Journal of Applied Mathematics and Physics, 2021, 9, 2038-2113
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Pythagoreans Figurative Numbers: The Beginning of Number Theory and Summation of Series

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How to cite this paper: Agarwal, R.P. (2021) Pythagoreans Figurative Numbers: The Beginning of Number Theory and Summation of Series. *Journal of Applied Mathematics and Physics*, **9**, 2038-2113.
https://doi.org/10.4236/jamp.2021.98132

Received: July 13, 2021
Accepted: August 28, 2021
Published: August 31, 2021

Abstract

In this article we shall examine several different types of figurative numbers which have been studied extensively over the period of 2500 years, and currently scattered on hundreds of websites. We shall discuss their computation through simple recurrence relations, patterns and properties, and mutual relationships which have led to curious results in the field of elementary number theory. Further, for each type of figurative numbers we shall show that the addition of first finite numbers and infinite addition of their inverses often require new/strange techniques. We sincerely hope that besides experts, students and teachers of mathematics will also be benefited with this article.

Keywords

Figurative Numbers, Patterns and Properties, Relations, Sums of Finite and Infinite Series, History

1. Introduction

Pythagoras of Samos (around 582-481 BC, Greece) and his several followers, especially, Hypsicles of Alexandria (around 190-120 BC, Greece), Plutarch of Chaeronea (around 46-120, Greece), Nicomachus of Gerasa (around 60-120, Jordan-Israel), and Theon of Smyrna (70-135, Greece) portrayed natural numbers in orderly geometrical configuration of points/dots/pebbles and labeled them as *figurative numbers*. From these arrangements, they deduced some astonishing number-theoretic results. This was indeed the beginning of the number theory, and an attempt to relate geometry with arithmetic. Nicomachus in his book, see [1], originally written about 100 A.D., collected earlier works of Pythagoreans on natural numbers, and presented cubic figurative numbers (solid numbers). Thus, figurate numbers had been studied by the ancient Greeks for polygonal numbers, pyramidal numbers, and cubes. The connection between regular geometric figures and the corresponding sequences of figurative numbers was profoundly significant in Plato's science, after Plato of Athens (around 427-347 BC, Greece), for example in his work *Timaeus*. The study of figurative numbers was further advanced by Diophantus of Alexandria (about 250, Greece). His main interest was in figurate numbers based on the Platonic solids (tetrahedron, cube, octahedron, dodecahedron, and icosahedron), which he documented in *De solidorum elementis*. However, this treatise was lost, and rediscovered only in 1860. Dicuilus (flourished 825, Ireland) wrote *Astronomical Treatise* in Latin about 814-816, which contains a chapter on triangular and square numbers, see Ross and Knott [2]. After Diophantus's work, several prominent mathematicians took interest in figurative numbers. The long list includes: Leonardo of Pisa/Fibonacci (around 1170-1250, Italy), Michael Stifel (1486-1567, Germany), Gerolamo Cardano (1501-1576, Italy), Johann Faulhaber (1580-1635, German), Claude Gaspard Bachet de Meziriac (1581-1638, France), René Descartes (1596-1650, France), Pierre de Fermat (1601-1665, France), John Pell (1611-1685, England). In 1665, Blaise Pascal (1623-1662, France) wrote the *Traité du triangle arithmétique*, *avec quelques autres petits traitez sur la mesme matiére* which contains some details of figurate numbers. Work of Leonhard Euler (1707-1783, Switzerland) and Joseph Louis Lagrange (1736-1813, France) on figurate numbers opened new avenues in number theory. Octahedral numbers were extensively examined by Friedrich Wilhelm Marpurg (1718-1795, German) in 1774, and Georg Simon Klügel (1739-1812, Germany) in 1808. The Pythagoreans could not have anticipated that figurative numbers would engage after 2000 years leading scholars such as Adrien-Marie Legendre (1752-1833, France), Karl Friedrich Gauss (1777-1855, Germany), Augustin-Louis Cauchy (1789-1857, France), Carl Guslov Jacob Jacobi (1804-1851, Germany), and Waclaw Franciszek Sierpiński (1882-1969, Poland). In 2011, Michel Marie Deza (1939-2016, Russia-France) and Elena Deza (Russia) in their book [3] had given an extensive information about figurative numbers.

In this article we shall systematically discuss most popular polygonal, centered polygonal, three dimensional numbers (including pyramidal numbers), and four dimensional figurative numbers. We shall begin with triangular numbers and end this article with pentatope numbers. For each type of polygonal figurative numbers, we shall provide definition in terms of a sequence, possible sketch, explicit formula, possible relations within the class of numbers through simple recurrence relations, properties of these numbers, generating function, sum of first finite numbers, sum of all their inverses, and relations with other types of polygonal figurative numbers. For each other type of figurative numbers mainly we shall furnish definition in terms of a sequence, possible sketch, explicit formula, generating function, sum of first finite numbers, and sum of all their inverses. The study of figurative numbers is interesting in its own sack, and often these numbers occur in real world situations. We sincerely hope after reading this article it will be possible to find new representations, patterns, relations with other types of popular numbers which are not discussed here, extensions, and real applications.

2. Triangular Numbers

In this arrangement rows contain $1,2,3,4,\cdots,n$ dots (see Figure 1).

From Figure 1 it follows that each new triangular number is obtained from the previous triangular number by adding another row containing one more dot than the previous row added, and hence t_n is the sum of the first n positive integers, *i.e.*,

$$t_n = t_{n-1} + n = t_{n-2} + (n-1) + n = \cdots = 1+2+3+\cdots+(n-1)+n, \quad (1)$$

i.e., the differences between successive triangular numbers produce the sequence of natural numbers. To find the sum in (1) we shall discuss two methods which are innovative.

Method 1. Since

$$t_n = 1+2+3+\cdots+(n-1)+n$$
$$t_n = n+(n-1)+(n-2)+\cdots+2+1$$

An addition of these two arrangements immediately gives

$$2t_n = (n+1)+(n+1)+\cdots+(n+1) = n(n+1)$$

and hence

$$t_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n. \quad (2)$$

Thus, it immediately follows that $t_1 = 1$, $t_2 = 1+2 = 3$, $t_3 = 1+2+3 = 6$, $t_4 = 1+2+3+4 = 10$, $t_5 = 15$, $t_6 = 21$, $t_7 = 28$, $\cdots$. This method was first employed by Gauss. The story is his elementary school teacher asked the class to add up the numbers from 1 to 100, expecting to keep them busy for a long time. Young Gauss found the Formula (2) instantly and wrote down the correct answer 5050.

Method 2. From Figure 2 *Proof without words* of (2) is immediate, see Alsina and Nelsen [4]. However, a needless explanation is a "stairstep" configuration made up of one block plus two blocks plus three blocks, etc, replicated it as the shaded section in Figure 2, and fit them together to form an $n \times (n+1)$ rectangular array. Because the rectangle is made of two identical stairsteps (each

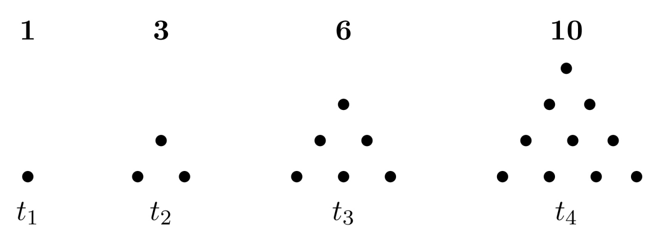

Figure 1. Triangular numbers.

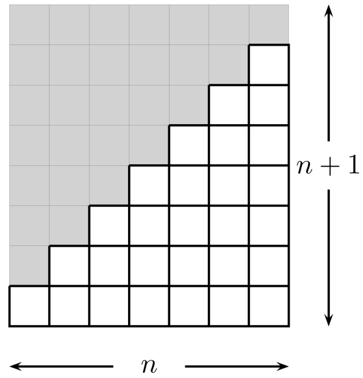

Figure 2. Proof of (2) without words.

representing t_n) and the rectangle's area is the product of base and height, that is, $n(n+1)$, then the stairstep's area must be half of the rectangle's, and hence (2) holds.

To prove (2) the *Principle of mathematical induction* is routinely used. The relation (1) is a special case of an arithmetic progression of the finite sequence $\{a_k\}, k = 0,1,\cdots,n-1$ where $a_k = a + kd$, or $a_k = a_\ell + (k-\ell)d, k \ge \ell \ge 0$, *i.e.*,

$$S = \sum_{k=0}^{n-1}(a+kd) = a + (a+d) + (a+2d) + \cdots + (a+(n-1)d). \tag{3}$$

For this, following the Method 1, it immediately follows that

$$S = \frac{n}{2}\left[2a + (n-1)d\right] = \frac{n}{2}\left[a_0 + a_{n-1}\right]. \tag{4}$$

Thus, the mean value of the series is $\bar{S} = S/n = (a_0 + a_{n-1})/2$, which is similar as in discrete uniform distribution. For $a = d = 1$, (3) reduces to (1), and (4) becomes same as (2). From (4), it is also clear that

$$\begin{aligned}\sum_{k=m}^{n-1}(a+kd) &= \sum_{k=0}^{n-1}(a+kd) - \sum_{k=0}^{m-1}(a+kd) \\ &= \frac{n}{2}\left[2a+(n-1)d\right] - \frac{m}{2}\left[2a+(m-1)d\right] \\ &= \frac{n-m}{2}\left[2a+(n+m-1)d\right].\end{aligned} \tag{5}$$

Ancient Indian Sulbas (see Agarwal and Sen [5]) contain several examples of arithmetic progression. Aryabhata (born 2765 BC) besides giving the Formula (4) also obtained *n* in terms of *S*, namely,

$$n = \frac{1}{2}\left[\frac{\sqrt{8Sd + (2a-d)^2} - 2a}{d} + 1\right] \tag{6}$$

He also provided elegant results for the summation of series of squares and cubes. In Rhind Papyruses (about 1850 and 1650 BC) out of 87 problems two problems deal with arithmetical progressions and seem to indicate that Egyptian scriber Ahmes (around 1680-1620 BC) knew how to sum such series. For example, Problem 40 concerns an arithmetic progression of five terms. It states: divide

100 loaves among 5 men so that the sum of the three largest shares is 7 times the sum of the two smallest
($x+(x+d)+(x+2d)+(x+3d)+(x+4d)=100$,
$7[x+(x+d)]=(x+2d)+(x+3d)+(x+4d)$, $x=10/6$, $d=55/6$). There is a discussion of arithmetical progression in the works of Archimedes of Syracuse (287-212 BC, Greece), Hypsicles, Brahmagupta (born 30 BC, India), Diophantus, Zhang Qiujian (around 430-490, China), Bhaskara II or Bhaskaracharya (working 486, India), Alcuin of York (around 735-804, England), Dicuil, Fibonacci, Johannes de Sacrobosco (around 1195-1256, England), Levi ben Gershon (1288-1344, France). Abraham De Moivre (1667-1754. England) predicted the day of his own death. He found that he slept 15 minutes longer each night, and summing the arithmetic progression, calculated that he would die on November 27, 1754, the day that he would sleep all 24 hours. Peter Gustav Lejeune Dirichlet (1805-1859, Germany) showed that there are infinitely many primes in the arithmetic progression $an+b$, where a and b are relatively prime. Enrico Bombieri (born 1940, Italy) is known for the distribution of prime numbers in arithmetic progressions. Terence Chi-Shen Tao (born 1975, Australia-USA) showed that there exist arbitrarily long arithmetic progressions of prime numbers.

The following equalities between triangular numbers can be proved rather easily.

$$t_n^2+t_{n-1}^2=t_{n^2}$$
$$3t_n+t_{n-1}=t_{2n}$$
$$3t_n+t_{n+1}=t_{2n+1}$$
$$t_n+t_m+nm=t_{n+m}$$
$$t_nt_m+t_{n-1}t_{m-1}t_{nm}$$

Instead of adding the above finite arithmetic series $\{a_k\}$, we can multiply its terms which in terms of Gamma function Γ can be written as

$$a_0a_1\cdots a_{n-1}=\prod_{k=0}^{n-1}(a+kd)=d^n\frac{\Gamma\left(\frac{a}{d}+n\right)}{\Gamma\left(\frac{a}{d}\right)}, \tag{7}$$

provided a/d is nonpositive.

- The triangular number t_n solves the *handshake problem* of counting the number of handshakes if each person in a room with $(n+1)$ people shakes hands once with each person. Similarly a fully connected network of $(n+1)$ computing devices requires t_n connections. The triangular number t_n also provides the number of games played by $(n+1)$ teams in a *Round-Robin Tournament* in which each team plays every other team exactly once and no ties are allowed. Further, the triangular number t_n is the number of ordered pairs (x,y), where $1\le x\le y\le n$. For an $(n+1)$ sided-polygon, the number of diagonals is $(n+1)(n-2)/2=2t_n-t_{n+1}, n\ge 2$. From **Figure 1**, it follows that the number of line segments between closest pairs of dots in the triangles is $\ell_n=3t_{n-1}=3(n-1)n/2$, or recursively, $\ell_n=\ell_{n-1}+3(n-1)$,

$\ell_1 = 0$. Thus, for example, $\ell_4 = 18$. A problem of Christoff Rudolff (1499-1545, Poland) reads: I am owed 3240 *florins*. The debtor pays me 1 *florin* the first day, 2 the second day, 3 the third day, and so on. How many days it takes to pay off the debt (80 days). For the Pythagoreans the fourth triangular number $t_4 = 10$ (decade) was most significant of all: it contains in itself the first four integers, one, two, three, and four $1+2+3+4=10$, it was considered to be a symbol of "perfection", being the sum of 1 (a point), 2 (a line), 3 (a plane) and 4 (a solid); it is the smallest integer n for which there are just as many primes between 1 and n as nonprimes, and it gives rise to the tetraktys (see **Figure 1** and its alternative form **Figure 3**). To them, the tetraktys was the sum of the divine influences that hold the universe together, or the sum of all the manifest laws of nature. They recognized tetraktys as fate, the universe, the heaven, and even God. Pythagoras also called the Deity a Tetrad or Tetracyts, meaning the "four sacred letters". These letters originated from the four sacrad letters JHVH, in which the ancient Jews called God our Father, the name "Jehovah". The tetraktys was so revered by the members of the brotherhood that they shared the following oath and their most jealously guarded secret, "I swear by him who has transmitted to our minds the holy tetraktys, the roots and source of ever-flowing nature". For Plato (Plato, meaning broad, is a nickname, his real name was Aristocles, he died at a wedding feast) number ten was the archetypal pattern of the universe. According to Eric Temple Bell (1883-1960, UK-USA), see [6], "Pythagoras asked a merchant if he could count. On the merchants's replying that he could, Pythagoras told him to go ahead. One, two, three, four ..., he began, when Pythagoras shouted Stop! What you name four is really what you would call ten. The fourth number is not four, but decade, our tetractys and inviolable oath by which we swear". Inadvertently, the tetractys occurs in the following: the arrangement of bowling pins in ten-pin bowling, the baryon decuplet, an archbishop's coat of arms, the "Christmas Tree" formation in association football, a Chinese checkers board, and the list continues. The number $t_5 = 15$ gives the number and arrangement of balls in Billiards. The 36th triangular number, *i.e.*, is 666 (The Beast of Revelation-Christians often seems to have difficulties with numbers). The 666th triangular number, *i.e.*, t_{666} is 222111. On triangular numbers an interesting article is due to Fearnehough [7].

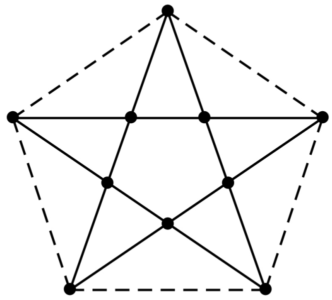

Figure 3. Alternative form of tetraktys.

- No triangular number has as its last digit (unit digit) 2, 4, 7 or 9. For this, let $n \equiv k \pmod{10}$, then $(n+1) \equiv (k+1) \pmod{10}$; here $0 \le k \le 9$. Thus, it follows that $t_n = n(n+1)/2 \equiv k(k+1)/2 \pmod{10}$. This relation gives only choices for k as $0,1,3,5,6$ and 8.
- We shall show that for an integer $k > 1$, $t_n \pmod{k}, n \ge 1$ repeats every k steps if k is odd, and every $2k$ steps if k is even, *i.e.*, if ℓ is the smallest positive integer such that for all integers n

$$\frac{(n+\ell)(n+\ell+1)}{2} \equiv \frac{n(n+1)}{2} \pmod{k}, \tag{8}$$

then $\ell = k$ if k is odd, and $\ell = 2k$ if k is even. For this, note that

$$\frac{(n+\ell)(n+\ell+1)}{2} - \frac{n(n+1)}{2} = n\ell + \frac{\ell(\ell+1)}{2},$$

and hence if (8) holds, then

$$n\ell + \frac{\ell(\ell+1)}{2} \equiv 0 \pmod{k}.$$

For $n = k$ and $n = 1$ the above equation respectively gives

$$0 + \frac{\ell(\ell+1)}{2} \equiv 0 \pmod{k} \text{ and } \ell + \frac{\ell(\ell+1)}{2} \equiv 0 \pmod{k}.$$

Combining these two relations, we find

$$\ell \equiv 0 \pmod{k}$$

and hence

$$\ell = ck \quad \text{for some positive integer } c. \tag{9}$$

Now if k is odd, then in view of $(k+1)/2$ is an integer, we have

$$nk + \frac{k(k+1)}{2} \equiv 0 \pmod{k}.$$

This implies that $k \ge \ell$, because ℓ is the smallest integer for which (8) holds. But, then from (9) it follows that $k = \ell$.

If k is even, then $k+1$ is odd, and so $k(k+1)/2 \not\equiv 0 \pmod{k}$. Thus, $\ell \ne k$, but

$$n(2k) + \frac{2k(k+1)}{2} \equiv 0 \pmod{k}$$

and so $2k$ satisfies (8). This implies that $2k \ge \ell$, which again from (9) gives $\ell = 2k$.

For example, for $t_n \pmod{3}, n \ge 1$, we have

$$1,0,0,\ 1,0,0,\ 1,0,0,\ 1,0,0,\cdots$$

and for $t_n \pmod{4}, n \ge 1$,

$$1,3,2,2,3,1,0,0,\ 1,3,2,2,3,1,0,0,\cdots.$$

- Triangular numbers and binomial coefficients are related by the relation

$$t_n = \binom{n+1}{2} = \binom{n+1}{n-1}.$$

Thus, triangular numbers are associated with Pascal's triangle

1

1 1

1 2 **1**

1 **3** **3** 1

1 4 **6** 4 1

1 5 **10** **10** 5 1

1 6 **15** 20 **15** 6 1

1 7 **21** 35 35 **21** 7 1

1 6 **28** 56 70 56 **28** 8 1

For the origin of Pascal triangle see Agarwal and Sen [5].

- The only triangular numbers which are the product of three consecutive integers are 6, 120, 210, 990, 185,136, 258, 474, 216, see Guy [8].
- A number is called palindromic if it is identical with its reverse, *i.e.*, reading the same forward as well as backward. There are 28 *palindromic triangular numbers* less than 10^{10}, namely, 1, 3, 6, 55, 66, 171, 595, 666, 3003, 5995, 8778, 15,051, 66,066, 617,716, 828,828, 1,269,621, 1,680,861, 3,544,453, 5,073,705, 5,676,765, 6,295,926, 351,335,153, 61,477,416, 178,727,871, 1,264,114,621, 1,634,004,361, 5,289,009,825, 6,172,882,716. The largest known palindromic triangular numbers containing only odd digits and even digits are $t_{32850970} = 539593131395935$ and $t_{128127032} = 8208268228628028$. It is known, see Trigg [9], that an infinity of palindromic triangular numbers exist in several different bases, for example, three, five, and nine; however, no infinite sequence of such numbers has been found in base ten.
- Let m be a given natural number, then it is n-th triangular number, *i.e.*, $m = t_n$ if and only if $n = \left(-1+\sqrt{1+8m}\right)/2$. This means if and only if $8m+1$ is a perfect square.
- If n is a triangular number, then $9n+1, 25n+3$ and $49n+6$ are also triangular numbers. This result of 1775 is due to Euler. Indeed, if $n = t_m$, then $9n+1 = t_{3m+1}$, $25n+3 = t_{5m+2}$ and $49n+6 = t_{7m+3}$. An extension of Euler's result is the identity

$$(2k+1)^2 t_m + t_k = t_{(2k+1)m+k}, \quad k = 1, 2, \cdots$$

i.e.,

$$(2k+1)^2 \cdot \frac{m(m+1)}{2} + \frac{k(k+1)}{2} = \frac{\left[(2k+1)m+k\right]\left[(2k+1)m+k+1\right]}{2}.$$

- From the identity

$$4\left(\frac{x(x+1)}{2} + \frac{y(y+1)}{2}\right) + 1 = (x+y+1)^2 + (x-y)^2$$

it follows that if n is the sum of two triangular numbers, then $4n+1$ is a sum of two squares.

- Differentiating the expansion $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$ twice, we get

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2} = \sum_{n=1}^{\infty}(n+1)(n)x^{n-1}, \tag{10}$$

and hence

$$\frac{x}{(1-x)^3} = 0x^0 + \sum_{n=1}^{\infty}\frac{(n+1)(n)}{2}x^n = \sum_{n=0}^{\infty}\frac{(n+1)(n)}{2}x^n = \sum_{n=0}^{\infty} t_n x^n.$$

Hence, $x(1-x)^{-3}$ is the *generating function* of all triangular numbers. In 1995, Sloane and Plouffe [10] have shown that

$$\left(1+2x+\frac{1}{2}x^2\right)e^x = \sum_{n=0}^{\infty} t_{n+1}\frac{x^n}{n!}.$$

- To find the sum of the first n triangular numbers, we need an expression for $\sum_{k=1}^{n} k^2$ (a general reference for the summation of series is Davis [11]). For this, we begin with Pascal's identity

$$k^3-(k-1)^3 = 3k^2-3k+1, \quad k \ge 1$$

and hence

$$\left(1^3-0^3\right)+\left(2^3-1^3\right)+\left(3^3-2^3\right)+\cdots+\left(n^3-(n-1)^3\right) = 3\sum_{k=1}^{n}k^2 - 3\sum_{k=1}^{n}k + \sum_{k=1}^{n}1,$$

which in view of (2) gives

$$\sum_{k=1}^{n}k^2 = \frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{2}n-\frac{1}{3}n = \frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n = \frac{1}{6}n(n+1)(2n+1). \tag{11}$$

Archimedes as proposition 10 in his text *On Spirals* stated the formula

$$(n+1)n^2 + (1+2+\cdots+n) = 3\left(1^2+2^2+\cdots+n^2\right) \tag{12}$$

from which (11) is immediate. It is believed that he obtained (12) by letting k the successive values $1,2,\cdots,n-1$ in the relation

$$n^2 = \left[k+(n-k)\right]^2 = k^2+2k(n-k)+(n-k)^2,$$

and adding the resulting $n-1$ equations, together with the identity $2n^2 = 2n^2$, to arrive at

$$(n+1)n^2 = 2\left(1^2+2^2+\cdots+n^2\right)+2\left[1(n-1)+2(n-2)+\cdots+(n-1)1\right]. \tag{13}$$

Next, letting $k=1,2,\cdots,n$ in the formula

$$k^2 = k+2\left[1+2+\cdots+(k-1)\right]$$

and adding n equations to get

$$1^2+2^2+\cdots+n^2 = (1+2+\cdots+n)+2\left[1(n-1)+2(n-2)+\cdots+(n-1)1\right]. \tag{14}$$

From (13) and (14), the Formula (12) follows.

Another proof of (11) is given by Fibonacci. He begins with the identity

$$k(k+1)(2k+1) = (k-1)k(2k-1)+6k^2.$$

and takes $k=1,2,3,\cdots,n$ to get the set of equations

$$\begin{aligned}
&1\cdot 2\cdot 3 = 6\cdot 1^2\\
&2\cdot 3\cdot 5 = 1\cdot 2\cdot 3 + 6\cdot 2^2\\
&3\cdot 4\cdot 7 = 2\cdot 3\cdot 5 + 6\cdot 3^2\\
&\qquad\vdots\\
&(n-1)n(2n-1) = (n-2)(n-1)(2n-3) + 6(n-1)^2\\
&n(n+1)(2n+1) = (n-1)n(2n-1) + 6n^2.
\end{aligned}$$

On adding these n equations and cancelling the common terms, (11) follows.

