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A Family of the Exponential Attractors and the Inertial Manifolds for a Class of Generalized Kirchhoff Equations

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Abstract

In this paper, we studied a family of the exponential attractors and the inertial manifolds for a class of generalized Kirchhoff-type equations with strong dissipation term. After making appropriate assumptions for Kirchhoff stress term and nonlinear term, the existence of exponential attractor is obtained by proving the discrete squeezing property of the equation, then according to Hadamard's graph transformation method, the spectral interval condition is proved to be true, therefore, the existence of a family of the inertial manifolds for the equation is obtained.

Keywords

Kirchhoff-Type Equation, Spectral Interval Condition, A Family of the Exponential Attractors, A Family of the Inertial Manifolds

1. Introduction

In the study of dynamic behavior for a long time in infinite dimensional dynamical system, the exponential attractors and inertial manifolds play a very important role. In 1994, Foias [1] puts forward the concept of exponential attractor, it is a positive invariant compact set which has finite fractal dimension and attracts solution orbits at an exponential rate. Inertial manifold is finite dimensional invariant smooth manifolds that contain the global attractor and attract all solution orbits at an exponential rate, their corresponding inertial manifold forms are powerful tools which could study the property of finite dynamical system about the dissipative evolution equation. Under the restriction of inertial manifold, a infinite dimension dynamical system could be transformed to finite dimension, therefore, the inertial manifolds become an important bridge which can contact finite dimensional dynamical system and infinite dimensional dynamical system, many scholars have done a great deal of research, we could refer to ([2]-[8]).

Guigui Xu, Libo Wang and Guoguang Lin [9] studied global attractor and inertial manifold for the strongly damped wave equations

$$\begin{cases} u_{tt} - \alpha\Delta u + \beta\Delta^2 u - \gamma\Delta u_t + g(u) = f(x,t), (x,t) \in \Omega \times R^+, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \\ u|_{\partial\Omega} = 0, \Delta u|_{\partial\Omega} = 0, (x,t) \in \partial\Omega \times R^+. \end{cases}$$

The assumption of $g(u)$ satisfies the following conditions:

(H_1) $\lim_{|s|\to\infty} \inf \frac{G(s)}{s^2} \ge 0, s \in R, G(s) = \int_0^s g(r)\mathrm{d}r$;

(H_2) There is a positive constant C_1, such that

$\lim_{|s|\to\infty} \inf \frac{sg(s) - C_1 G(s)}{s^2} \ge 0, s \in R$.

Under these reasonable assumptions, according to Hadamard's graph transformation method, the existence of the inertial manifolds for the equation is obtained.

Zhijian Yang and Zhiming Liu [10] studied the existence of exponential attractor for the Kirchhoff equations with strong nonlinear strongly dissipation and supercritical nonlinearity

$$u_{tt} - \sigma\left(\|\nabla u\|^2\right)\Delta u_t - \phi\left(\|\nabla u\|^2\right)\Delta u + f(u) = h(x).$$

The main result was that the nonlinearity $f(u)$ is of supercritical growth and they established an exponential attractor in natural energy space by using a new method based on the weak quasi-stability estimates.

Ruijin Lou, Penghui Lv, Guoguang Lin [11] studied the exponential attractor and inertial manifold of a higher-order kirchhoff equations

$$u_{tt} + M\left(\|\nabla^m u\|_p^p\right)(-\Delta)^{2m} u + \beta(-\Delta)^{2m} u_t + g(u) = f(x),$$

$$u(x,t) = 0, \frac{\partial^i u}{\partial v^i} = 0, i = 1,2,\cdots,2m-1, x \in \partial\Omega,$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, t > 0.$$

where Ω is finite region of R^n, $\partial\Omega$ is smooth boundary, $u_0(x)$ and $u_1(x)$ is initial value, $(-\Delta)^m u_t$ is strongly damped term, ϕ is stress term, $g(u)$ is nonlinear source term.

On the basis of reference [11], the stress term $\|D^m u\|^2$ is extended to $\|D^m u\|_p^p$, this paper studied the long-time dynamic behavior of a class of generalized Kirchhoff equation. Firstly, the existence of the exponential attractor of this equation is proved. Furthermore, the existence of a family of inertial manifold is proved by using Hadamard's graph transformation method, more relevant research can be referred to ([12]-[17]).

In this paper, we study the existence of exponential attractors and a family of the inertial manifolds for a class of generalized Kirchhoff-type equation with damping term:

$$u_{tt} + M\left(\|\nabla^m u\|_p^p\right)(-\Delta)^{2m} u + \beta(-\Delta)^{2m} u_t + g(u) = f(x), \tag{1.1}$$

$$u(x,t) = 0, \frac{\partial^i u}{\partial v^i} = 0, i = 1,2,\cdots,2m-1, x \in \partial\Omega, \tag{1.2}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, t > 0. \tag{1.3}$$

where $m > 1$, $p \geq 2$, $\Omega \subset R^n (n \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$, $M(s) \in C^2([0,+\infty); R^+)$ is a real function, $\beta(-\Delta)^{2m} u_t (\beta > 0)$ denotes strong damping term, $g(u)$ is nonlinear source term, $f(x)$ denotes the external force term. The assumption of $M(s)$ and $g(u)$ as follow:

(A1) $g(u) \in C^\infty(R)$

(A2) $M(s) \in C^2([0,+\infty), R^+), 1 \leq \mu_0 < M(s) < \mu_1, \mu = \begin{cases} \mu_0, \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^{2m} u\|^2 \geq 0, \\ \mu_1, \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^{2m} u\|^2 < 0. \end{cases}$

where μ, μ_0, μ_1 are constant, λ_1 is the first eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary conditions on Ω.

For convenience, define the following spaces and notations $H = L^2(\Omega)$, $H_0^{2m}(\Omega) = H^{2m}(\Omega) \cap H_0^1(\Omega)$, $H_0^{4m}(\Omega) = H^{4m}(\Omega) \cap H_0^1(\Omega)$, $H_0^{2m+k}(\Omega) = H^{2m+k}(\Omega) \cap H_0^1(\Omega)$, $E_0 = H^{2m}(\Omega) \times L^2(\Omega)$, $E_k = H_0^{2m+k}(\Omega) \times H_0^k(\Omega)$, ($k = 1,2,\cdots,2m$), $f(x) \in L^2(\Omega)$. $(\cdot,\cdot)$ and $\|\cdot\|$ represent the inner product and norms of H respectively, *i.e.* $(u,v) = \int_\Omega u(x)v(x)\mathrm{d}x$, $(u,u) = \|u\|^2$, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_P = \|\cdot\|_{L^P(\Omega)}$, $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\Omega)}$.

