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Generalized Harmonic Numbers $H_{n,k,r}(\alpha,\beta)$ with Combinatorial Sequences

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Abstract

In this paper, we observe the generalized Harmonic numbers $H_{n,k,r}(\alpha,\beta)$. Using generating function, we investigate some new identities involving generalized Harmonic numbers $H_{n,k,r}(\alpha,\beta)$ with Changhee sequences, Daehee sequences, Degenerate Changhee-Genoocchi sequences, Two kinds of degenerate Stirling numbers. Using Riordan arrays, we explore interesting relations between these polynomials, Apostol Bernoulli sequences, Apostol Euler sequences, Apostol Genoocchi sequences.

Keywords

Generating Function, Riordan Arrays, Generalized Harmonic Numbers, Changhee Sequences, Daehee Sequences, Apostol Bernoulli Sequences, Degenerate Changhee-Genoocchi Sequences

1. Introduction

The harmonic numbers play an important role in combinatorial problem and numbers theory, and they also frequently appear in the analysis of algorithms and probabilistic statistical calculation. The objective of this paper is using Riordan arrays and generating function to discover identities on the generalized harmonic numbers. The harmonic numbers $H_n(n\geq 0)$ are defined by

$$H_0=0,\ H_n=\sum_{k=1}^{\infty}\frac{1}{k}\quad (k=1,2,\cdots).$$

and the generating function of H_n is

$$\sum_{n=0}^{\infty}H_nt^n=\frac{-\ln(1-t)}{1-t}.$$

The first few harmonic numbers are $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \cdots$. The harmonic numbers H_n have been generalized by several authors. For other generalizations of the harmonic numbers, one can consult [1] [2]. One of them is the generalized harmonic numbers $H_{n,k,r}(\alpha,\beta)$ defined by see [3] [4]: $k, r \geq 1$ are integers, α, β are real numbers, and $\alpha\beta \neq 0$.

$$\sum_{n=0}^{\infty} H_{n,k,r}(\alpha,\beta) t^n = \frac{(-\ln(1-\alpha t))^r}{(1-\beta t)^k}. \tag{1}$$

For convenience, we recall some definitions involved in the paper as following [5]-[17].

High order Changhee polynomial of the first kind $Ch_n^{(k)}(x)$ and the second kind $\hat{C}h_n^{(k)}(x)$ has the following generating function

$$\sum_{n=0}^{\infty} Ch_n^{(k)}(x) \frac{t^n}{n!} = \left(\frac{2}{(1+t)+1}\right)^k (1+t)^x, \tag{2}$$

$$\sum_{n=0}^{\infty} \hat{C}h_n^{(k)}(x) \frac{t^n}{n!} = \left(\frac{2(1+t)}{(1+t)+1}\right)^k (1+t)^x. \tag{3}$$

when $x = 0$, $Ch_n^{(k)} = Ch_n^{(k)}(0)$ and $\hat{C}h_n^{(k)} = \hat{C}h_n^{(k)}(0)$ are called the high order Changhee numbers of the first kind and the second kind.

High order Daehee polynomial of the first kind $D_n^{(k)}(x)$ and the second kind $\hat{D}_n^{(k)}(x)$ has the following generating function

$$\sum_{n=0}^{\infty} D_n^{(k)}(x) \frac{t^n}{n!} = \left(\frac{\ln(1+t)}{t}\right)^k (1+t)^x, \tag{4}$$

$$\sum_{n=0}^{\infty} \hat{D}_n^{(k)}(x) \frac{t^n}{n!} = \left(\frac{(1+t)\ln(1+t)}{t}\right)^k (1+t)^x. \tag{5}$$

when $x = 0$, $D_n^{(k)} = D_n^{(k)}(0)$ and $\hat{D}_n^{(k)} = \hat{D}_n^{(k)}(0)$ are called the high order Daehee numbers of the first kind and the second kind.

High order Apostol Changhee polynomial $Ch_n^{(k)}(x:\lambda)$ has the following generating function

$$\sum_{n=0}^{\infty} Ch_n^{(k)}(x:\lambda) \frac{t^n}{n!} = \left(\frac{2}{\lambda(1+t)+1}\right)^k (1+t)^x. \tag{6}$$

when $x = 0$, $Ch_n^{(k)}(\lambda) = Ch_n^{(k)}(0:\lambda)$ are called the high order Apostol Changhee numbers.

High order Apostol Daehee polynomial $D_n^{(k)}(x:\lambda)$ has the following generating function

$$\sum_{n=0}^{\infty} D_n^{(k)}(x:\lambda) \frac{t^n}{n!} = \left(\frac{\ln(1+t)}{\lambda(1+t)-1}\right)^k (1+t)^x. \tag{7}$$

when $x = 0$, $D_n^{(k)}(\lambda) = D_n^{(k)}(0:\lambda)$ are called the high order Apostol Daehee numbers.

High order Apostol Bernoulli polynomial $B_n^{(k)}(x:\lambda)$ has the following generating function

$$\sum_{n=0}^{\infty} B_n^{(k)}(x:\lambda)\frac{t^n}{n!} = \left(\frac{t}{\lambda e^t - 1}\right)^k e^{xt}. \tag{8}$$

when $x=0$, $B_n^{(k)}(\lambda) = B_n^{(k)}(0:\lambda)$ are called the high order Apostol Bernoulli numbers.

High order Apostol Euler polynomial $E_n^{(k)}(x:\lambda)$ has the following generating function

$$\sum_{n=0}^{\infty} E_n^{(k)}(x:\lambda)\frac{t^n}{n!} = \left(\frac{2}{\lambda e^t + 1}\right)^k e^{xt}. \tag{9}$$

when $x=0$, $E_n^{(k)}(\lambda) = E_n^{(k)}(0:\lambda)$ are called the high order Apostol Euler numbers.

High order Apostol Genocchi polynomial $G_n^{(k)}(x:\lambda)$ has the following generating function

$$\sum_{n=0}^{\infty} G_n^{(k)}(x:\lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda e^t + 1}\right)^k e^{xt}. \tag{10}$$

when $x=0$, $G_n^{(k)}(\lambda) = G_n^{(k)}(0:\lambda)$ are called the high order Apostol Genocchi numbers.