Now from (2) and (11), we have

$$\sum_{k=1}^{n} t_k = \frac{1}{2}\sum_{k=1}^{n} k^2 + \frac{1}{2}\sum_{k=1}^{n} k = \frac{1}{6}n(n+1)(n+2). \tag{15}$$

Relation (15) is due to Aryabhata.

For an alternative proof of (15), we note that

$$\begin{aligned}
&(n+1)t_n - \sum_{k=0}^{n} t_k = \sum_{k=0}^{n}(t_n - t_k)\\
&= (1+2+3+\cdots+n) + (2+3+\cdots+n) + (3+\cdots+n) + \cdots + n\\
&= 1^2 + 2^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2,
\end{aligned}$$

and hence in view of (11), we have

$$\begin{aligned}
\sum_{k=1}^{n} t_k &= (n+1)t_n - \sum_{k=1}^{n} k^2\\
&= (n+1)\frac{n(n+1)}{2} - \frac{1}{6}n(n+1)(2n+1)\\
&= \frac{1}{6}n(n+1)(n+2).
\end{aligned}$$

From (15) it follows that

$$\sum_{k=m+1}^{n} t_k = \frac{1}{6}(n-m)\left[(n-m)^2 + 3(n+1)(m+1) - 1\right],$$

which in particular for $m=4, n=7$ gives $t_5 + t_6 + t_7 = 64 = 8^2$, *i.e.*, three successive triangular numbers whose sum is a perfect square. Similarly, we have $t_5 + t_6 + t_7 + t_8 = 10^2$.

From (15), we also have $\sum_{k=1}^{n} t_k = (1/3)(n+2)t_n$, which means t_n divides $\sum_{k=1}^{n} t_k$ if $n = 3m-2$, $m = 1, 2, \cdots$.

- The reciprocal of the $(n+1)$-th triangular number is related to the integral

$$\int_0^1\int_0^1 |x-y|^n \,\mathrm{d}x\mathrm{d}y = \frac{2}{(n+1)(n+2)} = \frac{1}{t_{n+1}}.$$

- The sum of reciprocals of the first n triangular numbers is

$$\sum_{k=1}^{n}\frac{1}{t_k} = \sum_{k=1}^{n}\frac{2}{k(k+1)} = 2\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = 2\left(1 - \frac{1}{n+1}\right), \tag{16}$$

and hence

$$\sum_{k=1}^{\infty}\frac{1}{t_k}=2\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=2. \tag{17}$$

Jacob Bernoulli (1654-1705, Switzerland) in 1689 summed numerous convergent series, the above is one of the examples. In the literature this procedure is now called *telescoping*, also see Lesko [12].

- Pythagoras theorem states that if a and b are the lengths of the two legs of a right triangle and c is the length of the hypothenuse, then the sum of the areas of the two squares on the legs equals the area of the square on the hypotenuse, *i.e.*,

$$a^2+b^2=c^2. \tag{18}$$

A set of three positive integers a, b and c which satisfy (18) is called *Pythagorean triple* and written as ordered triple (a,b,c). A Pythagorean triangle (a,b,c) is said to be *primitive* if a,b,c have no common divisor other than 1. For the origin, patterns, extensions, astonishing directions, and open problems, of Pythagoras theorem and his triples, see Agarwal [13] [14], and an interesting article of Beauregard and Suryanarayan [15]. There are Pythagorean triples (not necessarily primitive) each side of which is a triangular number, for example, $(t_{132},t_{143},t_{164})=(8778,10296,13530)$. It is not known whether infinitively many such triples exist.

- A number is called *perfect* if and only if it is equal to the sum of its positive divisors, excluding itself. For example, $t_3=6$ is perfect, because $6(1+2+3=6)$. The numbers $28,496,8128(t_7,t_{31},t_{127})$ are also perfect that Pythagoreans discovered. For mystical reasons, such numbers have been given considerable attention in the past. Especially, Pythagoreans praised the number six eulogistically, concluding that the universe is harmonized by it and from it comes wholeness, permanence, as well as perfect health. In fact, Plato asserted that the creation is perfect because the number 6 is perfect. They also realized that like squares, six equilateral triangles (see **Figure 4**) meeting at a point (add up to 360˚) leave no space in tilling a floor.

Till very recently only 51 even perfect numbers of the form $2^{p-1}(2^p-1)$ have been discovered. It is not known whether there are any odd perfect numbers, and if there exist infinitely many perfect numbers. The following result due

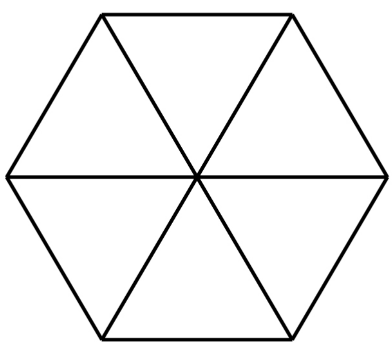

Figure 4. Tilling a floor.

to Euclid of Alexandria (around 325-265 BC, Greece) and Euler states that an even number is perfect if and only if it has the form $2^{p-1}\left(2^p-1\right)$, where 2^p-1 is a prime number (known as Pére Marin Mersenne's, 1588-1648, France, prime number). In 1575 it was observed that $2^{p-1}\left(2^p-1\right)=2^p\left(2^p-1\right)/2=t_{2^p-1}$, *i.e.*, every known perfect number is also a triangular number.

- Fermat numbers are defined as $F_n=2^{2^n}+1, n\ge 0$. First few Fermat's numbers are 3, 5, 17, 257, 65537. We shall show that for $n>0$, Fermat number F_n is never a triangular number, *i.e.*, there is no integer m which satisfies $2^{2^n}+1=m(m+1)/2$. This means the discriminant of the equation $m^2+m-2\left(2^{2^n}+1\right)=0$ is not an integer. Suppose to contrary that there exists an integer p such that $\sqrt{1+8\left(2^{2^n}+1\right)}=p$, but then $2^{2^n+3}=p^2-9=(p+3)(p-3)$, which implies that there exist integers r and s such that $p+3=2^r$ and $p-3=2^s$. Hence, we have $2^r-2^s=6$ for which the only solution is $r=3, s=1$. This means, $2^{2^n+3}=2^3\times 2$, or $2^{2^n}=2$, which is true only for $n=0$.
- We shall find all *square triangular numbers*, *i.e.*, all positive integers n and the corresponding m so that $n(n+1)/2=m^2$. This equation can be written as, so called Pell's Equation (for its origin, see Agarwal [5]) $b^2-2a^2=1$, where $b=2n+1$ and $a=2m$. We note that if $(a_{k-1},b_{k-1}), k\ge 1$ is an integer solution of $b^2-2a^2=\pm 1$, then (a_k,b_k) defined by the recurrence relations

$$a_k=a_{k-1}+b_{k-1},\quad b_k=2a_{k-1}+b_{k-1},\quad k\ge 1 \tag{19}$$

satisfy

$$b_k^2-2a_k^2=\left(2a_{k-1}+b_{k-1}\right)^2-2\left(a_{k-1}+b_{k-1}\right)^2=-\left(b_{k-1}^2-2a_{k-1}^2\right),$$

and hence $b^2-2a^2=\mp 1$. From this observation we conclude that if $(a_{k-1},b_{k-1}), k\ge 1$ is an integer solution of $b^2-2a^2=1$, then so is $(a_{k+1},b_{k+1})=(3a_{k-1}+2b_{k-1}, 4a_{k-1}+3b_{k-1})$. Since $(a_0,b_0)=(0,1)$ is a solution of $b^2-2a^2=1$ (its fundamental solution is $(a,b)=(2,3)$), it follows that the iterative scheme

$$\begin{aligned} x_k &= 3x_{k-1}+2y_{k-1} \\ y_k &= 4x_{k-1}+3y_{k-1},\quad x_0=0,\ y_0=1 \end{aligned} \tag{20}$$

gives all solutions of $b^2-2a^2=1$. System (20) can be written as

$$\begin{aligned} x_{k+1} &= 6x_k-x_{k-1},\quad x_0=0,\ x_1=2 \\ y_{k+1} &= 6y_k-y_{k-1},\quad y_0=1,\ y_1=3. \end{aligned} \tag{21}$$

Now in (21) using the substitution $x_k=2m_k, y_k=2n_k+1$, we get

$$\begin{aligned} m_{k+1} &= 6m_k-m_{k-1},\quad m_0=0,\ m_1=1 \\ n_{k+1} &= 6n_k-n_{k-1}+2,\quad n_0=0,\ n_1=1 \end{aligned} \tag{22}$$

Clearly, (22) generates all (infinite) solutions (m_k,n_k) of the equation $n(n+1)/2=m^2$. First few of these solutions are

$$(1,1),(6,8),(35,49),(204,288),(1189,1681),$$
$$(6930,9800),(40391,57121),(235416,332928).$$

For $k \ge 1$, explicit solution of the system (22) can be computed (for details see Agarwal [16] [17]) rather easily, and appears as

$$m_k = \frac{1}{4\sqrt{2}}\left[\left(3+2\sqrt{2}\right)^k - \left(3-2\sqrt{2}\right)^k\right]$$
$$n_k = \frac{1}{4}\left[\left(3+2\sqrt{2}\right)^k + \left(3-2\sqrt{2}\right)^k - 2\right] \tag{23}$$

This result is originally due to Euler which he obtained in 1730. While compare to the explicit solution (23) the computation of (m_k, n_k) from the recurrence relations (22) is very simple, the following interesting relation follows from (23) by direct substitution

$$m_k^2 - m_{k-1}^2 = m_{2k-1}. \tag{24}$$

Hence the difference between two consecutive square triangular numbers is the square root of another square triangular number.

Now we note that the system (19) can be written as

$$a_{n+1} = 2a_n + a_{n-1}, \quad a_0 = 0, a_1 = 1$$
$$b_{n+1} = 2b_n + b_{n-1}, \quad b_0 = 1, b_1 = 1$$

and its (integer) solution is

$$a_n = \frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n\right],$$
$$b_n = \frac{1}{2}\left[\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n\right]. \tag{25}$$

From this, and simple calculations the following relations follow

$$m_k = a_k b_k, \quad n_{2k} = b_{2k}^2 - 1 = 2\left(2m_k\right)^2, \quad n_{2k+1} = b_{2k+1}^2 = 2a_{2k+1}^2 - 1.$$

It is apparent that if (m_k, n_k) is a solution of $n(n+1)/2 = m^2$, then $((2p+1)m_k, (2p+1)n_k)$, $p \ge 1$ is a solution of $n(n+2p+1)/2 = m^2$. Now, if n is even, we have

$$t_n t_{n+1} \cdots t_{n+2p} = (n+1)^2 \left(\frac{n+2}{2}\right)^2 (n+3)^2 \left(\frac{n+4}{2}\right)^2$$
$$\cdots (n+2p-1)^2 \left(\frac{n+2p}{2}\right)^2 \left(\frac{n(n+2p+1)}{2}\right), \tag{26}$$

and, when n is odd,

$$t_n t_{n+1} \cdots t_{n+2p} = \left(\frac{n+1}{2}\right)^2 (n+2)^2 \left(\frac{n+3}{2}\right)^2 (n+4)^2$$
$$\cdots \left(\frac{n+2p-1}{2}\right)^2 (n+2p)^2 \left(\frac{n(n+2p+1)}{2}\right) \tag{27}$$

and hence the right side is a perfect square for $n = (2p+1)n_k$. Therefore, the product of $(2p+1)$ consecutive triangular numbers is a perfect square for each

$p \ge 1$ and $k \ge 1$. In particular, for $p = k = 2$, $n = 5n_2 = 40$, from (26) we have

$$t_{40}t_{41}t_{42}t_{43}t_{44} = (41)^2(21)^2(43)^2(22)^2(30)^2 = (24435180)^2$$

and for $p = k = 3, n = 7n_3 = 343$, from (27), we find

$$\begin{aligned} &t_{343}t_{344}t_{345}t_{346}t_{347}t_{348}t_{349} \\ &= (172)^2(345)^2(173)^2(347)^2(174)^2(349)^2(245)^2 \\ &= (52998536784979800)^2. \end{aligned}$$

Similarly, if n is even, we have

$$\begin{aligned} 2t_nt_{n+1}\cdots t_{n+2p-1} &= (n+1)^2\left(\frac{n+2}{2}\right)^2(n+3)^2\left(\frac{n+4}{2}\right)^2 \\ &\cdots\left(\frac{n+2p-2}{2}\right)^2(n+2p-1)^2\left(\frac{n(n+2p)}{2}\right), \end{aligned} \quad (28)$$

and hence the right side is a perfect square for $n = 2pn_k$ (which is always even). Therefore, two times the product of $2p$ consecutive triangular numbers is a perfect square for each $p \ge 1$ and $k \ge 1$. In particular, for $p = k = 2$, $n = 4n_2 = 32$, from (28) we have

$$2t_{32}t_{33}t_{34}t_{35} = (33)^2(17)^2(35)^2(24)^2 = (471240)^2$$

and for $p = k = 3, n = 6n_3 = 294$, we find

$$\begin{aligned} 2t_{294}t_{295}t_{296}t_{297}t_{298}t_{299} &= (295)^2(148)^2(297)^2(149)^2(299)^2(210)^2 \\ &= (121315678684200)^2. \end{aligned}$$

From the equality

$$\frac{(4n(n+1))(4n(n+1)+1)}{2} = 4\frac{n(n+1)}{2}(2n+1)^2$$

it follows that if the triangular number t_n is square, then $t_{4n(n+1)}$ is also square. Since t_1 is square, it follows that there are infinite number of square triangular numbers. This clever observation was reported in 1662, see Pietenpol *et al.* [18]. From this, the first four square triangular numbers, we get are t_1, t_8, t_{288} and t_{332928}.

- There are infinitely many triangular numbers that are simultaneously expressible as the sum of two cubes and the difference of two cubes. For this, Burton [19] begins with the identity

$$(27k^6)^2 - 1 = (9k^4 - 3k)^3 + (9k^3 - 1)^3 = (9k^4 + 3k)^3 - (9k^3 + 1)^3$$

and observed that if k is odd then this equality can be written as

$$(2n+1)^2 - 1 = (2a)^3 + (2b)^3 = (2c)^3 - (2d)^3,$$

which is the same as

$$t_n = a^3 + b^3 = c^3 - d^3.$$

For $k = 1, 3$ and 5 this gives

$$t_{13} = 3^3 + 4^3 = 6^3 - 5^3$$
$$t_{9841} = (360)^3 + (121)^3 = 369^3 - (122)^3$$
$$t_{210937} = (2805)^3 + (562)^3 = (2820)^3 - (563)^3$$

- In 1844, Eugéne Charles Catalan (1814-1894) conjectured that 8 and 9 are the only numbers which differ by 1 and are both exact powers $8 = 2^3$, $9 = 3^2$. This conjecture was proved by Preda Mihăilescu (Born 1955, Romania) after one hundred and fifty-eight years, and published two years later in [20]. Thus the only solution in natural numbers of the Diophantine equation $x^a - y^b = 1$ for $a, b > 1$, $x, y > 0$ is $x = 3$, $a = 2$, $y = 2$, $b = 3$. Now since $n(n+1)/2 = m^3$ can be written as $(2n+1)^2 - (2m)^3 = 1$, the only solution of this equation is $2n+1 = 3$, $2m = 2$, *i.e.*, $(1,1)$ is the only cubic triangular number.
- In 2001, Bennett [21] proved that if *a*, *b* and *n* are positive integers with $n \ge 3$, then the equation $|ax^n - by^n| = 1$, possesses at most one solution in positive integers *x* and *y*. This result is directly applicable to show that for the equation $n(n+1)/2 = m^p$, $p \ge 3$ the only solution is $(1,1)$. For this, first we note that integers $t, 2t+1$ and $t+1, 2t+1$ are coprime, *i.e.*, they do not have any common factor except 1. We also recall that if the product of co-prime numbers is a *p*-th power, then both are also of *p*-the power. Now let *n* be even, *i.e.*, $n = 2t$, then the equation $n(n+1)/2 = m^p$ is the same as $t(2t+1) = m^p$. Thus, it follows that $t = x^p$ and $2t+1 = y^p$, and hence $y^p - 2x^p = 1$, which has only one solution, namely, $x = 0, y = 1$ which gives $t = 0$, and hence $n = 0$ and so $(0,0)$ is the solution of $n(n+1)/2 = m^p$, but we are not interested in this solution. Now we assume that *n* is odd, *i.e.*, $n = 2t+1$, then the equation $n(n+1)/2 = m^p$ is the same as $(t+1)(2t+1) = m^p$. Thus, we must have $t+1 = x^p$ and $2t+1 = y^p$, which gives $y^p - 2x^p = -1$. The only solution of this equation is $x = y = 1$, and hence again $t = 0$ and so $(0,0)$ is the undesirable solution of $n(n+1)/2 = m^p$.
- Startling *generating function* of all square triangular numbers is recorded by Plouffe [22] as

$$\frac{x(1+x)}{(1-x)(x^2 - 34x + 1)} = x + 6^2 x^2 + 35^2 x^3 + \cdots. \tag{29}$$

3. Square Numbers S_n

In this arrangement rows as well as columns contain $1, 2, 3, 4, \cdots, n$ dots, (see **Figure 5**).

From **Figure 5** it is clear that a square made up of $(n+1)$ dots on a side can be divided into a smaller square of side *n* and an *L*, shaped border (a gnomon), which has $(n+1) + n = 2n+1$ dots (called $(n+1)$ *th gnomonic number and denoted as* g_{n+1}), and hence

$$S_{n+1} - S_n = (n+1)^2 - n^2 = (2n+1), \tag{30}$$

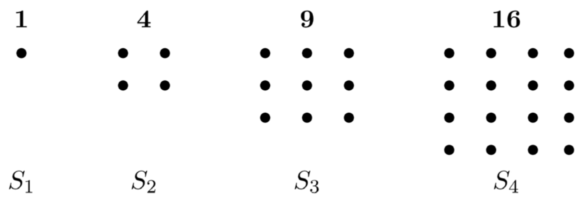

Figure 5. Square numbers.

i.e., the differences between successive nested squares produce the sequence of odd numbers. From (30) it follows that

$$\left(1^2-0^2\right)+\left(2^2-1^2\right)+\left(3^2-2^2\right)+\cdots+\left(n^2-(n-1)^2\right)=1+3+5+\cdots+(2n-1)$$

and hence

$$\sum_{k=1}^{n}(2k-1)=1+3+5+7+\cdots+(2n-1)=n^2=S_n. \quad (31)$$

An alternative proof of (31) is as follows

$$S_n=1+3+5+\cdots+(2n-3)+(2n-1)$$

$$S_n=(2n-1)+(2n-3)+(2n-5)+\cdots+3+1.$$

An addition of these two arrangements immediately gives

$$2S_n=2n+2n+\cdots+2n=2n^2.$$

Figure 6 provides proof of (31) without words. Here odd integers, one block, three blocks, five blocks, and so on, arranged in a special way. We begin with a single block in the lower left corner; three shaded blocks surrounded it to form a 2×2 square; five unshaded blocks surround these to form a 3×3 square; with the next seven shaded blocks we have a 4×4 square; and so on. The diagram makes clear that the sum of consecutive odd integers will always yield a (geometric) square.

Comparing **Figure 1** and **Figure 5** or **Figure 2** and **Figure 6**, it is clear that n-th square number is equal to the n-th triangular number increased by its predecessor, *i.e.*,

$$S_n=t_n+t_{n-1}=n^2. \quad (32)$$

Indeed, we have

$$t_n=1+2+3+\cdots+(n-1)+n$$

$$t_{n-1}=1+2+\cdots+(n-2)+(n-1).$$

An addition of these two arrangements in view of (31) gives

$$t_n+t_{n-1}=1+3+5+\cdots+(2n-1)=n^2=S_n.$$

Of course, directly from (1), (2), and (32), we also have

$$t_n+t_{n-1}=\frac{n(n+1)}{2}+\frac{(n-1)n}{2}=n^2=\left(t_n-t_{n-1}\right)^2=S_n,$$

or simply from (1) and (2),

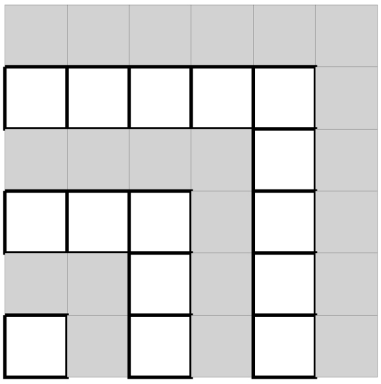

Figure 6. Proof of (31) without words.

$$t_n + t_{n-1} = 2t_{n-1} + n = n(n-1) + n = n^2 = S_n.$$

From (32), we find the identities

$$\sum_{k=1}^{2n} t_k = (t_2 + t_1) + (t_4 + t_3) + \cdots + (t_{2n} + t_{2n-1})$$
$$= 2^2 + 4^2 + \cdots + (2n)^2$$

and

$$\sum_{k=1}^{2n+1} t_k = t_1 + (t_3 + t_2) + (t_5 + t_4) + \cdots + (t_{2n+1} + t_{2n})$$
$$= 1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2.$$

It also follows that

$$t_{2n} - 2t_n = \frac{2n(2n+1)}{2} - 2\frac{n(n+1)}{2} = n^2 = S_n. \quad (33)$$

We also have equalities

$$t_{9n+4} - t_{3n+1} = [3(2n+1)]^2 = S_{3(2n+1)}, \quad (34)$$

$$S_1 - S_2 + S_3 - S_4 + \cdots + (-1)^{n+1} S_n = (-1)^{n+1} t_n, \quad (35)$$

and

$$\sum_{k=0}^{n} (t_{2n} + k)^2 = \sum_{k=1}^{n} (4t_n + k)^2, \quad (36)$$

which is the same as

$$\sum_{k=0}^{n} (2n^2 + n + k)^2 = \sum_{k=1}^{n} (2n^2 + 2n + k)^2 \quad \text{or} \quad \sum_{k=0}^{n} S_{2n^2+n+k} = \sum_{k=1}^{n} S_{2n^2+2n+k}$$

and, in particular, for $n = 4$ reduces to

$$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2.$$

The following equality is of exceptional merit

$$S_n + S_{n+1} = S_{n(n+1)+1} - S_{n(n+1)}, \quad (37)$$

which, in particular, for $n = 5$ gives $5^2 + 6^2 = 31^2 - 30^2$.

- Relation (30) reveals that every odd integer $(2n+1)$ is the difference of two consecutive square numbers S_{n+1} and S_n. Relation (32) shows that every square integer n^2 is a sum of two consecutive triangular numbers t_n and t_{n-1}, whereas (33) displays it is the difference of 2*n*-th and two times *n*-th triangular numbers.
- From the equalities

$$8t_n^2 = \left(n^2+n\right)^2 + \left(n^2+n\right)^2,$$

$$8t_n^2+1 = \left(n^2-1\right)^2 + \left(n^2+2n\right)^2,$$

$$8t_n^2+2 = \left(n^2+n-1\right)^2 + \left(n^2+n+1\right)^2$$

it follows that there are infinite triples of consecutive numbers which can be written as the sum of two squares.

- No square number has as its last digit (unit digit) 2, 3, 7 or 8.
- From (10) it follows that

$$\frac{2x}{(1-x)^3} = \sum_{n=0}^{\infty} n^2 x^n + x\sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} n^2 x^n + x\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x}$$

and hence

$$\frac{2x}{(1-x)^3} - \frac{x}{(1-x)^2} = \frac{x(1+x)}{(1-x)^3} = \sum_{n=0}^{\infty} n^2 x^n. \tag{38}$$

Therefore, $x(x+1)(1-x)^{-3}$ is the *generating function* of all square numbers. From (38) it also follows that the generating function for all gnomonic numbers is

$$\frac{x(1+x)}{(1-x)^2} = \sum_{n=1}^{\infty}(2n-1)x^n = \sum_{n=1}^{\infty} g_n x^n.$$

- The sum of the first *n* square numbers is given in (11). For the exact sum of the reciprocals of the first *n* square numbers no formula exists; however, the problem of summing the reciprocals of all square numbers has a long history and in the literature it is known as *the Basel problem*. Euler in 1748 considered $\sin x/x, x \neq 0$ which has roots at $\pm n\pi, n \geq 1$. Then, he wrote this function in terms of infinite product

$$\begin{aligned}\frac{\sin x}{x} &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \\ &= \left(1-\frac{x^2}{1^2\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\cdots,\end{aligned}$$

which on equating the coefficients of x^2, gives

$$\frac{1}{6} = \frac{1}{1^2\pi^2} + \frac{1}{2^2\pi^2} + \frac{1}{3^2\pi^2} + \cdots,$$

and hence

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \approx 1.6449340668. \tag{39}$$

The above demonstration of Euler is based on manipulations that were not justified at the time, and it was not until 1741 that he was able to produce a truly rigorous proof. Now in the literature for (39) several different proofs are known, e.g., for a recent elementary, but clever demonstration, see Murty [23].