2. Exponential Attractors

For brevity, define the inner product and norms as follow:

$\forall U_i = (u_i, v_i) \in E_0, i = 1,2$,

$$(U_1, U_2) = (\nabla^{2m} u_1, \nabla^{2m} u_2) + (v_1, v_2), \tag{2.1}$$

$$\|U\|_{E_0}^2 = (U,U)_{E_0} = \|\nabla^{2m} u\|^2 + \|v\|^2. \tag{2.2}$$

Let $U = (u,v) \in E_0$, $v = u_t + \varepsilon u$, $0 < \varepsilon \leq \min\left\{4-\beta, 1-\frac{\alpha}{\beta\lambda_1^{2m}}, \frac{\alpha}{1+\lambda_1^{-2m}}\right\}$, we can get the Equation (1.1) is equivalent to the following evolution equation

$$U_t + G(U) = F(U). \tag{2.3}$$

where

$$G = \begin{pmatrix} \varepsilon & -I \\ \varepsilon^2 - \beta\varepsilon(-\Delta)^{2m} + M\left(\|\nabla^m u\|_p^p\right)(-\Delta)^{2m} & \beta(-\Delta)^{2m} - \varepsilon \end{pmatrix},$$

$$F(U)=\begin{pmatrix}0\\ f(x)-g(u)\end{pmatrix}.$$

Then, we will use the following notations. Let E_0, E_k are two Hilbert spaces, we have $E_k \hookrightarrow E_0$ with dense and continuous injection, and $E_k \hookrightarrow E_0$ is compact. Let $S(t)$ is a map from $E_0(E_k)$ into $E_0(E_k)$.

In the following definitions, $k=1,2,\cdots,2m$.

Definition 2.1. [14] The semigroup $S(t)$ possesses a (E_k, E_0)-compact attractor A_k, If it exists a compact set $A_k \subset E_0$, A_k attracts all bounded subsets of E_k, and under the function of $S(t)$, A_k is an invariant set, *i.e.* $S(t)A_k = A_k, \forall t \ge 0$.

Definition 2.2. [14] If $A_k \subseteq M_k \subseteq B_k$ and 1) $S(t)M_k \subseteq M_k, \forall t \ge 0$; 2) M_k has finite fractal dimension, $d_F(M_k) < +\infty$; 3) there exist universal constants $c_1 > 0, c_2 > 0$, such that $\operatorname{dist}(S(t)B_k, M_k) \le c_1 \mathrm{e}^{-c_2 t}, t > 0$, where $\operatorname{dist}_{E_0}(A_k, B_k) = \sup_{x\in A_k}\inf_{y\in B_k}|x-y|_{E_0}$, $B_k \subset E_k$ is the positive invariant set of $S(t)$, the compact set $M_k \subset E_0$ is called a (E_k, E_0)-exponential attractor for the system $(S(t), B_k)$.

Definition 2.3. [14] if there exists limited function $l(t)$, such that

$$\|S(t)u - S(t)v\|_{E_0} \le l(t)\|u-v\|_{E_0}, \forall (u,v) \in B_k. \tag{2.4}$$

Then the semigroup $S(t)$ is Lipschitz continuous in B_k.

Definition 2.4. [14] If $\delta \in \left(0, \frac{1}{8}\right)$ and exists an orthogonal projection $P_N = P_N(\delta)$ of rank $N = N(\delta)$ such that for every $(u,v) \in B_k$,

$$\|S(t_*)u - S(t_*)v\|_{E_0} \le \delta\|u-v\|_{E_0}, \tag{2.5}$$

or

$$\|Q_N(S(t_*)u - S(t_*)v)\|_{E_0} \le \|P_N(S(t_*)u - S(t_*)v)\|_{E_0} \tag{2.6}$$

Then $S(t)$ is said to satisfy the discrete squeezing property, where $Q_N = I - P_N$.

Theorem 2.1. [15] Assume that 1) $S(t)$ possesses a (E_k, E_0)-compact attractor A_k; 2) it exists a positive invariant compact set $B_k \subset E_0$ of $S(t)$; 3) $S(t)$ is a Lipschitz continuous map with Lipschitz constant l on B_k, and satisfies the discrete squeezing property on B_k. Then $S(t)$ has a (E_k, E_0)-exponential attractor M_k, and $M_k \supseteq A_k$ on B_k, and $M_k = \bigcup_{0\le t\le t_*} S(t)M_*$, $M_* = A_k \cup \left(\bigcup_{j=1}^{\infty}\bigcup_{i=1}^{\infty} S(t_*)^j \left(E^{(i)}\right)\right)$. Moreover, the fractal dimension of M_k satisfies $d_F(M_k) \le c_3 N_0 + 1$, $\operatorname{dist}_{E_0}(S(t)B, M_k) \le c_1 \mathrm{e}^{-c_2 t}$, where N_0 is the smallest N which make the discrete squeezing property established.

Proposition 2.1. [15] There is $t_0(D_k)$ such that $B_k = \overline{\bigcup_{0\le t\le t_0(D_k)} S(t)D_k}$ is the positive invariant set of $S(t)$ in E_0, and B_k attracts all bounded subsets of E_k, where B_k is a closed bounded absorbing set for $S(t)$ in E_k.

Theorem 2.2. [16] Assuming the stress term $M(s)$ and the nonlinear term $g(u)$ satisfies the condition (A1)-(A2), $f \in H$, $(u_0, v_0) \in E_k$, then problem (1.1)-(1.3) admits a unique solution $(u,v) \in L^\infty(R^+; E_k)$. This solution possesses the following properties:

$$\|(u,v)\|_{E_0}^2 = \|\nabla^{2m} u\|^2 + \|v\|^2 \le c(r_0), \quad \|(u,v)\|_{E_k}^2 = \|\nabla^{2m+k} u\|^2 + \|\nabla^{2m} v\|^2 \le c(r_1).$$

We denote the solution in Theorem 2.1 by $S(t)(u_0, v_0) = (u(t), v(t))$. Then $S(t)$ composes a continuous semigroup in E_0. According to Theorem 2.1, we have the ball

$$D_0 = \left\{(u,v) \in E_0 : \|(u,v)\|_{E_0}^2 = \|\nabla^{2m} u\|^2 + \|v\|^2 \le c(r_0)\right\}, \tag{2.7}$$

$$D_k = \left\{(u,v) \in E_k : \|(u,v)\|_{E_k}^2 = \|\nabla^{2m+k} u\|^2 + \|\nabla^{2m} v\|^2 \le c(r_1)\right\}. \tag{2.8}$$

are absorbing sets of $S(t)$ in E_0 and E_k respectively. From Proposition 2.1

$$B_k = \overline{\bigcup_{0 \le t \le t_0(D_k)} S(t) D_k}. \tag{2.9}$$

is a positive invariant compact set of $S(t)$ in E_0, and absorbs all of the bounded subsets D_k in E_k. According to reference [15] and theorem 2.1, we can get the semigroup $\{S(t)\}_{t \ge 0}$ possesses (E_k, E_0)-compact global attractor $A_k = \bigcap_{s \ge 0} \overline{\bigcup_{t \ge s} S(t) D_k}$, where the bar means the closure in E_0, and A_k is bounded in E_k.

