}(t)$ is said to be ***strongly regular*** if the determinant $\det M(\vec{x}(t))$ is non-zero for all $t \in T$ (see, [6] [12]).

It is easily seen that each strongly regular path $\vec{x}(t)$ is obviously a regular path, but the opposite is usually not true. Let G be an arbitrary subgroup of the group $GL(4n, R)$.

Definition 7. Two paths $\vec{x}(t)$ and $y(t)$ are said to be ***G-equivalent*** if there exists an element $g \in G$ such that $\vec{y}(t) = \vec{x}(t) g$ for all $t \in T$ (see, [6] [12]).

In this case, it is obvious that $\vec{y}^{(r)}(t) = \vec{x}^{(r)}(t) g$, $r \in N$, and therefore the G-equivalence of the paths $x(t)$ and $\vec{y}(t)$ is equivalent to be equality $M(\vec{y}(t)) = M(\vec{x}(t)) g$ for all $t \in T$.

Usually the problem of finding necessary and sufficient conditions for the G-equivalence of paths $\vec{x}(t)$ and $\vec{y}(t)$ is called ***the problem of G-equivalence of paths***. The following we prove a theorem expressing the solution to this problem, which is in the case when $G = \mathfrak{Sp}(4n)$ and $\vec{x}(t), \vec{y}(t) \in V$ are strongly regular paths.

Theorem 1 *Two strongly regular paths* $\vec{x}(t)$ *and* $\vec{y}(t)$ *are* $\mathfrak{Sp}(4n)$*-equivalent if and only if the equalities*

1) $M'(x(t))\left[M(x(t))\right]^{-1} = M'(y(t))\left[M(y(t))\right]^{-1}$;
2) $M(x(t))\left[M(x(t))\right]^{\mathrm{T}} = M(y(t))\left[M(y(t))\right]^{\mathrm{T}}$;
3) $M(x(t)) I \left[M(x(t))\right]^{\mathrm{T}} = M(y(t)) I \left[M(y(t))\right]^{\mathrm{T}}$;
4) $M(x(t)) J \left[M(x(t))\right]^{\mathrm{T}} = M(y(t)) J \left[M(y(t))\right]^{\mathrm{T}}$;
5) $M(x(t)) K \left[M(x(t))\right]^{\mathrm{T}} = M(y(t)) K \left[M(y(t))\right]^{\mathrm{T}}$;
6) $\det M(x(t)) = \det M(y(t))$

are valid for all $t \in T$.

Proof. Suppose that the paths $\vec{x}(t)$ and $\vec{y}(t)$ are equivalent with respect to the action of the group $\mathfrak{Sp}(4n)$. Then the equality $M(\vec{y})(t) = M(\vec{x})(t) g$ is hold for them, where g is an element of $\mathfrak{Sp}(4n)$. In this case, it is no difficulty in showing that *Equalities* 1 - 6 hold, using the equality above. For example,

$$M'(\vec{y})(t)\left(M(\vec{y})(t)\right)^{-1} = \left(M(\vec{x})(t) g\right)' \left(M(\vec{x})(t) g\right)^{-1}$$
$$= \left(M(\vec{x})(t)\right)' g g^{-1} \left(M(\vec{x})(t)\right)^{-1} = \left(M(\vec{x})(t)\right)' \left(M(\vec{x})(t)\right)^{-1}.$$

Suppose now that *Equalities* 1 - 6 be true for the paths $\vec{x}(t)$ and $\vec{y}(t)$, on all $t \in T$. It is plain that the equality $\left(A^{-1}\right)' = -A^{-1} A' A^{-1}$ is true for any non-singular matrix A. From this statement and the operations defined matrices, *Equalities* 1 - 6 can be written as

1') $\left(\left(M(\vec{x})(t)\right)^{-1} M(\vec{y})(t)\right)' = 0$;

2') $\left(\left(M(\vec{x})(t)\right)^{-1}M(\vec{y})(t)\right)\left(\left(M(\vec{x})(t)\right)^{-1}M(\vec{y})(t)\right)^{\mathrm{T}}=E$;

3') $\left(\left(M(\vec{x})(t)\right)^{-1}M(\vec{y})(t)\right)I\left(\left(M(\vec{x})(t)\right)^{-1}M(\vec{y})(t)\right)^{\mathrm{T}}=I$;

4') $\left(\left(M(\vec{x})(t)\right)^{-1}M(\vec{y})(t)\right)J\left(\left(M(\vec{x})(t)\right)^{-1}M(\vec{y})(t)\right)^{\mathrm{T}}=J$;

5') $\left(\left(M(\vec{x})(t)\right)^{-1}M(\vec{y})(t)\right)K\left(\left(M(\vec{x})(t)\right)^{-1}M(\vec{y})(t)\right)^{\mathrm{T}}=K$;

6') $\det\left(\left(M(\vec{x})(t)\right)^{-1}M(\vec{y})(t)\right)=1$.

sing *Equality* 1' we get $\left(M(\vec{x})(t)\right)^{-1}M(\vec{y})(t)=g\in GL(4n,R)$; from this and *Equalities* 2' - 6', it follows the following

$$gg^{\mathrm{T}}=E,\ gIg^{\mathrm{T}}=I,\ gJg^{\mathrm{T}}=J,\ gKg^{\mathrm{T}}=K,\ \det g=1.$$

The above equalities are true, if and only if the matrix g is an element of $\mathfrak{Sp}(4n)$. Hence, the equality $\left(M(\vec{x})(t)\right)^{-1}M(\vec{y})(t)=g\in\mathfrak{Sp}(4n)$ is true. From here we get $M(\vec{y})(t)=M(\vec{x})(t)g$, $g\in\mathfrak{Sp}(4n)$. This equality shows that the paths $\vec{x}(t)$ and $\vec{y}(t)$ are equivalent with respect to the action of the group $\mathfrak{Sp}(4n)$. *Theoerem* 1 is proved. □

It is easily seen that, the matrix functions

$$M'(\vec{x}(t))\left[M(\vec{x}(t))\right]^{-1};\ M(\vec{x}(t))\left[M(\vec{x}(t))\right]^{\mathrm{T}};\ M(\vec{x}(t))I\left[M(\vec{x}(t))\right]^{\mathrm{T}};$$

$$M(\vec{x}(t))J\left[M(\vec{x}(t))\right]^{\mathrm{T}};\ M(\vec{x}(t))K\left[M(\vec{x}(t))\right]^{\mathrm{T}};\ \det M(\vec{x}(t))$$

which given conditions of *Theorem* 1, is represented an invariant functions with respect to the action of the group $\mathfrak{Sp}(4n)$, and denote by

$$A(t)=\left(a_{lm}(t)\right)_{l,m=1}^{4n};\ B(t)=\left(b_{lm}(t)\right)_{l,m=1}^{4n};\ C(t)=\left(c_{lm}(t)\right)_{l,m=1}^{4n};$$

$$D(t)=\left(d_{lm}(t)\right)_{l,m=1}^{4n};\ E(t)=\left(e_{lm}(t)\right)_{l,m=1}^{4n};\ F(t);$$

of them, respectively. Also, we define elements of this function. To do this, we use from the action, which a multiplication of the matrix and from the equality $\left[M(\vec{x}(t))\right]^{-1}M(\vec{x}(t))=E$. After some calculations, we would have the formula in the following

$$a_{lm}(t)=\begin{cases}0,\ \text{if equality } l\neq m-1 \text{ is hold for } l=\overline{1,4n-1} \text{ and } m=\overline{1,4n};\\ 1,\ \text{if } l=m-1 \text{ is hold for } l=\overline{1,4n-1} \text{ and } m=\overline{1,4n};\\ a_{l4n}(t)=\dfrac{\sum_{m=1}^{4n}(-1)^{l+m}x_m^{(4n)}(t)M_{lm}(\vec{x}(t))}{\det M(\vec{x}(t))}\end{cases}$$

$$b_{lm}(t)=\Omega_1\left(\vec{x}^{(l)},\vec{x}^{(m)}\right);\ c_{lm}(t)=\Omega_i\left(\vec{x}^{(l)},\vec{x}^{(m)}\right);$$

$$d_{lm}(t)=\Omega_j\left(\vec{x}^{(l)},\vec{x}^{(m)}\right);\ e_{lm}(t)=\Omega_k\left(\vec{x}^{(l)},\vec{x}^{(m)}\right);$$

$$f(t)=\left[\vec{x}\ \vec{x}^{(1)}\ \vec{x}^{(2)}\cdots\vec{x}^{(4n-1)}\right]$$

where $\left[\vec{x}\ \vec{x}^{(1)}\ \vec{x}^{(2)}\cdots\vec{x}^{(4n-1)}\right]$ is a determinant of the matrix $M(\vec{x}(t))$.

As well, the bilinear forms $\Omega_\alpha\left(\vec{x}^{(l)},\vec{x}^{(m)}\right)$, $\alpha\in\{1,i,j,k\}$ represents of invariant differential polynomial with respect to the action of the group $\mathfrak{Sp}(4n)$. This makes it possible to show the solution of the problem of *G*-equivalence of paths also through *G*-invariant differential polynomials or *G*-invariant differential rational functions. In follows will be express just about this.