- The following result provides a characterization of all Pythagorean triples, *i.e.*, solutions of (18): Let u and v be any two positive integers, with $u > v$, then the three numbers

$$a = u^2 - v^2,\quad b = 2uv,\quad c = u^2 + v^2 \tag{40}$$

form a Pythagorean triple. If in addition u and v are of opposite parity-one even and the other odd-and they are coprime, *i.e.*, that they do not have any common factor other than 1, then (a,b,c) is a primitive Pythagorean triple. The converse, *i.e.*, any Pythagorean triple is necessarily of the form (40) also holds. For the proof and history of this result see, Agarwal [14]. From (18), (32), and (40) the following relations hold

$$\begin{aligned} &S_a + S_b = S_c,\ \ (t_a + t_{a-1}) + (t_b + t_{b-1}) = (t_c + t_{c-1}),\\ &S_{u^2-v^2} + S_{2uv} = S_{u^2+v^2}. \end{aligned} \tag{41}$$

The relation (30) can be written as $n^2 + (2n+1) = (n+1)^2$. With the help of this relation we can find Pythagorean triples (a,b,c). For this, we let $2n+1 = m^2$, (and hence m is odd), then $n = m^2 - 1/2$, $n+1 = m^2 + 1/2$. Thus, it follows that

$$\begin{aligned} &m^2 + \left(\frac{m^2-1}{2}\right)^2 = \left(\frac{m^2+1}{2}\right)^2,\ \text{ i.e.,}\\ &S_m + S_{(m^2-1)/2} = S_{(m^2+1)/2},\ \ (m\ \text{odd}). \end{aligned} \tag{42}$$

For $m = 3,5,7,9,\cdots$ Equation (42) gives solutions of (18):

m	a	b	c
3	3	4	5
5	5	12	13
7	7	24	25
9	9	40	41

Similar to (42) for m even we also have the relation

$$\begin{aligned} &m^2 + \left(\frac{m^2}{4} - 1\right)^2 = \left(\frac{m^2}{4} + 1\right),\ \text{ i.e.,}\\ &S_m + S_{(m^2-4)/4} = S_{(m^2+4)/4}\ \ (m\ \text{even}). \end{aligned} \tag{43}$$

For $m = 4,6,8,10,\cdots$ Equation (43) gives solutions of (18):

m	a	b	c
4	4	3	5
6	6	8	10
8	8	15	17
10	10	24	26

In (40), letting $u=(m+2)^2$ and $v=(m+1)^2$, from (18) and (32), we get the relations

$$(2m+3)^2+\left(2(m+1)(m+2)\right)^2=\left((m+1)^2+(m+2)^2\right)^2, \text{ i.e.,}$$

$$S_{2m+3}+S_{2(m+1)(m+2)}=S_{(m+1)^2+(m+2)^2},$$

which is the same as

$$\left(t_{2m+3}+t_{2m+2}\right)+16t_{m+1}^2=\left(t_{m+2}+2t_{m+1}+t_m\right)^2.$$

- In 1875, Francois Edouard Anatole Lucas (1842-1891, French) challenged the mathematical community to prove that the only solution of the equation

$$\sum_{k=1}^{n}k^2=\frac{1}{6}n(n+1)(2n+1)=m^2$$

with $n>1$ is when $n=24$ and $m=70$. In the literature this has been termed as the cannonball problem, in fact, it can be visualized as the problem of taking a square arrangement of cannonballs on the ground and building a square pyramid out of them. It was only in 1918, George Neville Watson (1886-1965, Britain) used elliptic functions to provide correct (filling gaps in earlier attempts) proof of Lucas assertion. Simplified proofs of this result are available, e.g., in Ma [24] and Anglin [25].

4. Rectangular (Oblong, Pronic, Heteromecic) Numbers R_n

In this arrangement rows contain $(n+1)$ whereas columns contain *n* dots, see Figure 7.

From Figure 7 it is clear that the ratio $(n+1)/n$ of the sides of rectangles depends on *n*. Further, we have

$$\begin{aligned}R_n&=2+4+6+8+\cdots+2n\\&=2(1+2+3+4+\cdots+n)=2t_n=n(n+1)\end{aligned} \tag{44}$$

i.e., we add successive even numbers, or two times triangular numbers. It also follows that rectangular number R_{n+1} is made from R_n by adding an *L*--shaped border (a gnomon), with $2(n+1)$ dots, *i.e.*,

$$R_{n+1}-R_n=2(n+1), \tag{45}$$

i.e., the differences between successive nested rectangular numbers produce the sequence of even numbers.

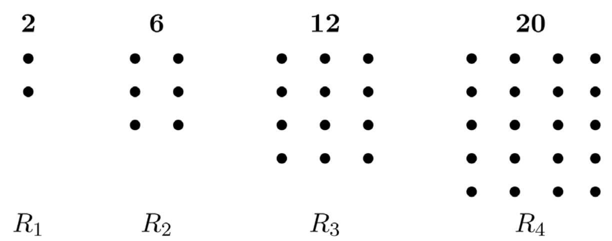

Figure 7. Rectangular numbers.

Thus the odd numbers generate a limited number of forms, namely, squares, while the even ones generate a multiplicity of rectangles which are not similar. From this the Pythagoreans deduced the following correspondence:

$$\text{odd} \leftrightarrow \text{limited} \quad \text{and} \quad \text{even} \leftrightarrow \text{unlimited}.$$

We also have the relations

$$R_n + S_n = 2t_n + (t_n + t_{n-1}) = 3\frac{n(n+1)}{2} + \frac{(n-1)n}{2} = \frac{2n(2n+1)}{2} = t_{2n}$$
$$R_n - S_n = 2t_n - (t_n + t_{n-1}) = t_n - t_{n-1} = n \tag{46}$$

$$2R_n + S_n + S_{n+1} = 6t_n + t_{n-1} + t_{n+1} = (2n+1)^2$$
$$S_{2n+1}^2 = (2n+1)^2 = 4n(n+1) + 1 = 8t_n + 1 = (4t_n + 1)^2 - (4t_n)^2 \tag{47}$$

$$(3S_{2n+1})^2 = t_{9n+4} - t_{3n+1}$$
$$R_n = t_{n+1} + t_{n-1} - 1.$$

From (31) and (46) it follows that

$$\sum_{k=1}^{2n-1} (-1)^{k+1} t_k = t_1 + (t_3 - t_2) + \cdots + (t_{2n-1} - t_{2n-2})$$
$$= 1 + 3 + \cdots + (2n-1) = n^2.$$

- Relation (44) shows that the product of two consecutive positive integers n and $(n+1)$ is the same as two times n-th triangular numbers. According to historians with this relation Pythagoreans' enthusiasm was endless. Relation (45) reveals that every even integer $2n$ is the difference of two consecutive rectangular numbers R_n and R_{n-1}. Relation (46) displays that every positive integer n is the difference of n-th and $(n-1)$-th triangular numbers. Relation (47) is due to Plutarch), it says an integer n is a triangular number if and only if $8n+1$ is a perfect odd square.
- Let m be a given natural number, then it is n-th rectangular number, *i.e.*, $m = R_n$ if and only if $n = \left(-1 + \sqrt{1+4m}\right)/2$.
- From (10) it is clear that $2x(1-x)^{-3}$ is the *generating function* of all rectangular numbers.
- From (15)-(17) and (44) it is clear that

$$\sum_{k=1}^{n} R_k = \frac{1}{3} n(n+1)(n+2),$$
$$\sum_{k=1}^{n} \frac{1}{R_k} = \left(1 - \frac{1}{n+1}\right), \quad \sum_{k=1}^{\infty} \frac{1}{R_k} = 1. \tag{48}$$

- There is no rectangular number which is also a perfect square, in fact, the equation $n(n+1) = m^2$ has no solutions (the product of two consecutive integers cannot be a prefect square).
- To find all *rectangular numbers which are also triangular numbers*, we need to find integer solutions of the equation $n(n+1) = m(m+1)/2$. This equation can be written as Pell's equation $b^2 - 2a^2 = -1$ (its fundamental solution is $(a,b) = (1,1)$) where $b = 2m+1$ and $a = 2n+1$. For this, corres-

ponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 34m_k - m_{k-1} + 16, \quad m_1 = 3, m_2 = 119 \\ n_{k+1} &= 34n_k - n_{k-1} + 16, \quad n_1 = 2, n_2 = 84. \end{aligned} \tag{49}$$

This system genetrates all (infinite) solutions (m_k, n_k) of the equation $n(n+1) = m(m+1)/2$. First few of these solutions are

$$(3,2),(119,84),(4059,2870),(137903,97512),(4684659,3312554).$$

For $k \ge 1$, explicit solution of the system (49) can be written as

$$m_k = \frac{1}{4}\left[\left(\sqrt{2}+1\right)^{4k-1} - \left(\sqrt{2}-1\right)^{4k-1} - 2\right]$$

$$n_k = \frac{\sqrt{2}}{8}\left[\left(\sqrt{2}+1\right)^{4k-1} + \left(\sqrt{2}-1\right)^{4k-1} - 2\sqrt{2}\right]$$

- *Fibonacci numbers* denoted as F_n are defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, \quad F_1 = 1$$

or the closed from expression

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

First few of these numbers are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144. For the origin of Fibonacci numbers, see Agarwal and Sen [5]. *Lucas numbers* denoted by L_n are defined by the same recurrence relation as Fibonacci numbers except first two numbers as $L_0 = 2, L_1 = 1$ or the closed from expression

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

First few of these numbers are 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571, 5778, 9349. Clearly, Fibonacci numbers 1, 3, 21, 55 are also triangular numbers t_1, t_2, t_6, t_{10}. In 1989, Luo [26] had used (47) to show that these are the only Fibonacci numbers which are also triangular. This conjecture was made by Verner Emil Hoggatt Jr. (1921-1980, USA) in 1971. Similarly, only Lucas numbers which are also triangular are 1, 3, 5778, *i.e.*, t_1, t_2, t_{107}. From the above explicit expressions the following relations can be obtained easily $L_n = F_{n-1} + F_{n+1}$ and $F_n = (L_{n-1} + L_{n+1})/5$.

5. Pentagonal Numbers P_n

The pentagonal numbers are defined by the sequence $1, 5, 12, 22, 35, 51, \cdots$, *i.e.*, beginning with 5 each number is formed from the previous one in the sequence by adding the next number in the related sequence $4, 7, 10, \cdots, (3n-2)$. Thus, $5 = 1 + 4$, $12 = 1 + 4 + 7 = 5 + 7$, $22 = 1 + 4 + 7 + 10 = 12 + 10$, and so on (see **Figure 8** and **Figure 9**).

Thus, *n*-th pentagonal number is defined as

$$P_n = P_{n-1} + (3n-2) = 1 + 4 + 7 + \cdots + (3n-2). \tag{50}$$

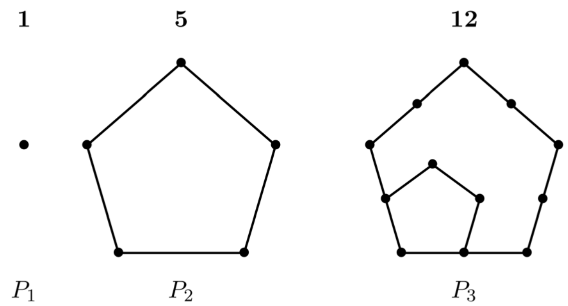

Figure 8. Pentagonal numbers.

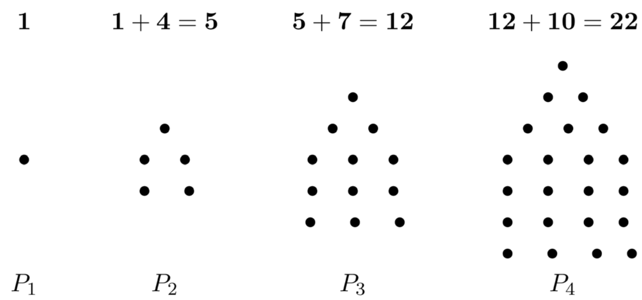

Figure 9. Pentagonal numbers.

Comparing (50) with (3), we have $a=1, d=3$ and hence from (4) it follows that

$$P_n = \frac{n}{2}(3n-1) = \frac{1}{3}\frac{(3n-1)(3n)}{2} = \frac{1}{3}t_{3n-1}. \tag{51}$$

It is interesting to note that P_n is the sum of n integers starting from n, *i.e.*,

$$P_n = n + (n+1) + (n+2) + \cdots + (2n-1), \tag{52}$$

whose sum from (4) is the same as in (51).

Note that from (50), we have

$$\begin{aligned} P_n &= 2P_{n-1} - P_{n-2} - (P_{n-1} - P_{n-2}) + (3n-2) \\ &= 2P_{n-1} - P_{n-2} - (3n-3-2) + (3n-2) \\ &= 2P_{n-1} - P_{n-2} + 3. \end{aligned}$$

From (32) and (51), we also have

$$\begin{aligned} P_n &= \frac{n(n-1)}{2} + n^2 = t_{n-1} + (t_n + t_{n-1}) \\ &= t_n + 2t_{n-1} = t_{2n-1} - t_{n-1}. \end{aligned} \tag{53}$$

- Relation (51) shows that pentagonal number P_n is the one-third of the $(3n-1)$-th triangular number, whereas relation (53) reveals that it is the sum of n-th triangular number and two times of $(n-1)$-th triangular number, and it is the difference of $(2n-1)$-th triangular number and $(n-1)$-th triangular number.
- Let m be a given natural number, then it is n-th pentagonal number, *i.e.*, $m = P_n$ if and only if $n = \left(1+\sqrt{1+24m}\right)/6$.

- As in (38), we have

$$\sum_{n=0}^{\infty} P_n x^n = \frac{3}{2}\frac{x(1+x)}{(1-x)^3} - \frac{1}{2}\frac{x}{(1-x)^2} = \frac{x(2x+1)}{(1-x)^3}$$

and hence $x(2x+1)(1-x)^{-3}$ is the *generating function* of all pentagonal numbers.

- From (2), (11) and (51) it is easy to find the sum of the first n pentagonal numbers

$$\sum_{k=1}^{n} P_k = \frac{1}{2}n^2(n+1). \tag{54}$$

- To find the sum of the reciprocals of all pentagonal numbers, we begin with the series

$$f(x) = \sum_{k=1}^{\infty} \frac{2}{k(3k-1)} x^{3k}$$

and note that

$$f(1) = \sum_{k=1}^{\infty} \frac{2}{k(3k-1)} = \sum_{k=1}^{\infty} \frac{1}{P_k},$$

$$f'(x) = 6\sum_{k=1}^{\infty} \frac{1}{3k-1} x^{3k-1},$$

$$f''(x) = 6\sum_{k=1}^{\infty} x^{3k-2} = \frac{6x}{1-x^3}.$$

Now since $f(0) = f'(0) = 0$, we have

$$f(x) = \int_0^x (x-t)\frac{6t}{1-t^3}\mathrm{d}t$$

and hence

$$f(1) = \int_0^1 (1-t)\frac{6t}{1-t^3}\mathrm{d}t$$
$$= 3\left[\int_0^1 \frac{2t+1}{t^2+t+1}\mathrm{d}t - \int_0^1 \frac{1}{(t+1/2)^2 + (\sqrt{3}/2)^2}\mathrm{d}t\right],$$

which immediately gives

$$\sum_{k=1}^{\infty} \frac{1}{P_k} = 3\ln 3 - \frac{\pi}{\sqrt{3}} \approx 1.4820375018. \tag{55}$$

- To find all *square pentagonal numbers*, we need to find integer solutions of the equation $n(3n-1)/2 = m^2$. This equation can be written as Pell's equation $b^2 - 6a^2 = 1$ (its fundamental solution is $(a,b) = (2,5)$), where $b = 6n-1$ and $a = 2m$. For this, corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 98m_k - m_{k-1}, \quad m_1 = 1, m_2 = 99 \\ n_{k+1} &= 98n_k - n_{k-1} - 16, \quad n_1 = 1, n_2 = 81 \end{aligned} \tag{56}$$

This system genetrates all (infinite) solutions (m_k, n_k) of the equation

$n(3n-1)/2 = m^2$. First few of these solutions are

$$(1,1),(99,81),(9701,7921),(950599,776161),(93149001,76055841).$$

For $k \geq 1$, explicit solution of the system (56) can be written as

$$m_k = \frac{1}{4\times 6^k}\left[\left(5\sqrt{6}+12\right)^{2k-1} - \left(5\sqrt{6}-12\right)^{2k-1}\right]$$
$$n_k = \frac{1}{2\times 6^{k+1/2}}\left[\left(5\sqrt{6}+12\right)^{2k-1} + \left(5\sqrt{6}-12\right)^{2k-1}\right] + \frac{1}{6}.$$

- To find all *pentagonal numbers which are also triangular numbers*, we need to find integer solutions of the equation $n(3n-1)/2 = m(m+1)/2$. This equation can be written as Pell's equation $b^2 - 3a^2 = -2$ (its fundamental solution is $(a,b) = (3,5)$) where $b = 6n-1$ and $a = 2m+1$. For this, corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 14m_k - m_{k-1} + 6, \quad m_1 = 1, m_2 = 20 \\ n_{k+1} &= 14n_k - n_{k-1} - 2, \quad n_1 = 1, n_2 = 12 \end{aligned} \tag{57}$$

This system generates all (infinite) solutions (m_k, n_k) of the equation $n(3n-1)/2 = m(m+1)/2$. First few of these solutions are

$$(1,1),(20,12),(285,165),(3976,2296),(55385,31977).$$

For $k \geq 1$, explicit solution of the system (57) can be written as

$$m_k = \frac{1}{12}\left[\left(3+\sqrt{3}\right)\left(2+\sqrt{3}\right)^{2k-1} + \left(3-\sqrt{3}\right)\left(2-\sqrt{3}\right)^{2k-1} - 6\right]$$
$$n_k = \frac{1}{12}\left[\left(1+\sqrt{3}\right)\left(2+\sqrt{3}\right)^{2k-1} + \left(1-\sqrt{3}\right)\left(2-\sqrt{3}\right)^{2k-1} + 2\right]$$

- To find all *pentagonal numbers which are also rectangular numbers*, we need to find integer solutions of the equation $n(3n-1)/2 = m(m+1)$. This equation can be written as Pell's equation $b^2 - 6a^2 = -5$ (its fundamental solution is $(a,b) = (1,1)$) where $b = 6n-1$ and $a = 2m+1$. For this, corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 98m_k - m_{k-1} + 48, \quad m_1 = 3, m_2 = 341 \\ n_{k+1} &= 98n_k - n_{k-1} - 16, \quad n_1 = 3, n_2 = 279 \end{aligned} \tag{58}$$

This system genetrates all (infinite) solutions (m_k, n_k) of the equation $n(3n-1)/2 = m(m+1)$. First few of these solutions are

$$(3,3),(341,279),(33463,27323),(3279081,2677359),(321316523,262353843).$$

For $k \geq 1$, explicit solution of the system (58) can be written as

$$m_k = \frac{1}{24}\left[\left(6+\sqrt{6}\right)\left(5+2\sqrt{6}\right)^{2k-1} + \left(6-\sqrt{6}\right)\left(5-2\sqrt{6}\right)^{2k-1} - 12\right]$$
$$n_k = \frac{1}{12}\left[\left(\sqrt{6}+1\right)\left(5+2\sqrt{6}\right)^{2k-1} + \left(\sqrt{6}-1\right)\left(5-2\sqrt{6}\right)^{2k-1} + 2\right]$$

6. Hexagonal Numbers H_n

The hexagonal numbers are defined by the sequence $1,6,15,28,45,\cdots$, *i.e.*, beginning with 6 each number is formed from the previous one in the sequence by adding the next number in the related sequence $5,9,13,17,21,\cdots,(4n-3)$. Thus, $6=1+5$, $15=1+5+9=6+9$, $28=1+5+9+13=15+13$, and so on (see Figure 10).

Thus, n-th hexagonal number is defined as

$$H_n = H_{n-1} + (4n-3) = 1+5+9+13+\cdots+(4n-3). \tag{59}$$

Comparing (59) with (3), we have $a=1, d=4$ and hence from (4) it follows that

$$H_n = \frac{n}{2}(4n-2) = \frac{(2n-1)(2n)}{2} = n(2n-1). \tag{60}$$

- From (60) it is clear that $H_n = t_{2n-1}$, *i.e.*, alternating triangular numbers are hexagonal numbers.
- Let m be a given natural number, then it is n-th hexagonal number, *i.e.*, $m = H_n$ if and only if $n = \left(1+\sqrt{1+8m}\right)/4$.
- As in (38), we have

$$\sum_{n=0}^{\infty} H_n x^n = 2\frac{x(1+x)}{(1-x)^3} - \frac{x}{(1-x)^2} = \frac{x(3x+1)}{(1-x)^3}$$

and hence $x(3x+1)(1-x)^{-3}$ is the *generating function* of all hexagonal numbers.

- From (2), (11), and (60) it is easy to find the sum of the first n hexagonal numbers

$$\sum_{k=1}^{n} H_k = \frac{1}{6} n(n+1)(4n-1). \tag{61}$$

- To find the sum of the reciprocals of all hexagonal numbers, as for pentagonal numbers we begin with the series $f(x) = \sum_{k=1}^{\infty} x^{2n} / \left[n(2n-1)\right]$, and get

$$f(1) = \sum_{k=1}^{\infty} \frac{1}{H_k} = 2\int_0^1 \frac{1-t}{1-t^2}\,\mathrm{d}t = 2\ln 2 \approx 1.3862943611. \tag{62}$$

- To find all *square hexagonal numbers*, we need to find integer solutions of the equation $n(2n-1) = m^2$. This equation can be written as Pell's equation $b^2 - 2a^2 = 1$ (its fundamental solution is $(a,b)=(2,3)$), where $b = 4n-1$ and $a = 2m$. For this, corresponding to (22) the system is

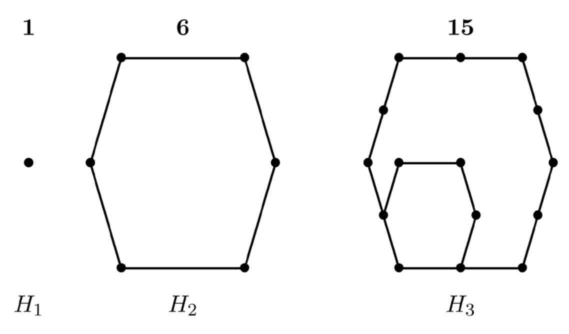

Figure 10. Hexagonal numbers.

$$m_{k+1} = 34m_k - m_{k-1}, \quad m_1 = 1, m_2 = 35$$
$$n_{k+1} = 34n_k - n_{k-1} - 8, \quad n_1 = 1, n_2 = 25 \tag{63}$$

This system generates all (infinite) solutions (m_k, n_k) of the equation $n(2n-1) = m^2$. First few of these solutions are

$$(1,1),(35,25),(1189,841),(40391,28561),(1372105,970225).$$

For $k \geq 1$, explicit solution of the system (63) appears as

$$m_k = a_{2k-1}b_{2k-1} = \frac{1}{4\sqrt{2}}\left[\left(3+2\sqrt{2}\right)^{2k-1} - \left(3-2\sqrt{2}\right)^{2k-1}\right]$$
$$n_k = a_{2k-1}^2 = \frac{1}{8}\left[\left(3+2\sqrt{2}\right)^{2k-1} + \left(3-2\sqrt{2}\right)^{2k-1} + 2\right]$$

here, a_n and b_n are as in (25).