Lemma 2.1. For any $U = (u,v) \in E_0$,

$$(G(U), U)_{E_0} \ge a_1 \|U\|_{E_0}^2 + a_2 \|\nabla^{2m} v\|^2. \tag{2.10}$$

Proof. By (2.1) and (2.2), we have

$$\begin{aligned}(G(U),U)_{E_0} &= \varepsilon \|\nabla^{2m} u\|^2 - (\nabla^{2m} v, \nabla^{2m} u) + (\alpha - \varepsilon)\|v\|^2 + \varepsilon^2 (u,v) - \alpha \|v\|^2 \\ &\quad - \beta\varepsilon(\nabla^{2m} u, \nabla^{2m} v) + \beta \|\nabla^m v\|^2 + M\left(\|\nabla^m u\|_p^p\right)(\nabla^{2m} u, \nabla^{2m} v).\end{aligned} \tag{2.11}$$

By using Holder's inequality, Young's inequality and Poincare's inequality and the condition (A2), we have,

$$\varepsilon^2 (u,v) = -\varepsilon^2 \lambda_1^{-m} \|\nabla^{2m} u\| \|v\| \ge -\frac{\varepsilon^2}{4} \|\nabla^{2m} u\|^2 - \varepsilon^2 \lambda_1^{-2m} \|v\|^2. \tag{2.12}$$

$$\begin{aligned} M\left(\|\nabla^m u\|_p^p - \beta\varepsilon - 1\right)(\nabla^{2m} u, \nabla^{2m} v) &\ge (\mu_0 - \beta\varepsilon - 1)\left(\frac{\|\nabla^{2m} u\|^2}{4} + \|\nabla^{2m} v\|^2\right) \\ &= -\beta\varepsilon\left(\frac{\|\nabla^{2m} u\|^2}{4} + \|\nabla^{2m} v\|^2\right). \end{aligned} \tag{2.13}$$

Substitute inequality (2.12)-(2.13) into Equation (2.11), we get

$$\left(G(U),U\right)_{E_0} \geq \left(\varepsilon-\frac{\beta\varepsilon}{4}-\frac{\varepsilon^2}{4}\right)\left\|\nabla^{2m}u\right\|^2+\left(\alpha-\varepsilon-\varepsilon^2\lambda_1^{-2m}\right)\|v\|^2 \\ +\left(\beta-\beta\varepsilon-\alpha\lambda_1^{-2m}\right)\left\|\nabla^{2m}v\right\|^2. \tag{2.14}$$

According to the assumption, we can get $\varepsilon-\frac{\beta\varepsilon}{4}-\frac{\varepsilon^2}{4}>0$, $\alpha-\varepsilon-\varepsilon^2\lambda_1^{-2m}>0$, $\beta-\beta\varepsilon-\alpha\lambda_1^{-2m}>0$. Let $a_1=\min\left\{\varepsilon-\frac{\beta\varepsilon}{4}-\frac{\varepsilon^2}{4},\alpha-\varepsilon-\varepsilon^2\lambda_1^{-2m}\right\}$, $a_2=\beta-\beta\varepsilon-\alpha\lambda_1^{-2m}$, so we can get

$$\left(G(U),U\right)_{E_0} \geq a_1\|U\|_{E_0}^2+a_2\left\|\nabla^{2m}v\right\|^2. \tag{2.15}$$

The Lemma 2.1 is proved. Then we prove the Lipschitz property and the discrete squeezing property of $S(t)$. ■

Set $S(t)U_0=U(t)=\left(u(t),v(t)\right)^{\mathrm{T}}$, where $v=u_t(t)+\varepsilon u(t)$; and $S(t)V_0=V(t)=\left(\hat{u}(t),\hat{v}(t)\right)^{\mathrm{T}}$, where $\hat{v}(t)=\hat{u}_t(t)+\varepsilon\hat{u}(t)$; let $Y(t)=S(t)U_0-S(t)V_0=U(t)-V(t)=\left(w(t),z(t)\right)^{\mathrm{T}}$, where $z(t)=w_t(t)+\varepsilon w(t)$, $w(t)=u(t)-\hat{u}(t)$, $w_t(t)=v(t)-\hat{v}(t)$, then $Y(t)$ satisfies

$$Y_t+G(U)-G(V)-\left(0,g(u)-g(\hat{u})\right)^{\mathrm{T}}=0, \tag{2.16}$$

$$Y(0)=U_0-V_0. \tag{2.17}$$

Lemma 2.2. (Lipschitz property). For $\forall U_0,V_0\in B_k$ and $t\geq 0$,

$$\left\|S(t)U_0-S(t)V_0\right\|_{E_0} \leq \mathrm{e}^{\gamma t}\left\|U_0-V_0\right\|_{E_0}^2. \tag{2.18}$$

Proof. Taking the inner product of the Equation (2.16) with $Y(t)$ in E_0, we can get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Y(t)\|^2+\left(G(U)-G(V),Y(t)\right)+\left(g(u)-g(\hat{u}),z(t)\right)=0. \tag{2.19}$$

Similar to Lemma 2.1, we have

$$\left(G(U)-G(V),Y(t)\right)_{E_0} \geq a_1\|Y(t)\|_{E_0}^2+a_2\left\|\nabla^{2m}z(t)\right\|_{E_0}^2. \tag{2.20}$$

By using the condition (A1) Young's inequality Poincare's inequality and differential mean value theorem, we get

$$\left|\left(g(u)-g(\hat{u}),z(t)\right)\right| \leq \left|g'(\xi)\right|\|w(t)\|\|z(t)\| \leq c_4\lambda_1^{-m}\left\|\nabla^{2m}w(t)\right\|\|z(t)\| \\ \leq \frac{c_4\lambda_1^{-m}}{2}\left(\left\|\nabla^{2m}w(t)\right\|^2+\|z(t)\|^2\right)=\frac{c_4\lambda_1^{-m}}{2}\|Y(t)\|^2. \tag{2.21}$$

Where $\xi=\theta+(1-\theta)\hat{u},0<\theta<1$.