3. The Ring of Invariant Polynomials with Respect to the Action of the Group $\mathfrak{Sp}(4n)$

In this section, we study the ring of *G*-invariant polynomials under the action of the group $\mathfrak{Sp}(4n)$ and describe its generators. Moreover, we define the relationship between them.

Let V be a $4n$ dimensional real vector space, which the realification of H^n. The elements of the space V will be represented as $4n$ dimensional row-vector, and denoted by $\vec{x}$. Let be $G \subset GL(4n, R)$. As an action of the group G to the space V is defined as right multiplication of the matrix $g \in G$ to the row-vector $\vec{x} \in V$, *i.e.*, $(g, \vec{x}) = \vec{x}g$. Let $R[\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{4n}]$ be the ring of real polynomials in $4n$ vector arguments, where $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{4n} \in V$.

Definition 8. The polynomial function $f[\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{4n}]$ is called ***G-invariant***, if the equality

$$f[\vec{x}_1 g, \vec{x}_2 g, \cdots, \vec{x}_{4n} g] = f[\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{4n}]$$

is true for any $g \in G$.

We denote the set of all invariant polynomials with $R[\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{4n}]^G$. It is plain, this set is a subring of the ring $R[\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{4n}]^G$ with respect to operations defined on the ring $R[\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{4n}]$, *i.e.*,

$$R[\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{4n}]^G \subset R[\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{4n}].$$

Let the set $\Sigma = \{\varepsilon_l\}_{l \in \Delta}$ consists of the certain elements in $R[\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{4n}]$, where Δ is a finite ordered set natural numbers.

Definition 9. The elements $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_s \in \Sigma$ are called ***algebraical dependent***, if such that exist polynomial $P[\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_s]$ in $R[\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{4n}]$, then be $P(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_s) = 0$, otherwise these elements are called ***algebraical independent*** (see, [6]).

The set $\Sigma = \{\varepsilon_l\}_{l \in \Delta}$ is called *a system generators* of the ring $R[\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{4n}]^G$, if an arbitrary element

$$f[\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{4n}] \in R[\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{4n}]^G$$

can be generating by applying a finite number of operations of the ring $R[\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{4n}]^G$ to the elements in $\Sigma = \{\varepsilon_l\}_{l \in \Delta}$. A system of algebraical independent generators is called ***a integrity basis*** of the ring $R[\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{4n}]^G$. In the following, we consider the problem of ***describing a system of generators*** in $R[\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{4n}]^G$ for $G = \mathfrak{Sp}(4n)$.

Let there be given several *G*-invariant polynomials of some vector arguments $u_1, u_2, \cdots$, *i.e.*,

$$\varphi_1[u_1, u_2, \cdots], \varphi_2[u_1, u_2, \cdots], \cdots \tag{8}$$

System (8) will be a complete table of typical basic invariants for *m* arguments, if it changes into an integrity basis for invariants of *m* arguments $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_m$, by substituting for $u_1, u_2, \cdots$ these arguments in all possible combinations (repetitions included). Also, for the table of typical basic invariants of a linear group of *n*-th degree to be complete with respect to any *m* argument, it is sufficient that it is true for *n* arguments (see, [23]). Using these facts, we prove the following theorem.

Theorem 2. Let *G* be group $\mathfrak{Sp}(4n)$. Any $\mathfrak{Sp}(4n)$-invariant polynomial is expressed an integrally rational manner by the elements of the system

$$\Omega_\alpha\left(\vec{x}_l, \vec{\xi}_m\right), \ \alpha \in \{1, i, j, k\}, \tag{9}$$

where $\vec{x}_l \in V$, $\vec{\xi}_m \in V^*$, V^* is the adjoint space for the space *V*.

Proof. To prove *Theorem* 2, it suffices to show that, according to the above facts, the statement of the theorem is true for the sets of vectors $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{4n} \in V$ and $\vec{\xi}_1, \vec{\xi}_2, \cdots, \vec{\xi}_{4n} \in V^*$. In other, suffices to show that arbitrary $\mathfrak{Sp}(4n)$-invariant polynomial of these vector arguments, be generated through forms (9).

Obviously, we can express any polynomial $p\left[\vec{x}_1, \vec{x}_2, \cdots \middle| \vec{\xi}_1, \vec{\xi}_2, \cdots\right]$ by the form $P\left\{\left\langle \vec{x}_l \middle| \vec{\xi}_m \right\rangle\right\}$, where $\left\langle \vec{x}_l \middle| \vec{\xi}_m \right\rangle = \vec{x}_l \vec{\xi}_m = \sum_{n=1}^{4n} x_{ln} \xi_{mn}$.

Let $p\left[\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{4n} \middle| \vec{\xi}_1, \vec{\xi}_2, \cdots, \vec{\xi}_{4n}\right]$ be any $\mathfrak{Sp}(4n)$-invariant polynomial. Using the transformation $g \in \mathfrak{Sp}(4n)$, we can pass the set of vector arguments $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{4n} \in V$ to the set of vectors $\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_{4n}$, which the set of standard basic of vectors in *V*.

Let be $g \in \mathfrak{Sp}(4n)$. Then we have the system of equations $\vec{x}_l g = \vec{e}_l$, ($l = \overline{1, 4n}$). Also, considering that $\vec{e}_{4t+2} = \vec{e}_{4t+1} I$, $\vec{e}_{4t+3} = \vec{e}_{4t+1} J$, $\vec{e}_{4t+4} = \vec{e}_{4t+1} K$, ($t = \overline{0, n-1}$) and $g \in \mathfrak{Sp}(4n)$, we get the system of equations

$$\vec{x}_{4t+2} = \vec{x}_{4t+1} I, \ \vec{x}_{4t+3} = \vec{x}_{4t+1} J, \ \vec{x}_{4t+4} = \vec{x}_{4t+1} K. \tag{10}$$

In this case, the coordinates of each vector of the set vectors $\{\vec{x}_l, \vec{x}_{l+1}, \vec{x}_{l+2}, \vec{x}_{l+3}\}$ are determined by the coordinates of the vector $\vec{x}_l$, where $l = 4t + 1$, $t = \overline{0, n-1}$. In what follows, we denote by *X* the matrix in the form $\left(x_{lm}\right)_{l,m=1}^{4n}$. Then, the matrix *X* with the help of *Equations* (10) is defined in the following form

$$X = \begin{bmatrix} X_{11} & X_{15} & \cdots & X_{1(4n-3)} \\ X_{51} & X_{55} & \cdots & X_{5(4n-3)} \\ \vdots & \vdots & \ddots & \vdots \\ X_{(4n-3)1} & X_{(4n-3)5} & \cdots & X_{(4n-3)(4n-3)} \end{bmatrix},$$

where X_{lm} will be as follows:

$$X_{lm} = \begin{bmatrix} x_{lm} & x_{lm+1} & x_{lm+2} & x_{lm+3} \\ -x_{lm+1} & x_{lm} & -x_{lm+3} & x_{lm+2} \\ -x_{lm+2} & x_{lm+3} & x_{lm} & -x_{lm+1} \\ -x_{lm+3} & -x_{lm+2} & x_{lm+1} & x_{lm} \end{bmatrix}.$$

In addition, we have the equality $g = X^{-1}$ from the equation $Xg = E$. Also, we obtain to the equality $g = X^{\mathrm{T}}$ from $g \in \mathfrak{Sp}(4n)$. In turn, during the transformation the set $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{4n}\}$ to the set $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_{4n}\}$ respectively the set of vectors $\{\vec{\xi}_1, \vec{\xi}_2, \cdots, \vec{\xi}_{4n}\}$ will changed into the set of vectors $\{\vec{\xi}'_1, \vec{\xi}'_2, \cdots, \vec{\xi}'_{4n}\}$ and defined them as follows.

We denote of the matrices $(\xi_{lm})_{l,m=1}^{4n}$ and $(\xi'_{lm})_{l,m=1}^{4n}$ respectively by Ξ and Ξ'. Then, we have to the equation $\Xi' = g^{\mathrm{T}}\Xi = X\Xi$. From this equation followed of the equality $\xi'_{lm} = \Omega_\alpha(\vec{x}_s, \vec{\xi}_m)$, where $l, m = \overline{1, 4n}$,

$$\alpha = \begin{cases} 1, \text{ if } l = 4s-3, \text{ then be } s = \overline{1,n}; \\ i, \text{ if } l = 4s-2, \text{ then be } s = \overline{1,n}; \\ j, \text{ if } l = 4s-1, \text{ then be } s = \overline{1,n}; \\ k, \text{ if } l = 4s, \text{ then be } s = \overline{1,n}. \end{cases} \tag{11}$$

Then, we can see that each vector $\vec{\xi}'_l$ is defined as an algebraic expression of the bilinear form $\Omega_\alpha(\vec{x}_s, \vec{x}_m)$. In addition, the equation

$$p\left[\vec{x}_1, \cdots, \vec{x}_{4n} \middle| \vec{\xi}_1, \cdots, \vec{\xi}_{4n}\right] = p\left[\vec{e}_1, \cdots, \vec{e}_{4n} \middle| \vec{\xi}'_1, \cdots, \vec{\xi}'_{4n}\right] = P\left[\xi'_{lm}\right],$$

is true for any $\mathfrak{Sp}(4n)$-invariant polynomial $p\left[\vec{x}_1, \cdots, \vec{x}_{4n} \middle| \vec{\xi}_1, \cdots, \vec{\xi}_{4n}\right]$. Then it follows that any $\mathfrak{Sp}(4n)$-invariant polynomial $p\left[\vec{x}_1, \cdots, \vec{x}_{4n} \middle| \vec{\xi}_1, \cdots, \vec{\xi}_{4n}\right]$ algebraically expressed by the forms $\Omega_\alpha(\vec{x}_s, \vec{x}_m)$. *Theorem* 2 is proved. □

It is known that ***the second main problem*** in the course of invariant theory is ***the definition of the relationship the between generators of the ring of G-invariant polynomials***. Accordingly, below we define the relationship between the elements of *System* (9).