- To find all *hexagonal numbers which are also rectangular numbers*, we need to find integer solutions of the equation $n(2n-1) = m(m+1)$. This equation can be written as Pell's equation $b^2 - 2a^2 = -1$ (its fundamental solution is $(a,b) = (1,1)$) where $b = 4n-1$ and $a = 2m+1$. For this, corresponding to (22) the system is

$$m_{k+1} = 34m_k - m_{k-1} + 16, \quad m_1 = 2, m_2 = 84$$
$$n_{k+1} = 34n_k - n_{k-1} - 8, \quad n_1 = 2, n_2 = 60 \tag{64}$$

This system generates all (infinite) solutions (m_k, n_k) of the equation $n(2n-1) = m(m+1)$. First few of these solutions are

$$(2,2),(84,60),(2870,2030),(97512,68952),(3312554,2342330).$$

For $k \geq 1$, explicit solution of the system (64) can be written as

$$m_k = \frac{\sqrt{2}}{8}\left[\left(\sqrt{2}+1\right)^{4k-1} + \left(\sqrt{2}-1\right)^{4k-1} - 2\sqrt{2}\right]$$
$$n_k = \frac{1}{8}\left[\left(\sqrt{2}+1\right)^{4k-1} - \left(\sqrt{2}-1\right)^{4k-1} + 2\right]$$

- To find all *hexagonal numbers which are also pentagonal numbers*, we need to find integer solutions of the equation $n(2n-1) = m(3m-1)/2$. This equation can be written as Pell's equation $b^2 - 3a^2 = -2$ (its fundamental solution is $(a,b) = (1,1)$) where $b = 6m-1$ and $a = 4n-1$. For this, corresponding to (22) the system is

$$m_{k+1} = 194m_k - m_{k-1} - 32, \quad m_1 = 1, m_2 = 165$$
$$n_{k+1} = 194n_k - n_{k-1} - 48, \quad n_1 = 1, n_2 = 143 \tag{65}$$

This system generates all (infinite) solutions (m_k, n_k) of the equation $n(2n-1) = m(3m-1)/2$. First few of these solutions are

$$(1,1),(165,143),(31977,27693),(6203341,5372251),$$
$$(1203416145,1042188953).$$

For $k \geq 1$, explicit solution of the system (65) can be written as

$$m_k = \frac{1}{12}\left[\left(\sqrt{3}-1\right)\left(2+\sqrt{3}\right)^{4k-2} - \left(\sqrt{3}+1\right)\left(2-\sqrt{3}\right)^{4k-2} + 2\right]$$
$$n_k = \frac{1}{24}\left[\left(9+5\sqrt{3}\right)\left(2+\sqrt{3}\right)^{4k-4} + \left(9-5\sqrt{3}\right)\left(2-\sqrt{3}\right)^{4k-4} + 6\right]$$

7. Generalized Pentagonal Numbers (Centered Hexagonal Numbers, Hex Numbers) $(GP)_n$

The generalized pentagonal numbers are defined by the sequence $1,7,19,37,61,\cdots$, *i.e.*, beginning with 7 each number is formed from the previous one in the sequence by adding the next number in the related sequence $6,12,18,\cdots,6(n-1)$. Thus, $7=1+6$, $19=1+6+12=7+12$, $37=1+6+12+18=19+18$, and so on (see **Figure 11**). These numbers are also called centered hexagonal numbers as these represent hexagons with a dot in the center and all other dots surrounding the center dot in a hexagonal lattice. These numbers have practical applications in materials logistics management, for example, in packing round items into larger round containers, such as Vienna sausages into round cans, or combining individual wire strands into a cable.

Thus, n-th generalized pentagonal number is defined as

$$\begin{aligned}(GP)_n &= (GP)_{n-1}+6(n-1)=1+6+12+\cdots+6(n-1)\\ &=1+6\left[1+2+\cdots+(n-1)\right].\end{aligned} \tag{66}$$

Hence, from (2) it follows that

$$\begin{aligned}(GP)_n &= 1+6\frac{(n-1)n}{2}=1+3n(n-1)\\ &=t_1+6t_{n-1}=t_n+4t_{n-1}+t_{n-2}.\end{aligned} \tag{67}$$

- Incidentally, $(GP)_2=7$ occurs in uds baryon octet, whereas $(GP)_5=61$ makes a part of a Chinese checkers board.
- Since $1+3n(n-1)=n^3-(n-1)^3$, generalized pentagonal numbers are differences of two consecutive cubes, so that the $(GP)_n$ are the gnomon of the cubes.
- Clearly, $(2n-1)^2-(GP)_n^2=n(n-1)=R_{n-1}=2t_{n-1}$.
- Let m be a given natural number, then it is n-th generalized pentagonal number, *i.e.*, $m=(GP)_n$ if and only if $n=\left(3+\sqrt{12m-3}\right)/6$.
- From (10) and (67), we have

$$\sum_{n=0}^{\infty}(GP)_n x^n=\frac{x}{1-x}+\frac{6x^2}{(1-x)^3}=\frac{x\left(1+4x+x^2\right)}{(1-x)^3}$$

and hence $x\left(1+4x+x^2\right)(1-x)^{-3}$ is the *generating function* of all generalized pentagonal numbers.

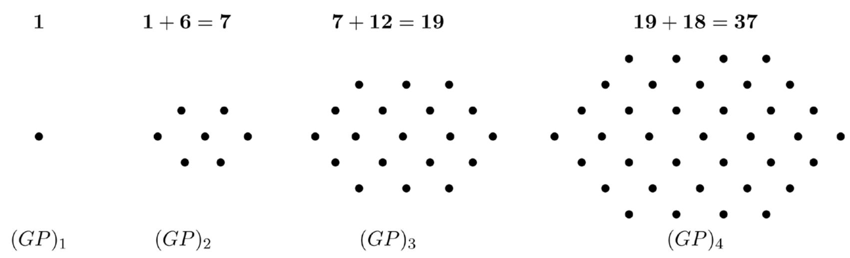

Figure 11. Generalized pentagonal numbers (centered hexagonal numbers).

- From (15) and (67) it is easy to find the sum of the first n generalized pentagonal numbers

$$\sum_{k=1}^{n}(GP)_k = n+6\sum_{k=1}^{n}t_{k-1} = n+6\sum_{k=1}^{n-1}t_k = n+(n-1)n(n+1) = n^3. \quad (68)$$

- Since from (32) and (46), we have

$$t_n^2 - t_{n-1}^2 = (t_n + t_{n-1})(t_n - t_{n-1}) = n^3$$

from (68) it follows that

$$\sum_{k=1}^{n}(GP)_k = t_n^2 - t_{n-1}^2 = n^3. \quad (69)$$

Thus the equation $c^2 = a^3 + b^2$ has an infinite number of integer solutions. In fact, for each $n \ge 1$ equations $c^2 = a^2 + b^n$ and $c^2 = a^n + b^2$ have infinite number of solutions (see Agarwal [14]).

- To find the sum of the reciprocals of all generalized pentagonal numbers we need the following well-known result, e.g., see Andrews *et al.* [27], page 536, and Efthimiou [28]

$$\frac{1}{s} + 2\sum_{k=1}^{\infty}\frac{1}{k^2+s} = \frac{\pi}{\sqrt{s}}\frac{1+e^{-2\pi\sqrt{s}}}{1-e^{-2\pi\sqrt{s}}}. \quad (70)$$

Now from (70), we have

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{1}{(GP)_k} &= \sum_{k=1}^{\infty}\frac{1}{3k^2-3k+1} = \frac{4}{3}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2+1/3} \\
&= \frac{4}{3}\left[\sum_{k=1}^{\infty}\frac{1}{k^2+1/3} - \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2+1/12}\right] \\
&= \frac{4}{3}\left[\frac{1}{2}\left(\pi\sqrt{3}\frac{1+e^{-2\pi/\sqrt{3}}}{1-e^{-2\pi/\sqrt{3}}} - 3\right) - \frac{1}{8}\left(2\pi\sqrt{3}\frac{1+e^{-\pi/\sqrt{3}}}{1-e^{-\pi/\sqrt{3}}} - 12\right)\right] \\
&= \frac{\pi}{\sqrt{3}\left(1-e^{-\pi/\sqrt{3}}\right)}\left[2\frac{1+e^{-2\pi/\sqrt{3}}}{1+e^{-\pi/\sqrt{3}}} - \left(1+e^{-\pi/\sqrt{3}}\right)\right] \\
&= \frac{\pi}{\sqrt{3}}\frac{1-e^{-\pi/\sqrt{3}}}{1+e^{-\pi/\sqrt{3}}}
\end{aligned}$$

and hence

$$\sum_{k=1}^{\infty}\frac{1}{(GP)_k} = \frac{\pi}{\sqrt{3}}\tanh\frac{\pi}{2\sqrt{3}} \approx 1.3052841530. \quad (71)$$

- To find all *square generalized pentagonal numbers*, we need to find integer solutions of the equation $1+3n(n-1) = m^2$. This equation can be written as Pell's equation $b^2 - 3a^2 = 1$ (its fundamental solution is $(a,b) = (1,2)$), where $b = 2m$ and $a = 2n-1$. For this, corresponding to (22) the system is

$$\begin{aligned}
m_{k+1} &= 14m_k - m_{k-1}, \quad m_1 = 1, m_2 = 13 \\
n_{k+1} &= 14n_k - n_{k-1} - 6, \quad n_1 = 1, n_2 = 8
\end{aligned} \quad (72)$$

This system genetrates all (infinite) solutions (m_k, n_k) of the equation

$1+3n(n-1)=m^2$. First few of these solutions are

$$(1,1),(13,8),(181,105),(2521,1456),(35113,20273).$$

For $k \ge 1$, explicit solution of the system (72) appears as

$$m_k = \frac{1}{4}\left[\left(2+\sqrt{3}\right)^{2k-1} + \left(2-\sqrt{3}\right)^{2k-1}\right]$$
$$n_k = \frac{\sqrt{3}}{12}\left[\left(2+\sqrt{3}\right)^{2k-1} - \left(2-\sqrt{3}\right)^{2k-1}\right] + \frac{1}{2}$$

- To find all *generalized pentagonal numbers which are also triangular numbers*, we need to find integer solutions of the equation $1+3n(n-1)=m(m+1)/2$. This equation can be written as Pell's equation $b^2-6a^2=3$ (its fundamental solution is $(a,b)=(1,3)$), where $b=2m+1$ and $a=2n-1$. For this, corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 10m_k - m_{k-1} + 4, \quad m_1 = 1, m_2 = 13 \\ n_{k+1} &= 10n_k - n_{k-1} - 4, \quad n_1 = 1, n_2 = 6 \end{aligned} \tag{73}$$

This system genetrates all (infinite) solutions (m_k, n_k) of the equation $1+3n(n-1)=m(m+1)/2$. First few of these solutions are

$$(1,1),(13,6),(133,55),(1321,540),(13081,5341).$$

For $k \ge 1$, explicit solution of the system (73) can be written as

$$m_k = \frac{1}{4}\left[\left(3+\sqrt{6}\right)\left(5+2\sqrt{6}\right)^{k-1} + \left(3-\sqrt{6}\right)\left(5-2\sqrt{6}\right)^{k-1} - 2\right]$$
$$n_k = \frac{1}{8}\left[\left(2+\sqrt{6}\right)\left(5+2\sqrt{6}\right)^{k-1} + \left(2-\sqrt{6}\right)\left(5-2\sqrt{6}\right)^{k-1} + 4\right]$$

- There is no generalized pentagonal number which is also a rectangular number, in fact, the equation $1+3n(n-1)=m(m+1)$ has no solutions. For this, we note that this equation can be written as Pell's equation $b^2-3a^2=2$, where $b=2m+1$ and $a=2n-1$. Now reducing this equation to $(\text{mod}\,3)$ gives $b^2=2(\text{mod}\,3)$, which is impossible since all squares $(\text{mod}\,3)$ are either 0 or 1 $(\text{mod}\,3)$.
- To find all *generalized pentagonal numbers which are also pentagonal numbers*, we need to find integer solutions of the equation $1+3n(n-1)=m(3m-1)/2$. This equation can also be written as Pell's equation $b^2-18a^2=7$ (its fundamental solution is $(a,b)=(1,5)$), where $b=6m-1$ and $a=2n-1$. For this, corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 1154m_k - m_{k-1} - 192, \quad m_1 = 1, m_2 = 889 \\ n_{k+1} &= 1154n_k - n_{k-1} - 576, \quad n_1 = 1, n_2 = 629 \end{aligned} \tag{74}$$

This system genetrates all (infinite) solutions (m_k, n_k) of the equation $1+3n(n-1)=m(3m-1)/2$. First few of these solutions are

$$(1,1),(889,629),(1025713,725289),(1183671721,836982301),$$
$$(1365956140129,965876849489).$$

For $k \ge 1$, explicit solution of the system (74) can be written as

$$m_k = \frac{1}{10404}\Big[\big(378879-267903\sqrt{2}\big)\big(577+408\sqrt{2}\big)^k + \big(378879+267903\sqrt{2}\big)\big(577-408\sqrt{2}\big)^k + 1734\Big]$$

$$n_k = \frac{1}{6936}\Big[\big(126293\sqrt{2}-178602\big)\big(577+408\sqrt{2}\big)^k - \big(126293\sqrt{2}+178602\big)\big(577-408\sqrt{2}\big)^k + 3468\Big].$$

- To find all *generalized pentagonal numbers which are also hexagonal numbers*, we need to find integer solutions of the equation $1+3n(n-1)=m(2m-1)$. This equation can also be written as Pell's equation $b^2-6a^2=3$ (its fundamental solution is $(a,b)=(1,3)$), where $b=4m-1$ and $a=2n-1$. For this, corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 10m_k - m_{k-1} - 2, \quad m_1 = 1, m_2 = 7 \\ n_{k+1} &= 10n_k - n_{k-1} - 4, \quad n_1 = 1, n_2 = 6 \end{aligned} \tag{75}$$

This system generates all (infinite) solutions (m_k, n_k) of the equation $1+3n(n-1)=m(2m-1)$. First few of these solutions are

$$(1,1),(7,6),(67,55),(661,540),(6541,5341).$$

For $k \ge 1$, explicit solution of the system (75) can be written as

$$m_k = \frac{1}{8}\Big[\big(3+\sqrt{6}\big)\big(5+2\sqrt{6}\big)^{k-1} + \big(3-\sqrt{6}\big)\big(5-2\sqrt{6}\big)^{k-1} + 2\Big]$$

$$n_k = \frac{1}{8}\Big[\big(2+\sqrt{6}\big)\big(5+2\sqrt{6}\big)^{k-1} + \big(2-\sqrt{6}\big)\big(5-2\sqrt{6}\big)^{k-1} + 4\Big]$$

8. Heptagonal Numbers (Heptagon Numbers) $(HEP)_n$

These numbers are defined by the sequence $1,7,18,34,55,81,\cdots$, *i.e.*, beginning with 7 each number is formed from the previous one in the sequence by adding the next number in the related sequence $6,11,16,21,\cdots,(5n-4)$. Thus, $7=1+6$, $18=1+6+11=7+11$, $34=1+6+11+16=18+16$, and so on (see Figure 12).

Thus, n-th heptagonal number is defined as

$$\begin{aligned} (HEP)_n &= (HEP)_{n-1} + (5n-4) = 1+6+11+16+\cdots+(5n-4) \\ &= 1+(1+5)+(1+2\times 5)+\cdots+(1+(n-1)5). \end{aligned} \tag{76}$$

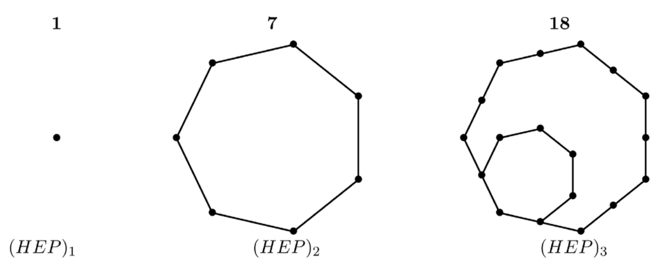

Figure 12. Heptagonal numbers.

Comparing (76) with (3), we have $a=1, d=5$, and hence from (4) it follows that

$$(HEP)_n = \frac{n}{2}(5n-3) = \frac{1}{2}n\left[(n+1)+4(n-1)\right] = t_n + 4t_{n-1}. \quad (77)$$

- For all integers $k \ge 0$ it follows that $(HEP)_{4k+1}$ and $(HEP)_{4k+2}$ are odd, whereas $(HEP)_{4k+3}$ and $(HEP)_{4k+4}$ are even.
- From (77) the following equality holds

$$5(HEP)_n + 1 = 5t_n + 20t_{n-1} + 1 = \frac{(5n-2)(5n-1)}{2} = t_{5n-2}.$$

- Let m be a given natural number, then it is n-th heptagonal number, *i.e.*, $m=(HEP)_n$ if and only if $n = \left(3+\sqrt{9+40m}\right)/10$.
- From (10) and (77), we have

$$\frac{x(4x+1)}{(1-x)^3} = x + 7x^2 + 18x^3 + 34x^4 + \cdots$$

and hence $x(4x+1)(1-x)^{-3}$ is the *generating function* of all heptagonal numbers.

- In view of (15) and (77), we have

$$\sum_{k=1}^{n}(HEP)_k = \frac{1}{6}n(n+1)(5n-2). \quad (78)$$

- The sum of reciprocals of all heptagonal numbers is (see https://en.wikipedia.org/wiki/Heptagonal_number)

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{(HEP)_k} &= \frac{1}{15}\pi\sqrt{25-10\sqrt{5}} + \frac{2}{3}\ln(5) + \frac{1+\sqrt{5}}{3}\ln\left(\frac{1}{2}\sqrt{10-2\sqrt{5}}\right) \\ &\quad + \frac{1-\sqrt{5}}{3}\ln\left(\frac{1}{2}\sqrt{10+2\sqrt{5}}\right) \\ &\approx 1.3227792531.\end{aligned} \quad (79)$$

- To find all *square heptagonal numbers*, we need to find integer solutions of the equation $n(5n-3)/2 = m^2$. This equation can be written as Pell's equation $b^2 - 40a^2 = 9$ (its fundamental solutions are $(a,b) = (1,7), (2,13)$ and $(9,57)$), where $b = 10n-3$ and $a = m$. For $(1,7)$, corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 1442m_k - m_{k-1}, m_1 = 1, m_2 = 1519 \\ n_{k+1} &= 1442n_k - n_{k-1} - 432, n_1 = 1, n_2 = 961 \end{aligned} \quad (80)$$

This system genetrates infinite number of solutions (m_k, n_k) of the equation $n(5n-3)/2 = m^2$. First four of these solutions are

$$(1,1), (1519,961), (2190397,1385329), (3158550955,1997643025).$$

For $(2,13)$ recurrence relations remain the same as in (80) with $m_1 = 77, m_2 = 111035$ and $n_1 = 49, n_2 = 70225$. This leads to another set of infinite number of solutions (m_k, n_k) of the equation $n(5n-3)/2 = m^2$. First four of these solutions are

$$(77,49),(111035,70225),(160112393,101263969),$$
$$(230881959671,146022572641).$$

For $(9,57)$ also recurrence relations remain the same as in (80) with $m_1=9, m_2=12987$ and $n_1=6, n_2=8214$. This leads to further set of infinite number of solutions (m_k,n_k) of the equation $n(5n-3)/2=m^2$. First four of these solutions are

$$(9,6),(12987,8214),(18727245,11844150),(27004674303,17079255654).$$

- To find all *heptagonal numbers which are also triangular numbers*, we need to find integer solutions of the equation $n(5n-3)/2=m(m+1)/2$. This equation can be written as Pell's equation $b^2-5a^2=4$ (its fundamental solutions are $(a,b)=(3,7)$ and $(1,3)$), where $b=10n-3$ and $a=2m+1$. For $(3,7)$ corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 322m_k - m_{k-1} + 160, \quad m_1=1, m_2=493 \\ n_{k+1} &= 322n_k - n_{k-1} - 96, \quad n_1=1, n_2=221 \end{aligned} \tag{81}$$

This system genetrates infinite number of solutions (m_k,n_k) of the equation $n(5n-3)/2=m(m+1)/2$. First four of these solutions are

$$(1,1),(493,221),(158905,71065),(51167077,22882613).$$

For $(1,3)$ recurrence relations remain the same as in (81) with $m_1=10, m_2=3382$ and $n_1=5, n_2=1513$. This leads to another set of infinite number of solutions (m_k,n_k) of the equation $n(5n-3)/2=m(m+1)/2$. First four of these solutions are

$$(10,5),(3382,1513),(1089154,487085),(350704366,156839761).$$

- To find all *heptagonal numbers which are also rectangular numbers*, we need to find integer solutions of the equation $n(5n-3)/2=m(m+1)$. This equation can be written as Pell's equation $b^2-10a^2=-1$ (its fundamental solution is $(a,b)=(1,3)$), where $b=10n-3$ and $a=2m+1$. For this, corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 1442m_k - m_{k-1} + 720, \quad m_1=18, m_2=26676 \\ n_{k+1} &= 1442n_k - n_{k-1} - 432, \quad n_1=12, n_2=16872 \end{aligned} \tag{82}$$

This system genetrates infinite number of solutions (m_k,n_k) of the equation $n(5n-3)/2=m(m+1)$. First four of these solutions are

$$(18,12),(26676,16872),(38467494,24328980),(55470100392,35082371856).$$

To find all *heptagonal numbers which are also pentagonal numbers*, we need to find integer solutions of the equation $n(5n-3)/2=m(3m-1)/2$. This equation can be written as Pell's equation $b^2-15a^2=66$ (its fundamental solution is $(a,b)=(-1,9)$), where $b=3(10n-3)$ and $a=6m-1$. For this, corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 62m_k - m_{k-1} - 10, \quad m_1=1, m_2=54 \\ n_{k+1} &= 62n_k - n_{k-1} - 18, \quad n_1=1, n_2=42 \end{aligned} \tag{83}$$

This system genetrates infinite number of solutions (m_k, n_k) of the equation $n(5n-3)/2 = m(3m-1)/2$. First few of these solutions are

$$(1,1),(54,42),(3337,2585),(206830,160210),(12820113,9930417).$$

- To find all *heptagonal numbers which are also hexagonal numbers*, we need to find integer solutions of the equation $n(5n-3)/2 = m(2m-1)$. This equation can be written as Pell's equation $b^2 - 5a^2 = 4$ (its fundamental solution is $(a,b) = (-1,3)$), where $b = 10n-3$ and $a = 4m-1$. For this, corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 322m_k - m_{k-1} - 80, \quad m_1 = 1, m_2 = 247 \\ n_{k+1} &= 322n_k - n_{k-1} - 96, \quad n_1 = 1, n_2 = 221 \end{aligned} \tag{84}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(5n-3)/2 = m(2m-1)$. First few of these solutions are

$$(1,1),(247,221),(79453,71065),(25583539,22882613),$$
$$(8237820025,7368130225).$$

- To find all *heptagonal numbers which are also generalized pentagonal numbers*, we need to find integer solutions of the equation $n(5n-3)/2 = 1+3m(m-1)$. This equation can be written as Pell's equation $b^2 - 30a^2 = 19$ (its fundamental solution is $(a,b) = (1,7)$), where $b = 10n-3$ and $a = 2m-1$. For this, corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 22m_k - m_{k-1} - 10, \quad m_1 = 1, m_2 = 13 \\ n_{k+1} &= 22n_k - n_{k-1} - 6, \quad n_1 = 1, n_2 = 14 \end{aligned} \tag{85}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(5n-3)/2 = 1+3m(m-1)$. First few of these solutions are

$$(1,1),(13,14),(275,301),(6027,6602),(132309,144937).$$

9. Octagonal Numbers O_n

These numbers are defined by the sequence $1, 8, 21, 40, 65, 96, 133, 176, \cdots$, *i.e.*, beginning with 8 each number is formed from the previous one in the sequence by adding the next number in the related sequence $7, 13, 19, 25, \cdots, (6n-5)$. Thus, $8 = 1+7$, $21 = 1+7+13 = 8+13$, $40 = 1+7+13+19 = 21+19$, and so on (see **Figure 13**).