When $\lambda = 1$ in (10), we get the generating function of high order Genocchi polynomials $G_n^{(k)}(x)$

$$\sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!} = \left(\frac{2t}{e^t + 1}\right)^k e^{xt}. \tag{11}$$

The degenerate Changhee polynomials of the second kind $Ch_{n,\lambda}(x)$ have the following generating function

$$\sum_{n=0}^{\infty} Ch_{n,\lambda}(x)\frac{t^n}{n!} = \frac{2}{1+\left(1+\lambda\ln(1+t)\right)^{\frac{1}{\lambda}}}\left(1+\lambda\ln(1+t)\right)^{\frac{x}{\lambda}}. \tag{12}$$

The degenerate Euler polynomial $E_{n,\lambda}(x)$ has the following generating function

$$\sum_{n=0}^{\infty} E_{n,\lambda}(x)\frac{t^n}{n!} = \frac{2}{1+(1+\lambda t)^{\frac{1}{\lambda}}}(1+\lambda t)^{\frac{x}{\lambda}}. \tag{13}$$

The degenerate Genocchi polynomial $G_{n,\lambda}(x)$ has the following generating function

$$\sum_{n=0}^{\infty} G_{n,\lambda}(x)\frac{t^n}{n!} = \frac{2t}{1+(1+\lambda t)^{\frac{1}{\lambda}}}(1+\lambda t)^{\frac{x}{\lambda}}. \tag{14}$$

The degenerate Changhee-Genocchi polynomial of the second kind $CG_{n,\lambda}(x)$ has the following generating function

$$\sum_{n=0}^{\infty} CG_{n,\lambda}(x)\frac{t^n}{n!} = \frac{2\ln(1+t)}{1+\left(1+\lambda\ln(1+t)\right)^{\frac{1}{\lambda}}}\left(1+\lambda\ln(1+t)\right)^{\frac{x}{\lambda}}. \tag{15}$$

The degenerate stirling numbers of first kind and second kind have following generating function

$$\sum_{n\geq k} s_{1,\lambda}(n,k)\frac{t^n}{n!}=\frac{\left(\frac{1}{\lambda}\left((1+t)^{\lambda}-1\right)\right)^k}{k!}, \tag{16}$$

$$\sum_{n\geq k} S_{2,\lambda}(n,k)\frac{t^n}{n!}=\frac{\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)^k}{k!}. \tag{17}$$

Generalized Harmonic polynomial $H_n^{(r)}(z)$ has the following generating function

$$\sum_{n=r}^{\infty} H_n^{(r)}(z)t^n=\frac{(-\ln(1-t))^{r+1}}{t(1-t)}(1-t)^z. \tag{18}$$

with $H_0^{(r)}(z)=1$, and we also obtain when $r=0, z=0$, $H_{n-1}^{(0)}(0)=H_n \quad (n\geq 1)$.

Let $n\geq k+r$, the combinatorial numbers $P(r,n+k,k)$ has the following generating function

$$\sum_{n=0}^{\infty}\binom{n+k}{k}P(r,n+k,k)t^n=\frac{(-\ln(1-t))^r}{(1-t)^{k+1}}. \tag{19}$$

Lemma 1. If $D(g(t),f(t))=(d_{n,k})_{n.k\in N}$ is a Riordan array and $h(t)$ is the generating function of the sequence $\{(h_k)_{k\in N}\}$, then we have ([18])

$$\sum_{k=0}^{n} d_{n,k}h_k=\left[t^n\right]g(t)h(f(t)). \tag{20}$$

2. Some Identities Involving Generalized Harmonic Numbers $H_{n,k,r}(\alpha,\beta)$

In this part, using generating functions and coefficient method we discuss some interesting relationships of generalized harmonic numbers $H_{n,k,r}(\alpha,\beta)$.

Theorem 2.1. Let n is a nonnegative integer, we have

$$\sum_{j=0}^{n} H_{j,k,r}(\alpha,\beta)\frac{(-2\beta)^{n-j}Ch_{n-j}^{(m)}}{(n-j)!}=H_{n,k+m,r}(\alpha,\beta), \tag{21}$$

$$\sum_{j=0}^{n} H_{j,k,r}(\alpha,\beta)\frac{(-2\beta)^{n-j}\hat{C}h_{n-j}^{(m)}}{(n-j)!}=\sum_{i=0}^{n}(-2\beta)^{n-i}\binom{m}{n-i}H_{i,k+m,r}(\alpha,\beta). \tag{22}$$

Proof. By (1) and (2), we get

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n} H_{j,k,r}(\alpha,\beta)\frac{(-2\beta)^{n-j}Ch_{n-j}^{(m)}}{(n-j)!}t^n=\frac{(-\ln(1-\alpha t))^r}{(1-\beta t)^{k+m}}=\sum_{n=0}^{\infty}H_{n,k+m,r}(\alpha,\beta)t^n.$$

Comparing the coefficients of t^n in both sides of the last equation, we get the identity. (22) can be obtained in the same way.

Corollary 2.1. For $m=1$ in Theorem 2.1, we obtain the following identities

$$\sum_{j=0}^{n} H_{j,k,r}(\alpha,\beta)\frac{(-2\beta)^{n-j} Ch_{n-j}}{(n-j)!} = H_{n,k+1,r}(\alpha,\beta), \tag{23}$$

$$\sum_{j=0}^{n} H_{j,k,r}(\alpha,\beta)\frac{(-2\beta)^{n-j} \hat{C}h_{n-j}}{(n-j)!} = H_{n,k+1,r}(\alpha,\beta) - 2\beta H_{n-1,k+1,r}(\alpha,\beta). \tag{24}$$

Corollary 2.2. For $\beta = \alpha$ in Corollary 2.1, we obtain the following identities

$$\sum_{j=0}^{n} \frac{(-2)^{n-j} H_{j,k,r}(\alpha,\alpha) Ch_{n-j}}{\alpha^{j}(n-j)!} = \binom{n+k}{k} P(r,n+k,k), \tag{25}$$

$$\begin{aligned}&\sum_{j=0}^{n} \frac{(-2)^{n-j} H_{j,k,r}(\alpha,\alpha) \hat{C}h_{n-j}}{\alpha^{j}(n-j)!} \\ &= \binom{n+k}{k} P(r,n+k,k) - 2\beta \binom{n+k-1}{k} P(r,n+k-1,k).\end{aligned} \tag{26}$$

Theorem 2.2. Let n is a nonnegative integer, we have

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{j-h,k,r}(\alpha,\beta)\frac{(-2\alpha)^{n-j} Ch_{n-j}^{(m)}}{(n-j)!} = H_{n,m,r}(\alpha,\alpha), \tag{27}$$

$$\begin{aligned}&\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{j-h,k,r}(\alpha,\beta)\frac{(-2\alpha)^{n-j} \hat{C}h_{n-j}^{(m)}}{(n-j)!} \\ &= \sum_{i=0}^{n}(-2\alpha)^{n-i}\binom{m}{n-i} H_{i,m,r}(\alpha,\alpha).\end{aligned} \tag{28}$$

Proof. By (1) and (2), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{j-h,k,r}(\alpha,\beta)\frac{(-2\alpha)^{n-j} Ch_{n-j}^{(m)}}{(n-j)!} t^{n} \\ &= \frac{(-\ln(1-\alpha t))^{r}}{(1-\alpha t)^{m}} = \sum_{n=0}^{\infty} H_{n,m,r}(\alpha,\alpha) t^{n}.\end{aligned}$$

Comparing the coefficients of t^n in both sides of the last equation, we get the identity. (28) can be obtained in the same way.