Let the scalar product in the space H^n be given by the form $\langle , \rangle$. In this case, consider the product

$$\langle x_1, x_2 \rangle \langle x_3, x_4 \rangle \cdots \langle x_{2n-1}, x_{2n} \rangle, \tag{12}$$

corresponding to the vectors $x_1, x_2, x_3, \cdots, x_{2n} \in H^n$, where $\vec{x}_l \neq \theta$, $l = \overline{1, 2n}$. Replacing each $\langle , \rangle$ bilinear form given in *Product* (12) by (5), we obtain a formula of the following form

$$\langle x_1, x_2 \rangle \cdots \langle x_{2n-1}, x_{2n} \rangle = F^{\alpha_1, \cdots, \alpha_n} + F^{\beta_1, \cdots, \beta_n} i + F^{\gamma_1, \cdots, \gamma_n} j + F^{\nu_1, \cdots, \nu_n} k, \tag{13}$$

where

$$F^{\alpha_1, \cdots, \alpha_n} = \sum c_\alpha \Omega_{\alpha_1}(\vec{x}_1, \vec{x}_2) \cdots \Omega_{\alpha_n}(\vec{x}_{2n-1}, \vec{x}_{2n}), \ \alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_n = \pm 1;$$

$$F^{\beta_1, \cdots, \beta_n} = \sum c_\beta \Omega_{\beta_1}(\vec{x}_1, \vec{x}_2) \cdots \Omega_{\beta_n}(\vec{x}_{2n-1}, \vec{x}_{2n}), \ \beta_1 \cdot \beta_2 \cdot \cdots \cdot \beta_n = \pm i;$$

$$F^{\gamma_1, \cdots, \gamma_n} = \sum c_\gamma \Omega_{\gamma_1}(\vec{x}_1, \vec{x}_2) \cdots \Omega_{\gamma_n}(\vec{x}_{2n-1}, \vec{x}_{2n}), \ \gamma_1 \cdot \gamma_2 \cdot \cdots \cdot \gamma_n = \pm j;$$

$$F^{\nu_1, \cdots, \nu_n} = \sum c_\nu \Omega_{\nu_1}(\vec{x}_1, \vec{x}_2) \cdots \Omega_{\nu_n}(\vec{x}_{2n-1}, \vec{x}_{2n}), \ \nu_1 \cdot \nu_2 \cdot \cdots \cdot \nu_n = \pm k;$$

$$c_\alpha = (-1)^q \, sign(\alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_n), \ c_\beta = (-1)^q \, sign(\beta_1 \cdot \beta_2 \cdot \cdots \cdot \beta_n);$$

$$c_\gamma = (-1)^q \, sign(\gamma_1 \cdot \gamma_2 \cdot \cdots \cdot \gamma_n), \ c_\nu = (-1)^q \, sign(\nu_1 \cdot \nu_2 \cdot \cdots \cdot \nu_n);$$

$\alpha_\ell, \beta_\ell, \gamma_\ell, \nu_\ell \in \{1, i, j, k\}$, *q*: the number of imaginary units in the product $\{\omega_1 \cdot \omega_2 \cdot \cdots \cdot \omega_n\}$, $\omega_\ell \in \{\alpha_\ell, \beta_\ell, \gamma_\ell, \nu_\ell\}$, $\ell = \overline{1, n}$.

For example, (see, [24])

$$
\begin{aligned}
&\langle x_1,x_2\rangle\langle x_3,x_4\rangle\\
&=\big[\Omega_1(x_1,x_2)\Omega_1(x_3,x_4)-\Omega_i(x_1,x_2)\Omega_i(x_3,x_4)-\Omega_j(x_1,x_2)\Omega_j(x_3,x_4)\\
&\quad-\Omega_k(x_1,x_2)\Omega_k(x_3,x_4)\big]+i\big[-\Omega_1(x_1,x_2)\Omega_i(x_3,x_4)-\Omega_i(x_1,x_2)\Omega_1(x_3,x_4)\\
&\quad+\Omega_j(x_1,x_2)\Omega_k(x_3,x_4)-\Omega_k(x_1,x_2)\Omega_j(x_3,x_4)\big]+j\big[-\Omega_1(x_1,x_2)\Omega_j(x_3,x_4)\\
&\quad-\Omega_j(x_1,x_2)\Omega_1(x_3,x_4)+\Omega_k(x_1,x_2)\Omega_i(x_3,x_4)-\Omega_i(x_1,x_2)\Omega_k(x_3,x_4)\big]\\
&\quad+k\big[-\Omega_1(x_1,x_2)\Omega_k(x_3,x_4)-\Omega_k(x_1,x_2)\Omega_1(x_3,x_4)+\Omega_i(x_1,x_2)\Omega_j(x_3,x_4)\\
&\quad-\Omega_j(x_1,x_2)\Omega_i(x_3,x_4)\big]\\
&=F^{\alpha_1,\alpha_2}(x_1,x_2,x_3,x_4)+F^{\beta_1,\beta_2}(x_1,x_2,x_3,x_4)i+F^{\gamma_1,\gamma_2}(x_1,x_2,x_3,x_4)j\\
&\quad+F^{\nu_1,\nu_2}(x_1,x_2,x_3,x_4).
\end{aligned}
$$

Using the above definition and formulas, we will define the relationship between the elements of system (10). To do this, we use an n linear independent vectors $x_1,\cdots,x_n\in H^n$ and the properties of the determinant of the Gram matrix composed of them. Further, let us denote the Gram matrix as $\Gamma(x_1,\cdots,x_n)$. It is known that the Gram matrix $\Gamma(x_1,\cdots,x_n)$ is a Hermitian quaternion matrix of order n and its determinant defines with formula in the following

$$
\begin{aligned}
\det{}_l\Gamma(x_1,\cdots,x_n)=\sum_{\sigma\in S_n}&(-1)^{n-\kappa}\left\langle x_l,x_{l_{m_1}}\right\rangle\left\langle x_{l_{m_1}},x_{l_{m_1+1}}\right\rangle\cdots\left\langle x_{l_{m_1+\delta_1}},x_l\right\rangle\\
&\times\cdots\times\left\langle x_{l_{m_\kappa}},x_{l_{m_\kappa+1}}\right\rangle\cdots\left\langle x_{l_{m_\kappa+\delta_\kappa}},x_n\right\rangle,
\end{aligned}
\tag{14}
$$

where S_n is a group of the permutation of the set $\{1,2,\cdots,n\}$, l is number of rows,

$$
\sigma=\left(l,l_{m_1},l_{m_1+1},\cdots,l_{m_1+\delta_1}\right)\cdots\left(l_{m_\kappa},l_{m_\kappa+1},\cdots,l_{m_\kappa+\delta_\kappa}\right)\in S_n,
$$

κ is number of cyclic; in addition, the following properties hold for the Gram determinant:

Proposition 3. The vectors $\{x_1,x_2,\cdots,x_n\}$ is linearly independent in H^n, then the relations $\det_l\Gamma(x_1,\cdots,x_n)\in R$ and $\det_l\Gamma(x_1,\cdots,x_n)\neq 0$ hold, where l is the number of rows;

Proposition 4. Let $\{x_1,\cdots,x_n,x\}$ be a set of the first n linearly independent vectors in the space H^n. Then for the Gram matrix consisting of them the equality $\det_l\Gamma(x_1,\cdots,x_n,x)=0$ is true, where

$$
x=\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_nx_n;
$$

Proposition 5. Let $\{x_1,\cdots,x_{n+1},x_{n+2}\}$ be a set of the first n linearly independent vectors in the space H^n. Then, the minor determinant $M_{n+2,1}$ of order $(n+1)$-th of $\det_l\Gamma(x_1,\cdots,x_n,x_{n+1},x_{n+2})$ is equal zero, *i.e.*,

$$
\begin{aligned}
M_{n+2,1}=\sum_{\sigma_1'\in\tilde S_{n+1}}&(-1)^{n+1-\kappa}\left\langle x_1,x_{l_{m_1}}\right\rangle\left\langle x_{l_{m_1}},x_{l_{m_1+1}}\right\rangle\cdots\widetilde{\left\langle x_{l_{m_1+\delta_1}},x_1\right\rangle}\cdots\\
&\times\left\langle x_{l_{m_\kappa}},x_{l_{m_\kappa+1}}\right\rangle\cdots\left\langle x_{l_{m_\kappa+\delta_\kappa}},x_{l_{m_\kappa}}\right\rangle\\
=0&
\end{aligned}
$$

where the permutation σ_1' and the set $\overline{S}_{n+1}$ are obtained as follows:

Let $\sigma' \in S_{n+2}$ be a permutation in form as the follows

$$\sigma' = \left(n+2\ 1\ l_{m_1}\ \cdots\ l_{m_1+\delta_1}\right)\cdots\left(l_{m_\kappa}\ l_{m_\kappa+1}\ \cdots\ l_{m_\kappa+\delta_\kappa}\right);$$

denote by σ_1' that the decomposition

$$\left(1\ l_{m_1}\ \cdots\ l_{m_1+\delta_1}\right)\cdots\left(l_{m_\kappa}\ l_{m_\kappa+1}\ \cdots\ l_{m_\kappa+\delta_\kappa}\right)$$

Also, we denote by $\overline{S}_{n+1}$ that the set of all the decomposition σ_1'; $\widetilde{\left\langle x_{l_{m_1+\delta_1}}, x_1\right\rangle} = \left\langle x_{l_{m_1+\delta_1}}, x_{n+2}\right\rangle$.