Substitute inequality (2.20)-(2.21) into equation (2.19), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Y(t)\|^2+2a_1\|Y(t)\|_{E_0}^2+2a_2\left\|\nabla^{2m}z(t)\right\|_{E_0}^2 \leq c_4\lambda_1^{-m}\|Y(t)\|^2. \tag{2.22}$$

We can get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Y(t)\|^2 \le c_4\lambda_1^{-m}\|Y(t)\|^2. \tag{2.23}$$

According to Gronwall's inequality, we have

$$\|Y(t)\|^2 \le \mathrm{e}^{c_4\lambda_1^{-m}t}\|Y(0)\|^2 = \mathrm{e}^{\gamma t}\|Y(0)\|^2. \tag{2.24}$$

where $\gamma = c_4\lambda_1^{-m}$. Therefore, we get

$$\|S(T)U_0 - S(T)V_0\|_{E_0} \le \mathrm{e}^{\gamma t}\|U_0 - V_0\|_{E_0}^2. \tag{2.25}$$

The Lemma 2.2 is proved. ■

Now, we define the operator $-\Delta$: $D(-\Delta)\to H^{4m}$, the domain of definition is $D(-\Delta) = H^2(\Omega)\cap H_0^1(\Omega)$, obviously, $-\Delta$ is an unbounded self-adjoint closed positive operator, and $(-\Delta)^{-1}$ is compact, we find by elementary spectral theory the existence of an orthonormal basis of H consisting of eigenvectors w_j of $-\Delta$, such that:

$$\begin{cases} (-\Delta)w_j = \lambda_j w_j, & j = 1,2,\cdots, \\ 0 < \lambda_1 \le \lambda_2 \le \cdots & \lambda_j \to \infty \text{ as } j\to\infty. \end{cases} \tag{2.26}$$

For a given integer *n*, $0 < n \le N$ we denote by P_n the orthogonal projection of H^{4m} onto the space spanned by $w_1,\cdots,w_n$ *i.e.* $p = p_n = H^{4m} \to \operatorname{span}\{w_1, w_2,\cdots,w_n\}$, let $Q_n = I - P_n$. Then we have

$$\left\|(-\Delta)^{2m}u\right\| \ge \lambda_{n+1}^{2m}\|u\|, \quad \forall u\in Q = Q_n\left(H^{4m}(\Omega)\cap H_0^1(\Omega)\right), \tag{2.27}$$

$$\|Q_n u\| \le \|u\|, \quad u\in H. \tag{2.28}$$

where $\|u\|^2 \le \lambda_{n+1}^{-4m}\|u\|_{D\left((-\Delta)^{2m}\right)}^2$.

Lemma 2.3. For any $U_0, V_0\in B_k$, $\forall n_0\in N^*$, $n_0\le N$, Let

$$Q_{n_0}(t) = Q_{n_0}(U(t)-V(t)) = Q_{m_0}Y(t) = \left(\omega_{n_0}, z_{n_0}\right)^{\mathrm{T}}, \tag{2.29}$$

then we have

$$\left\|Y_{n_0}(t)\right\|_{E_0}^2 \le \left(\mathrm{e}^{-2a_1 t} + \frac{c_2\lambda_{n_0+1}^{-m}}{2a_1+\gamma}\mathrm{e}^{\gamma t}\right)\|Y(0)\|_{E_0}^2, \tag{2.30}$$

Proof. Taking projection operator Q_{n_0} in (2.16), we have

$$Y_{n_0 t}(t) + Q_{n_0}(G(U)-G(V)) + \left(0, Q_{n_0}\left(g(u)-g(\hat{u})\right)\right)^{\mathrm{T}} = 0. \tag{2.31}$$

Taking the inner product $(\cdot,\cdot)_{E_0}$ in (2.31) with $Y_{n_0}(t)$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|Y_{n_0}(t)\right\|^2 + a_1\left\|Y_{n_0}(t)\right\|^2 + a_2\left\|\nabla^{2m}z_{n_0}(t)\right\|^2 + Q_{n_0}\left(g(u)-g(\hat{u}), z_{n_0}(t)\right) = 0. \tag{2.32}$$

According to (A1) and Young inequality, we have

$$\begin{aligned} &\left|Q_{n_0}\left(g(u)-g(\hat{u}), z_{n_0}(t)\right)\right| \\ &\le \left|g'(\xi')\right|\left\|w_{n_0}(t)\right\|\left\|z_{n_0}(t)\right\| \le c_5\lambda_{n_0+1}^{-m}\left\|\nabla^{2m}w_{n_0}(t)\right\|\left\|z_{n_0}(t)\right\| \\ &\le \frac{c_5\lambda_{n_0+1}^{-m}}{2}\left(\left\|\nabla^{2m}w_{n_0}(t)\right\|^2 + \left\|z_{n_0}(t)\right\|^2\right) = \frac{c_5\lambda_{n_0+1}^{-m}}{2}\left\|Y_{n_0}(t)\right\|^2. \end{aligned} \tag{2.33}$$

where $\xi' = \theta_{n_0} + \left(1-\theta_{n_0}\right)\hat{u}, 0 < \theta_{n_0} < 1$.

Together with (2.32)-(2.33) and Lemma 2.2, it follows

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left\|Y_{n_0}\right\|_{E_0}^2 + 2a_1\left\|Y_{n_0}(t)\right\|^2 &\le c_5\lambda_{n_0+1}^{-m}\left\|Y_{n_0}(t)\right\|^2 = c_5\lambda_{n_0+1}^{-m}\left\|S(t)U_0 - S(t)V_0\right\|^2 \\ &\le c_5\lambda_{n_0+1}^{-m}\mathrm{e}^{\gamma t}\left\|U_0 - V_0\right\|^2 = c_5\lambda_{n_0+1}^{-m}\mathrm{e}^{\gamma t}\left\|Y(0)\right\|^2. \end{aligned} \tag{2.34}$$

By using Gronwall's inequality, we get

$$\left\|Y_{n_0}(t)\right\|^2 \le \left\|Y(0)\right\|^2\mathrm{e}^{-2a_1 t} + \frac{c_5\lambda_{n_0+1}^{-m}}{2a_1+\gamma}\mathrm{e}^{\gamma t}\left\|Y(0)\right\|^2 = \left(\mathrm{e}^{-2a_1 t} + \frac{c_5\lambda_{n_0+1}^{-m}}{2a_1+\gamma}\mathrm{e}^{\gamma t}\right)\left\|Y(0)\right\|^2. \tag{2.35}$$

The Lemma 2.3 is proved. ■

Lemma 2.4. (Discrete squeezing property). For any $U_0, V_0 \in B_k$, $\tau^* \ge 0$, if

$$\left\|P_{n_0}\left(S(\tau^*)U_0 - S(\tau^*)V_0\right)\right\|_{E_0} \le \left\|\left(I-P_{n_0}\right)\left(S(\tau^*)U_0 - S(\tau^*)V_0\right)\right\|_{E_0}, \tag{2.36}$$

then

$$\left\|S(\tau^*)U_0 - S(\tau^*)V_0\right\|_{E_0} \le \frac{1}{8}\left\|U_0 - V_0\right\|_{E_0}. \tag{2.37}$$