Corollary 2.3 For $m = 1$ in Theorem 2.2, we obtain the following identities

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{j-h,k,r}(\alpha,\beta)\frac{(-2\alpha)^{n-j} Ch_{n-j}}{(n-j)!} = \alpha^{n} H(n,r-1), \tag{29}$$

$$\begin{aligned}&\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{j-h,k,r}(\alpha,\beta)\frac{(-2\alpha)^{n-j} \hat{C}h_{n-j}}{(n-j)!} \\ &= \alpha^{n}\left(H(n,r-1) - 2\alpha H(n-1,r-1)\right).\end{aligned} \tag{30}$$

For $\beta = \alpha = m = 1$ in (27), the Theorem 2.3 in [5] is as follows

$$\begin{aligned}&\sum_{j=0}^{n}(-2)^{n-k}\frac{Ch_{n-j}}{(n-j)!}\binom{n+k-1}{k-1} P(r,n+k-1,k-1) \\ &= \binom{n+k+r-1}{k+r-1} P(r,n+k+r-1,k+r-1).\end{aligned} \tag{31}$$

Theorem 2.3. Let n be a nonnegative integer, we have

$$\sum_{j=0}^{n} H_{j,k,r}(\alpha,\beta)\frac{(-\alpha)^{n-j}D_{n-j}^{(m)}}{(n-j)!}=\frac{H_{n+m,k,r+m}(\alpha,\beta)}{\alpha^{m}}, \tag{32}$$

$$\sum_{j=0}^{n} H_{j,k,r}(\alpha,\beta)\frac{(-\alpha)^{n-j}\hat{D}_{n-j}^{(m)}}{(n-j)!}=\sum_{i=0}^{n}(-1)^{n-i}\alpha^{n-m-i}\binom{m}{n-i}H_{i+m,k,r+m}(\alpha,\beta). \tag{33}$$

Proof. By (1) and (4), we get

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n} H_{j,k,r}(\alpha,\beta)\frac{(-\alpha)^{n-j}D_{n-j}^{(m)}}{(n-j)!}t^{n}=\frac{(-\ln(1-\alpha t))^{r}}{(1-\beta t)^{k}}\left(\frac{\ln(1-\alpha t)}{-\alpha t}\right)^{m}$$

$$=\frac{1}{(\alpha t)^{m}}\frac{(-\ln(1-\alpha t))^{r+m}}{(1-\beta t)^{k}}=\frac{1}{\alpha^{m}}\sum_{n=m}^{\infty}H_{n+m,k,r+m}(\alpha,\beta)t^{n}.$$

Comparing the coefficients of t^{n} in both sides of the last equation, we get the identity. (33) can be obtained in the same way.

Corollary 2.4. For $m=1$ in Theorem 2.3, we obtain the following identities

$$\sum_{j=0}^{n} H_{j,k,r}(\alpha,\beta)\frac{(-\alpha)^{n-j}D_{n-j}}{(n-j)!}=\frac{H_{n+1,k,r+1}(\alpha,\beta)}{\alpha}, \tag{34}$$

$$\sum_{j=0}^{n} H_{j,k,r}(\alpha,\beta)\frac{(-\alpha)^{n-j}\hat{D}_{n-j}}{(n-j)!}=\frac{H_{n+1,k,r+1}(\alpha,\beta)}{\alpha}-H_{n,k,r+1}(\alpha,\beta). \tag{35}$$

Corollary 2.5. For $\beta=\alpha$ in Corollary 2.4, we obtain the following identities

$$\sum_{j=0}^{n} H_{j,k,r}(\alpha,\alpha)\frac{(-\alpha)^{n-j}D_{n-j}}{(n-j)!}=\alpha^{n}\binom{n+k}{k-1}P(r+1,n+k,k-1), \tag{36}$$

$$\sum_{j=0}^{n} H_{j,k,r}(\alpha,\alpha)\frac{(-\alpha)^{n-j}\hat{D}_{n-j}}{(n-j)!}=\alpha^{n}\binom{n+k-1}{k-2}P(r+1,n+k-1,k-2). \tag{37}$$

Theorem 2.4. Let n be a nonnegative integer, we have

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h}H_{j-h,k,r+1}(\alpha,\beta)\frac{(-\alpha)^{n-j}D_{n-j}^{(m)}(x)}{(n-j)!}=\alpha^{n}H_{n+m-1}^{(m+r)}(1+x), \tag{38}$$

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h}H_{j-h,k,r+1}(\alpha,\beta)\frac{(-\alpha)^{n-j}\hat{D}_{n-j}^{(m)}(x)}{(n-j)!}=\alpha^{n}H_{n+m-1}^{(m+r)}(1+x+m). \tag{39}$$

Proof. By (1) and (4), we get

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h}H_{j-h,k,r+1}(\alpha,\beta)\frac{(-\alpha)^{n-j}D_{n-j}^{(m)}(x)}{(n-j)!}t^{n}$$

$$=(-\ln(1-\alpha t))^{r+1}\left(\frac{\ln(1-\alpha t)}{-\alpha t}\right)^{m}(1-\alpha t)^{x}$$

$$=\frac{1}{(\alpha t)^{m-1}}\frac{(-\ln(1-\alpha t))^{m+r+1}}{\alpha t(1-\alpha t)^{-x}}=\sum_{n=r+1}^{\infty}\alpha^{n}H_{n+m-1}^{(m+r)}(1+x)t^{n}.$$

Comparing the coefficients of t^{n} in both sides of the last equation, we get the identity. (39) can be obtained in the same way.

Corollary 2.7. For $m=1$ in Theorem 2.4, we obtain the following identities

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h}H_{j-h,k,r+1}(\alpha,\beta)\frac{(-\alpha)^{n-j}D_{n-j}(x)}{(n-j)!}=\alpha^{n}H_{n}^{(r+1)}(1+x), \tag{40}$$

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h}H_{j-h,k,r+1}(\alpha,\beta)\frac{(-\alpha)^{n-j}\hat{D}_{n-j}(x)}{(n-j)!}=\alpha^{n}H_{n}^{(r+1)}(2+x). \tag{41}$$

Corollary 2.8. For $x=0$ in Corollary 2.7, we obtain the following identities

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}\frac{(-1)^{n+h-j}\beta^{h}H_{j-h,k,r+1}(\alpha,\beta)D_{n-j}}{\alpha^{j}(n-j)!}=\frac{(r+2)!}{(n+1)!}\left|s(n+1,r+2)\right|, \tag{42}$$

$$\begin{aligned}&\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}\frac{(-1)^{n+h-j}\beta^{h}H_{j-h,k,r+1}(\alpha,\beta)\hat{D}_{n-j}}{\alpha^{j}(n-j)!}\\&=\frac{(r+2)!}{n!}\left(\frac{\left|s(n+1,r+2)\right|}{n+1}-\left|s(n,r+2)\right|\right).\end{aligned} \tag{43}$$