Proposition 3 - 5 follows from the properties of Hermitian quaternion matrices (see, [25]).

Now, we define the relations corresponding to *Propositions* 3 - 5 for the set of strongly linearly independent vectors $\vec{x}_1, \cdots, \vec{x}_n$ given in space *V*.

Let a set $B_1 = \{1, 2, \cdots, n\}$ and a permutation group S_n, consisting of elements of the set B_1 be given. It is known that any permutation

$$\sigma = \begin{pmatrix} 1 & \cdots & l_{m_1} & \cdots & l_{m_1+\delta_1} & \cdots & l_{m_\kappa} & \cdots & n \\ l_{m_1} & \cdots & l_{m_1+1} & \cdots & 1 & \cdots & l_{m_\kappa+1} & \cdots & l_{m_\kappa} \end{pmatrix} \in S_n$$

can be represented as a decomposition

$$\left(1, l_{m_1}, \cdots, l_{m_1+\delta_1}\right)\left(l_{m_2}, l_{m_2+1}, \cdots, l_{m_2+\delta_2}\right)\cdots\left(l_{m_\kappa}, l_{m_\kappa+1}, \cdots, n\right) \tag{15}$$

where $l_{m_s+\delta_s} = \overline{1,n}$, $\delta_s \in Z_0^+$, $s = \overline{1,\kappa}$, $l_{m_2} < l_{m_3} < \cdots < l_{m_\kappa}$, κ is number of cyclic. We denote by υ the set of ordered pairs corresponding to decomposition (15), *i.e.*,

$$\left\{\left(1, l_{m_1}\right), \cdots, \left(l_{m_1+\delta_1}, 1\right), \cdots, \left(l_{m_\kappa}, l_{m_\kappa+1}\right), \cdots, \left(n, l_{m_\kappa}\right)\right\},$$

and denote by ρ a bijective mapping from B_1 to υ. We also denote the set of all mappings ρ by A_ρ and by $F_{\rho_\tau}^{\alpha_1,\cdots,\alpha_n}$ the product

$$\Omega_{\alpha_1}\left(\vec{x}_{l_{m_{s_1}}}, \vec{x}_{l'_{m_{s_1}}}\right)\Omega_{\alpha_2}\left(\vec{x}_{l_{m_{s_2}}}, \vec{x}_{l'_{m_{s_2}}}\right)\cdots\cdot\Omega_{\alpha_n}\left(\vec{x}_{l_{m_{s_n}}}, \vec{x}_{l'_{m_{s_n}}}\right),$$

where $\alpha_1, \cdots, \alpha_n \in \{1, i, j, k\}$, $\left\{l_{m_s}, l'_{m_s}\right\} = \rho_\tau^{-1}(m_s)$, $l_{m_s} < l'_{m_s}$, $m_s \in B_1$, $\tau = \overline{1, n!}$.

Lemma 6. Let $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n\}$ be a set of strong linearly independent vectors in *V*. If the condition $\alpha_1\alpha_2\cdots\cdot\alpha_n = \pm 1$ holds, then the relation

$$F\left(\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n\right) = \sum_{\rho_\tau \in A_\rho} (-1)^{n-\kappa} c_{\rho_\tau}^{\alpha} F_{\rho_\tau}^{\alpha_1\alpha_2\cdots\cdot\alpha_n} \neq 0 \tag{16}$$

is valid for the set of vectors $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n$ where κ is number of the permutation σ_τ ($\upsilon_\tau \leftrightarrow \sigma_\tau$), also

$$c_{\bar{\rho}_\tau}^{\alpha} = (-1)^q\, sign\{\alpha_1 \cdot \alpha_2 \cdots\cdot \alpha_n\}$$
$$\times sign\left\{\Omega_{\alpha_1}\left(\vec{x}_{l_{m_{s_1}}}, \vec{x}_{l'_{m_{s_1}}}\right)\Omega_{\alpha_2}\left(\vec{x}_{l_{m_{s_2}}}, \vec{x}_{l'_{m_{s_2}}}\right)\cdots\cdot\Omega_{\alpha_2}\left(\vec{x}_{l_{m_{s_n}}}, \vec{x}_{l'_{m_{s_n}}}\right)\right\},$$

q is number of imaginary units in the product $\alpha_1 \cdot \alpha_2 \cdots\cdot \alpha_n$.

Proof. To prove *Lemma* 6, we use *Proposition* 3 and equality (14). According to *Proposition* 3, the relations $\det_l \Gamma\left(x_1, x_2, \cdots, x_n\right) \in R$ and

$\det_l \Gamma(x_1, x_2, \cdots, x_n) \neq 0$ are valid for the linearly independent vectors $x_1, x_2, \cdots, x_n \in H^n$; from equality (14) we have that, for $l = 1$, the formula

$$\det_1 \Gamma(x_1, \cdots, x_n) = \sum_{\sigma \in S_n} (-1)^{n-\kappa} \left\langle x_1, x_{l_{m_1}} \right\rangle \left\langle x_{l_{m_1}}, x_{l_{m_1+1}} \right\rangle \cdots \cdot \left\langle x_{l_{m_{\kappa+\delta_\kappa}}}, x_n \right\rangle. \tag{17}$$

Taking into account the definition of the mapping ρ and the equality

$$\Omega_1\left(x_{l_{m_\kappa}}, x_{l'_{m_\kappa}}\right) = \Omega_1\left(x_{l'_{m_\kappa}}, x_{l_{m_\kappa}}\right), \; \Omega_{\alpha_1}\left(x_{l_{m_\kappa}}, x_{l'_{m_\kappa}}\right) = -\Omega_{\alpha_1}\left(x_{l'_{m_\kappa}}, x_{l_{m_\kappa}}\right),$$

$(\alpha_1 \in \{i, j, k\})$ we obtain the following relation by applying equality (14) to the right of formula (17):

$$\begin{aligned} \det_1 \Gamma(x_1, \cdots, x_n) = & \sum_{\rho_\tau \in A_\rho} (-1)^{n-\kappa} c^{\alpha}_{\rho_\tau} F^{\alpha_1 \cdots \cdot \alpha_n}_{\rho_\tau} + i \sum_{\rho_\tau \in A_\rho} (-1)^{n-\kappa} c^{\beta}_{\rho_\tau} F^{\beta_1 \cdots \cdot \beta_n}_{\rho_\tau} \\ & + j \sum_{\rho_\tau \in A_\rho} (-1)^{n-\kappa} c^{\gamma}_{\rho_\tau} F^{\gamma_1 \cdots \cdot \gamma_n}_{\rho_\tau} + k \sum_{\rho_\tau \in A_\rho} (-1)^{n-\kappa} c^{\nu}_{\rho_\tau} F^{\nu_1 \cdots \cdot \nu_n}_{\rho_\tau}, \end{aligned} \tag{18}$$

where the vectors $\vec{x}_1, \cdots, \vec{x}_n$ corresponds to the vectors $x_1, \cdots, x_n$ with respect to the action of realification.

In this case, since the vectors $x_1, \cdots, x_n$ are linearly independent, it is clear that the vectors $\vec{x}_1, \cdots, \vec{x}_n$ are strongly linearly independent; since $\det_1 \Gamma(x_1, \cdots, x_n) \in R$, in equality (18) the coefficients in front of the imaginary units i, j, k are equal to zero; From $\det_1 \Gamma(x_1, \cdots, x_n) \neq 0$ it follows that statements of *Lemma* 6. □

Lemma 7. Let $\{\vec{x}_l\}_{l=1}^{n+1}$ be a sequence of vectors $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n, \vec{x}_{n+1}$, the firs n of which are strongly linearly independent. If the condition $\alpha_1 \alpha_2 \cdots \cdot \alpha_{n+1} = \pm 1$ holds, then the relation

$$F(\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{n+1}) = \sum_{\rho_\tau \in A_\rho} (-1)^{n+1-\kappa} c^{\alpha}_{\rho_\tau} F^{\alpha_1 \alpha_2 \cdots \cdot \alpha_{n+1}}_{\rho_\tau} = 0 \tag{19}$$

is valid for the sequence $\{\vec{x}_l\}_{l=1}^{n+1}$, where κ is a number of cycles in permutation σ, which the corresponding to the mapping ρ_τ, also

$$c^{\alpha}_{\rho_\tau} = (-1)^q \, sign\{\alpha_1 \cdots \cdot \alpha_{n+1}\} \times sign\left\{\Omega_{\alpha_1}\left(x_{l_{m_1}}, x_{l'_{m_1}}\right) \cdots \cdot \Omega_{\alpha_{n+1}}\left(x_{l_{m_{n+1}}}, x_{l'_{m_{n+1}}}\right)\right\}.$$

The statement of *Lemma* 7 follows from *Proposition* 4, equalities (14) and (18). Only in this case is considered the permutation of $\sigma \in S_{n+1}$, the corresponding set υ that it, and the set of mapping $\rho: \{1, 2, \cdots, n+1\} \to \upsilon$.