Proof. If $\left\|P_{n_0}\left(S(\tau^*)U_0 - S(\tau^*)V_0\right)\right\|_{E_0} \le \left\|\left(I-P_{n_0}\right)\left(S(\tau^*)U_0 - S(\tau^*)V_0\right)\right\|_{E_0}$, then

$$\begin{aligned} &\left\|S(\tau^*)U_0 - S(\tau^*)V_0\right\|_{E_0}^2 \\ &\le \left\|\left(I-P_{n_0}\right)\left(S(\tau^*)U_0 - S(\tau^*)V_0\right)\right\|_{E_0}^2 + \left\|P_{n_0}\left(S(\tau^*)U_0 - S(\tau^*)V_0\right)\right\|_{E_0}^2 \\ &\le 2\left\|\left(I-P_{n_0}\right)\left(S(\tau^*)U_0 - S(\tau^*)V_0\right)\right\|_{E_0}^2 \le 2\left(\mathrm{e}^{-2a_1\tau^*} + \frac{c_5\lambda_{n_0+1}^{-m}}{2a_1+\gamma}\mathrm{e}^{\gamma\tau^*}\right)\left\|U_0 - V_0\right\|_{E_0}^2. \end{aligned} \tag{2.38}$$

Let τ^* be large enough,

$$\mathrm{e}^{-2a_1\tau^*} \le \frac{1}{256}. \tag{2.39}$$

Also let n_0 be large enough, we get

$$\frac{c_5\lambda_{n_0+1}^{-m}}{2a_1+\gamma}\mathrm{e}^{\gamma\tau^*} \le \frac{1}{256}. \tag{2.40}$$

Substitute inequality (2.39)-(2.40) into Equation (2.38), we get

$$\left\|S(\tau^*)U_0 - S(\tau^*)V_0\right\|_{E_0}^2 \le \frac{1}{8}\left\|U_0 - V_0\right\|_{E_0}^2. \tag{2.41}$$

The Lemma 2.4 is proved. ■

Theorem 2.3. Let (A1), (A2) be in force, assume that $f \in H$, $(u_0, v_0) \in E_0(E_k)$, ($k = 1, 2, \cdots, 2m$), then the semigroup $S(t)$ determined by (1.1)-(1.3) possesses an (E_k, E_0)-exponential attractor M_k on *B*,

$$M_k = \bigcup_{0\le t\le\tau^*} S(t)\left(A_k \cup \left(\bigcup_{j=1}^{\infty}\bigcup_{i=1}^{\infty} S(\tau^*)^j\left(E^{(i)}\right)\right)\right), \tag{2.42}$$

The fractal dimension of M_k satisfies

$$d_F\left(M_k\right)\le c_3N_0+1. \tag{2.43}$$

Proof. According to Theorem 2.1, Lemma 2.2 and Lemma 2.4, Theorem 2.2 is easily proven. ■

3. Inertial Manifolds

Next, we will prove the existence of inertial manifolds when N is large enough by using graph norm transformation method.

Definition 3.1. [17] Assume $S=S(t)_{t\ge 0}$ is a solution semigroup of Banach space $E_k=H_0^{2m+k}(\Omega)\times H_0^k(\Omega)\left(k=1,2,\cdots,2m\right)$, then a family of inertial manifolds μ_k is a subset of E_k and satisfies the following three properties:

1) μ_k is finite dimensional Lipschitz manifold of E_k;

2) μ_k is positively invariant for the semigroup $\left\{S(t)\right\}_{t\ge 0}$, *i.e.* $\forall u_0\in\mu_k$, $S(t)u_0\subset\mu_k$, $\forall t\ge 0$;

3) μ_k attracts exponentially all the orbits of the solution, *i.e.* $\exists\vartheta>0$, for $\forall u\in E_k$, $\exists k>0$, such that

$$\operatorname{dist}\left(S(t)u,u\right)\le k\cdot \mathrm{e}^{-\vartheta t},t\ge 0. \tag{3.1}$$

Lemma 3.1. Let $\Lambda:E_k\to E_k$ be an operator and assume that $F\in C_b\left(E_k,E_k\right)$ satisfies the Lipschitz condition

$$\left\|F(U)-F(U)\right\|_{E_k}\le l_F\left\|U-V\right\|_{E_k},\quad U,V\in E_k. \tag{3.2}$$

The operator Λ is said to satisfy the spectral gap condition relative to *F*, if the point spectrum of the operator Λ can be divided into two parts σ_1 and σ_2, of which σ_1 is finite, and we have

$$\Lambda_1=\sup\left\{\operatorname{Re}\lambda\mid\lambda\in\sigma_1\right\},\quad \Lambda_2=\inf\left\{\operatorname{Re}\lambda\mid\lambda\in\sigma_2\right\}, \tag{3.3}$$

and $E_{k_i}=\operatorname{span}\left\{\omega_j\mid j\in\sigma_i,i=1,2\right\}$.

Then

$$\Lambda_2-\Lambda_1>4l_F, \tag{3.4}$$

and the orthogonal decomposition

$$E_k=E_{k_1}\oplus E_{k_2}, \tag{3.5}$$

Then $P_1:E_k\to E_{k_1}$ and $P_2:E_k\to E_{k_2}$ are both continuous orthogonal projections . The Lemma 3.1 is proved.

Lemma 3.2. Let the eigenvalues $\mu_j^{\pm}\left(j\ge 1\right)$ is non-decreasing, and for $m\in N^*$, there exists $N\ge m$, such that μ_{N+1}^- and μ_N^- are consecutive adjacent values.

Lemma 3.3. The function $g(u)$ satisfies $g:H_0^k(\Omega)\to L^2(\Omega)$ which is uniformly bounded and globally Lipschitz continuous, and *l* is the Lipschitz coefficient.

Proof. For $\forall u_1,u_2\in H_0^k(\Omega)$, we have

$$\left\|g(u_1)-g(u_2)\right\|=\left\|g'(\eta)(u_1-u_2)\right\|\le\left\|g'(\eta)\right\|_\infty\left\|u_1-u_2\right\|_{H_k}, \tag{3.6}$$

where $\eta\in\left(u_1,u_2\right)$, From the hypothesis (A1) and the differential mean value theorem, we know

$$\left\| g\left(u_1\right)-g\left(u_2\right)\right\| \le C_6 \left\| u_1-u_2\right\|_{H_0^k}, \tag{3.7}$$