Thus, n-th octagonal number is defined as

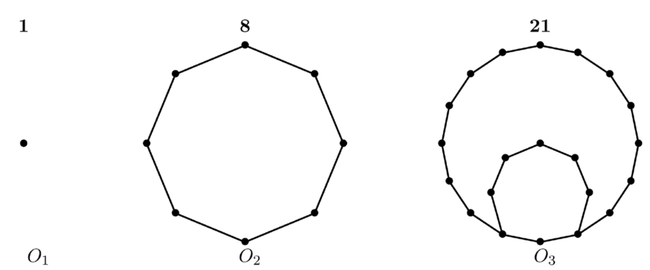

Figure 13. Octagonal numbers.

$$O_n = O_{n-1} + (6n-5) = 1+7+13+19+\cdots+(6n-5)$$
$$= 1+(1+6)+(1+2\times 6)+\cdots+(1+(n-1)6). \tag{86}$$

Comparing (86) with (3), we have $a = 1, d = 6$, and hence from (4) it follows that

$$O_n = \frac{n}{2}(6n-4) = n(3n-2) = t_n + 5t_{n-1}. \tag{87}$$

- For all integers $k \ge 0$ it follows that O_{2k+1} are odd, whereas O_{2k+2} are even (in fact divisible by 4).
- Let m be a given natural number, then it is n-th octagonal number, *i.e.*, $m = O_n$ if and only if $n = \left(1+\sqrt{1+3m}\right)/3$.
- From (10) and (87), we have

$$\frac{x(5x+1)}{(1-x)^3} = x + 8x^2 + 21x^3 + 40x^4 + \cdots$$

and hence $x(5x+1)(1-x)^{-3}$ is the *generating function* of all octagonal numbers.
- In view of (15) and (87), we have

$$\sum_{k=1}^{n} O_k = \frac{1}{2} n(n+1)(2n-1). \tag{88}$$

- To find the sum of the reciprocals of all octagonal numbers, following Downey [29] we begin with the series

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k(3k-2)} x^{3k-2}$$

and note that

$$f(1) = \sum_{k=1}^{\infty} \frac{1}{k(3k-2)} = \sum_{k=1}^{\infty} \frac{1}{O_k}, \quad f'(x) = \sum_{k=1}^{\infty} \frac{1}{k} x^{3k-3} = -\frac{\ln(1-x^3)}{x^3}.$$

Thus, we have

$$f(x) = \int_{0^+}^{x} -\frac{\ln(1-t^3)}{t^3} \mathrm{d}t = \frac{\ln(1-x^3)}{2x^2} - \frac{3}{2}\int_{0^+}^{x} \frac{1}{t^3-1} \mathrm{d}t$$
$$= \frac{\ln(1-x^3)}{2x^2} - \frac{1}{4}\int_{0^+}^{x} \left[\frac{2}{t-1} - \frac{2t+1}{t^2+t+1} - 3\frac{1}{(t+1/2)^2+3/4}\right] \mathrm{d}t \quad \left(\lim_{t\to 0^+} \frac{\ln(1-t^3)}{2t^2} = 0\right)$$
$$= \frac{\ln(1+x+x^2)}{2x^2} + \frac{\ln(1-x)}{2x^2} - \frac{1}{2}\ln|x-1| + \frac{1}{4}\ln(x^2+x+1) + \frac{\sqrt{3}}{2}\tan^{-1}\frac{2x+1}{\sqrt{3}} - \frac{\sqrt{3}\pi}{12}.$$

Now since

$$\lim_{x\to 1^-} \frac{1}{2}\ln(1-x)\left(\frac{1}{x^2} - 1\right) = 0$$

it follows that

$$\sum_{k=1}^{\infty}\frac{1}{O_k}=\frac{3}{4}\ln 3+\frac{\sqrt{3}}{12}\pi\approx 1.2774090576. \tag{89}$$

- To find all *square octagonal numbers*, we need to find integer solutions of the equation $n(3n-2)=m^2$. This equation can be written as Pell's equation $b^2-3a^2=1$ (its fundamental solution is $(a,b)=(1,2)$), where $b=3n-1$ and $a=m$. For this, corresponding to (22) the system is

$$\begin{aligned} m_{k+1}&=14m_k-m_{k-1},\quad m_1=1, m_2=15\\ n_{k+1}&=14n_k-n_{k-1}-4,\quad n_1=1, n_2=9 \end{aligned} \tag{90}$$

This system generates infinite number of solutions (m_k,n_k) of the equation $n(3n-2)=m^2$. First few of these solutions are

$$(1,1),(15,9),(209,121),(2911,1681),(40545,23409).$$

- To find all *octagonal numbers which are also triangular numbers*, we need to find integer solutions of the equation $n(3n-2)=m(m+1)/2$. This equation can be written as Pell's equation $b^2-6a^2=10$ (its fundamental solutions are $(a,b)=(1,4)$ and $(3,8)$), where $b=4(3n-1)$ and $a=2m+1$. For $(1,4)$ corresponding to (22) the system is

$$\begin{aligned} m_{k+1}&=98m_k-m_{k-1}+48,\quad m_1=6, m_2=638\\ n_{k+1}&=98n_k-n_{k-1}-82,\quad n_1=3, n_2=261 \end{aligned} \tag{91}$$

This system generates infinite number of solutions (m_k,n_k) of the equation $n(3n-2)=m(m+1)/2$. First few of these solutions are

$$(6,3),(638,261),(62566,25543),(6130878,2502921),$$
$$(600763526,245260683).$$

For $(3,8)$ recurrence relations remain the same as in (91) with $m_1=1, m_2=153$ and $n_1=1, n_2=63$. This leads to another set of infinite number of solutions (m_k,n_k) of the equation $n(3n-2)/2=m(m+1)/2$. First few of these solutions are

$$(1,1),(153,63),(15041,6141),(1473913,601723),(144428481,58962681).$$

- There is no octagonal number which is also a rectangular number, in fact, the equation $n(3n-2)=m(m+1)$ has no solutions. For this, we note that this equation can be written as Pell's equation $b^2-3a^2=1$ (its fundamental solution is $(a,b)=(1,2)$), where $b=2(3n-1)$ and $a=2m+1$. For this, Pell's equation all solutions can be generated by the system (corresponding to (21))

$$\begin{aligned} a_{k+2}&=4a_{k+1}-a_k,\quad a_1=1, a_2=4\\ b_{k+2}&=4b_{k+1}-b_k,\quad b_1=2, b_2=7 \end{aligned} \tag{92}$$

Now an explicit solution of the second equation of (92) can be written as

$$b_k=\frac{1}{2}\left[\left(2+\sqrt{3}\right)^k+\left(2-\sqrt{3}\right)^k\right].$$

Next, if $\left(2+\sqrt{3}\right)^k = s_k + t_k\sqrt{3}$, then $\left(2-\sqrt{3}\right)^k = s_k - t_k\sqrt{3}$, and hence it follows that $b_k = s_k$. We note that $s_1 \equiv 2 \pmod 6$ and $s_2 \equiv 1 \pmod 6$. Thus, from the second equation of (92) mathematical induction immediately gives $s_{2\ell-1} \equiv 2 \pmod 6$ and $s_{2\ell} \equiv 1 \pmod 6$ for all $\ell \ge 1$. In conclusion $b_k = s_k \equiv 1$ or $2 \pmod 6$. Finally, reducing the relation $b = 2(3n-1)$ to $\pmod 6$ gives $b \equiv -2 \pmod 6$. Hence, in view of $b > 0$, we conclude that $b \ne s_k$ for all integers k, and therefore, the equation $n(3n-2) = m(m+1)$ has no solution.

- To find all *octagonal numbers which are also pentagonal numbers*, we need to find integer solutions of the equation $n(3n-2) = m(3m-1)/2$. This equation can be written as Pell's equation $b^2 - 8a^2 = -7$ (its fundamental solutions are $(a,b) = (1,1)$ and $(2,5)$), where $b = 6m-1$ and $a = 3n-1$. For $(1,1)$ corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 1154m_k - m_{k-1} - 192, \quad m_1 = 1, m_2 = 1025 \\ n_{k+1} &= 1154n_k - n_{k-1} - 384, \quad n_1 = 1, n_2 = 725 \end{aligned} \tag{93}$$

This system genetrates infinite number of solutions (m_k, n_k) of the equation $n(3n-2) = m(3m-1)/2$. First four of these solutions are

$$(1,1), (1025,725), (1182657,836265), (1364784961,965048701).$$

For $(2,5)$ recurrence relations remain the same as in (93) with $m_1 = 11, m_2 = 12507$ and $n_1 = 8, n_2 = 8844$. This leads to another set of infinite number of solutions (m_k, n_k) of the equation $n(3n-2) = m(3m-1)/2$. First four of these solutions are

$$(11,8), (12507,8844), (14432875,10205584), (16655525051,11777234708).$$

- To find all *octagonal numbers which are also hexagonal numbers*, we need to find integer solutions of the equation $n(3n-2) = m(2m-1)$. This equation can be written as Pell's equation $b^2 - 6a^2 = 10$ (its fundamental solutions are $(a,b) = (1,4)$ and $(3,8)$), where $b = 4(3n-1)$ and $a = 4m-1$. For $(3,8)$ corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 98m_k - m_{k-1} - 24, \quad m_1 = 1, m_2 = 77 \\ n_{k+1} &= 98n_k - n_{k-1} - 32, \quad n_1 = 1, n_2 = 63 \end{aligned} \tag{94}$$

This system genetrates infinite number of solutions (m_k, n_k) of the equation $n(3n-2) = m(2m-1)$. First few of these solutions are

$$(1,1), (77,63), (7521,6141), (736957,601723), (72214241,58962681).$$

With $(a,b) = (1,4)$ the system corresponding to (21) is

$$\begin{aligned} a_{k+2} &= 10a_{k+1} - a_k, \quad a_1 = 1, a_2 = 13 \\ b_{k+2} &= 10b_{k+1} - b_k, \quad b_1 = 4, b_2 = 32 \end{aligned} \tag{95}$$

Now note that $a_1 \equiv 1 \pmod 4$ and $a_2 \equiv 1 \pmod 4$. Thus, from the first equation of (95) mathemtical induction immediately gives $a_{k+2} \equiv 10 \pmod 4 - 1 \pmod 4 \equiv 1 \pmod 4$ for all $k \ge 1$. Next reducing the relation $a = 4m-1$ to $\pmod 4$ gives $a \equiv -1 \pmod 4$. Hence, in view of $b > 0$, we conclude that $a \ne a_k$ for all integers k, and therefore, the equation

$n(3n-2)=m(2m-1)$ has no solution.

- To find all *octagonal numbers which are also generalized pentagonal numbers*, we need to find integer solutions of the equation $n(3n-2)=1+3m(m-1)$. This equation can be written as Pell's equation $b^2-a^2=7$, where $b=2(3n-1)$ and $a=3(2m-1)$. For the equation $b^2-a^2=7$ the only meaningful integer solution is $b=4, a=3$ and it gives $(m,n)=(1,1)$.
- To find all *octagonal numbers which are also heptagonal numbers*, we need to find integer solutions of the equation $n(3n-2)=m(5m-3)/2$. This equation can be written as Pell's equation $b^2-30a^2=-39$ (its fundamental solution is $(a,b)=(2,-9)$), where $b=3(10m-3)$ and $a=2(3n-1)$. For this, corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 482m_k - m_{k-1} - 144, \quad m_1 = 1, m_2 = 345 \\ n_{k+1} &= 482n_k - n_{k-1} - 160, \quad n_1 = 1, n_2 = 315 \end{aligned} \tag{96}$$

This system genetrates infinite number of solutions (m_k, n_k) of the equation $n(3n-2)=m(5m-3)/2$. First few of these solutions are

$$(1,1),(345,315),(166145,151669),(80081401,73103983),$$
$$(38599068993,35235967977).$$

10. Nonagonal Numbers N_n

These numbers are defined by the sequence $1,9,24,46,75,111,154,\cdots$, *i.e.*, beginning with 9 each number is formed from the previous one in the sequence by adding the next number in the related sequence $8,15,22,29,\cdots,(7n-6)$. Thus, $9=1+8$, $24=1+8+15=9+15$, $46=1+8+15+22=24+22$, and so on (see Figure 14).

Thus, n-th nonagonal number is defined as

$$\begin{aligned} N_n &= N_{n-1}+(7n-6) = 1+8+15+22+\cdots+(7n-6) \\ &= 1+(1+7)+(1+2\times 7)+\cdots+(1+(n-1)7). \end{aligned} \tag{97}$$

Comparing (97) with (3), we have $a=1, d=7$, and hence from (4) it follows that

$$N_n = \frac{n}{2}(7n-5) = t_n + 6t_{n-1}. \tag{98}$$

- For all integers $k \ge 0$ it follows that N_{4k+1}, N_{4k+2} are odd, whereas N_{4k+3}, N_{4k+4} are even.

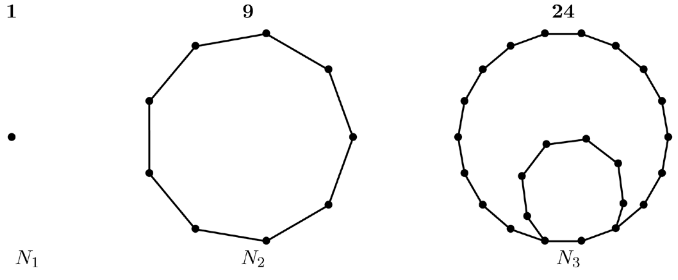

Figure 14. Nonagonal numbers.

- Let m be a given natural number, then it is n-th nonagonal number, *i.e.*, $m = N_n$ if and only if $n = \left(5+\sqrt{25+56m}\right)/14$.
- From (10) and (98), we have

$$\frac{x(6x+1)}{(1-x)^3} = x + 9x^2 + 24x^3 + 46x^4 + \cdots$$

and hence $x(6x+1)(1-x)^{-3}$ is the *generating function* of all nonagonal numbers.

- In view of (15) and (98), we have

$$\sum_{k=1}^{n} N_k = \frac{1}{6} n(n+1)(7n-4). \tag{99}$$

- The sum of reciprocals of all nonagonal numbers is

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{N_k} = \sum_{k=1}^{\infty} \frac{2}{k(7k-5)} &= \sum_{k=1}^{\infty} \left(\frac{14}{5(7k-5)} - \frac{2}{5n}\right) \\ &= -\frac{2}{25}\left(5\Psi\left(-\frac{5}{7}\right) - 7 + 5\gamma\right) \\ &\approx 1.2433209262;\end{aligned} \tag{100}$$

here, $\Psi(x)$ is the *digamma function* defined as the logarithmic derivative of the *gamma function* $\Gamma(x)$, *i.e.*, $\Psi(x) = \Gamma'(x)/\Gamma(x)$, and $\gamma = 0.5772156649$ is the *Euler-Mascheroni constant*.

- To find all *square nonagonal numbers*, we need to find integer solutions of the equation $n(7n-5)/2 = m^2$. This equation can be written as Pell's equation $b^2 - 14a^2 = 25$ (its fundamental solution are $(a,b) = (2,9)$ and $(6,23)$), where $b = 14n-5$ and $a = 2m$. For $(2,9)$ corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 30m_k - m_{k-1}, \quad m_1 = 1, m_2 = 33 \\ n_{k+1} &= 30n_k - n_{k-1} - 10, \quad n_1 = 1, n_2 = 18 \end{aligned} \tag{101}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(7n-5)/2 = m^2$. First few of these solutions are

$$(1,1),(33,18),(989,529),(29637,15842),(888121,474721).$$

For $(6,23)$ recurrence relations remain the same as in (101) with $m_1 = 3, m_2 = 91$ and $n_1 = 2, n_2 = 49$. This leads to another set of infinite number of solutions (m_k, n_k) of the equation $n(7n-5)/2 = m^2$. First few of these solutions are

$$(3,2),(91,49),(2727,1458),(81719,43681),(2448843,1308962).$$

- To find all *nonagonal numbers which are also triangular numbers*, we need to find integer solutions of the equation $n(7n-5)/2 = m(m+1)/2$. This equation can be written as Pell's equation $b^2 - 7a^2 = 18$ (its fundamental solutions are $(a,b) = (1,5),(3,9)$ and $(7,19)$), where $b = 14n-5$ and $a = 2m+1$. For $(3,9)$ corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 16m_k - m_{k-1} + 7, \quad m_1 = 1, m_2 = 25 \\ n_{k+1} &= 16n_k - n_{k-1} - 5, \quad n_1 = 1, n_2 = 10 \end{aligned} \tag{102}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(7n-5)/2 = m(m+1)/2$. First few of these solutions are

$$(1,1),(25,10),(406,154),(6478,2449),(1032249,39025).$$

For $(1,5)$ and $(7,19)$ there are no integer solutions.

- There is no nonagonal number which is also a rectangular number, in fact, the equation $n(7n-5)/2 = m(m+1)$ has no solutions.
- To find all *nonagonal numbers which are also pentagonal numbers*, we need to find integer solutions of the equation $n(7n-5)/2 = m(3m-1)/2$. This equation can be written as Pell's equation $b^2 - 21a^2 = 204$ (its fundamental solutions are $(a,b) = (5,27)$ and $(125,573)$), where $b = 3(14n-5)$ and $a = 6m-1$. For $(5,27)$ corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 12098m_k - m_{k-1} - 2016, \quad m_1 = 1, m_2 = 10981 \\ n_{k+1} &+ 12098n_k - n_{k-1} - 4320, \quad n_1 = 1, n_2 = 7189 \end{aligned} \tag{103}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(7n-5)/2 = m(3m-1)/2$. First four of these solutions are

$$(1,1),(10981,7189),(132846121,86968201),$$
$$(1607172358861,1052141284189).$$

For $(125,573)$ recurrence relations remain the same as in (103) with $m_1 = 21, m_2 = 252081$ and $n_1 = 14, n_2 = 165026$. This leads to another set of infinite number of solutions (m_k, n_k) of the equation $n(3n-2) = m(3m-1)/2$. First four of these solutions are

$$(21,14),(252081,165026),(3049673901,1996480214),$$
$$(36894954600201,24153417459626).$$

- To find all *nonagonal numbers which are also hexagonal numbers*, we need to find integer solutions of the equation $n(7n-5)/2 = m(2m-1)$. This equation can be written as Pell's equation $b^2 - 7a^2 = 18$ (its fundamental solutions are $(a,b) = (1,5),(3,9)$ and $(7,19)$), where $b = 14n-5$ and $a = 4m-1$. For $(3,9)$ corresponding to (20) the system is

$$\begin{aligned} b_{k+1} &= 8b_k + 21a_k \\ a_{k+1} &= 8a_k + 3b_k, \quad b_1 = 9, a_1 = 3 \end{aligned}$$

This system gives first four integer solutions (m_k, n_k) of the equation $n(7n-5)/2 = m(2m-1)$ rather easily, which appear as $(1,1),(13,10),(51625,39025),(822757,621946)$. Now the following system generates infinite number of solutions $(m_{2k+1}, n_{2k+1}), k \ge 2$

$$\begin{aligned} m_{2k+1} &= 64514m_{2k-1} - m_{2k-3} - 16128, \quad m_1 = 1, m_3 = 51625 \\ n_{2k+1} &= 64514n_{2k-1} - n_{2k-3} - 23040, \quad n_1 = 1, n_3 = 39025 \end{aligned} \tag{104}$$

Similarly, the following system generates infinite number of solutions $(m_{2k}, n_{2k}), k \ge 2$

$$\begin{aligned} m_{2k+2} &= 64514 m_{2k} - m_{2k-2} - 16128, m_2 = 13, m_4 = 822757 \\ n_{2k+2} &= 64514 n_{2k} - n_{2k-2} - 23040, n_2 = 10, n_4 = 621946 \end{aligned} \quad (105)$$

The first eight solutions (m_k, n_k) are

$(1,1),(13,10),(51625,39025),(822757,621946),$
$(3330519121,2517635809),(53079328957,40124201194),$
$(214865110504441,162422756519761),$
$(3424359827493013,2588572715184730).$

With $(a,b)=(1,5)$ and $(7,19)$ there are no integer solutions of the required equation.

- To find all *nonagonal numbers which are also generalized pentagonal numbers*, we need to find integer solutions of the equation $n(7n-5)/2 = 1+3m(m-1)$. This equation can be written as Pell's equation $b^2 - 42a^2 = 39$ (its fundamental solutions are $(a,b)=(1,9)$ and $(5,33)$), where $b = 14n-5$ and $a = 2m-1$. For $(1,9)$, corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 674 m_k - m_{k-1} - 336, m_1 = 1, m_2 = 403 \\ n_{k+1} &= 674 n_k - n_{k-1} - 240, n_1 = 1, n_2 = 373 \end{aligned} \quad (106)$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(5n-3)/2 = 1+3m(m-1)$. First few of these solutions are

$(1,1),(403,373),(271285,251161),(182845351,169281901),$
$(123237494953,114095749873).$

For $(5,33)$ recurrence relations remain the same as in (106) with $m_1 = 66, m_2 = 44148$ and $n_1 = 61, n_2 = 40873$. This leads to another set of infinite number of solutions (m_k, n_k) of the equation $n(7n-5)/2 = 1+3m(m-1)$. First four of these solutions are

$(66,61),(44148,40873),(29755350,27548101),(20055061416,18567378961).$

- To find all *nonagonal numbers which are also heptagonal numbers*, we need to find integer solutions of the equation $n(7n-5)/2 = m(5m-3)/2$. This equation can be written as Pell's equation $b^2 - 35a^2 = 310$ (its fundamental solution is $(a,b)=(7,45)$), where $b = 5(14n-5)$ and $a = 10m-3$. For this, corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 142 m_k - m_{k-1} - 42, m_1 = 1, m_2 = 104 \\ n_{k+1} &= 142 n_k - n_{k-1} - 50, n_1 = 1, n_2 = 88 \end{aligned} \quad (107)$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(7n-5)/2 = m(5m-3)/2$. First few of these solutions are

$(1,1),(104,88),(14725,12445),(2090804,1767052),(296879401,250908889).$

- To find all *nonagonal numbers which are also octagonal numbers*, we need to find integer solutions of the equation $n(7n-5)/2 = m(3m-2)$. This equation can be written as Pell's equation $b^2 - 42a^2 = 57$ (its fundamental solution is $(a,b) = (4,27)$), where $b = 3(14n-5)$ and $a = 2(3m-1)$. For this, corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 674m_k - m_{k-1} - 224, m_1 = 1, m_2 = 459 \\ n_{k+1} &= 674n_k - n_{k-1} - 240, n_1 = 1, n_2 = 425 \end{aligned} \tag{108}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(7n-5)/2 = m(3m-2)$. First few of these solutions are

$$(1,1),(459,425),(309141,286209),(208360351,192904201),$$
$$(140434567209,130017145025).$$

11. Decagonal Numbers *D_n*

These numbers are defined by the sequence $1,10,27,52,85,126,175,\cdots$, *i.e.*, beginning with 10 each number is formed from the previous one in the sequence by adding the next number in the related sequence $9,17,25,33,\cdots,(8n-7)$. Thus, $10 = 1+9$, $27 = 1+9+17 = 10+17$, $52 = 1+9+17+25 = 27+25$ and so on (see **Figure 15**).

Hence, *n*-th decagonal number is defined as

$$\begin{aligned} D_n &= D_{n-1} + (8n-7) = 1+9+17+25+\cdots+(8n-7) \\ &= 1+(1+8)+(1+2\times 8)+\cdots+(1+(n-1)8). \end{aligned} \tag{109}$$

Comparing (109) with (3), we have $a = 1, d = 8$, and hence from (4) it follows that

$$D_n = \frac{n}{2}(8n-6) = n(4n-3) = t_n + 7t_{n-1}. \tag{110}$$

- For all integers $k \ge 0$ it follows that D_{2k+1} are odd, whereas D_{2k} are even.
- Let *m* be a given natural number, then it is *n*-th decagonal number, *i.e.*, $m = D_n$ if and only if $n = \left(3+\sqrt{9+16m}\right)/8$.
- From (10) and (110), we have

$$\frac{x(7x+1)}{(1-x)^3} = x + 10x^2 + 27x^3 + 52x^4 + \cdots$$

and hence $x(7x+1)(1-x)^{-3}$ is the *generating function* of all decagonal numbers.