Let us now define the relation that follows from *Proposition* 5.

Let a set $B_2 = \{1, 2, \cdots, n+1, n+2\}$ and a group S_{n+2} be the permutation group consisting of the elements of B_2 be given. We obtain the elements of the group S_{n+2} in the following form

$$\sigma = \begin{pmatrix} n+2 & 1 & 2 & \cdots & n+1 \\ 1 & l_{m_{s_1}} & l_{m_{s_2}} & \cdots & l_{m_{s_{n+1}}} \end{pmatrix}.$$

Also, we can be represented of this permutations in form decomposition of independent cycles, *i.e.*,

$$\sigma = \left(n+2, 1, l_{m_1}, \cdots, l_{m_1+\delta_1}\right) \cdots \left(l_{m_\kappa}, l_{m_\kappa+1}, \cdots, l_{m_\kappa+\delta_\kappa}\right), \tag{20}$$

where $l_{m_s+\delta_s}=\overline{1,n+2}$, $\delta_s\in Z_0^+$, $s=\overline{1,\kappa}$, $l_{m_2}<l_{m_3}<\cdots<l_{m_\kappa}$, κ is the number of cycles.

We denote by υ the set of ordered pairs corresponding to *Decomposition* (20), *i.e.*,

$$\left\{(n+2,1)(1,l_{m_1}),\cdots,(l_{m_1+\delta_1},n+2),\cdots,(l_{m_\kappa},l_{m_\kappa+1}),\cdots,(l_{m_\kappa+\delta_\kappa},l_{m_\kappa})\right\},$$

also, denote by υ' that the subset υ in the form

$$\left\{(1,l_{m_1}),\cdots,(l_{m_1+\delta_1},n+2),\cdots,(l_{m_\kappa},l_{m_\kappa+1}),\cdots,(l_{m_\kappa+\delta_\kappa},l_{m_\kappa})\right\}.$$

Further, we denote the bijective mapping from the set $\{1,2,\cdots,n+1\}$ to σ' by ρ', and also by $A_{\rho'}$ the set of all mappings ρ' that define.

Lemma 8. Let $\{\vec{x}_l\}_{l=1}^{n+2}$ be a sequence of vectors $\vec{x}_1,\vec{x}_2,\cdots,\vec{x}_{n+1},\vec{x}_{n+2}$, the first n of which are strongly linearly independent. If the product $\omega_1\omega_2\cdot\cdots\cdot\omega_{n+1}$ is equal to one of the values $\pm1;\pm i;\pm j;\pm k$, then the relation

$$F(\vec{x}_1,\cdots,\vec{x}_{n+1},\vec{x}_{n+2})=\sum_{\rho'_\tau\in A_{\rho'}}(-1)^{n+1-\kappa}c^{\omega}_{\rho'_\tau}F^{\omega_1,\omega_2,\cdots,\omega_{n+1}}_{\rho'_\tau}=0 \tag{21}$$

is valid for the sequence $\{\vec{x}_l\}_{l=1}^{n+2}$, where

$$c^{\omega}_{\rho'}=(-1)^q\,sign\{\omega_1\cdot\omega_2\cdot\cdots\cdot\omega_{n+1}\}$$
$$\times sign\left\{\Omega_{\omega_1}\left(\vec{x}_{l_{m_{s_1}}},\vec{x}_{l'_{m_{s_1}}}\right)\cdot\cdots\cdot\Omega_{\omega_{n+1}}\left(\vec{x}_{l_{m_{s_{n+1}}}},\vec{x}_{l'_{m_{s_{n+1}}}}\right)\right\},$$

q is the number of imaginary units in the product $\omega_1\cdot\omega_2\cdot\cdots\cdot\omega_{n+1}$,
$\omega_\ell\in\{1,i,j,k\}$, $\{l_{m_s},l'_{m_s}\}=(\rho'_\tau)^{-1}(m_s)$, $l_{m_s}<l'_{m_s}$, $\tau=\overline{1,n+1!}$, $m_s=\overline{1,n+1}$.

Lemma 8 proves by *Proposition* 3, formula (14), and the definition of the bijective mapping $\rho'_\tau\in A_{\rho'}$. From in above theorem and lemmas, we obtain a corollary the following:

Corollary 1. *System* 9 is a complete table of typical basic invariants for the group $G=\mathfrak{Sp}(4n)$ and the statements of Lemmas 6-8 are represented the relations between of them.

4. The Differential Field $\mathfrak{Sp}(4n)$-Invariant Rational Functions

In this section, we study the differential ring that the corresponding to the ring $R\left[\vec{x}_1,\vec{x}_2,\cdots,\vec{4n}\right]^{\mathfrak{Sp}(4n)}$. Also, we solve the problems describing the system of *d*-generators and finding the relations in between of them.

Let $\mathbb{K}$ be a commutative ring and d a derivation in $\mathbb{K}$, *i.e.*,

$$d(x+y)=d(x)+d(y),\ d(x\cdot y)=d(x)\cdot y+x\cdot d(y)$$

for any $x,y\in\mathbb{K}$.

It is known ([13]) that a derivation d in an integral domain $\mathbb{K}$ admits a unique extension to a derivation of the corresponding field of fractions.

A commutative ring $\mathbb{K}$ with unity (respectively, a field $\mathbb{P}$) in which a fixed derivation is specified is called ***a differential ring*** (*d*-ring), (respectively ***a differential field***, *d*-field). A subfield $\mathbb{F}$ in a *d*-field $\mathbb{P}$ is called a ***d-subfield*** if

$d(\mathbb{F}) \subset \mathbb{F}$.

We will use the following examples of *d*-rings and *d*-fields.

We fix a natural number $4n \in N$ and consider the ring of polynomials of countable number of variables

$$x_1^{(0)}, x_2^{(0)}, \cdots, x_{4n}^{(0)}, x_1^{(1)}, x_2^{(1)}, \cdots, x_{4n}^{(1)}, \cdots, x_1^{(r)}, x_2^{(r)}, \cdots, x_{4n}^{(r)}, \cdots, \ r = 0,1,2,\cdots$$

of the form

$$R\left[x_1^{(0)}, x_2^{(0)}, \cdots, x_{4n}^{(0)}, x_1^{(1)}, x_2^{(1)}, \cdots, x_{4n}^{(1)}, \cdots, x_1^{(r)}, x_2^{(r)}, \cdots, x_{4n}^{(r)}, \cdots\right]$$

with coefficients from the field of real numbers *R*; we denote this ring by $R\{\vec{x}\}$ (we assume that $x_l = x_l^{(0)}$, $l = \overline{1,4n}$). We set $d\left(x_l^{(r)}\right) = x_l^{(r+1)}$, $d(c) = 0$, $c \in R$, for all $l = \overline{1,4n}$, $r \in Z_0^+$. The mapping d can be uniquely extended to a differentiation $\mathfrak{d}$ in the ring $R\{\vec{x}\}$. Then this ring becomes ***a differential ring***, its elements are called ***d-polynomials***; we denote them $f\{\vec{x}\}$, where $\vec{x} = \{x_l\}_{l=1}^{4n} \in V$.

We denote by $R\langle\vec{x}\rangle$ the field of fractions for the ring by $R\{\vec{x}\}$, *i.e.*, $R\langle\vec{x}\rangle$ is the field of all rational functions of the same variables $x_l^{(r)}$, $r \in Z_0^+$, $l = \overline{1,4n}$. Also, the differentiation $\mathfrak{d}$ can be naturally extended from the ring $R\{\vec{x}\}$ to a differentiation on the field $R\langle\vec{x}\rangle$. Then this field becomes a differential field, and elements of the *d*-field $R\langle\vec{x}\rangle$ are called ***d-rational functions***; we denote them $f\langle\vec{x}\rangle$.

Let G be a subgroup of the group $GL(4n, R)$.

Definition 10. A differential polynomial $f\{\vec{x}\}$ (respectively, a d-rational function $f\langle\vec{x}\rangle$) is said to be ***G-invariant*** if

$$f\{\vec{x}g\} = f\{\vec{x}\}, \ \left(f\langle\vec{x}g\rangle = f\langle\vec{x}\rangle\right)$$

for all $g \in G$, (see, [14]).