Theorem 2.5. Let $n\geq r\geq 1$ be a nonnegative integer, we have

$$\sum_{i=0}^{n-r}\binom{n-r}{i}\frac{(-1)^{n-r}\alpha^{n-i}(2\beta)^{i}Ch_{i}^{(k)}D_{n-r-i}^{(r)}}{(n-r)!}=H_{n,k,r}(\alpha,\beta). \tag{44}$$

Proof. By (1), (2) and (4), we get

$$\begin{aligned}\sum_{n=0}^{\infty}H_{n,k,r}(\alpha,\beta)t^{n}&=\frac{(-\ln(1-\alpha t))^{r}}{(1-\beta t)^{k}}=\left(\frac{1}{1-\beta t}\right)^{k}\left(\frac{-\ln(1-\alpha t)}{\alpha t}\right)^{r}(\alpha t)^{r}\\&=\sum_{n=0}^{\infty}Ch_{n}^{(k)}\frac{(-2\beta)^{n}t^{n}}{n!}\sum_{n=0}^{\infty}D_{n}^{(r)}\frac{(-\alpha)^{n}t^{n}}{n!}(\alpha t)^{r}\\&=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}(-1)^{n}\alpha^{n+r-i}(2\beta)^{i}Ch_{i}^{(k)}D_{n-i}^{(r)}\frac{t^{n+r}}{n!}\\&=\sum_{n=r}^{\infty}\sum_{i=0}^{n-r}\binom{n-r}{i}\frac{(-1)^{n-r}\alpha^{n-i}(2\beta)^{i}Ch_{i}^{(k)}D_{n-r-i}^{(r)}}{(n-r)!}t^{n}.\end{aligned}$$

Comparing the coefficients of t^{n} in both sides of the last equation, we get the identity.

Corollary 2.9. For $\alpha=\beta=k=r=1$ in Theorem 2.5, we obtain the following identities

$$\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{(-1)^{n-1}2^{i}Ch_{i}D_{n-i-1}}{(n-1)!}=H_{n},(n>1),H_{1}=0. \tag{45}$$

3. Identities about Generalized Harmonic Number $H_{n,k,r}(\alpha,\beta)$

In this part, using Riordan arrays, we derive some new equalities between Generalized Harmonic number $H_{n,k,r}(\alpha,\beta)$ and Apostol Bernoulli polynomials, Apostol Euler polynomials, Apostol Genocchi polynomials.

Theorem 3.1. Let n be a nonnegative integer, we have

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{j-h,k,r}(\alpha,\beta)\frac{(-1)^{j} B_{j}^{(m)}(x:\lambda)}{j!}$$
$$=\alpha^{n}\sum_{l=0}^{n}\lambda^{l}\binom{l+m-1}{l}H_{n-1}^{(m-1)}(x+l+1)=\frac{(-\alpha)^{n} D_{n}^{(m)}(x:\lambda)}{n!}. \tag{46}$$

Proof. An interesting Riordan arrays, associated with the $H_{n,k,r}(\alpha,\beta)$ are defined by

$$\Re\left\{\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{j-h,k,r}(\alpha,\beta)\right\}=\left(1,-\ln(1-\alpha t)\right). \tag{47}$$

On the one hand, by (8), (47) and Lemma 1, we get

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{j-h,k,r}(\alpha,\beta)\frac{(-1)^{j} B_{j}^{(m)}(x:\lambda)}{j!}$$
$$=\left[t^{n}\right]\left[\left(\frac{-y}{\lambda e^{-y}-1}\right)^{m} e^{-xy} \mid y=-\ln(1-\alpha t)\right]$$
$$=\left[t^{n}\right]\left(\frac{\ln(1-\alpha t)}{\lambda(1-\alpha t)-1}\right)^{m}(1-\alpha t)^{x}=\frac{(-\alpha)^{n} D_{n}^{(m)}(x:\lambda)}{n!}.$$

On the other hand, we get

$$\left[t^{n}\right]\left(\frac{\ln(1-\alpha t)}{\lambda(1-\alpha t)-1}\right)^{m}(1-\alpha t)^{x}$$
$$=\left[t^{n}\right]\frac{\left(-\ln(1-\alpha t)\right)^{m}}{(1-\alpha t)^{-x}}\left(1-\lambda(1-\alpha t)\right)^{-m}$$
$$=\left[t^{n}\right]\frac{\left(-\ln(1-\alpha t)\right)^{m}}{(1-\alpha t)^{-x-l}}\sum_{l=0}^{m}\lambda^{l}\binom{l+m-1}{l}$$
$$=\alpha\left[t^{n-1}\right]\frac{\left(-\ln(1-\alpha t)\right)^{m}}{\alpha t(1-\alpha t)^{-x-l}}\sum_{l=0}^{m}\lambda^{l}\binom{l+m-1}{l}$$
$$=\alpha^{n}\sum_{l=0}^{m}\lambda^{l}\binom{l+m-1}{l}H_{n-1}^{(m-1)}(x+l+1).$$

which completes the proof.

Corollary 3.1. For $\lambda=1$ in Theorem 3.1, we obtain the following identities

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{j-h,k,r}(\alpha,\beta)\frac{(-1)^{j} B_{j}^{(m)}(x)}{j!}$$
$$=\alpha^{n}\sum_{l=0}^{n}\binom{l+m-1}{l}H_{n-1}^{(m-1)}(x+l+1)=\frac{(-\alpha)^{n} D_{n}^{(m)}(x)}{n!}. \tag{48}$$

Corollary 3.2. For $m=1$ in Corollary 3.1, we obtain the following identities

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{j-h,k,r}(\alpha,\beta)\frac{(-1)^{j} B_{j}(x)}{j!}$$
$$=\alpha^{n}\sum_{l=0}^{n}H_{n-1}(x+l+1)=\frac{(-\alpha)^{n} D_{n}(x)}{n!}. \tag{49}$$

Corollary 3.3. For $x=0$ in Corollary 3.2, we obtain the following identities

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{j-h,k,r}(\alpha,\beta)\frac{(-1)^{j} B_{j}}{j!}=\alpha^{n}\sum_{l=0}^{n} H_{n}^{(-l)}=\frac{(-\alpha)^{n} D_{n}}{n!}. \quad (50)$$

Theorem 3.2. Let n be a nonnegative integer, we have

$$\begin{aligned}&\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{j-h,k,r}(\alpha,\beta)\frac{(-2)^{j} B_{j}^{(m)}(x:\lambda^{2})}{j!}\\&=\frac{(-\alpha)^{n}}{n!}\sum_{i=0}^{n}\binom{n}{i} Ch_{i}^{(m)}(x:\lambda) D_{n-i}^{(m)}(x:\lambda).\end{aligned} \quad (51)$$