Let $l=C_6$, l is the Lipschitz coefficient. ■

Then we prove the existence of a family of the inertial manifold of this equation, Equation (1.1) is equivalent to the following first-order evolution equation:

$$U_t+\Lambda U=F\left(U\right), \tag{3.8}$$

where

$$U=\left(u,v\right)^{\mathrm{T}}=\left(u,u_t\right)^{\mathrm{T}},\Lambda=\begin{pmatrix} 0 & -I \\ M\left(\left\|\nabla^m u\right\|_p^p\right)\left(-\Delta\right)^{2m} & \beta\left(-\Delta\right)^{2m}\end{pmatrix},$$

$$F\left(U\right)=\begin{pmatrix} 0 \\ f\left(x\right)-g\left(u\right)\end{pmatrix},$$

$$D\left(\Lambda\right)=\left\{u\in H^{4m}\left(\Omega\right)\mid u\in H,\left(-\Delta\right)^{2m}u\in H^{4m}\left(\Omega\right)\right\}\times H^{2m}\left(\Omega\right).$$

We consider in E_k the usual graph norm, induced by the scalar product

$$\left\langle U,V\right\rangle_{E_k}=\left(M\cdot\nabla^{2m+k}u,\nabla^{2m+k}\overline{y}\right)+\left(v,\overline{z}\right). \tag{3.9}$$

where $U=\left(u,v\right)^{\mathrm{T}}$, $V=\left(y,z\right)^{\mathrm{T}}\in E_k$, and $\overline{y},\overline{z}$ respectively denote the conjugation of *y* and *z*, and $v,z\in H_0^{2m+k}\left(\Omega\right)$, $u,y\in H_0^{2m+k}\left(\Omega\right)$. Moreover, the operator Λ is monotone, indeed, for $\forall U\in D\left(\Lambda\right)$, we have

$$\begin{aligned}
&\left\langle \Lambda U,U\right\rangle_{E_k}\\
&=-\left(M\cdot\nabla^{2m+k}u_t,\nabla^{2m+k}\overline{u}\right)+\left(M\left(\left\|\nabla^m u\right\|_p^p\right)\left(-\Delta\right)^{2m}u+\beta\left(-\Delta\right)^{2m}u_t,\overline{v}\right)\\
&\ge-\left(M\cdot\nabla^{2m+k}u_t,\nabla^{2m+k}\overline{u}\right)+M\left(\nabla^{2m+k}u,\nabla^{2m+k}v\right)+\beta\left(-\Delta^m v,-\Delta^m\overline{v}\right)\\
&\ge\beta\left\|\nabla^{2m}v\right\|^2>0.
\end{aligned} \tag{3.10}$$

so that Λ is a Monotonically increasing operator and $\left\langle \Lambda U,U\right\rangle_{E_k}$ is real and nonnegative. To determine the eigenvalues of Λ, we observe that the eigenvalue equation

$$\Lambda U=\lambda U,\quad U=\left(u,v\right)^{\mathrm{T}}\in E_k \tag{3.11}$$

is equivalent to the system

$$\begin{cases} -v=\lambda u, \\ M\left(\left\|\nabla^m u\right\|_p^p\right)\left(-\Delta\right)^{2m}u+\beta\left(-\Delta\right)^{2m}v=0. \end{cases} \tag{3.12}$$

Thus, we can get the eigenvalue problem

$$\begin{cases} \lambda^2 u+M\left(\left\|\nabla^m u\right\|_p^p\right)\left(-\Delta\right)^{2m}u-\beta\lambda\left(-\Delta\right)^{2m}u=0, \\ u\big|_{\partial\Omega}=\left(-\Delta\right)^{2m}u\big|_{\partial\Omega}=0. \end{cases} \tag{3.13}$$

Using $\left(-\Delta\right)^k u$ with the first formula of (3.13) to take the inner product, and bring u_j to the position of *u*, we can get

$$\lambda^2\left\|\nabla^k u\right\|^2+M\left(\left\|\nabla^m u\right\|_p^p\right)\left\|\nabla^{2m+k}u\right\|^2-\beta\lambda\left\|\nabla^{2m+k}u\right\|^2=0. \tag{3.14}$$

Regarding Equation (3.14) as a quadratic equation of one variable with respect to λ, for $\forall j\in N^+$ and let $s=\left\|\nabla^m u\right\|_p^p$, $M=M(s)$, the corresponding eigenvalues of equation (3.11) are as follows:

$$\lambda_j^{\pm}=\frac{\beta\mu_j\pm\sqrt{\beta^2\mu_j^2-4M\mu_j}}{2}. \tag{3.15}$$

where $\mu_j(j\ge 1)$ is the eigenvalue of $(-\Delta)^{2m}$ in $H_0^{2m}(\Omega)$, and $\mu_j=\lambda_1 j^{\frac{2m}{n}}$. Because of β is large enough, the eigenvalue of Λ are all positive and real numbers, the corresponding eigenvalues have the form

$$U_j^{\pm}=\left(u_j,-\lambda_j^{\pm}u_j\right). \tag{3.16}$$

For formula (3.15), for the convenience of later use, define the following formula

$$\left\|\nabla^{2m+k}u_j\right\|=\sqrt{\mu_j},\left\|\nabla^k u_j\right\|=1,\left\|\nabla^{-2m-k}u_j\right\|=\frac{1}{\sqrt{\mu_j}},k=1,2,\cdots,2m. \tag{3.17}$$

Next, it will be proved that the eigenvalue of the operator Λ satisfies the spectral interval condition.

Theorem 3.1 let l is the Lipschitz constant of $g(u)$, assume $\mu_j\ge\frac{4M(s)}{\beta^2}$, if $N_1\in Z^+$ is large enough, when $N\ge N_1$, the following inequality holds

$$\left(\mu_{N+1}+\mu_N\right)\left(\beta-\sqrt{\beta^2\mu_j^2-4M(s)}\right)\ge\frac{8l}{\sqrt{\beta^2\mu_j^2-4M(s)}}+1. \tag{3.18}$$

Then, the operator Λ satisfies the spectral gap condition of Lemma 3.1.

Proof. Because of all the eigenvalues of the operator Λ are positive real numbers, $\beta\ge 2\sqrt{\frac{M}{\mu_j}}$ and the sequence $\left\{\lambda_j^-\right\}_{j\ge1}$ and $\left\{\lambda_j^+\right\}_{j\ge1}$ are monotonically increasing. The theorem is proved in four steps below.

step 1 Since $\lambda_j^{\pm}$ is a non-decreasing sequence, according to Lemma 3.2, given N, so that λ_N^- and λ_{N+1}^- are consecutive adjacent eigenvalues, the eigenvalues of the operator Λ are decomposed into σ_1 and σ_2, where σ_1 is the finite parts, which are expressed as follows

$$\sigma_1=\left\{\lambda_h^-,\lambda_j^+\mid\max\left\{\lambda_h^-,\lambda_j^+\right\}\le\lambda_N^-\right\}, \tag{3.19}$$

$$\sigma_2=\left\{\lambda_h^+,\lambda_j^{\pm}\mid\lambda_h^-\le\lambda_N^-\le\min\left\{\lambda_h^+,\lambda_j^{\pm}\right\}\right\}. \tag{3.20}$$

step 2 The corresponding E_k is decomposed into

$$E_{k_1}=\operatorname{span}\left\{U_h^-,U_j^{\pm}\mid\lambda_h^-,\lambda_j^+\in\sigma_1\right\}, \tag{3.21}$$

$$E_{k_2}=\operatorname{span}\left\{U_h^+,U_j^+\mid\lambda_h^-,\lambda_j^{\pm}\in\sigma_2\right\}. \tag{3.22}$$