The set of all *G*-invariant *d*-polynomials (respectively, *G*-invariant *d*-rational functions) is denoted by $R\{\vec{x}\}^G$ (respectively, $R\langle\vec{x}\rangle^G$). It is known that $R\{\vec{x}\}^G \subset R\{\vec{x}\}$ (respectively, $R\langle\vec{x}\rangle^G \subset R\langle\vec{x}\rangle$).

Let the set $\Sigma' = \{\varepsilon_l'\}_{l \in L}$ is consisted from elements of $R\{\vec{x}\}^G$, where *L* is a set in finite number, the ordered of natural number.

A subset Σ' of $R\langle\vec{x}\rangle^G$ is called ***a generating system*** of the *d*-field $R\langle\vec{x}\rangle^G$ if an arbitrary element $f\langle\vec{x}\rangle$ of $R\langle\vec{x}\rangle^G$ can be generating by applying a finite number of operations of the *d*-field $R\langle\vec{x}\rangle^G$ to the elements of Σ', and the elements of Σ' are called ***d-generators*** of the *d*-field $R\langle\vec{x}\rangle^G$.

Elements $\{\varepsilon_1', \varepsilon_2', \cdots, \varepsilon_s'\}$ of Σ' are said to be ***d-algebraical dependent*** over *d*-field $R\langle\vec{x}\rangle^G$ if there exists a non zero *d*-polynomial $P\{y_1, y_2, \cdots, y_s\} \in R\{\vec{x}\}$ such that $P\{\varepsilon_1', \varepsilon_2', \cdots, \varepsilon_s'\} = 0$. Otherwise, the system of elements $\varepsilon_1', \varepsilon_2', \cdots, \varepsilon_s'$ is said to be ***d-algebraically independent*** over $R\langle\vec{x}\rangle^G$. Also, a finite system of *d*-generators in $R\langle\vec{x}\rangle^G$ that is *d*-algebraically independent is called ***d-rational basis*** of the *d*-generators of the *d*-field $R\langle\vec{x}\rangle^G$ (see, [14]).

We consider the following ***problem constructing a finite system of d-generators of the d-field*** $R\langle\vec{x}\rangle^G$ ***in case*** $G = \mathfrak{Sp}(4n)$.

Theorem 9. A system of d-generators of the d-field $R\langle\vec{x}\rangle^{\mathfrak{Sp}(4n)}$ is formed be the polynomials

$$\Omega_1\left(\vec{x}^{(r)},\vec{x}^{(r)}\right),\Omega_{\alpha_1}\left(\vec{x}^{(r)},\vec{x}^{(r+1)}\right),\left(r=\overline{0,n-1},\ \alpha_1\in\{i,j,k\}\right). \tag{22}$$

Proof. To prove *Theorem* 9, we use the following claims and propositions.

Claim 1. Any $\mathfrak{Sp}(4n)$-invariant *d*-rational function is the ratio of two $\mathfrak{Sp}(4n)$-invariant *d*-polynomials.

This claim follows from *Proposition* 1 in [6] (see, also the proof of *Theorem* 2.1.1 in [14]).

Claim 2. Any $\mathfrak{Sp}(4n)$-invariant *d*-polynomial is represented by the *d*-polynomials

$$\Omega_\alpha\left(\vec{x}^{(l)},\vec{x}^{(m)}\right),\left(\alpha\in\{1,i,j,k\}\right).$$

Claim 2 is differential analogy of *Theorem* 2.

Claim 3. If $\alpha\in\{1,i,j,k\}$ and $\vec{x}\in V$, then the following affirmations are hold for the $\mathfrak{Sp}(4n)$-invariant *d*-polynomials $\Omega_\alpha\left(\vec{x}^{(l)},\vec{x}^{(m)}\right)$:

i$_1$) any *d*-polynomial $\Omega_1\left(\vec{x}^{(l)},\vec{x}^{(m)}\right), l,m\in Z_0^+$ is expressed in terms of the *d*-polynomials $\Omega_1\left(\vec{x}^{(r_1)},\vec{x}^{(r_1)}\right)$ with use of finite number of operations in the *d*-ring $R\{\vec{x}\}^{\mathfrak{Sp}(4n)}$, where $r_1\le\left[\frac{l+m}{2}\right]$;

i$_2$) any *d*-polynomial $\Omega_1\left(\vec{x}^{(r_1)},\vec{x}^{(r_1)}\right)$ is expressed *d*-rational by the *d*-polynomials $\Omega_1\left(\vec{x}^{(r_1')},\vec{x}^{(r_1')}\right)$, where $r_1\ge 4n$, $r_1'=\overline{0,4n-1}$;

i$_3$) any *d*-polynomial $\Omega_i\left(\vec{x}^{(l)},\vec{x}^{(m)}\right), l,m\in Z_0^+, l<m$ is expressed in terms of the *d*-polynomials $\Omega_i\left(\vec{x}^{(r_2)},\vec{x}^{(r_2+1)}\right)$ with the use of finite number of operations in the *d*-ring $R\{\vec{x}\}^{\mathfrak{Sp}(4n)}$, where $l+m\ge 2r_2+1$;

i$_4$) any *d*-polynomial $\Omega_i\left(\vec{x}^{(r_2)},\vec{x}^{(r_2+1)}\right), r_2\ge 4n$ is expressed *d*-rationally in term of the *d*-polynomials $\Omega_i\left(\vec{x}^{(r_2')},\vec{x}^{(r_2'+1)}\right)$, where $r_2'=\overline{0,4n-1}$;

i$_5$) any *d*-polynomials $\Omega_j\left(\vec{x}^{(l)},\vec{x}^{(m)}\right)$ and $\Omega_k\left(\vec{x}^{(l)},\vec{x}^{(m)}\right), l,m\in Z_0^+, l<m$ are expressed in terms of the *d*-polynomials $\Omega_j\left(\vec{x}^{(r_3)},\vec{x}^{(r_3+1)}\right)$ and $\Omega_k\left(\vec{x}^{(r_3)},\vec{x}^{(r_2+1)}\right)$ with the use of finite number of operations in the *d*-ring $R\{\vec{x}\}^{\mathfrak{Sp}(4n)}$, where $l+m\ge 2r_3+1$;

i$_6$) any *d*-polynomial $\Omega_j\left(\vec{x}^{(r_3)},\vec{x}^{(r_3+1)}\right)$ and $\Omega_k\left(\vec{x}^{(r_3)},\vec{x}^{(r_3+1)}\right), r_3\ge 4n$ is expressed *d*-rationally in term of the *d*-polynomials $\Omega_j\left(\vec{x}^{(r_3')},\vec{x}^{(r_3'+1)}\right)$ and $\Omega_k\left(\vec{x}^{(r_3')},\vec{x}^{(r_3'+1)}\right)$ where $r_3'=\overline{0,4n-1}$.

Parts *i*$_1$) and *i*$_2$) of *Claim* 3 have been proved by Aripov R.G and Xadjiyev Dj [11] in case generally; parts *i*$_3$) and *i*$_4$) was proved by Muminov K.K [14]; parts *i*$_5$) and *i*$_6$) are follow by applying the equalities

$$\Omega_j\left(\vec{x}^{(l)},\vec{x}^{(m)}\right)=\Omega_i\left(\vec{x}^{(l)}A_1,\vec{x}^{(m)}A_1\right) \text{ and } \Omega_k\left(\vec{x}^{(l)},\vec{x}^{(m)}\right)=\Omega_i\left(\vec{x}^{(l)}A_2,\vec{x}^{(m)}A_2\right)$$

to the proof of parts *i*$_3$) and *i*$_4$).

Note. *Since the equality* $\Omega_\alpha\left(\vec{x}^{(l)},\vec{x}^{(m)}\right)=-\Omega_\alpha\left(\vec{x}^{(m)},\vec{x}^{(l)}\right)$ *is hold, the case* $l>m$ *of parts i*$_3$) - *i*$_6$) *follows from those case* $l<m$. *From Claims* 1 - 3, *we get the following corollary for the* $\mathfrak{Sp}(4n)$-*invariant d-polynomial* $\Omega_\alpha\left(\vec{x}^{(l)},\vec{x}^{(m)}\right)$.

Corollary 2. Any $\mathfrak{Sp}(4n)$-invariant *d*-rational function is expressed *d*-rationally with *d*-polynomials

$$\Omega_1\left(\vec{x}^{(r')},\vec{x}^{(r')}\right),\Omega_{\alpha_1}\left(\vec{x}^{(r')},\vec{x}^{(r'+1)}\right),\ r'=\overline{0,4n-1},\ \alpha_1\in\{i,j,k\}. \tag{23}$$

It follows from *Corollary* 2 that to prove *Theorem* 2 it suffices to show that elements of *System* (23) will be *d*-rationally expressed by elements of system (22). In other words, we study the problem of minimizing the number of elements of the system (23). To do this, we widely use the following propositions:

Proposition 10. For any non-zero $\mathfrak{Sp}(4n)$-invariant *d*-polynomial $\Omega_\alpha\left(\vec{x}^{(l)},\vec{x}^{(m)}\right)$, $(\alpha\in\{1,i,j,k\})$, the following equality holds:

$$d\left[\Omega_\alpha\left(\vec{x}^{(l)},\vec{x}^{(m)}\right)\right]=\Omega_\alpha\left(\vec{x}^{(l+1)},\vec{x}^{(m)}\right)+\Omega_\alpha\left(\vec{x}^{(l)},\vec{x}^{(m+1)}\right); \tag{24}$$

where $(\alpha\in\{1,i,j,k\})$.