Proof. By (8), (47) and Lemma 1, we get

$$\begin{aligned}&\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{j-h,k,r}(\alpha,\beta)\frac{(-2)^{j} B_{j}^{(m)}(x:\lambda^{2})}{j!}\\&=\left[t^{n}\right]\left[\left(\frac{-2y}{\lambda^{2}\mathrm{e}^{-2y}-1}\right)^{m}\mathrm{e}^{-2xy}\mid y=-\ln(1-\alpha t)\right]\\&=\left[t^{n}\right]\left(\frac{2\ln(1-\alpha t)}{\lambda^{2}(1-\alpha t)^{2}-1}\right)^{m}(1-\alpha t)^{2x}\\&=\left[t^{n}\right]\left(\frac{2}{\lambda(1-\alpha t)+1}\right)^{m}\left(\frac{\ln(1-\alpha t)}{\lambda(1-\alpha t)-1}\right)^{m}(1-\alpha t)^{2x}\\&=\sum_{i=0}^{n}\left[t^{i}\right]\left(\frac{2}{\lambda(1-\alpha t)+1}\right)^{m}(1-\alpha t)^{x}\left[t^{n-i}\right]\left(\frac{\ln(1-\alpha t)}{\lambda(1-\alpha t)-1}\right)^{m}(1-\alpha t)^{x}\\&=\sum_{i=0}^{n}(-\alpha)^{n}\frac{Ch_{i}^{(m)}(x:\lambda)}{i!}\frac{D_{n-i}^{(m)}(x:\lambda)}{(n-i)!}=\frac{(-\alpha)^{n}}{n!}\sum_{i=0}^{n}\binom{n}{i} Ch_{i}^{(m)}(x:\lambda) D_{n-i}^{(m)}(x:\lambda).\end{aligned}$$

which completes the proof.

Corollary 3.4. For $\lambda=1$ in Theorem 3.2, we obtain the following identities

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{j-h,k,r}(\alpha,\beta)\frac{(-2)^{j} B_{j}^{(m)}(x)}{j!}=\frac{(-\alpha)^{n}}{n!}\sum_{i=0}^{n}\binom{n}{i} Ch_{i}^{(m)}(x) D_{n-i}^{(m)}(x). \quad (52)$$

Corollary 3.5. For $m=1$ in Corollary 3.4, we obtain the following identities

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{j-h,k,r}(\alpha,\beta)\frac{(-2)^{j} B_{j}(x)}{j!}=\frac{(-\alpha)^{n}}{n!}\sum_{i=0}^{n}\binom{n}{i} Ch_{i}(x) D_{n-i}(x). \quad (53)$$

Corollary 3.6. For $x=0$ in Corollary 3.5, we obtain the following identities

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{j-h,k,r}(\alpha,\beta)\frac{(-2)^{j} B_{j}}{j!}=\frac{(-\alpha)^{n}}{n!}\sum_{i=0}^{n}\binom{n}{i} Ch_{i} D_{n-i}. \quad (54)$$

Theorem 3.3. Let $n\geq i\geq 1$ be a nonnegative integer, we have

$$\begin{aligned}&\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{j-h,k,r}(\alpha,\beta)\frac{(-1)^{j} E_{j}^{(m)}(x:\lambda)}{j!}\\&=2^{m}\sum_{i=0}^{n}\frac{(-1)^{n-i}\alpha^{n}\lambda^{i}}{(\lambda+1)^{m+i}}\binom{i+m-1}{i}\binom{x}{n-i}=\frac{(-\alpha)^{n} Ch_{n}^{(m)}(x:\lambda)}{n!}.\end{aligned} \quad (55)$$

Proof. On the one hand, by (9), (47) and Lemma 1, we get

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h}H_{j-h,k,r}(\alpha,\beta)\frac{(-1)^{j}E_{j}^{(m)}(x:\lambda)}{j!}$$
$$=\left[t^{n}\right]\left[\left(\frac{2}{\lambda e^{-y}+1}\right)^{m}e^{-xy}\mid y=-\ln(1-\alpha t)\right]$$
$$=\left[t^{n}\right]\left(\frac{2}{\lambda(1-\alpha t)+1}\right)^{m}(1-\alpha t)^{x}=\frac{(-\alpha)^{n}Ch_{n}^{(m)}(x:\lambda)}{n!}.$$

On the other hand, we get

$$\left[t^{n}\right]\left(\frac{2}{\lambda(1-\alpha t)+1}\right)^{m}(1-\alpha t)^{x}$$
$$=\left[t^{n}\right]\left(\frac{2}{\lambda}\right)^{m}\left(\frac{\lambda+1}{\lambda}-\alpha t\right)^{-m}(1-\alpha t)^{x}$$
$$=\left(\frac{2}{\lambda}\right)^{m}\sum_{i=0}^{n}\left[t^{i}\right]\left(\frac{\lambda+1}{\lambda}-\alpha t\right)^{-m}\left[t^{n-i}\right](1-\alpha t)^{x}$$
$$=\left(\frac{2}{\lambda+1}\right)^{m}\sum_{i=0}^{n}\left[t^{i}\right]\left(1-\frac{\lambda\alpha}{\lambda+1}t\right)^{-m}\left[t^{n-i}\right](1-\alpha t)^{x}$$
$$=\left(\frac{2}{\lambda+1}\right)^{m}\sum_{i=0}^{n}\binom{i+m-1}{i}\left(\frac{\lambda\alpha}{\lambda+1}\right)^{i}\binom{x}{n-i}(-\alpha)^{n-i}$$
$$=2^{m}\sum_{i=0}^{n}\frac{(-1)^{n-i}\alpha^{n}\lambda^{i}}{(\lambda+1)^{m+i}}\binom{i+m-1}{i}\binom{x}{n-i}.$$

which completes the proof.

Corollary 3.7. For $\lambda=1$ in Theorem 3.3, we obtain the following identities

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h}H_{j-h,k,r}(\alpha,\beta)\frac{(-1)^{j}E_{j}^{(m)}(x)}{j!}$$
$$=\sum_{i=0}^{n}\frac{(-1)^{n-i}\alpha^{n}}{2^{i}}\binom{i+m-1}{i}\binom{x}{n-i}=\frac{(-\alpha)^{n}Ch_{n}^{(m)}(x)}{n!}. \tag{56}$$

Corollary 3.8. For $x=0$ in Corollary 3.7, we obtain the following identities

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h}H_{j-h,k,r}(\alpha,\beta)\frac{(-1)^{j}E_{j}^{(m)}}{j!}=\frac{\alpha^{n}}{2^{n}}\binom{n+m-1}{n}=\frac{(-\alpha)^{n}Ch_{n}^{(m)}}{n!}. \tag{57}$$

Corollary 3.9. For $m=1$ in Corollary 3.8, we obtain the following identities

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h}H_{j-h,k,r}(\alpha,\beta)\frac{(-1)^{j}E_{j}}{j!}=\frac{\alpha^{n}}{2^{n}}=\frac{(-\alpha)^{n}Ch_{n}}{n!}. \tag{58}$$