We aim at madding two orthogonal subspaces of E_k and verifying the spectral gap condition (3.4) is true when $\Lambda_1 = \lambda_N^-, \Lambda_2 = \lambda_{N+1}^-$. Therefore, we further decompose $E_{k_2} = E_S + E_R$, *i.e.*

$$E_S = \operatorname{span}\left\{U_h^- \mid \lambda_h^- \le \lambda_N^- \le \lambda_h^+\right\}, \tag{3.23}$$

$$E_R = \operatorname{span}\left\{U_R^+ \mid \lambda_N^- \le \lambda_j^\pm\right\}. \tag{3.24}$$

And set $E_N = E_{k_1} \oplus E_S$. Note that E_{k_1} and E_S are finite dimensional, that $\lambda_N^- \in E_{k_1}$, $\lambda_{N+1}^- \in E_R$, and that the reason why E_{k_1} is not orthogonal to E_{k_2} is that, while it is orthogonal to E_R, E_{k_1} is not orthogonal to E_S. We now introduce two functions $\Psi : E_N \to R$ and $\psi : E_R \to R$, defined by

$$\begin{aligned}\Phi(U,V) &= \beta\left(\nabla^{2m+k}u, \nabla^{2m+k}\overline{y}\right) + 2\beta\left(\nabla^{-2m-k}\overline{z}, \nabla^{2m}u\right) \\ &\quad + 2\beta\left(\nabla^{-2m-k}v, \nabla^{2m}\overline{y}\right) + 4\left(\nabla^{-2m-k}v, \nabla^{-2m-k}z\right) \\ &\quad - 4M\left(\left\|\nabla^m u\right\|_p^p\right)\left(\nabla^k\overline{u}, \nabla^k y\right) + \left(2\beta^2 - \beta\right)\left(\nabla^{2m+k}\overline{u}, \nabla^{2m+k}y\right).\end{aligned} \tag{3.25}$$

$$\begin{aligned}\Psi(U,V) &= \left(\nabla^{2m+k}u, \nabla^{2m+k}\overline{y}\right) + \left(\nabla^{-2m-k}\overline{z}, \nabla^{2m+k}u\right) - \left(\nabla^{-2m-k}v, \nabla^{2m+k}\overline{y}\right) \\ &\quad - 4M\left(\left\|\nabla^m u\right\|_p^p\right)\left(\nabla^k\overline{u}, \nabla^k y\right) + \left(\beta^2 - 1\right)\left(\nabla^{2m+k}\overline{u}, \nabla^{2m+k}y\right).\end{aligned} \tag{3.26}$$

where $U = (u,v)^{\mathrm{T}}, V = (y,z)^{\mathrm{T}} \in E_N$, and $\overline{y}, \overline{z}$ are respectively the conjugates of y, z. We now show that Φ and Ψ are positive definite. For $\forall U = (u,v) \in E_N$, we have

$$\begin{aligned}\Phi(U,V) &= \beta\left(\nabla^{2m+k}u, \nabla^{2m+k}\overline{u}\right) + 2\beta\left(\nabla^{-2m-k}\overline{v}, \nabla^{2m}u\right) + 2\beta\left(\nabla^{-2m-k}v, \nabla^{2m}\overline{u}\right) \\ &\quad + 4\left(\nabla^{-2m-k}v, \nabla^{-2m-k}v\right) - 4M\left(\left\|\nabla^m u\right\|_p^p\right)\left(\nabla^k\overline{u}, \nabla^k u\right) \\ &\quad + \left(2\beta^2 - \beta\right)\left(\nabla^{2m+k}\overline{u}, \nabla^{2m+k}u\right) \\ &\ge \beta\left\|\nabla^{2m+k}u\right\|^2 - 4\left\|\nabla^{-2m-k}\overline{v}\right\|^2 - \beta^2\left\|\nabla^{2m+k}u\right\|^2 - 4M(s)\left\|\nabla^{m+k}u\right\|^2 \\ &\quad + \left(2\beta^2 - \beta\right)\left\|\nabla^{2m+k}u\right\|^2 + 4\left\|\nabla^{-2m-k}\overline{v}\right\|^2 \\ &= \beta^2\left\|\nabla^{2m+k}u\right\|^2 - 4\mu_1\left\|\nabla^k u\right\|^2 \\ &\ge \left(\beta^2\mu_j^2 - 4M(s)\right)\left\|\nabla^k u\right\|^2.\end{aligned} \tag{3.27}$$

When β is large enough, we conclude that $\Phi(U,U) \ge 0$, *i.e.* Φ is positive definite. Similarly, for $\forall U = (u,v) \in E_R$, we have

$$\begin{aligned}\Psi(U,V) &= \left(\nabla^{2m+k}u, \nabla^{2m+k}\overline{u}\right) + \left(\nabla^{-2m-k}\overline{v}, \nabla^{2m+k}u\right) - \left(\nabla^{-2m-k}v, \nabla^{2m+k}\overline{u}\right) \\ &\quad - 4M\left(\left\|\nabla^m u\right\|_p^p\right)\left(\nabla^k\overline{u}, \nabla^k u\right) + \left(\beta^2 - 1\right)\left(\nabla^{2m+k}\overline{u}, \nabla^{2m+k}u\right) \\ &\ge \left\|\nabla^{2m+k}u\right\|^2 - 4M(s)\left\|\nabla^{m+k}u\right\|^2 + \left(\beta^2 - 1\right)\left\|\nabla^{2m+k}u\right\|^2 \\ &\ge \left(\beta^2\mu_j^2 - 4M(s)\right)\left\|\nabla^k u\right\|^2.\end{aligned} \tag{3.28}$$

When β is large enough, we conclude that $\Psi(U,U) \ge 0$, *i.e.* Ψ is positive definite.