Proposition 11. For any sequence of the strong linear independent vectors $\vec{x},\vec{x}^{(1)},\cdots,\vec{x}^{(n-1)}$ in *V*, the following relation is true:

$$F\left(\vec{x},\vec{x}^{(1)},\cdots,\vec{x}^{(n-1)}\right)=\sum_{\rho_\tau\in A_\rho}(-1)^{n-\kappa}c_{\rho_\tau}^{\alpha}F_{\rho_\tau}^{\alpha_1,\alpha_2,\cdots,\alpha_n}\neq 0; \tag{25}$$

where $\alpha_1\cdot\alpha_2\cdot\cdots\cdot\alpha_n=\pm1$, $\alpha_s\in\{1,i,j,k\}$,

$$c_{\rho_\tau}^{\alpha}=(-1)^q\,sign\{\alpha_1\cdot\alpha_2\cdot\ldots\cdot\alpha_n\}\times sign\left\{\Omega_{\alpha_1}\cdot\Omega_{\alpha_2}\cdot\cdots\cdot\Omega_{\alpha_n}\right\};$$

Proposition 12. Let be a set $\left\{\vec{x},\vec{x}^{(1)},\cdots,\vec{x}^{(n-1)}\right\}$ of strong linearly independent vectors in *V*. Then, the following relation

$$F\left(\vec{x},\vec{x}^{(1)},\cdots,\vec{x}^{(n)}\right)=\sum_{\rho_\tau\in A_\rho}(-1)^{n+1-\kappa}c_{\rho_\tau}^{\alpha}F_{\rho_\tau}^{\alpha_1,\alpha_2,\cdots,\alpha_{n+1}}=0 \tag{26}$$

is true for any sequence of vectors $\vec{x},\vec{x}^{(1)},\cdots,\vec{x}^{(n)}$ in *V*, where $\alpha_1\cdot\alpha_2\cdot\cdots\cdot\alpha_{n+1}=\pm1$, $\alpha_s\in\{1,i,j,k\}$,

$$c_{\rho_\tau}^{\alpha}=(-1)^q\,sign\{\alpha_1\cdot\alpha_2\cdot\cdots\cdot\alpha_n\}\times sign\left\{\Omega_{\alpha_1}\cdot\Omega_{\alpha_2}\cdot\cdots\cdot\Omega_{\alpha_n}\right\};$$

Proposition 13. Let be a set $\left\{\vec{x},\vec{x}^{(1)},\cdots,\vec{x}^{(n-1)}\right\}$ of strong linearly independent vectors in *V*. Then, the following relation

$$F\left(\vec{x},\vec{x}^{(1)},\vec{x}^{(2)},\cdots,\vec{x}^{(n+1)}\right)=\sum_{\rho'_\tau\in A_{\rho'}}(-1)^{n+1-\kappa}c_{\rho'_\tau}^{\omega}F_{\rho'_\tau}^{\omega_1,\omega_2,\cdots,\omega_{n+1}}=0 \tag{27}$$

is true for the $\vec{x},\vec{x}^{(1)},\cdots,\vec{x}^{(n)},\vec{x}^{(n+1)}$ in *V*, where $w_1\cdot w_2\cdot\cdots\cdot w_n=\beta$, $\beta\in\{\pm1,\pm i,\pm j,\pm k\}$,

$$c_{\rho_\tau}^{w}=(-1)^q\,sign\{w_1\cdot w_2\cdot\cdots\cdot w_{n+1}\}\times sign\left\{\Omega_{w_1}\cdot\Omega_{w_2}\cdot\cdots\cdot\Omega_{w_{n+1}}\right\}.$$

Proposition 10 follows from definition of the operation differential; *Propositions* 11 - 13 represents the differential analogy of *Lemmas* 6 - 8.

We first minimize the number of *d*-polynomials $\Omega_1\left(\vec{x}^{(r')},\vec{x}^{(r')}\right)$, ($r'=\overline{0,4n-1}$) using the above propositions and claims. To do this, we use the method of mathematical induction:

Step 1. Let be $r'=\overline{0,n-1}$. Then the *d*-polynomials $\Omega_1\left(\vec{x}^{(r')},\vec{x}^{(r')}\right)$ is elements of *System* (22);

Step 2. Let be $r'=n$. There expressing equality (26) of *Proposition* 12 in the form

$$F\left(\vec{x},\cdots,\vec{x}^{(n)}\right)=F_1\left(\vec{x},\cdots,\vec{x}^{(n)}\right)+\Omega_1\left(\vec{x}^{(n)},\vec{x}^{(n)}\right)F\left(\vec{x},\cdots,\vec{x}^{(n-1)}\right)=0,$$

we obtain the following

$$\Omega_1\left(\vec{x}^{(n)},\vec{x}^{(n)}\right)=-\frac{F_1\left(\vec{x},\cdots,\vec{x}^{(n)}\right)}{F\left(\vec{x},\cdots,\vec{x}^{(n-1)}\right)},\tag{28}$$

where the expression $F_1\left(\vec{x},\cdots,\vec{x}^{(n)}\right)$ will be the sum of such terms of the expression $F\left(\vec{x},\cdots,\vec{x}^{(n)}\right)$ that it doesn't contain *d*-polynomial $\Omega_1\left(\vec{x}^{(n)},\vec{x}^{(n)}\right)$; it is known from *Proposition* 11 that $F\left(\vec{x},\cdots,\vec{x}^{(n-1)}\right)$ isn't equal to zero; therefore the fraction is well defined, which on the right side of formula (28); also the expression $F_1\left(\vec{x},\cdots,\vec{x}^{(n)}\right)$ and $F\left(\vec{x},\cdots,\vec{x}^{(n-1)}\right)$ consist of *d*-polynomials $\Omega_\alpha\left(\vec{x}^{(l)},\vec{x}^{(m)}\right)$, $l,m=\overline{0,n}$, $\alpha\in\{1,i,j,k\}$; In this case, under the conditions i_1), i_2), i_3) of *Claim* 3, we have that the $\Omega_1\left(\vec{x}^{(r_1)},\vec{x}^{(r_1)}\right)$, $\Omega_{\alpha_1}\left(\vec{x}^{(r_2)},\vec{x}^{(r_2+1)}\right)$, ($\alpha_1\in\{i,j,k\}$) are expressed *d*-rationally in terms of elements of System (22), since
$l+m\le\max\{l+m\}=2n-1$,

$$r_1\le\left[\frac{2n-1}{2}\right]=\left[n-\frac{1}{2}\right]=n-1\Rightarrow r_1=\overline{0,n-1};$$

$$2r_2+1\le 2n-1\Rightarrow r_2\le n-1\Rightarrow r_2=\overline{0,n-1}.$$

Hence, the expressions $F_1\left(\vec{x},\cdots,\vec{x}^{(n)}\right)$ and $F\left(\vec{x},\cdots,\vec{x}^{(n-1)}\right)$ are also expressed *d*-rationally in terms of elements of *System* (22). From this and the equality (28) follows that *d*-polynomial $\Omega_1\left(\vec{x}^{(n)},\vec{x}^{(n)}\right)$ is also expressed *d*-rationally by the elements of *System* (22);

Now, we shall show that the $\mathfrak{Sp}(4n)$-invariant *d*-polynomials $\Omega_{\alpha_1}\left(\vec{x}^{(n)},\vec{x}^{(n+1)}\right),\left(\alpha_1=\{i,j,k\}\right)$ are expressed *d*-rationally in terms of elements of the system (22). To do this we use *Equality* (25), as a result we have the system,

$$\begin{cases}\sum\limits_{\rho'_\tau\in A_{\rho'}}(-1)^{n+1-\kappa}c^{\alpha}_{\rho'_\tau}F^{\alpha_1\cdots\alpha_{n+1}}_{\rho'_\tau}=0,\ \alpha_1\cdots\alpha_{n+1}=\pm1;\\ \sum\limits_{\rho'_\tau\in A_{\rho'}}(-1)^{n+1-\kappa}c^{\beta}_{\rho'_\tau}F^{\beta_1\cdots\beta_{n+1}}_{\rho'_\tau}=0,\ \beta_1\cdots\beta_{n+1}=\pm i;\\ \sum\limits_{\rho'_\tau\in A_{\rho'}}(-1)^{n+1-\kappa}c^{\gamma}_{\rho'_\tau}F^{\gamma_1\cdots\gamma_{n+1}}_{\rho'_\tau}=0,\ \gamma_1\cdots\gamma_{n+1}=\pm j;\\ \sum\limits_{\rho'_\tau\in A_{\rho'}}(-1)^{n+1-\kappa}c^{\nu}_{\rho'_\tau}F^{\nu_1\cdots\nu_{n+1}}_{\rho'_\tau}=0,\ \nu_1\cdots\nu_{n+1}=\pm k\end{cases}\tag{29}$$

where $\alpha_\ell,\beta_\ell,\gamma_\ell,\nu_\ell\in\{1,i,j,k\}$, $l=\overline{1,n+1}$.