Theorem 3.4. Let n be a nonnegative integer, we have

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}\frac{(-1)^{m+h+j}\beta^{h}H_{j-h,k,r}(\alpha,\beta)G_{j}^{(m)}(x:\lambda)}{j!}$$
$$=\sum_{i=0}^{n}\frac{(-\alpha)^{n}Ch_{n-i}^{(m)}(x+1,\lambda)H(i,m-1)}{(n-i)!}. \tag{59}$$

Proof. By (10), (47) and Lemma 1, we get

$$\begin{aligned}
&\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{j-h,k,r}(\alpha,\beta)\frac{(-1)^{j} G_{j}^{(m)}(x:\lambda)}{j!}\\
&=\left[t^{n}\right]\left[\left(\frac{-2y}{\lambda e^{-y}+1}\right)^{m} e^{-xy} \mid y=-\ln(1-\alpha t)\right]\\
&=\left[t^{n}\right]\left(\frac{2\ln(1-\alpha t)}{\lambda(1-\alpha t)+1}\right)^{m}(1-\alpha t)^{x}\\
&=\left[t^{n}\right]\left(\frac{2}{\lambda(1-\alpha t)+1}\right)^{m}(1-\alpha t)^{x+1}(-1)^{m}\frac{(-\ln(1-\alpha t))^{m}}{1-\alpha t}\\
&=(-1)^{m}\sum_{i=0}^{n}\left[t^{n-i}\right]\left(\frac{2}{\lambda(1-\alpha t)+1}\right)^{m}(1-\alpha t)^{x+1}\left[t^{i}\right]\frac{(-\ln(1-\alpha t))^{m}}{1-\alpha t}\\
&=(-1)^{m}\sum_{i=0}^{n}\frac{(-\alpha)^{n} Ch_{n-i}^{m}(x+1,\lambda)H(i,m-1)}{(n-i)!}.
\end{aligned}$$

which completes the proof.

Corollary 3.10. For $\lambda=1$ in Theorem 3.4, we obtain the following identities

$$\begin{aligned}
&\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}\frac{(-1)^{m+h+j}\beta^{h} H_{j-h,k,r}(\alpha,\beta) G_{j}^{(m)}(x)}{j!}\\
&=\sum_{i=0}^{n}\frac{(-\alpha)^{n} Ch_{n-i}^{(m)}(x+1)H(i,m-1)}{(n-i)!}.
\end{aligned}\tag{60}$$

Corollary 3.11. For $m=1$ in Corollary 3.10, we obtain the following identities

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}\frac{(-1)^{h+j+1}\beta^{h} H_{j-h,k,r}(\alpha,\beta) G_{j}(x)}{j!}=\sum_{i=0}^{n}\frac{(-\alpha)^{n} Ch_{n-i}(x+1)Hi}{(n-i)!}.\tag{61}$$

Corollary 3.12. For $x=0$ in Corollary 3.11, we obtain the following identities

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}\frac{(-1)^{h+j+1}\beta^{h} H_{j-h,k,r}(\alpha,\beta) G_{j}}{j!}=\sum_{i=0}^{n}\frac{(-\alpha)^{n}\hat{C}h_{n-i}Hi}{(n-i)!}.\tag{62}$$

Theorem 3.5. Let n be a nonnegative integer, we have

$$\begin{aligned}
&\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{j-h,k,r}(\alpha,\beta)\frac{(-1)^{j} G_{j}^{(m)}(2x)}{j!}\\
&=\frac{(-\alpha)^{n}}{(n-m)!}\sum_{i=0}^{n-m}\binom{n-m}{i} Ch_{i}^{(m)}(x) D_{n-m-i}^{(m)}(x).
\end{aligned}\tag{63}$$

Proof. By (11), (47) and Lemma 1, we get

$$\begin{aligned}
&\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{j-h,k,r}(\alpha,\beta)\frac{(-1)^{j} G_{j}^{(m)}(2x)}{j!}\\
&=\left[t^{n}\right]\left[\left(\frac{-2y}{e^{-y}+1}\right)^{m} e^{-2xy} \mid y=-\ln(1-\alpha t)\right]
\end{aligned}$$

$$=\left[t^n\right]\left(\frac{2\ln(1-\alpha t)}{2-\alpha t}\right)^m(1-\alpha t)^{2x}$$
$$=\left[t^n\right]\left(\frac{2}{2-\alpha t}\right)^m(1-\alpha t)^x\left(\frac{-\ln(1-\alpha t)}{\alpha t}\right)^m(1-\alpha t)^x(-\alpha t)^m$$
$$=\left[t^{n-m}\right](-\alpha)^m\left(\frac{2}{2-\alpha t}\right)^m(1-\alpha t)^x\left(\frac{\ln(1-\alpha t)}{-\alpha t}\right)^m(1-\alpha t)^x$$
$$=(-\alpha)^m\sum_{i=0}^{n}\left[t^i\right]\left(\frac{2}{2-\alpha t}\right)^m(1-\alpha t)^x\left[t^{n-m-i}\right]\left(\frac{\ln(1-\alpha t)}{-\alpha t}\right)^m(1-\alpha t)^x$$
$$=\sum_{i=0}^{n}(-\alpha)^n\frac{Ch_i^{(m)}(x)}{i!}\frac{D_{n-m-i}^{(m)}(x)}{(n-m-i)!}$$
$$=\frac{(-\alpha)^n}{(n-m)!}\sum_{i=0}^{n-m}\binom{n-m}{i}Ch_i^{(m)}(x)D_{n-m-i}^{(m)}(x).$$

which completes the proof.

Corollary 3.13. For $x=0$ in Theorem 3.5, we obtain the following identities

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{j-h,k,r}(\alpha,\beta)\frac{(-1)^j G_j^{(m)}}{j!}=\frac{(-\alpha)^n}{(n-m)!}\sum_{i=0}^{n-m}\binom{n-m}{i}Ch_i^{(m)}D_{n-m-i}^{(m)}. \quad (64)$$

Corollary 3.14. For $m=1$ in Corollary 3.13, we obtain the following identities

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{j-h,k,r}(\alpha,\beta)\frac{(-1)^j G_j}{j!}=\frac{(-\alpha)^n}{(n-1)!}\sum_{i=0}^{n-1}\binom{n-1}{i}Ch_iD_{n-i-1}. \quad (65)$$

4. Identities about Generalized Harmonic Number $H_{n,k,r}(\alpha,\beta)$

In this part, using generating functions and coefficient method, we derive the new identities involving Generalized Harmonic number $H_{n,k,r}(\alpha,\beta)$, with Degenerate Changhee polynomials, Degenerate Genocchi polynomials, Degenerate Changhee-Genocchi polynomials, Two kinds of degenerate Stirling numbers and so on.