- In view of (15) and (110), we have

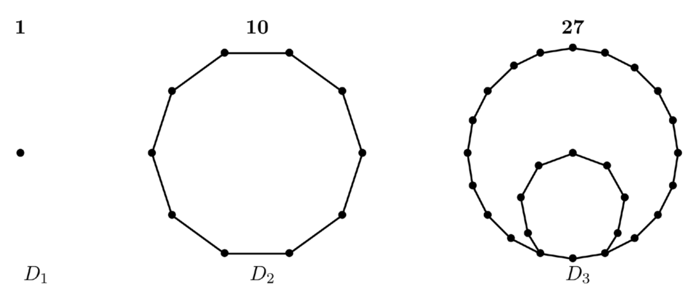

Figure 15. Decagonal numbers.

$$\sum_{k=1}^{n} D_k = \frac{1}{6} n(n+1)(8n-5). \tag{111}$$

- To find the sum of the reciprocals of all decagonal numbers, as in (89) we begin with the series

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k(4k-3)} x^{4k-3}$$

and following the same steps Downey [29] obtained

$$f(1) = \sum_{k=1}^{\infty} \frac{1}{k(4k-3)} = \sum_{k=1}^{\infty} \frac{1}{D_k} = \frac{\pi}{6} + \ln 2 \approx 1.2167459562. \tag{112}$$

- To find all *square decagonal numbers*, we need to find integer solutions of the equation $n(4n-3) = m^2$. This equation can be written as Pell's equation $b^2 - a^2 = 9$, where $b = 8n-3$ and $a = 4m$. For the equation $b^2 - a^2 = 9$ the only meaningful integer solution is $b = 5, a = 4$ and it gives $(m,n) = (1,1)$.
- To find all *decagonal numbers which are also triangular numbers*, we need to find integer solutions of the equation $n(4n-3) = m(m+1)/2$. This equation can be written as Pell's equation $b^2 - 2a^2 = 7$ (its fundamental solutions are $(a,b) = (1,3)$ and $(3,5)$), where $b = 8n-3$ and $a = 2m+1$. For $(3,5)$ corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 34m_k - m_{k-1} + 16, m_1 = 1, m_2 = 55 \\ n_{k+1} &= 34n_k - n_{k-1} - 12, n_1 = 1, n_2 = 20 \end{aligned} \tag{113}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(4n-3) = m(m+1)/2$. First few of these solutions are

$$(1,1), (55,20), (1885,667), (64051,22646), (2175865,769285).$$

For $(1,3)$ recurrence relations remain the same as in (113) with $m_1 = 4, m_2 = 154$ and $n_1 = 2, n_2 = 55$. This leads to another set of infinite number of solutions (m_k, n_k) of the equation $n(4n-3) = m(m+1)/2$. First few of these solutions are

$$(4,2), (154,55), (5248,1856), (178294,63037), (6056764,2141390).$$

- There is no decagonal number which is also a rectangular number, in fact, the equation $n(4n-3) = m(m+1)$ has no solutions.
- To find all *decagonal numbers which are also pentagonal numbers*, we need to find integer solutions of the equation $n(4n-3) = m(3m-1)/2$. This equation can be written as Pell's equation $b^2 - 6a^2 = 75$ (its fundamental solution is $(a,b) = (5,15)$), where $b = 3(8n-3)$ and $a = 6m-1$. For $(5,15)$ corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 98m_k - m_{k-1} - 16, m_1 = 1, m_2 = 91 \\ n_{k+1} &= 98n_k - n_{k-1} - 36, n_1 = 1, n_2 = 56 \end{aligned} \tag{114}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(4n-3) = m(3m-1)/2$. First few of these solutions are

$$(1,1), (91,56), (8901,5451), (872191,534106), (85465801,52336901).$$

- To find all *decagonal numbers which are also hexagonal numbers*, we need to find integer solutions of the equation $n(4n-3)=m(2m-1)$. This equation can be written as Pell's equation $b^2-2a^2=7$ (its fundamental solution is $(a,b)=(3,5)$), where $b=8n-3$ and $a=4m-1$. For this, corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 34m_k - m_{k-1} - 8, \quad m_1 = 1, m_2 = 28 \\ n_{k+1} &= 34n_k - n_{k-1} - 12, \quad n_1 = 1, n_2 = 20 \end{aligned} \tag{115}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(4n-3)=m(2m-1)$. First few of these solutions are

$$(1,1),(28,20),(943,667),(32026,22646),(1087933,769285).$$

- To find all *decagonal numbers which are also generalized pentagonal numbers*, we need to find integer solutions of the equation $n(4n-3)=1+3m(m-1)$. This equation can be written as Pell's equation $b^2-12a^2=13$ (its fundamental solution is $(a,b)=(1,5)$), where $b=8n-3$ and $a=2m-1$. For this, corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 194m_k - m_{k-1} - 96, \quad m_1 = 1, m_2 = 119 \\ n_{k+1} &= 194n_k - n_{k-1} - 72, \quad n_1 = 1, n_2 = 103 \end{aligned} \tag{116}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(4n-3)=1+3m(m-1)$. First few of these solutions are

$$(1,1),(119,103),(22989,19909),(4459651,3862171),$$
$$(865149209,749241193).$$

- To find all *decagonal numbers which are also heptagonal numbers*, we need to find integer solutions of the equation $n(4n-3)=m(5m-3)/2$. This equation can be written as Pell's equation $b^2-10a^2=540$ (its fundamental solution is $(a,b)=(14,50)$), where $b=10(8n-3)$ and $a=2(10m-3)$. For this, corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 1442m_k - m_{k-1} - 432, \quad m_1 = 1, m_2 = 1075 \\ n_{k+1} &= 1442n_k - n_{k-1} - 540, \quad n_1 = 1, n_2 = 850 \end{aligned} \tag{117}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(4n-3)=m(5m-3)/2$. First few of these solutions are

$$(1,1),(1075,850),(1549717,1225159),(2234690407,1766677888),$$
$$(3222422016745,2547548288797).$$

- To find all *decagonal numbers which are also octagonal numbers*, we need to find integer solutions of the equation $n(4n-3)=m(3m-2)$. This equation can be written as Pell's equation $b^2-3a^2=33$ (its fundamental solution is $(a,b)=(8,15)$), where $b=3(8n-3)$ and $a=4(3m-1)$. For this, corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 194m_k - m_{k-1} - 64, \quad m_1 = 1, m_2 = 135 \\ n_{k+1} &= 194n_k - n_{k-1} - 72, \quad n_1 = 1, n_2 = 117 \end{aligned} \tag{118}$$

This system generates infinite number of solutions (m_k, n_k) of the equation

$n(4n-3)=m(3m-2)$. First few of these solutions are

$$(1,1),(135,117),(26125,22625),(5068051,4389061),$$
$$(983175705,851455137).$$

- To find all *decagonal numbers which are also nonagonal numbers*, we need to find integer solutions of the equation $n(4n-3)=m(7m-5)/2$. This equation can be written as Pell's equation $b^2-14a^2=91$ (its fundamental solution is $(a,b)=(9,35)$), where $b=7(8n-3)$ and $a=14m-5$. For this, corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 898m_k - m_{k-1} - 320, \quad m_1 = 1, m_2 = 589 \\ n_{k+1} &= 898n_k - n_{k-1} - 336, \quad n_1 = 1, n_2 = 551 \end{aligned} \tag{119}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(4n-3)=m(7m-5)/2$. First few of these solutions are

$$(1,1),(589,551),(528601,494461),(474682789,444025091),$$
$$(426264615601,398734036921).$$

12. Tetrakaidecagonal Numbers $(TET)_n$

These numbers are defined by the sequence $1,14,39,76,125,\cdots$, *i.e.*, beginning with 14 each number is formed from the previous one in the sequence by adding the next number in the related sequence $13,25,37,49,\cdots,(12n-11)$. Thus, $14=1+13$, $39=1+13+25=14+25$, $76=1+13+25+37=39+37$, and so on (see Figure 16).

Hence, n-th tetrakaidecagonal number is defined as

$$\begin{aligned} (TET)_n &= (TET)_{n-1} + (12n-11) = 1+13+25+37+\cdots+(12n-11) \\ &= 1+(1+12)+(1+2\times 12)+\cdots+(1+(n-1)12). \end{aligned} \tag{120}$$

Comparing (120) with (3), we have $a=1, d=12$, and hence from (4) it follows that

$$(TET)_n = \frac{n}{2}(12n-10) = n(6n-5) = t_n + 11t_{n-1}. \tag{121}$$

- For all integers $k \ge 0$ it follows that $(TET)_{2k+1}$ are odd, whereas $(TET)_{2k}$ are even.
- Let m be a given natural number, then it is n-th tetrakaidecagonal number, *i.e.*, $m=(TET)_n$ if and only if $n=\left(5+\sqrt{25+24m}\right)/12$.

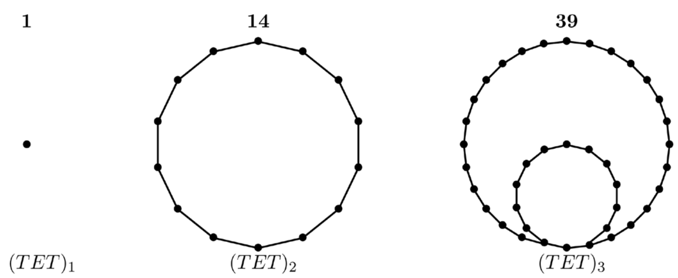

Figure 16. Tetrakaidecagonal numbers.

- From (10) and (121), we have

$$\frac{x(11x+1)}{(1-x)^3} = x + 14x^2 + 39x^3 + 76x^4 + \cdots$$

and hence $x(11x+1)(1-x)^{-3}$ is the *generating function* of all tetrakaidecagonal numbers.

- In view of (15) and (121), we have

$$\sum_{k=1}^{n} (TET)_k = \frac{1}{2} n(n+1)(4n-3). \tag{122}$$

- To find the sum of the reciprocals of all tetrakaidecagonal numbers, as in (89) we begin with the series

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k(6k-5)} x^{6k-5}$$

and following the same steps Downey [29] obtained

$$f(1) = \sum_{k=1}^{\infty} \frac{1}{k(6k-5)} = \sum_{k=1}^{\infty} \frac{1}{(TET)_k} = \frac{1}{10}\left(4\ln 2 + 3\ln 3 + \sqrt{3}\pi\right) \approx 1.1509823681. \tag{123}$$

- To find all *square tetrakaidecagonal numbers*, we need to find integer solutions of the equation $n(6n-5) = m^2$. This equation can be written as Pell's equation $b^2 - 6a^2 = 25$ (its fundamental solution are $(a,b) = (2,7)$ and $(4,11)$), where $b = 12n-5$ and $a = 2m$. For (2,7) corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 98m_k - m_{k-1}, \quad m_1 = 1, m_2 = 119 \\ n_{k+1} &= 98n_k - n_{k-1} - 40, \quad n_1 = 1, n_2 = 49 \end{aligned} \tag{124}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(6n-5) = m^2$. First few of these solutions are

$$(1,1), (119,49), (11661,4761), (1142659,466489), (111968921,45711121).$$

For $(4,11)$ recurrence relations remain the same as in (124) with $m_1 = 21, m_2 = 2059$ and $n_1 = 2, n_2 = 841$. This leads to another set of infinite number of solutions (m_k, n_k) of the equation $n(6n-5) = m^2$. First few of these solutions are

$$(21,9), (2059,841), (201761,82369), (19770519,8071281), (1937309101,790903129).$$

- To find all *tetrakaidecagonal numbers which are also triangular numbers*, we need to find integer solutions of the equation $n(6n-5) = m(m+1)/2$. This equation can be written as Pell's equation $b^2 - 3a^2 = 22$ (its fundamental solutions are $(a,b) = (1,5)$ and $(3,7)$), where $b = 12n-5$ and $a = 2m+1$. For $(3,7)$ corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 194m_k - m_{k-1} + 96, m_1 = 1, m_2 = 341 \\ n_{k+1} &= 194n_k - n_{k-1} - 80, n_1 = 1, n_2 = 99 \end{aligned} \tag{125}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(6n-5) = m(m+1)/2$. First few of these solutions are

$$(1,1),(341,99),(66249,19125),(12852061,3710071),$$
$$(2493233681,719734569).$$

For $(1,5)$ recurrence relations remain the same as in (125) with $m_1 = 50, m_2 = 9798$ and $n_1 = 15, n_2 = 2829$. This leads to another set of infinite number of solutions (m_k, n_k) of the equation $n(6n-5) = m^2$. First few of these solutions are

$$(50,15),(9798,2829),(1900858,548731),$$
$$(368756750,106450905),(71536908738,20650926759).$$

- There is no tetrakaidecagonal number which is also a rectangular number, in fact, the equation $n(6n-5) = m(m+1)$ has no solutions.
- To find all *tetrakaidecagonal numbers which are also pentagonal numbers*, we need to find integer solutions of the equation $n(6n-5) = m(3m-1)/2$. This equation can be written as Pell's equation $b^2 - a^2 = 24$, where $b = 12n-5$ and $a = 6m-1$. For the equation $b^2 - a^2 = 24$ the only meaningful integer solution is $b = 7, a = 5$ and it gives $(m,n) = (1,1)$.
- To find all *tetrakaidecagonal numbers which are also hexagonal numbers*, we need to find integer solutions of the equation $n(6n-5) = m(2m-1)$. This equation can be written as Pell's equation $b^2 - 3a^2 = 22$ (its fundamental solutions are $(a,b) = (1,5)$ and $(3,7)$), where $b = 12n-5$ and $a = 4m-1$. For $(3,7)$ corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 194m_k - m_{k-1} - 48, m_1 = 1, m_2 = 171 \\ n_{k+1} &= 194n_k - n_{k-1} - 80, n_1 = 1, n_2 = 99 \end{aligned} \tag{126}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(6n-5) = m(2m-1)$. First few of these solutions are

$$(1,1),(171,99),(33125,19125),(6426031,3710071),$$
$$(1246616841,719734569).$$

With $(a,b) = (1,5)$ there are no integer solutions of the required equation.

- To find all *tetrakaidecagonal numbers which are also generalized pentagonal numbers*, we need to find integer solutions of the equation $n(6n-5) = 1 + 3m(m-1)$. This equation can be written as Pell's equation $b^2 - 18a^2 = 31$ (its fundamental solutions are $(a,b) = (1,7)$ and $(11,47)$), where $b = 12n-5$ and $a = 2m-1$. For $(1,7)$, corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 1154m_k - m_{k-1} - 576, m_1 = 1, m_2 = 765 \\ n_{k+1} &= 1154n_k - n_{k-1} - 480, n_1 = 1, n_2 = 541 \end{aligned} \tag{127}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(6n-5) = 1 + 3m(m-1)$. First few of these solutions are

$$(1,1),(765,541),(882233,623833),(1018095541,719902261),$$
$$(1174881371505,830766584881).$$

For $(11,47)$ recurrence relations remain the same as in (127) with $m_1 = 188, m_2 = 216376$ and $n_1 = 133, n_2 = 151001$. This leads to another set of infinite number of solutions (m_k, n_k) of the equation $n(6n-5) = 1+3m(m-1)$. First four of these solutions are

$$(188,133),(216376,153001),(249697140,176562541),$$
$$(288150282608,203753018833).$$

- To find all *tetrakaidecagonal numbers which are also heptagonal numbers*, we need to find integer solutions of the equation $n(6n-5) = m(5m-3)/2$. This equation can be written as Pell's equation $b^2 - 15a^2 = 490$ (its fundamental solutions are $(a,b) = (3,25)$ and $(7,35)$), where $b = 5(12n-5)$ and $a = 10m-3$. For (7,35), corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 3842m_k - m_{k-1} - 1152, m_1 = 1, m_2 = 3081 \\ n_{k+1} &= 3842n_k - n_{k-1} - 1600, n_1 = 1, n_2 = 1989 \end{aligned} \tag{128}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(6n-5) = m(5m-3)/2$. First few of these solutions are

$$(1,1),(3081,1989),(11836049,7640137),(45474096025,29353402765),$$
$$(174711465090849,112775765781393).$$

With $(a,b) = (3,25)$ there are no integer solutions of the required equation.

- To find all *tetrakaidecagonal numbers which are also octagonal numbers*, we need to find integer solutions of the equation $n(6n-5) = m(3m-2)$. This equation can be written as Pell's equation $b^2 - 2a^2 = 17$ (its fundamental solutions are $(a,b) = (2,5)$ and $(4,7)$), where $b = 12n-5$ and $a = 6m-2$. For (4,7), corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 1154m_k - m_{k-1} - 384, m_1 = 1, m_2 = 861 \\ n_{k+1} &= 1154n_k - n_{k-1} - 480, n_1 = 1, n_2 = 609 \end{aligned} \tag{129}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(6n-5) = m(3m-2)$. First few of these solutions are

$$(1,1),(861,609),(993209,702305),(1146161941,810458881),$$
$$(1322669886321,935268845889).$$

With $(a,b) = (2,5)$ there are no integer solutions of the required equation.

- To find all *tetrakaidecagonal numbers which are also nonagonal numbers*, we need to find integer solutions of the equation $n(6n-5) = m(7m-5)/2$. This equation can be written as Pell's equation $b^2 - 21a^2 = 700$ (its fundamental solutions are $(a,b) = (2,28),(5,35)$ and $(9,49)$), where $b = 7(12n-5)$ and $a = 14m-5$. For (9,49), corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 12098m_k - m_{k-1} - 4320, \quad m_1 = 1, m_2 = 8509 \\ n_{k+1} &= 12098n_k - n_{k-1} - 5040, \quad n_1 = 1, n_2 = 6499 \end{aligned} \tag{130}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(6n-5) = m(7m-5)/2$. First four of these solutions are

$$(1,1),(8509,6499),(102937561,78619861),(1245338600149,951143066839).$$

With $(a,b)=(2,28)$ and $(5,35)$ there are no integer solutions of the required equation.

- To find all *tetrakaidecagonal numbers which are also decagonal numbers*, we need to find integer solutions of the equation $n(6n-5)=m(4m-3)$. This equation can be written as Pell's equation $b^2-6a^2=46$ (its fundamental solutions are $(a,b)=(3,10)$ and $(5,14)$), where $b=2(12n-5)$ and $a=8m-3$. For (5,14), corresponding to (22) the system is

$$\begin{aligned} m_{k+1} &= 98m_k - m_{k-1} - 36, \quad m_1 = 1, m_2 = 66 \\ n_{k+1} &= 98n_k - n_{k-1} - 40, \quad n_1 = 1, n_2 = 54 \end{aligned} \tag{131}$$

This system generates infinite number of solutions (m_k,n_k) of the equation $n(6n-5)=m(4m-3)$. First few of these solutions are

$$(1,1),(66,54),(6431,5251),(630136,514504),(61746861,50416101).$$

With $(a,b)=(3,10)$ there are no integer solutions of the required equation.

13. Centered Triangular Numbers $(ct)_n$

These numbers are defined by the sequence $1,4,10,19,31,46,64,85,109,\cdots$, *i.e.*, beginning with 4 each number is formed from the previous one in the sequence by adding the next number in the related sequence $3,6,9,12,\cdots,3(n-1)$. Thus, $4=1+3$, $10=1+3+6=4+6$, $19=1+3+6+9=10+9$, and so on (see Figure 17).

Hence, n-th centered triangular number is defined as

$$\begin{aligned} (ct)_n &= (ct)_{n-1} + 3(n-1) = 1+3+6+9+\cdots+(3n-3) \\ &= 1+3\big(1+2+3+\cdots+(n-1)\big). \end{aligned} \tag{132}$$

Thus, from (2) it follows that

$$(ct)_n = 1+3\frac{(n-1)n}{2} = 1+3t_{n-1} = t_n + t_{n-1} + t_{n-2}. \tag{133}$$

- Let m be a given natural number, then it is n-th centered triangular number, *i.e.*, $m=(ct)_n$ if and only if $n=\left(3+\sqrt{9+24(m-1)}\right)/6$.
- From (10) and (133), we have

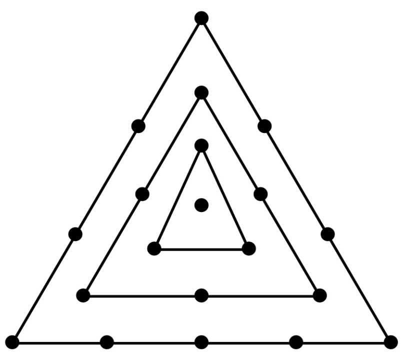

Figure 17. Centered triangular numbers.

$$\frac{x\left(x^2+x+1\right)}{\left(1-x\right)^3}=x+4x^2+10x^3+19x^4+\cdots$$

and hence $x\left(x^2+x+1\right)\left(1-x\right)^{-3}$ is the *generating function* of all centered triangular numbers.

In view of (15) and (133), we have

$$\sum_{k=1}^{n}(ct)_k=\frac{1}{2}n\left(n^2+1\right). \tag{134}$$

- To find the sum of the reciprocals of all centered triangular numbers we follow as in (71), to obtain

$$\sum_{k=1}^{\infty}\frac{1}{(ct)_k}=\frac{2\pi}{\sqrt{15}}\tanh\left(\frac{\pi}{2}\sqrt{\frac{5}{3}}\right)\approx 1.5670651313. \tag{135}$$

14. Centered Square Numbers $(cS)_n$

These numbers are defined by the sequence $1,5,13,25,41,61,85,113,\cdots$, *i.e.*, beginning with 5 each number is formed from the previous one in the sequence by adding the next number in the related sequence $4,8,12,16,\cdots,4(n-1)$. Thus, $5=1+4$, $13=1+4+8=5+8$, $25=1+4+8+12=13+12$, and so on (see Figure 18).

Hence, n-th centered square number is defined as

$$\begin{aligned}(cS)_n&=(cS)_{n-1}+4(n-1)=1+4+8+12+\cdots+(4n-4)\\&=1+4\left(1+2+3+\cdots+(n-1)\right).\end{aligned} \tag{136}$$

Thus, from (2) it follows that

$$\begin{aligned}(cS)_n&=1+4\frac{(n-1)n}{2}=1+4t_{n-1}=1+2n^2-2n=n^2+(n-1)^2\\&=S_n+S_{n-1}=t_n+2t_{n-1}+t_{n-2}.\end{aligned} \tag{137}$$

- Let m be a given natural number, then it is n-th centered square number, *i.e.*, $m=(cS)_n$ if and only if $n=\left(1+\sqrt{2m-1}\right)/2$.
- From (10) and (137), we have

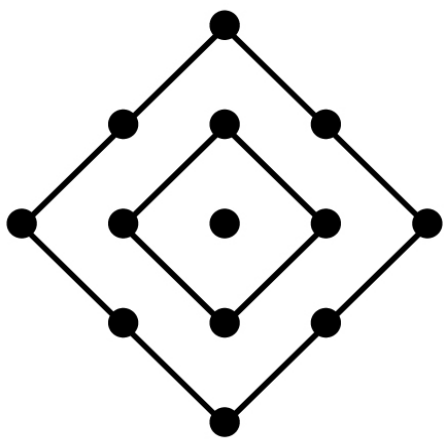

Figure 18. Centered square numbers.

$$\frac{x(x+1)^2}{(1-x)^3} = x + 5x^2 + 13x^3 + 25x^4 + \cdots$$

and hence $x(x+1)^2(1-x)^{-3}$ is the *generating function* of all centered square numbers.

- In view of (15) and (137), we have

$$\sum_{k=1}^{n} (cS)_k = \frac{1}{3}n(2n^2+1). \quad (138)$$

- To find the sum of the reciprocals of all centered triangular numbers we follow as in (71), to obtain

$$\sum_{k=1}^{\infty} \frac{1}{(cS)_k} = \frac{\pi}{2}\tanh\left(\frac{\pi}{2}\right) \approx 1.44065952. \quad (139)$$

15. Centered Pentagonal Numbers $(cP)_n$

These numbers are defined by the sequence $1, 6, 16, 31, 51, 76, 106, 141, 181, \cdots$, *i.e.*, beginning with 6 each number is formed from the previous one in the sequence by adding the next number in the related sequence $5, 10, 15, 20, \cdots, 5(n-1)$. Thus, $6 = 1 + 5$, $16 = 1 + 5 + 10 = 6 + 10$, $31 = 1 + 5 + 10 + 15 = 16 + 15$, and so on (see **Figure 19**).

Hence, n-th centered pentagonal number is defined as

$$\begin{aligned}(cP)_n &= (cP)_{n-1} + 5(n-1) = 1 + 5 + 10 + 15 + \cdots + (5n-5) \\ &= 1 + 5(1 + 2 + 3 + \cdots + (n-1)).\end{aligned} \quad (140)$$

Thus, from (2) it follows that

$$(cP)_n = 1 + 5\frac{(n-1)n}{2} = 1 + 5t_{n-1} = t_n + 3t_{n-1} + t_{n-2}. \quad (141)$$

- Let m be a given natural number, then it is n-th centered pentagonal number, *i.e.*, $m = (cP)_n$ if and only if $n = \left(5 + \sqrt{25 + 40(m-1)}\right)/10$.
- From (10) and (141), we have

$$\frac{x(x^2+3x+1)}{(1-x)^3} = x + 6x^2 + 16x^3 + 31x^4 + \cdots$$

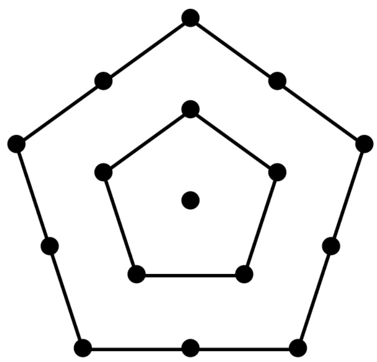

Figure 19. Centered pentagonal numbers.

and hence $x\left(x^2+3x+1\right)(1-x)^{-3}$ is the *generating function* of all centered pentagonal numbers.