Thus Φ and Ψ define a scalar product, respectively on E_N and E_R, and we can define an equivalent scalar product in E_k, by

$$\langle\langle U,V\rangle\rangle_{E_k} = \Phi(P_N U, P_N V) + \Psi(P_R U, P_R V). \tag{3.29}$$

where P_N and P_R are respectively the projections of $E_k \to E_N$ and $E_k \to E_R$. Rewrite (3.29) as follows

$$\langle\langle U,V\rangle\rangle_{E_k} = \Phi(U,V) + \Psi(U,V). \tag{3.30}$$

We proceed then to show that the subspaces E_{k_1} and E_{k_2} defined in (3.21), (3.22) are orthogonal with respect to the scalar product (3.29). In fact, it is sufficient to show that E_N is orthogonal to E_S, in turn, this reduces to showing that $\langle\langle U_h^-, U_h^+\rangle\rangle_{E_k} = 0$ if $U_h^- \in E_N$ and $U_h^+ \in E_S$. Recalling (3.27) and (3.28), we immediately compute that

$$\begin{aligned}
\langle\langle U_j^+, U_j^-\rangle\rangle_{E_k} &= \Phi(U_j^+, U_j^-) \\
&= \beta\left(\nabla^{2m+k}u_j, \nabla^{2m+k}\overline{u}_j\right) + 2\beta\left(-\lambda_j^-\nabla^{-2m-k}\overline{u}_j, \nabla^{2m}u_j\right) \\
&\quad + 2\beta\left(-\lambda_j^+\nabla^{-2m-k}u_j, \nabla^{2m}\overline{u}_j\right) + 4\left(-\lambda_j^+\nabla^{-2m-k}u_j, -\lambda_j^-\nabla^{-2m-k}u_j\right) \\
&\quad - 4M\left(\|\nabla^m u\|_p^p\right)\left(\nabla^k\overline{u}_j, \nabla^k u_j\right) + \left(2\beta^2 - \beta\right)\left(\nabla^{2m+k}\overline{u}_j, \nabla^{2m+k}u_j\right) \\
&= \beta\left\|\nabla^{2m+k}u_j\right\|^2 - 2\beta\left(\lambda_j^- + \lambda_j^+\right)\left\|\nabla^{-k}u_j\right\|^2 + 4\lambda_j^-\lambda_j^+\left\|\nabla^{-2m-k}u_j\right\|^2 \\
&\quad - 4M(s)\left\|\nabla^k u_j\right\|^2 + \left(2\beta^2 - \beta\right)\left\|\nabla^{2m+k}u_j\right\|^2 \\
&= 2\beta^2\mu_j - 2\beta\left(\lambda_j^- + \lambda_j^+\right) + 4\lambda_j^-\lambda_j^+\frac{1}{\mu_j} - 4M(s).
\end{aligned} \tag{3.31}$$

According to (3.15), we have

$$\lambda_j^- + \lambda_j^+ = \beta\mu_j. \tag{3.32}$$

$$\lambda_j^-\lambda_j^+ = M\mu_j. \tag{3.33}$$

Therefore

$$\langle\langle U_j^+, U_j^-\rangle\rangle_{E_k} = \Phi(U_j^+, U_j^-) = 0. \tag{3.34}$$

step 3 Further, we estimate the Lipschitz constant l_F of $F(U) = (0, f(x) - g(u))^{\mathrm{T}}$, according to Lemma 3.3 we can get $g: H_0^{2m}(\Omega) \to L^2(\Omega)$ is uniformly bounded and globally Lipschitz continuous. For $\forall U(u,v)^{\mathrm{T}} \in E_k$, $U_i = (u_i, v_i)^{\mathrm{T}} \in P_i U\ (i = 1,2)$, we have

$$\begin{aligned}
\|U\|_{E_k}^2 &= \Phi(P_1U, P_1U) + \Psi(P_2U, P_2U) \\
&\geq \left(\beta^2\mu_j^2 - 4M(s)\right)\left\|\nabla^k P_1 u\right\|^2 + \left(\beta^2\lambda_1^m - 4\mu_1\right)\left\|\nabla^k P_2 u\right\|^2 \\
&\geq \left(\beta^2\mu_j^2 - 4M(s)\right)\left\|\nabla^k u\right\|^2.
\end{aligned} \tag{3.35}$$

Given $U = (u,v)^{\mathrm{T}}, V = (\tilde{u}, \tilde{v})^{\mathrm{T}} = (y,z)^{\mathrm{T}} \in E_k$, we have

$$\|F(U) - F(V)\|_{E_k} = \|g(u) - g(\tilde{u})\| \leq l\|u - \tilde{u}\| \leq \frac{1}{\sqrt{\beta^2\mu_j^2 - 4M(s)}}\|U - V\|_{E_k}. \tag{3.36}$$

Thus, we have

$$l_F \le \frac{1}{\sqrt{\beta^2\mu_j^2 - 4M(s)}}. \tag{3.37}$$

step 4 Now, we will show the spectral gap condition (3.4) holds.

Since $\Lambda_1 = \lambda_N^-, \Lambda_2 = \lambda_{N+1}^-$, then

$$\Lambda_2 - \Lambda_1 = \lambda_{N+1}^- - \lambda_N^- = \frac{\beta}{2}(\mu_{N+1} - \mu_N) + \frac{1}{2}\left(\sqrt{R(N)} - \sqrt{R(N+1)}\right). \tag{3.38}$$

where $R(N) = \beta^2\mu_N^2 - 4M\mu_N^2$.

There exists $N_1 \ge 0$, such that for $\forall N \ge N_1$,

$R_1(N) = 1 - \sqrt{\frac{\beta^2}{\beta^2\mu_j^2 - 4M(s)} - \frac{4M}{\beta^2\mu_j^2 - 4M(s)}}$. We can get

$$\begin{aligned} &\sqrt{R(N)} - \sqrt{R(N+1)} + \sqrt{\beta^2\mu_j^2 - 4M(s)}(\mu_{N+1} - \mu_N) \\ &= \sqrt{\beta^2\mu_j^2 - 4M(s)}\left(\mu_{N+1}R_1(N+1) - \mu_N R_1(N)\right), \end{aligned} \tag{3.39}$$

According to assumption (A2), we can easily see that

$$\lim_{N\to+\infty}\left(\sqrt{R(N)} - \sqrt{R(N+1)} + \sqrt{\beta^2\mu_j^2 - 4M(s)}(\mu_{N+1} - \mu_N)\right) = 0, \tag{3.40}$$

Then according to (3.18) and (3.37)-(3.40), we have

$$\Lambda_2 - \Lambda_1 \ge \frac{1}{2}\left((\mu_{N+1} - \mu_N)\left(\beta - \sqrt{\beta^2\mu_j^2 - 4M(s)}\right) - 1\right) \ge \frac{4l}{\sqrt{\beta^2\mu_j^2 - 4M(s)}} \ge 4l_F. \tag{3.41}$$

The Theorem 3.1 is proved.

Theorem 3.2. Under the conclusion of Theorem 3.1, the problem (1.1)-(1.3) exists a family of inertial manifolds μ_k in E_k

$$\mu_k = \text{graph}(m) := \left\{\zeta_k + \gamma(\zeta_k) : \zeta_k \in E_{k_1}\right\} \tag{3.42}$$

where E_{k_1}, E_{k_2} defined in (3.21)-(3.22), and $\chi : E_{k_1} \to E_{k_2}$ is Lipschitz continuous function. ■

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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