Theorem 4.1. Let n be a nonnegative integer, we have

$$\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{j-h,k,r}(\alpha,\beta)\frac{(-\alpha)^{n-j}Ch_{n-j,\lambda}^{(r)}(x)}{(n-j)!}=\frac{(-1)^{n+r}\alpha^n CG_{n,\lambda}^{(r)}(x)}{n!}. \quad (66)$$

Proof. By (1) and (12), we get

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{j-h,k,r}(\alpha,\beta)\frac{(-\alpha)^{n-j}Ch_{n-j,\lambda}^{(r)}(x)}{(n-j)!}t^n$$
$$=\left(-\ln(1-\alpha t)\right)^r\left(\frac{2}{1+\left(1+\lambda\ln(1-\alpha t)\right)^{\frac{1}{\lambda}}}\right)^r\left(1+\lambda\ln(1-\alpha t)\right)^{\frac{x}{\lambda}}$$

$$=(-1)^r\left(\frac{2\ln(1-\alpha t)}{1+(1+\lambda\ln(1-\alpha t))^{\frac{1}{\lambda}}}\right)^r(1+\lambda\ln(1-\alpha t))^{\frac{x}{\lambda}}$$

$$=(-1)^r\sum_{n=0}^{\infty}CG_{n,\lambda}^{(r)}(x)\frac{(-\alpha t)^n}{n!}.$$

Comparing the coefficients of t^n in both sides of the last equation, we get the identity.

Theorem 4.2. Let n be a nonnegative integer, we have

$$\sum_{j=0}^{n}\sum_{m=0}^{j}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{j-h,k,m}(\alpha,\beta)\frac{(-1)^m(-\alpha)^{n-j}(x)_{m,\lambda}CG_{n-j,\lambda}}{m!(n-j)!}=\frac{(-\alpha)^n CG_{n,\lambda}(x)}{n!}. \tag{67}$$

The λ-analogue of falling factorial sequence is given by $(x)_{m\lambda}=x(x-\lambda)\cdots(x-(m-1)\lambda),(m\geq 0)$, $(x)_{0,\lambda}=1$.

Proof. By (14), we get

$$\sum_{n=0}^{\infty}CG_{n,\lambda}(x)\frac{(-\alpha t)^n}{n!}$$

$$=\frac{2\ln(1-\alpha t)}{1+(1+\lambda\ln(1-\alpha t))^{\frac{1}{\lambda}}}(1+\lambda\ln(1-\alpha t))^{\frac{x}{\lambda}}$$

$$=\sum_{n=0}^{\infty}CG_{n,\lambda}\frac{(-\alpha t)^n}{n!}\sum_{m=0}^{\infty}\binom{\frac{x}{\lambda}}{m}\lambda^m\ln^m(1-\alpha t)$$

$$=\sum_{n=0}^{\infty}CG_{n,\lambda}\frac{(-\alpha t)^n}{n!}\sum_{m=0}^{\infty}\frac{(-1)^m(x)_{m,\lambda}}{m!}(-\ln(1-\alpha t))^m$$

$$=\sum_{n=0}^{\infty}CG_{n,\lambda}\frac{(-\alpha t)^n}{n!}\sum_{m=0}^{\infty}\frac{(-1)^m(x)_{m,\lambda}}{m!}\sum_{n=0}^{\infty}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{n-h,k,m}(\alpha,\beta)t^n$$

$$=\sum_{n=0}^{\infty}CG_{n,\lambda}\frac{(-\alpha t)^n}{n!}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{n-h,k,m}(\alpha,\beta)\frac{(-1)^m(x)_{m,\lambda}}{m!}t^n$$

$$=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{m=0}^{j}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{j-h,k,m}(\alpha,\beta)\frac{(-1)^m(-\alpha)^{n-j}(x)_{m,\lambda}CG_{n-j,\lambda}}{m!(n-j)!}t^n.$$

Comparing the coefficients of t^n in both sides of the last equation, we get the identity.

Theorem 4.3. Let n be a nonnegative integer, we have

$$\sum_{m=0}^{n}\sum_{h=0}^{k}\binom{k}{h}\frac{(-1)^{m+h+1}\beta^h(x)_{m,\lambda}H_{n-h,k,n+m+1}(\alpha,\beta)}{m!}=\frac{(-\alpha)^n(CG_{n,\lambda}(x+1)+CG_{n,\lambda}(x))}{2n!}. \tag{68}$$

Proof. By (14), we get

$$\sum_{n=0}^{\infty}\left(CG_{n,\lambda}(x+1)+CG_{n,\lambda}(x)\right)\frac{(-\alpha t)^n}{n!}$$
$$=\frac{2\ln(1-\alpha t)}{1+(1+\lambda\ln(1-\alpha t))^{\frac{1}{\lambda}}}(1+\lambda\ln(1-\alpha t))^{\frac{x+1}{\lambda}}$$
$$+\frac{2\ln(1-\alpha t)}{1+(1+\lambda\ln(1-\alpha t))^{\frac{1}{\lambda}}}(1+\lambda\ln(1-\alpha t))^{\frac{x}{\lambda}}$$
$$=2\ln(1-\alpha t)(1+\lambda\ln(1-\alpha t))^{\frac{x}{\lambda}}$$
$$=2\ln(1-\alpha t)\sum_{m=0}^{\infty}\binom{\frac{x}{\lambda}}{m}\lambda^m\ln^m(1-\alpha t)$$
$$=2\sum_{m=0}^{\infty}\binom{\frac{x}{\lambda}}{m}\lambda^m(-1)^{m+1}(-\ln(1-\alpha t))^{m+1}$$
$$=2\sum_{m=0}^{\infty}\frac{(-1)^{m+1}(x)_{m,\lambda}}{m!}\sum_{n=0}^{\infty}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{n-h,k,m+1}(\alpha,\beta)t^n$$
$$=2\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{h=0}^{k}\binom{k}{h}\frac{(-1)^{m+h+1}\beta^h(x)_{m,\lambda}H_{n-h,k,m+1}(\alpha,\beta)}{m!}t^n.$$

Comparing the coefficients of t^n in both sides of the last equation, we get the identity.