- In view of (15) and (141), we have

$$\sum_{k=1}^{n}(cP)_k=\frac{1}{6}n\left(5n^2+1\right). \tag{142}$$

- To find the sum of the reciprocals of all centered pentagonal numbers we follow as in (71), to obtain

$$\sum_{k=1}^{\infty}\frac{1}{(cP)_k}=\frac{2\pi}{\sqrt{15}}\tanh\left(\frac{\pi}{2}\sqrt{\frac{3}{5}}\right)\approx 1.36061317. \tag{143}$$

16. Centered Heptagonal Numbers $(cHEP)_n$

These numbers are defined by the sequence $1,8,22,43,71,106,148,197,253,\cdots$, *i.e.*, beginning with 8 each number is formed from the previous one in the sequence by adding the next number in the related sequence $7,14,21,28,\cdots,7(n-1)$. Thus, $8=1+7$, $22=1+7+14=8+14$, $43=1+7+14+21=22+21$, and so on (see Figure 20).

Hence, n-th centered heptagonal number is defined as

$$\begin{aligned}(cHEP)_n&=(cHEP)_{n-1}+7(n-1)=1+7+14+21+\cdots+(7n-7)\\&=1+7(1+2+3+\cdots+(n-1)).\end{aligned} \tag{144}$$

Thus, from (2) it follows that

$$(cHEP)_n=1+7\frac{(n-1)n}{2}=1+7t_{n-1}=t_n+5t_{n-1}+t_{n-2}. \tag{145}$$

- Let m be a given natural number, then it is n-th centered heptagonal number, *i.e.*, $m=(cHEP)_n$ if and only if $n=\left(7+\sqrt{49+56(m-1)}\right)/14$.
- From (10) and (145), we have

$$\frac{x\left(x^2+5x+1\right)}{(1-x)^3}=x+8x^2+22x^3+43x^4+\cdots$$

and hence $x\left(x^2+5x+1\right)(1-x)^{-3}$ is the *generating function* of all centered heptagonal numbers.

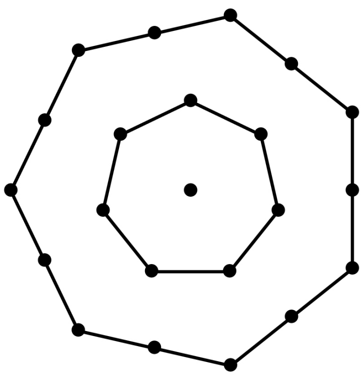

Figure 20. Centered heptagonal numbers.

- In view of (15) and (145), we have

$$\sum_{k=1}^{n}(cHEP)_k = \frac{1}{6}n\left(7n^2-1\right). \tag{146}$$

- To find the sum of the reciprocals of all centered heptagonal numbers we follow as in (71), to obtain

$$\sum_{k=1}^{\infty}\frac{1}{(cHEP)_k} = \frac{2\pi}{\sqrt{7}}\tanh\left(\frac{\pi}{2\sqrt{7}}\right) \approx 1.264723172. \tag{147}$$

17. Centered Octagonal Numbers $(cO)_n$

These numbers are defined by the sequence $1,9,25,49,81,121,169,225,289,361,\cdots$, *i.e.*, beginning with 9 each number is formed from the previous one in the sequence by adding the next number in the related sequence $8,16,24,32,\cdots,8(n-1)$. Thus, $9=1+8$, $25=1+8+14=9+16$, $49=1+8+16+24=25+24$, and so on (see **Figure 21**).

Hence, n-th centered octagonal number is defined as

$$\begin{aligned}(cO)_n &= (cO)_{n-1} + 8(n-1) = 1+8+16+24+\cdots+(8n-8)\\ &= 1+8\left(1+2+3+\cdots+(n-1)\right).\end{aligned} \tag{148}$$

Thus, from (2) it follows that

$$(cO)_n = 1+8\frac{(n-1)n}{2} = 1+8t_{n-1} = (2n-1)^2 = S_{2n-1} = t_n + 6t_{n-1} + t_{n-2}. \tag{149}$$

Hence, the centered octagonal numbers are the same as the odd square numbers.

- Let m be a given natural number, then it is n-th centered octagonal number, *i.e.*, $m=(cO)_n$ if and only if $n=\left(1+\sqrt{m}\right)/2$.
- From (10) and (149), we have

$$\frac{x\left(x^2+6x+1\right)}{(1-x)^3} = x + 9x^2 + 25x^3 + 49x^4 + \cdots$$

and hence $x\left(x^2+6x+1\right)(1-x)^{-3}$ is the *generating function* of all centered octagonal numbers.

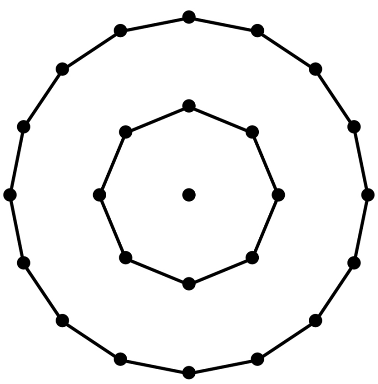

Figure 21. Centered octagonal numbers.

- In view of (15) and (149), we have

$$\sum_{k=1}^{n}(cO)_k = \frac{1}{3}n\left(4n^2-1\right). \tag{150}$$

- To find the sum of the reciprocals of all centered octagonal numbers we use (39), to obtain

$$\sum_{k=1}^{\infty}\frac{1}{(cO)_k} = \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \sum_{k=1}^{\infty}\frac{1}{k^2} - \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{8} \approx 1.2337005501. \tag{151}$$

18. Centered Nonagonal Numbers $(cN)_n$

These numbers are defined by the sequence $1, 10, 28, 55, 91, 136, 190, 253, 325, \cdots$, *i.e.*, beginning with 10 each number is formed from the previous one in the sequence by adding the next number in the related sequence $9, 18, 27, 36, \cdots, 9(n-1)$. Thus, $10 = 1+9$, $28 = 1+9+18 = 10+18$, $55 = 1+9+18+27 = 28+27$, and so on (see Figure 22).

Hence, n-th centered nonagonal number is defined as

$$\begin{aligned}(cN)_n &= (cN)_{n-1} + 9(n-1) = 1+9+18+27+\cdots+9(n-1)\\ &= 1+9\left(1+2+3+\cdots+(n-1)\right).\end{aligned} \tag{152}$$

Thus, from (2) it follows that

$$\begin{aligned}(cN)_n &= 1+9\frac{(n-1)n}{2} = 1+9t_{n-1} = \frac{(3n-2)(3n-1)}{2}\\ &= t_{3n-2} = t_n + 7t_{n-1} + t_{n-2}.\end{aligned} \tag{153}$$

- In 1850, Frederick Pollock (1783-1870, England) conjectured that every natural number is the sum of at most eleven centered nonagonal numbers, which has not been proved.
- Let m be a given natural number, then it is n-th centered nonagonal number, *i.e.*, $m = (cN)_n$ if and only if $n = \left(9+\sqrt{81+72(m-1)}\right)/18$.
- From (10) and (153), we have

$$\frac{x\left(x^2+7x+1\right)}{(1-x)^3} = x + 10x^2 + 28x^3 + 55x^4 + \cdots$$

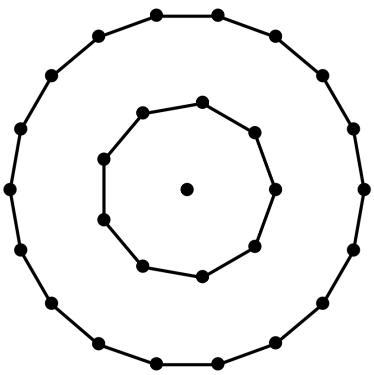

Figure 22. Centered nonagonal numbers.

and hence $x\left(x^2+7x+1\right)\left(1-x\right)^{-3}$ is the *generating function* of all centered nonagonal numbers.

- In view of (15) and (153), we have

$$\sum_{k=1}^{n}\left(cN\right)_k=\frac{1}{2}n\left(3n^2-1\right). \tag{154}$$

- To find the sum of the reciprocals of all centered heptagonal numbers we follow as in (71), to obtain

$$\sum_{k=1}^{\infty}\frac{1}{\left(cN\right)_k}=\frac{2\pi}{3}\tan\left(\frac{\pi}{6}\right)=\frac{2\sqrt{3}\pi}{9}\approx 1.2091995762. \tag{155}$$

19. Centered Decagonal Numbers $(cD)_n$

These numbers are defined by the sequence $1,11,31,61,101,151,211,281,\cdots$, *i.e.*, beginning with 11 each number is formed from the previous one in the sequence by adding the next number in the related sequence $10,20,30,40,\cdots,10(n-1)$. Thus, $11=1+10$, $31=1+10+20=11+20$, $61=1+10+20+30=31+30$, and so on (see **Figure 23**).

Hence, n-th centered decagonal number is defined as

$$\begin{aligned}(cD)_n&=(cD)_{n-1}+10(n-1)=1+10+20+30+\cdots+10(n-1)\\&=1+10\left(1+2+3+\cdots+(n-1)\right).\end{aligned} \tag{156}$$

Thus, from (2) it follows that

$$(cD)_n=1+10\frac{(n-1)n}{2}=1+10t_{n-1}=5n^2-5n+1=t_n+8t_{n-1}+t_{n-2}. \tag{157}$$

- For each $(cD)_n$ the last digit is 1.
- Let m be a given natural number, then it is n-th centered decagonal number, *i.e.*, $m=(cD)_n$ if and only if $n=\left(5+\sqrt{25+20(m-1)}\right)/10$.
- From (10) and (157), we have

$$\frac{x\left(x^2+8x+1\right)}{\left(1-x\right)^3}=x+11x^2+31x^3+61x^4+\cdots$$

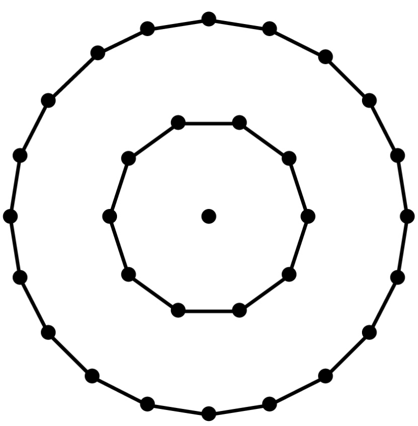

Figure 23. Centered decagonal numbers.

and hence $x\left(x^2+8x+1\right)\left(1-x\right)^{-3}$ is the *generating function* of all centered decagonal numbers.

- In view of (15) and (157), we have

$$\sum_{k=1}^{n}\left(cD\right)_k=\frac{1}{3}n\left(5n^2-2\right). \tag{158}$$

- To find the sum of the reciprocals of all centered heptagonal numbers we follow as in (71), to obtain

$$\sum_{k=1}^{\infty}\frac{1}{\left(cD\right)_k}=\frac{\pi}{\sqrt{5}}\tan\left(\frac{\pi}{2\sqrt{5}}\right)\approx 1.189356247. \tag{159}$$

20. Star Numbers $(ST)_n$

These numbers are defined by the sequence $1,13,37,73,121,181,253,337,\cdots$, *i.e.*, beginning with 13 each number is formed from the previous one in the sequence by adding the next number in the related sequence $12,24,36,48,\cdots,12\left(n-1\right)$. Thus, $13=1+12$, $37=1+12+24=13+24$, $73=1+12+24+36=37+36$, and so on (see **Figure 24**).

Hence, n-th star number is defined as

$$\begin{aligned}\left(ST\right)_n&=\left(ST\right)_{n-1}+12\left(n-1\right)=1+12+24+36+\cdots+12\left(n-1\right)\\&=1+12\left(1+2+3+\cdots+\left(n-1\right)\right).\end{aligned} \tag{160}$$

Thus, from (2) it follows that

$$\left(ST\right)_n=1+12\frac{\left(n-1\right)n}{2}=1+12t_{n-1}=6n^2-6n+1=t_n+10t_{n-1}+t_{n-2}. \tag{161}$$

- All star numbers are odd. The star number $\left(ST\right)_{77}=35113$ is unique, since its prime factors 13, 37, 73 are also consecutive star numbers. There are infinite number of star numbers which are also triangular numbers, also square numbers.
- Let m be a given natural number, then it is n-th star number, *i.e.*, $m=\left(ST\right)_n$ if and only if $n=\left(3+\sqrt{9+6\left(m-1\right)}\right)/6$.

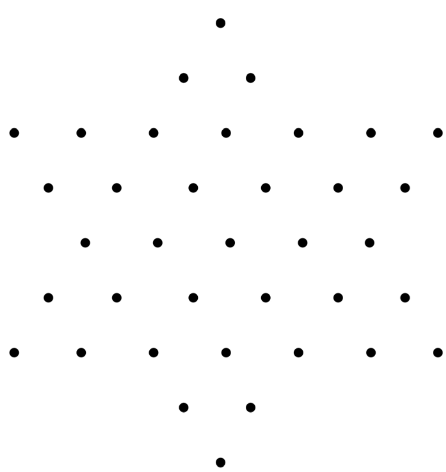

Figure 24. Star number $(ST)_3$.

- From (10) and (161), we have

$$\frac{x\left(x^2+10x+1\right)}{\left(1-x\right)^3}=x+13x^2+37x^3+73x^4+\cdots$$

and hence $x\left(x^2+10x+1\right)\left(1-x\right)^{-3}$ is the *generating function* of all star numbers.

- In view of (15) and (161), we have

$$\sum_{k=1}^{n}\left(ST\right)_k=n\left(2n^2-1\right). \tag{162}$$

- To find the sum of the reciprocals of all star numbers we follow as in (71), to obtain

$$\sum_{k=1}^{\infty}\frac{1}{\left(ST\right)_k}=\frac{\pi}{2\sqrt{3}}\tan\left(\frac{\pi}{2\sqrt{3}}\right)\approx 1.15917332. \tag{163}$$

21. Centered Tetrakaidecagonal Numbers (*cTET*)*n*

These numbers are defined by the sequence $1,15,43,85,141,\cdots$, *i.e.*, beginning with 15 each number is formed from the previous one in the sequence by adding the next number in the related sequence $14,28,42,56,\cdots,14\left(n-1\right)$. Thus, $15=1+14$, $43=1+14+28=15+28$, $85=1+14+28+42=43+42$, and so on (see **Figure 25**).

Hence, *n*-th centered tetrakaidecagonal number is defined as

$$\begin{aligned}\left(cTET\right)_n&=\left(cTET\right)_{n-1}+14\left(n-1\right)=1+14+28+42+\cdots+14\left(n-1\right)\\&=1+14\left(1+2+3+\cdots+\left(n-1\right)\right).\end{aligned} \tag{164}$$

Thus, from (2) it follows that

$$\left(cTET\right)_n=1+14\frac{\left(n-1\right)n}{2}=1+14t_{n-1}=7n^2-7n+1=t_n+12t_{n-1}+t_{n-2}. \tag{165}$$

- Each $\left(cTET\right)_n$ is odd.
- Let *m* be a given natural number, then it is *n*-th centered tetrakaidecagonal number, *i.e.*, $m=\left(cTET\right)_n$ if and only if $n=\left(7+\sqrt{49+28\left(m-1\right)}\right)/14$.
- From (10) and (165), we have

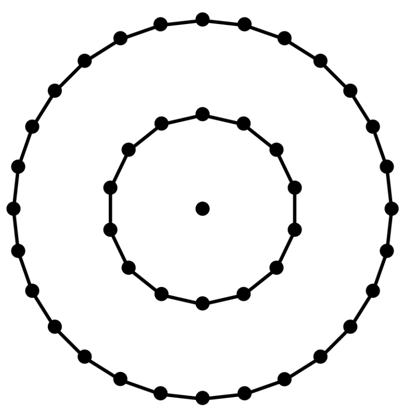

Figure 25. Centered tetrakaidecagonal numbers.

$$\frac{x\left(x^2+12x+1\right)}{(1-x)^3}=x+15x^2+43x^3+85x^4+\cdots$$

and hence $x\left(x^2+12x+1\right)(1-x)^{-3}$ is the *generating function* of all centered tetrakaidecagonal numbers.

- In view of (15) and (165), we have

$$\sum_{k=1}^{n}(cTET)_k=\frac{1}{3}n\left(7n^2-4\right). \quad (166)$$

- To find the sum of the reciprocals of all centered tetrakaidecagonal numbers we follow as in (71), to obtain

$$\sum_{k=1}^{\infty}\frac{1}{(cTET)_k}=\frac{\pi}{\sqrt{21}}\tan\left(\frac{\pi}{2}\sqrt{\frac{3}{7}}\right)\approx 1.1372969963. \quad (167)$$

22. Cubic Numbers C_n

A cubic number can be written as a product of three equal factors of natural numbers. Thus, $1,8,27,64,\cdots,n^3$ are first few cubic numbers (see Figure 26).

- Last digit of a number is the same as the last digit of its cube, except that 2 becomes 8 (and 8 becomes 2) and 3 becomes 7 (and 7 becomes 3).
- Nicomachus considered the following infinite triangle of odd numbers. It is clear that the sum of the numbers in the nth row is n^3.

									1	1^3
								3	5	2^3
							7	9	11	3^3
						13	15	17	19	4^3
					21	23	25	27	29	5^3
				31	33	35	37	39	41	6^3
			43	45	47	49	51	53	55	7^3
		57	59	61	63	65	67	69	71	8^3
	73	75	77	79	81	83	85	87	89	9^3
91	93	95	97	99	101	103	105	107	109	10^3
.	.	.	.	.	.	.	.	.	.	.

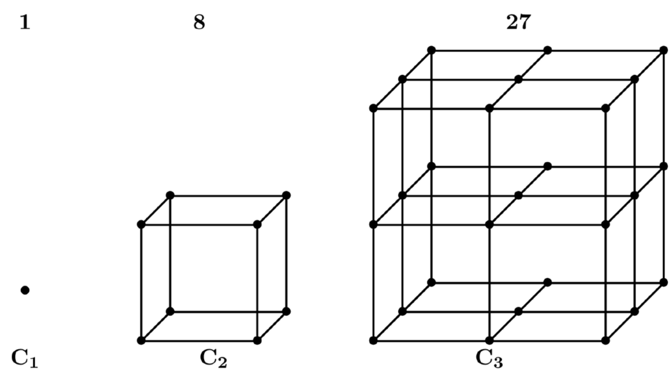

Figure 26. Cubic numbers.

In the literature often the above representation is referred to as Pascal's triangle. Now noting that numbers in each row are odd, so the general term in view of (31) can be written as

$$\begin{aligned}&\left[k(k-1)+1\right]+\left[k(k-1)+3\right]+\left[k(k-1)+5\right]+\cdots+\left[k(k-1)+(2k-1)\right]\\&=k\times k(k-1)+\left[1+3+5+\cdots+(2k-1)\right]=k^3-k^2+k^2=k^3.\end{aligned}$$

Taking successively $k=1,2,3,\cdots,n$ in the above relation, adding these n equations, and observing that $n(n-1)+(2n-1)=2n(n+1)/2-1$, we find

$$1+3+5+7+9+11+\cdots+\left[2\frac{n(n+1)}{2}-1\right]=1^3+2^3+3^3+\cdots+n^3,$$

i.e., the number of terms in the left side are $n(n+1)/2$. Now again from (31) we find that the left hand side is the same as $\left[n(n+1)/2\right]^2$, and hence it follows that

$$1^3+2^3+3^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2=\frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2=t_n^2, \qquad (168)$$

i.e., a perfect square number. This identity is sometimes called Nicomachus's theorem.

- From (69), the relation (168) follows immediately. In fact, we have

$$\sum_{k=1}^{n}k^3=\sum_{k=1}^{n}\left(t_k^2-t_{k-1}^2\right)=t_n^2.$$

- The *generating function* for all cubic numbers is

$$\frac{x\left(x^2+4x+1\right)}{(1-x)^4}=x+8x^2+27x^3+\cdots.$$

- From the relations $(k+1)^2-k^2=2k+1$ and $(k+1)^2-(k-1)^2=4k$ it follows that all odd and multiple of 4 integers can be expressed as the difference of two squares. However, $4k+2$ cannot be expressed as the difference of two squares. Indeed, if $a^2-b^2=(a+b)(a-b)=4k+2$, then letting $x=a+b, y=a-b$ gives $a=(x+y)/2, b=(x-y)/2$, which implies that both x and y must be of the same parity. If both x and y are odd (even) then xy is odd (multiple of 4), hence in either case we have a contradiction. Now since cube of any odd (even) integer is odd (multiple of 4), we can conclude that every cube is a difference of two squares. Clearly, in the conclusion cube can be replaced by any power greater than 3. For example, $3^3=14^2-13^2$, $4^3=17^2-15^2$, $5^7=39063^2-39062^2$.
- There are infinite number of *square cubic numbers*, in fact, $\left(k^2\right)^3=\left(k^3\right)^2, k=1,2,\cdots$.
- The well known *Riemann zeta function* after George Friedrich Bernhard Riemann (1826-1866, Germany) is defined as

$$\zeta(s)=1+\frac{1}{2^s}+\frac{1}{3^s}+\cdots, \quad s=\sigma+it.$$

In 1979, Roger Apéry (1916-1994, Greek-French) published an unexpected proof of the irrationality of $\zeta(3)$. In the literature $\zeta(3)$ is known as Apéry constant. In July 2020, Seungmin Kim (Korea) has computed the value of $\zeta(3)$ to one trillion two hundred billion and one hundred decimal places.

- Related with cubic numbers there are *centered cubic numbers* $(cC)_n = n^3 + (n-1)^3 = (2n-1)(n^2 - n + 1)$. Thus, $(cC)_n$ is the count of number of points in a body-centered cubic pattern within a cube that has $n+1$ points along each of its edges. First few centered cubic numbers are 1, 9, 35, 91, 189, 341. Clearly, no centered cubic number is prime. Further, the only centered cube number that is also a square number is 9. The *generating function* for all centered cube numbers is

$$\frac{x(x^3 + 5x^2 + 5x + 1)}{(1-x)^4} = x + 9x^2 + 35x^3 + 91x^4 + \cdots.$$

From (168) it follows that

$$\sum_{k=1}^{n} (cC)_k = \frac{1}{2} n^2 (n^2 + 1). \tag{169}$$

Further, from (39) and (70), we find

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{(cC)_k} &= \sum_{k=1}^{\infty} \frac{2}{k^2(k^2+1)} = 2\sum_{k=1}^{\infty}\left(\frac{1}{k^2} - \frac{1}{k^2+1}\right) \\ &= \frac{\pi^2}{3} + 1 - \pi \coth \pi \approx 1.1365200388. \end{aligned} \tag{170}$$

23. Tetrahedral Numbers (Triangular Pyramidal Numbers) T_n

These numbers count the number of dots in pyramids built up of triangular numbers. If the base is the triangle of side *n*, then the pyramid is formed by placing similarly situated triangles upon it, each of which has one less in its sides than that which precedes it (see Figure 27).

In general, the *n*th tetrahedral number T_n is given in terms of the sum of the first *n* triangular numbers, *i.e.*,

$$T_n = T_{n-1} + t_n = t_1 + t_2 + t_3 + \cdots + t_n,$$

which in view of (15) is the same as

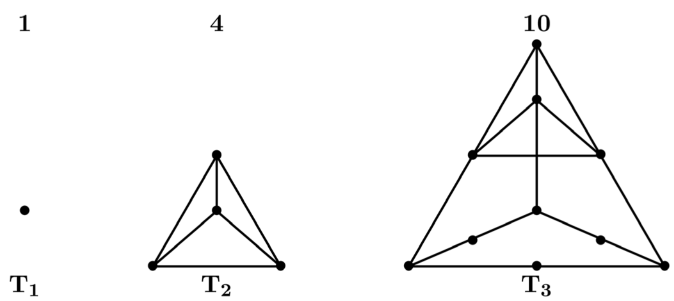

Figure 27. Tetrahedral numbers.

$$T_n = \frac{n(n+1)(n+2)}{6} = \frac{n+2}{3} t_n. \tag{171}$$

Thus, first few tetrahedral numbers are 1, 4, 10, 20, 35, 56, 84, 120, 165.