Let us system (29) with respect to the polynomials $\Omega_\alpha\left(\vec{x}^{(n)},\vec{x}^{(n+1)}\right)$ $\left(\alpha\in\{1,i,j,k\}\right)$. To do this, denote by ρ''_τ a bijective mapping that satisfies the conditions $\left(\rho'_\tau\right)^{(-1)}(n+1)=\{n,n+1\}$, and we obtain the following

$$F^{\omega_1,\cdots,\omega_{n+1}}_{\rho''_\tau}=F^{\omega_1,\cdots,\omega_n}_{\rho''_\tau}\Omega_{\omega_{n+1}}\left(\vec{x}^{(n)},\vec{x}^{(n+1)}\right)$$

where $\omega_\ell\in\{\alpha_\ell,\beta_\ell,\gamma_\ell,\nu_\ell\}$;

Also, we introduce notations in the following:

$$z_1=\Omega_1\left(\vec{x}^{(n)},\vec{x}^{(n+1)}\right),\ z_2=\Omega_i\left(\vec{x}^{(n)},\vec{x}^{(n+1)}\right),$$

$$z_3 = \Omega_j\left(\vec{x}^{(n)}, \vec{x}^{(n+1)}\right),\ z_4 = \Omega_k\left(\vec{x}^{(n)}, \vec{x}^{(n+1)}\right)$$

$$D_1 = \sum_{\rho''_\tau \in A_{\rho'}} (-1)^{n+1-\kappa} c^{\alpha}_{\rho''_\tau} F^{\alpha_1,\cdots,\alpha_n}_{\rho''_\tau}, (\alpha_1 \cdot \cdots \cdot \alpha_n = \pm 1);$$

$$D_2 = \sum_{\rho''_\tau \in A_{\rho'}} (-1)^{n+1-\kappa} c^{\beta}_{\rho''_\tau} F^{\beta_1,\cdots,\beta_n}_{\rho''_\tau}, (\beta_1 \cdot \cdots \cdot \beta_n = \pm i);$$

$$D_3 = \sum_{\rho''_\tau \in A_{\rho'}} (-1)^{n+1-\kappa} c^{\gamma}_{\rho''_\tau} F^{\gamma_1,\cdots,\gamma_n}_{\rho''_\tau}, (\gamma_1 \cdot \cdots \cdot \gamma_n = \pm j);$$

$$D_4 = \sum_{\rho''_\tau \in A_{\rho'}} (-1)^{n+1-\kappa} c^{\nu}_{\rho''_\tau} F^{\nu_1,\cdots,\nu_n}_{\rho''_\tau}, (\nu_1 \cdot \cdots \cdot \nu_n = \pm k);$$

Moreover, we denote by N_1, N_2, N_3, N_4 the sum of the products $F^{\omega_1,\cdots,\omega_{n+1}}_{\rho'_\tau}$ corresponding to the mapping $\rho'_\tau \in A_{\rho'}$ that satisfies the conditions $(\rho'_\tau)^{(-1)}(n+1) \neq \{n, n+1\}$. From these notations and from system (29), we have a system of the following form:

$$\begin{cases} -D_1 z_1 + D_2 z_2 + D_3 z_3 + D_4 z_4 = N_1 \\ D_2 z_1 + D_1 z_2 + D_4 z_3 - D_3 z_4 = N_2 \\ D_3 z_1 - D_4 z_2 + D_1 z_3 + D_2 z_4 = N_3 \\ D_4 z_1 + D_3 z_2 - D_2 z_3 + D_1 z_4 = N_4. \end{cases} \tag{30}$$

It is easily shown that a determinant of coefficient matrix of *System* (30) is defined by formula

$$\Delta' = -\left(D_1^2 + D_2^2 + D_3^2 + D_4^2\right)^2,$$

where Δ' is determinant of coefficient matrix.

Hence, the equal $\Delta' = 0$ is true if and only if $D_1 = D_2 = D_3 = D_4 = 0$. According to *Lemme* 6, at least one of the coefficients D_1, D_2, D_3, D_4 is non-zero. Hence, $\Delta' \neq 0$, and *System* (30) has a unique solution. In this case, we can be represented by *d*-polynomials $\Omega_\omega\left(\vec{x}^{(n)}, \vec{x}^{(n+1)}\right)$, $\left(\omega \in \{1, i, j, k\}\right)$ in terms of D_1, D_2, D_3, D_4 and $N_t\left(t = \overline{1,4}\right)$ in a unique form. It is easily shown that the conditions $l + m \leq 2n$ is hold for the *d*-polynomials in these expressions. From this and from *i*₃) it is follows that *d*-polynomials $\Omega_\omega\left(\vec{x}^{(l)}, \vec{x}^{(m)}\right)$, $\left(\omega \in \{1, i, j, k\}\right)$ are expressed *d*-rationally in terms of the elements of *System* (22). This shown that the *d*-polynomials $\Omega_\omega\left(\vec{x}^{(n)}, \vec{x}^{(n+1)}\right)$, $\left(\omega \in \{1, i, j, k\}\right)$ are also expressed *d*-rationally by the elements of *System* (23).

Step 3. Let be $r' = n + s, \left(s = \overline{2, 3n-2}\right)$. In this case, we assume that the statement of *Theorem* 9 is true;

Now, we prove that the statement of *Theorem* 9 is true by using the above assumption, for $r' = n + s + 1 = 4n - 1$. To do this, we write of equality (26) in *Proposition* 12 for a set of vectors $\vec{x}^{(3n-1)}, \cdots, \vec{x}^{(4n-1)}$, and have to the following

$$\begin{aligned} &F\left(\vec{x}^{(3n-1)}, \cdots, \vec{x}^{(4n-1)}\right) \\ &= F_1\left(\vec{x}^{(3n-1)}, \cdots, \vec{x}^{(4n-1)}\right) + \Omega_1\left(\vec{x}^{(4n-1)}, \vec{x}^{(4n-1)}\right) F\left(\vec{x}^{(3n-1)}, \cdots, \vec{x}^{(4n-2)}\right) \\ &= 0 \end{aligned}$$

From the above equality, it follows

$$\Omega_1\left(\vec{x}^{(4n-1)},\vec{x}^{(4n-1)}\right)=-\frac{F_1\left(\vec{x}^{(3n-1)},\cdots,\vec{x}^{(4n-1)}\right)}{F\left(\vec{x}^{(3n-1)},\cdots,\vec{x}^{(4n-2)}\right)},$$

$$\left(F\left(\vec{x}^{(3n-1)},\cdots,\vec{x}^{(4n-2)}\right)\neq 0.\right)$$

The *d*-polynomials $\Omega_\alpha\left(\vec{x}^{(l)},\vec{x}^{(m)}\right)$ in the expressions $F_1\left(\vec{x}^{(3n-1)},\cdots,\vec{x}^{(4n-1)}\right)$ and $F\left(\vec{x}^{(3n-1)},\cdots,\vec{x}^{(4n-2)}\right)$, are expressed *d*-rationally in terms of the elements of System (22) under *Claim* 3 and the assumption, because $l,m=\overline{3n-1,4n-1}$. This implies that the *d*-polynomials $\Omega_1\left(\vec{x}^{(4n-1)},\vec{x}^{(4n-1)}\right)$ is also expressed *d*-rationally in terms of the elements of *System* (22);

As above, it can be shown that the *d*-polynomials $\Omega_{\alpha_1}\left(\vec{x}^{(4n-1)},\vec{x}^{(4n)}\right)$ are also expressed *d*-rationally of the elements of *System* (22) in accordance with *Claim* 3 and assumption. To do this, it suffices to repeat the calculation for *d*-polynomial $\Omega_{\alpha_1}\left(\vec{x}^{(n)},\vec{x}^{(n+1)}\right)$ by a set of vectors $\vec{x}^{(3n-1)},\cdots,\vec{x}^{(4n)}$. Hence, from the principle of Mathematical Induction, it follows that the *d*-polynomials $\Omega_1\left(\vec{x}^{(r')},\vec{x}^{(r')}\right)$, $\Omega_{\alpha_1}\left(\vec{x}^{(r')},\vec{x}^{(r'+1)}\right)$ are expressed *d*-rationally of the elements of the *System* (22) for all values of $r'=\overline{0,4n-1}$. That is exactly what we wanted to show. *Theorem* 9 is proved. □

5. Conclusions

In conclusion, we can state the following corollary from *Theorems* 1 and 9.

Corollary 3. Two strongly regular paths $\vec{x}(t)$ and $\vec{y}(t)$ are $\mathfrak{Sp}(4n)$-equivalent if and only if the equalities

1) $\Omega_1\left(\vec{x}^{(r)}(t),\vec{x}^{(r)}(t)\right)=\Omega_1\left(\vec{y}^{(r)}(t),\vec{y}^{(r)}(t)\right)$;

2) $\Omega_{\alpha_1}\left(\vec{x}^{(r)}(t),\vec{x}^{(r+1)}(t)\right)=\Omega_{\alpha_1}\left(\vec{y}^{(r)}(t),\vec{y}^{(r+1)}(t)\right)$

are valid for all $t\in T$, where $r=\overline{0,n-1}$, $\alpha_1\in\{i,j,k\}$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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