Theorem 4.4. Let n be a nonnegative integer, we have

$$\sum_{m=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{n-h,k,m}(\alpha,\beta)\frac{(-1)^m E_{m,\lambda}^{(l)}(x)}{m!}=\frac{(-\alpha)^n Ch_{n,\lambda}^{(l)}(x)}{n!}. \quad (69)$$

Proof. By (1) and (13), we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{n-h,k,m}(\alpha,\beta)\frac{(-1)^m E_{m,\lambda}^{(l)}(x)}{m!}t^n$$
$$=\sum_{m=0}^{\infty}E_{m,\lambda}^{(l)}(x)\frac{(-1)^m}{m!}\sum_{n=0}^{\infty}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{n-h,k,m}(\alpha,\beta)t^n$$
$$=\sum_{m=0}^{\infty}E_{m,\lambda}^{(l)}(x)\frac{(-1)^m}{m!}(-\ln(1-\alpha t))^m=\sum_{m=0}^{\infty}E_{m,\lambda}^{(l)}(x)\frac{\ln^m(1-\alpha t)}{m!}$$
$$=\left(\frac{2}{1+(1+\lambda\ln(1-\alpha t))^{\frac{1}{\lambda}}}\right)^l(1+\lambda\ln(1-\alpha t))^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}Ch_{n,\lambda}^{(l)}(x)\frac{(-\alpha)^n t^n}{n!}.$$

Comparing the coefficients of t^n in both sides of the last equation, we get the identity.

Theorem 4.5. Let n be a nonnegative integer, we have

$$\sum_{m=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{n-h,k,m}(\alpha,\beta)\frac{(-1)^m G_{m,\lambda}^{(l)}(x)}{m!}=\frac{(-\alpha)^n CG_{n,\lambda}^{(l)}(x)}{n!}. \quad (70)$$

Proof. By (1) and (14), we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{n-h,k,m}(\alpha,\beta)\frac{(-1)^m G_{m,\lambda}^{(l)}(x)}{m!}t^n$$
$$=\sum_{m=0}^{\infty}G_{m,\lambda}^{(l)}(x)\frac{(-1)^m}{m!}\sum_{n=0}^{\infty}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{n-h,k,m}(\alpha,\beta)t^n$$
$$=\sum_{m=0}^{\infty}G_{m,\lambda}^{(l)}(x)\frac{(-1)^m}{m!}\left(-\ln(1-\alpha t)\right)^m=\sum_{m=0}^{\infty}G_{m,\lambda}^{(l)}(x)\frac{\ln^m(1-\alpha t)}{m!}$$
$$=\left(\frac{2\ln(1-\alpha t)}{1+\left(1+\lambda\ln(1-\alpha t)\right)^{\frac{1}{\lambda}}}\right)^l\left(1+\lambda\ln(1-\alpha t)\right)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}CG_{n,\lambda}^{(l)}(x)\frac{(-\alpha)^n t^n}{n!}.$$

Comparing the coefficients of t^n in both sides of the last equation, we get the identity.

The combined inversion relations are introduced below (see [19]):

$$f_n=\sum_{k=0}^{n}S(n,k)g_k \Leftrightarrow g_n=\sum_{k=0}^{n}s(n,k)f_k. \qquad (*)$$

where f_n and g_n are two sequences, expressed as following

$$f=\sum_{n=0}^{\infty}f_n\frac{t^n}{n!},\quad g=\sum_{n=0}^{\infty}g_n\frac{t^n}{n!}.$$

Theorem 4.6. Let $m\ge r\ge 1$ be a nonnegative integer, we have

$$\sum_{n=0}^{m}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{n-h,k,r}(\alpha,\beta)\frac{(-\lambda)^{n-r}n!s_{1,\lambda}(m,n)}{\alpha^n}=(m)_r D_{m-r}^{(r)}, \qquad (71)$$

$$\sum_{h=0}^{k}(-\beta)^h H_{n-h,k,r}(\alpha,\beta)\frac{(-\lambda)^{m-r}m!}{\alpha^m}=\sum_{n=0}^{m}(n)_r D_{n-r}^{(r)}S_{2,\lambda}(m,n). \qquad (72)$$

Proof. Let $t=\frac{1-(1+t)^\lambda}{\alpha}$, by (1) and (16), we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{n-h,k,r}(\alpha,\beta)\frac{(-\lambda)^n n!s_{1,\lambda}(m,n)}{\alpha^n}\frac{t^m}{m!}$$
$$=\sum_{n=0}^{\infty}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{n-h,k,r}(\alpha,\beta)\frac{(-\lambda)^n n!}{\alpha^n}\sum_{m=0}^{\infty}s_{1,\lambda}(m,n)\frac{t^m}{m!}$$
$$=\sum_{n=0}^{\infty}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{n-h,k,r}(\alpha,\beta)\frac{(-\lambda)^n n!}{\alpha^n}\frac{\left(\frac{1}{\lambda}\left((1+t)^\lambda-1\right)\right)^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^h H_{n-h,k,r}(\alpha,\beta)\left(\frac{1-(1+t)^\lambda}{\alpha}\right)^n$$
$$=\left(-\lambda\ln(1+t)\right)^r=(-\lambda)^r\left(\frac{\ln(1+t)}{t}\right)^r t^r=(-\lambda)^r\sum_{m=r}^{\infty}(m)_r D_{m-r}^{(r)}\frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ in both sides of the last equation, we get the identity.

In addition, from the inversion formula (*), we can get (72).

Theorem 4.7. Let $n \geq k \geq 1$ be a nonnegative integer, we have

$$\sum_{l=k}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{n-h,k,l}(\alpha,\beta)\frac{(-1)^{l}\lambda^{l-k}}{l!}S(l,k)=\frac{(-\alpha)^{n} s_{1,\lambda}(n,k)}{n!}, \tag{73}$$

$$\sum_{l=k}^{n}\frac{(-\alpha)^{l} s_{1,\lambda}(l,k)}{l!}s(l,k)=\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{n-h,k,l}(\alpha,\beta)\frac{(-1)^{n}\lambda^{n-k}}{n!}. \tag{74}$$

Proof. By (16), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} s_{1,\lambda}(n,k)\frac{(-\alpha)^{n} t^{n}}{n!} &= \frac{\left(\frac{1}{\lambda}\left((1-\alpha t)^{\lambda}-1\right)\right)^{k}}{k!} = \frac{1}{\lambda^{k}}\frac{\left(e^{\lambda \ln(1-\alpha t)}-1\right)^{k}}{k!} \\
&= \frac{1}{\lambda^{k}}\sum_{l=k}^{\infty} S(l,k)\frac{\left(\lambda \ln(1-\alpha t)\right)^{l}}{l!} = \sum_{l=k}^{\infty}\frac{(-1)^{l}\lambda^{l-k}S(l,k)}{l!}\left(-\ln(1-\alpha t)\right)^{l} \\
&= \sum_{l=k}^{\infty}\frac{(-1)^{l}\lambda^{l-k}S(l,k)}{l!}\sum_{n=0}^{\infty}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{n-h,k,l}(\alpha,\beta)t^{n} \\
&= \sum_{n=0}^{\infty}\sum_{l=k}^{n}\sum_{h=0}^{k}\binom{k}{h}(-\beta)^{h} H_{n-h,k,l}(\alpha,\beta)\frac{(-1)^{l}\lambda^{l-k}S(l,k)}{l!}t^{n}.
\end{aligned}$$

Comparing the coefficients of t^{n} in both sides of the last equation, we get the identity.

In addition, from the inversion formula (*), we can get (74).

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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