- In 1850, Pollock conjectured that every natural number is the sum of at most five tetrahedral numbers, which has not been proved. Tetrahedral numbers are even, except for $T_{4n+1}, n = 0, 1, 2, \cdots$, which are odd, see Conway and Guy [30]. The only numbers which are simultaneously square and tetrahedral are $T_1 = 1, T_2 = 4$, and $T_{48} = 19600$, see Meyl [31].
- The *generating function* for all tetrahedral numbers is

$$\frac{x}{(1-x)^4} = x + 4x^2 + 10x^3 + 20x^4 + \cdots.$$

- From (2), (11), and (168) it follows that

$$\sum_{k=1}^{n} T_k = \frac{1}{6}\sum_{k=1}^{n}\left(k^3 + 3k^2 + 2k\right) = \frac{1}{24} n\left(n^3 + 6n^2 + 11n + 6\right). \tag{172}$$

- To find the sum of the reciprocals of all tetrahedral numbers we follow as in (16) and (17), to obtain

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{6}{k(k+1)(k+2)} &= \lim_{n\to\infty}\left[3\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) - 3\sum_{k=1}^{n}\left(\frac{1}{k+1} - \frac{1}{k+2}\right)\right] \\ &= \lim_{n\to\infty}\left[3\left(1 - \frac{1}{n+1}\right) - 3\left(\frac{1}{2} - \frac{1}{n+2}\right)\right] = \frac{3}{2}.\end{aligned} \tag{173}$$

- As in (173), we find

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{6(-1)^{k-1}}{k(k+1)(k+2)} &= 3\sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{1}{k} - \frac{1}{k+1}\right) - 3\sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{1}{k+1} - \frac{1}{k+2}\right) \\ &= 3\left(1 + 2\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{k}\right) - 3\left(\frac{1}{2} - 2\sum_{k=3}^{\infty}\frac{(-1)^{k-1}}{k}\right) \\ &= 3(1 + 2\ln 2 - 2) - 3\left(\frac{1}{2} - 2\ln 2 + 1\right) \\ &= 12\ln 2 - \frac{15}{2} \approx 0.8177661667.\end{aligned} \tag{174}$$

- The numbers $T_{3n-2} - 4T_{n-1} = n\left(23n^2 - 27n + 10\right)/6$ are called *truncated tetrahedral numbers* and denoted as $(TT)_n$. These numbers are assembled by removing the $(n-1)$ th tetrahedral number from each of the four corners from the $(3n-2)$ th tetrahedral number. First few of these numbers are 1, 16, 68, 180, 375, 676, 1106. The *generating function* for all truncated tetrahedral numbers is

$$\frac{x\left(10x^2 + 12x + 1\right)}{(1-x)^4} = x + 16x^2 + 68x^3 + 180x^4 + \cdots.$$

24. Square Pyramidal Numbers $(SP)_n$

These numbers count the number of dots in pyramids built up of square numbers. First few square pyramidal numbers are 1, 5, 14, 30, 55, 91, 140. In general, the *n*th square pyramidal number $(SP)_n$ is given in terms of the sum of the first *n* square numbers, *i.e.*,

$$(SP)_n = (SP)_{n-1} + n^2 = 1^2 + 2^2 + \cdots + n^2,$$

which in view of (11) and (171) is the same as

$$(SP)_n = \frac{n(n+1)(2n+1)}{6} = \frac{2n(2n+2)(2n+1)}{4\times 6} = \frac{1}{4}T_{2n} = \frac{1}{6}(n+1)t_{2n}. \quad (175)$$

- In 1918, George Neville Watson (1886-1965, England) proved that besides 1, there is only one other number that is both a square and a pyramid number, 4900, (as conjectured by Lucas in 1875), the 70th square number and the 24th square pyramidal number, *i.e.*, $S_{70} = (SP)_{24}$.
- The *generating function* for all square pyramidal numbers is

$$\frac{x(x+1)}{(1-x)^4} = x + 5x^2 + 14x^3 + 30x^4 + \cdots.$$

- From (2), (11), and (168) it follows that

$$\sum_{k=1}^{n}(SP)_k = \frac{1}{6}\sum_{k=1}^{n}\left(2k^3 + 3k^2 + k\right) = \frac{1}{12}n(n+1)^2(n+2). \quad (176)$$

- To find the sum of the reciprocals of all square pyramidal numbers, we note that

$$\begin{aligned}\sum_{k=1}^{n}\frac{6}{k(k+1)(2k+1)} &= 12\sum_{k=1}^{n}\left(\frac{1}{2k} + \frac{1}{2(k+1)} - \frac{2}{2k+1}\right)\\ &= 12\sum_{k=1}^{n}\int_0^1\left(x^{2k-1} + x^{2k+1} - 2x^{2k}\right)\mathrm{d}x\\ &= 12\int_0^1 x(1-x)^2\left(\sum_{k=0}^{n-1}x^{2k}\right)\mathrm{d}x\\ &= 12\int_0^1 x(1-x)^2\frac{1-x^{2n}}{1-x^2}\mathrm{d}x\\ &= 12\int_0^1\frac{x(1-x)}{1+x}\left(1-x^{2n}\right)\mathrm{d}x.\end{aligned}$$

Now since

$$\int_0^1\frac{x(1-x)}{1+x}x^{2n}\mathrm{d}x < \int_0^1 x^{2n}\mathrm{d}x = \frac{1}{2n+1} \to 0 \text{ as } n\to\infty$$

it follows that

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{(SP)_k} &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{6}{k(k+1)(2k+1)} = 12\int_0^1\frac{x(1-x)}{1+x}\mathrm{d}x\\ &= 12\int_0^1\left(2 - x - \frac{2}{1+x}\right)\mathrm{d}x = 6(3-4\ln 2) \quad (177)\\ &\approx 1.364467667.\end{aligned}$$

- In 2006, Fearnehough [32] used Madhava of Sangamagramma (1340-1425, India) series

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\cdots,$$

known in the literature after James Gregory (1638-1675, England) and Gottfried Wilhelm von Leibniz (1646-1716, Germany) in the above partial fractions, to obtain

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{6(-1)^{k-1}}{k(k+1)(2k+1)}\\&=6\left[\left(\frac{1}{1}+\frac{1}{2}-\frac{4}{3}\right)-\left(\frac{1}{2}+\frac{1}{3}-\frac{4}{5}\right)+\left(\frac{1}{3}+\frac{1}{4}-\frac{4}{7}\right)-\left(\frac{1}{4}+\frac{1}{5}-\frac{4}{9}\right)+\cdots\right]\\&=6\left[1+4\left(-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots\right)\right]\\&=6\left[1+4\left(\frac{\pi}{4}-1\right)\right]=6(\pi-3).\end{aligned}\tag{178}$$

- The sum of two consecutive square pyramidal numbers, *i.e.*, $(SP)_n+(SP)_{n-1}=n(2n^2+1)/3, n\ge 1$ are called *octahedral numbers*, and denoted as $(OH)_n$. The first few octahedral numbers are 1, 6, 19, 44, 85, 146, 231, 344. These numbers represent the number of spheres in an octahedral formed from close-packed spheres. Descartes initiated the study of octahedral numbers around 1630. In 1850, Pollock conjectured that every positive integer is the sum of at most 7 octahedral numbers, which for finitely many numbers have been proved by Brady [33]. The difference between two consecutive octahedral numbers is a centered square number, *i.e.*, $(OH)_n-(OH)_{n-1}=n^2+(n-1)^2=(cS)_n$. The *generating function* for all octahedral numbers is

$$\frac{x(x+1)^2}{(1-x)^4}=x+6x^2+19x^3+44x^4+\cdots.$$

From (2) and (168) it follows that

$$\sum_{k=1}^{n}(OH)_k=\frac{1}{6}n(n+1)(n^2+n+1).\tag{179}$$

Further, as in (100), we have

$$\sum_{k=1}^{\infty}\frac{1}{(OH)_k}=\frac{3}{2}\left(2\gamma+\Psi\left(\frac{1}{2}\left(2-i\sqrt{2}\right)\right)+\Psi\left(\frac{1}{2}\left(2+i\sqrt{2}\right)\right)\right)\approx 1.2781850979.\tag{180}$$

- The sum of two consecutive octahedral numbers, *i.e.*, $(OH)_n+(OH)_{n-1}=(2n-1)(2n^2-2n+3)/3$ is called *centered octahedral number* or *Haüy octahedral numbers* (named after René Just Haüy, 1743-1822, France) and denoted as $(cOH)_n$. The first few centered octahedral numbers are 1, 7, 25, 63, 129, 231, 377. The *generating function* for all centered octahedral numbers is

$$\frac{x(x+1)^3}{(1-x)^4}=x+7x^2+25x^3+63x^4+\cdots.$$

As earlier, we have

$$\sum_{k=1}^{n}(cOH)_k = \frac{1}{3}n^2\left(n^2+2\right). \tag{181}$$

- From the definitions of $(cC)_n$ and $(SP)_n$ it follows that $(cC)_n + 6(SP)_{n-1} = (2n-1)\left(2n^2-2n+1\right)$. These numbers are called *Haüy rhombic dodecahedral numbers*. First few of these numbers are 1, 15, 65, 175, 369, 671. These numbers are constructed as a centered cube with a square pyramid appended to each face. The *generating function* for all Haüy rhombic dodecahedral numbers is

$$\frac{x(x+1)\left(x^2+10x+1\right)}{(1-x)^4} = x + 15x^2 + 65x^3 + 175x^4 + \cdots.$$

- Haüy also gave construction of another set of numbers involving cubes and odd square numbers, namely,

$$\begin{aligned}&(2n-1)^3 + 6\left[1^2+3^2+\cdots+(2n-3)^2\right]\\ &=(2n-1)^3 + 2(n-1)(2n-1)(2n-3)\\ &=(2n-1)\left(8n^2-14n+7\right).\end{aligned}$$

First few of these numbers are 1, 33, 185, 553, 1233. These numbers are called *Haüy rhombic dodecahedron numbers*. The *generating function* for all of these numbers is

$$\frac{x\left(1+29x+59x^2-13x^3\right)}{(1-x)^4} = x + 33x^2 + 185x^3 + 553x^4 + \cdots.$$

- From the definitions of $(OH)_n$ and $(SP)_n$ it follows that $(OH)_{3n-2} - 6(SP)_{n-1} = 16n^3 - 33n^2 + 24n - 6$. These numbers are called *truncated octahedral numbers*. First few of these numbers are 1, 38, 201, 586, 1289, 2406. These numbers are obtained by truncating all six vertices of octahedron. The *generating function* for all truncated octahedral numbers is

$$\frac{x\left(6x^3+55x^2+34x+1\right)}{(1-x)^4} = x + 38x^2 + 201x^3 + 586x^4 + \cdots.$$

25. Pentagonal Pyramidal Numbers $(PP)_n$

These numbers count the number of dots in pyramids built up of pentagonal numbers. First few pentagonal pyramidal numbers are 1, 6, 18, 40, 75, 126, 196, 288. In general, the *n*th pentagonal pyramidal number $(PP)_n$ is given in terms of the sum of the first *n* pentagonal numbers, *i.e.*,

$$(PP)_n = (PP)_{n-1} + \frac{n}{2}(3n-1) = 1+5+12+22+35+\cdots+\frac{n}{2}(3n-1),$$

which in view of (54) is the same as

$$(PP)_n = \frac{1}{2}n^2(n+1) = nt_n. \tag{182}$$

- The *generating function* for all pentagonal pyramidal numbers is

$$\frac{x(2x+1)}{(1-x)^4} = x + 6x^2 + 18x^3 + 40x^4 + \cdots.$$

- From (11) and (168) it follows that

$$\sum_{k=1}^{n}(PP)_k = \frac{1}{2}\sum_{k=1}^{n}\left(k^3 + k^2\right) = \frac{1}{24}n(n+1)\left(3n^2 + 7n + 2\right). \qquad (183)$$

- To find the sum of the reciprocals of all pentagonal pyramidal numbers, from (16), (17), and (39), we have

$$\sum_{k=1}^{\infty}\frac{1}{(PP)_k} = \sum_{k=1}^{\infty}\frac{2}{k^2(k+1)} = 2\sum_{k=1}^{\infty}\frac{1}{k^2} - 2\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{\pi^2}{3} - 2. \qquad (184)$$

- Similar to (174) it follows that

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(PP)_k} = \sum_{k=1}^{\infty}\frac{2(-1)^{k-1}}{k^2(k+1)} = \frac{\pi^2}{6} + 2 - 4\ln 2 \approx 0.8723453446. \qquad (185)$$

26. Hexagonal Pyramidal Numbers $(HP)_n$

These numbers count the number of dots in pyramids built up of hexagonal numbers. First few pentagonal pyramidal numbers are 1, 7, 22, 50, 95, 161, 252, 372. In general, the nth hexagonal pyramidal number $(HP)_n$ is given in terms of the sum of the first n hexagonal numbers, *i.e.*,

$$(HP)_n = (HP)_{n-1} + n(2n-1) = 1 + 6 + 15 + 28 + 45 + \cdots + n(2n-1),$$

which in view of (61) is the same as

$$(HP)_n = \frac{1}{6}n(n+1)(4n-1) = \frac{1}{3}(4n-1)t_n. \qquad (186)$$

- The *generating function* for all hexagonal pyramidal numbers is

$$\frac{x(3x+1)}{(1-x)^4} = x + 7x^2 + 22x^3 + 50x^4 + \cdots.$$

- From (11) and (168) it follows that

$$\sum_{k=1}^{n}(HP)_k = \frac{1}{6}\sum_{k=1}^{n}\left(4k^3 + 3k^2 - k\right) = \frac{1}{6}n^2(n+1)(n+2). \qquad (187)$$

- To find the sum of the reciprocals of all hexagonal pyramidal numbers, we follow as in (177) and (184), to obtain

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{(HP)_k} &= 6\sum_{k=1}^{\infty}\left(-\frac{1}{k} + \frac{1}{5(k+1)} + \frac{16}{5(4k-1)}\right)\\ &= \frac{6}{5}(12\ln 2 - 2\pi - 1) \approx 1.2414970314.\end{aligned} \qquad (188)$$

- Similarly, we have

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(HP)_k} &= 6\sum_{k=1}^{\infty}(-1)^{k-1}\left(-\frac{1}{k} + \frac{1}{5(k+1)} + \frac{16}{5(4k-1)}\right)\\ &= \frac{6}{5}\left(1 + 4\Phi\left(-1,1,\frac{3}{4}\right)\right) - 6\ln 2 \approx 0.8892970462.\end{aligned} \qquad (189)$$

In (189), the function Φ is the Dirichlet beta function.

27. Generalized Pentagonal Pyramidal Numbers $(GPP)_n$

These numbers count the number of dots in pyramids built up of generalized pentagonal numbers. First few generalized pentagonal pyramidal numbers are 1, 8, 27, 64, 125, 216, 343, 512. In general, the *n*th generalized pentagonal pyramidal number $(GPP)_n$ is given in terms of the sum of the first *n* generalized pentagonal numbers, *i.e.*,

$$(GPP)_n = (GPP)_{n-1} + [1+3n(n-1)] = 1+7+19+37+\cdots+[1+3n(n-1)],$$

which in view of (68) is the same as

$$(GPP)_n = n^3. \tag{190}$$

Thus, generalized pentagonal pyramidal numbers are the same as cubic numbers.

28. Heptagonal Pyramidal Numbers $(HEPP)_n$

These numbers count the number of dots in pyramids built up of heptagonal numbers. First few heptagonal pyramidal numbers are 1, 8, 26, 60, 115, 196, 308, 456. In general, the *n*th heptagonal pyramidal number $(HEPP)_n$ is given in terms of the sum of the first *n* heptagonal numbers, *i.e.*,

$$(HEPP)_n = (HEPP)_{n-1} + \frac{n}{2}(5n-3) = 1+7+18+34+\cdots+\frac{n}{2}(5n-3),$$

which in view of (78) is the same as

$$(HEPP)_n = \frac{1}{6}n(n+1)(5n-2) = \frac{1}{3}t_n(5n-2). \tag{191}$$

- The *generating function* for all heptagonal pyramidal numbers is

$$\frac{x(4x+1)}{(1-x)^4} = x + 8x^2 + 26x^3 + 60x^4 + \cdots.$$

- From (2), (11), and (168) it follows that

$$\sum_{k=1}^{n}(HEPP)_k = \frac{1}{24}n(n+1)(n+2)(5n-1). \tag{192}$$

- The sum of reciprocals of all heptagonal pyramidal numbers appears as

$$\sum_{k=1}^{\infty}\frac{1}{(HEPP)_k} = \frac{15}{14}\left(2\ln 5 - \frac{4}{5} - \pi\sqrt{1-\frac{2\sqrt{5}}{5}} + (\sqrt{5}+1)\ln\left(\sqrt{\frac{5-\sqrt{5}}{2}}\right) - (\sqrt{5}-1)\ln\left(\sqrt{\frac{5+\sqrt{5}}{2}}\right)\right) \approx 1.2072933193. \tag{193}$$

- Similarly, as in (189), we have

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(HEPP)_k} = \frac{3}{7}\left(2 + 5\Phi\left(-1,1,\frac{3}{5}\right) - 9\ln 2\right) \approx 0.9023419344. \tag{194}$$

29. Octagonal Pyramidal Numbers $(OP)_n$

These numbers count the number of dots in pyramids built up of octagonal numbers. First few octagonal pyramidal numbers are 1, 9, 30, 70, 135, 231, 364, 540. In general, the *n*th octagonal pyramidal number $(OP)_n$ is given in terms of the sum of the first *n* octagonal numbers, *i.e.*,

$$(OP)_n = (OP)_{n-1} + n(3n-2) = 1+8+21+40+\cdots+n(3n-2),$$

which in view of (88) is the same as

$$(OP)_n = \frac{1}{2}n(n+1)(2n-1) = t_n(2n-1). \tag{195}$$

- The *generating function* for all octagonal pyramidal numbers is

$$\frac{x(5x+1)}{(1-x)^4} = x + 9x^2 + 30x^3 + 70x^4 + \cdots.$$

- From (2), (11), and (168) it follows that

$$\sum_{k=1}^{n} (OP)_k = \frac{1}{12}n(n+1)(n+2)(3n-1). \tag{196}$$

- The sum of reciprocals of all heptagonal pyramidal numbers appears as

$$\sum_{k=1}^{\infty} \frac{1}{(OP)_k} = \frac{2}{3}(4\ln 2 - 1) \approx 1.1817258148. \tag{197}$$

- Similarly, we find

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(OP)_k} = \frac{2}{3}(1+\pi-4\ln 2) \approx 0.9126692876. \tag{198}$$

30. Nonagonal Pyramidal Numbers $(NP)_n$

These numbers count the number of dots in pyramids built up of nonagonal numbers. First few nonagonal pyramidal numbers are 1, 10, 34, 80, 155, 266, 420, 624. In general, the *n*th nonagonal pyramidal number $(NP)_n$ is given in terms of the sum of the first *n* nonagonal numbers, *i.e.*,

$$(NP)_n = (NP)_{n-1} + \frac{n}{2}(7n-5) = 1+9+24+46+\cdots+\frac{n}{2}(7n-5),$$

which in view of (99) is the same as

$$(NP)_n = \frac{1}{6}n(n+1)(7n-4) = \frac{1}{3}t_n(7n-4). \tag{199}$$

- The *generating function* for all nonagonal pyramidal numbers is

$$\frac{x(6x+1)}{(1-x)^4} = x + 10x^2 + 34x^3 + 80x^4 + \cdots.$$

- From (2), (11), and (168) it follows that

$$\sum_{k=1}^{n} (NP)_k = \frac{1}{24}n(n+1)(n+2)(7n-3). \tag{200}$$

- The sum of reciprocals of all heptagonal pyramidal numbers appears as

$$\sum_{k=1}^{\infty}\frac{1}{(NP)_k}=\frac{3}{88}\left(33-28\Psi\left(-\frac{4}{7}\right)-28\gamma\right)\approx 1.6184840638. \quad (201)$$

- Similarly, we find

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(NP)_k}=\frac{3}{22}\left(4+7\Phi\left(-1,1,\frac{3}{7}\right)-15\ln 2\right)\approx 0.9210386965. \quad (202)$$

31. Decagonal Pyramidal Numbers $(DP)_n$

These numbers count the number of dots in pyramids built up of decagonal numbers. First few decagonal pyramidal numbers are 1, 11, 38, 90, 175, 301, 476, 708. In general, the n-th decagonal pyramidal number $(DP)_n$ is given in terms of the sum of the first n decagonal numbers, *i.e.*,

$$(DP)_n=(DP)_{n-1}+n(4n-3)=1+10+27+52+\cdots+n(4n-3),$$

which in view of (111) is the same as

$$(DP)_n=\frac{1}{6}n(n+1)(8n-5)=\frac{1}{3}t_n(8n-5). \quad (203)$$

- The *generating function* for all decagonal pyramidal numbers is

$$\frac{x(7x+1)}{(1-x)^4}=x+11x^2+38x^3+90x^4+\cdots.$$

- From (2), (11), and (168) it follows that

$$\sum_{k=1}^{n}(DP)_k=\frac{1}{6}n(n+1)(n+2)(2n-1). \quad (204)$$

- The sum of reciprocals of all decagonal pyramidal numbers appears as

$$\sum_{k=1}^{\infty}\frac{1}{(DP)_k}=\frac{6}{325}\left(39-40\Psi\left(-\frac{5}{8}\right)-40\gamma\right)\approx 1.1459323453. \quad (205)$$

- Similarly, we find

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(DP)_k}=\frac{6}{65}\left(5+8\Phi\left(-1,1,\frac{3}{8}\right)-18\ln 2\right)\approx 0.9279541642. \quad (206)$$

32. Tetrakaidecagonal Pyramidal Numbers $(TETP)_n$

These numbers count the number of dots in pyramids built up of tetrakaidecagonal numbers. First few tetrakaidecagonal pyramidal numbers are 1, 15, 54, 130, 255, 441, 700, 1044, 1485. In general, the n-th tetrakaidecagonal pyramidal number $(TETP)_n$ is given in terms of the sum of the first n tetrakaidecagonal numbers, *i.e.*,

$$(TETP)_n=(TETP)_{n-1}+n(6n-5)=1+14+39+76+\cdots+n(6n-5),$$

which in view of (122) is the same as

$$(TETP)_n=\frac{1}{2}n(n+1)(4n-3)=t_n(4n-3). \quad (207)$$

- The *generating function* for all tetrakaidecagonal pyramidal numbers is

$$\frac{x(11x+1)}{(1-x)^4} = x + 15x^2 + 54x^3 + 130x^4 + \cdots.$$

- From (2), (11), and (168) it follows that

$$\sum_{k=1}^{n} (TETP)_k = \frac{1}{6} n(n+1)(n+2)(3n-2). \quad (208)$$

- The sum of reciprocals of all tetrakaidecagonal pyramidal numbers appears as

$$\sum_{k=1}^{\infty} \frac{1}{(TETP)_k} = \frac{2}{21}(2\pi + 12\ln 2 - 3) \approx 1.1048525213. \quad (209)$$

- Similarly, we find

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(TETP)_k} = \frac{2}{21}\left(3 + 4\Phi\left(-1, 1, \frac{1}{4}\right) - 10\ln 2\right) \approx 0.9466758087. \quad (210)$$

33. Stella Octangula Numbers $(SO)_n$

The word octangula for eight-pointed star was given by Johannes Kepler (1571-1630, Germany) in 1609. Stella octangula numbers count the number of dots in pyramids built up of star numbers. These numbers also arise in a parametric family of instances to the crossed ladders problem in which the lengths and heights of the ladders and the height of their crossing point are all integers. The ratio between the heights of the two ladders is a stella octangula number. First few stella octangula numbers are 1, 14, 51, 124, 245, 426, 679, 1016, 1449. In general, the *n*th stella octangula number $(SO)_n$ is given in terms of the sum of the first *n* star numbers, *i.e.*,

$$(SO)_n = (SO)_{n-1} + [1 + 6n(n-1)] = 1 + 13 + 37 + 73 + \cdots + [1 + 6n(n-1)],$$

which in view of (162) is the same as

$$(SO)_n = n(2n^2 - 1) = (OH)_n + 8T_{n-1}. \quad (211)$$

- The only known square stella octangula numbers are 1 and $9653449 = 3107^2 = (SO)_{169}$, see Conway and Guy [30].
- The *generating function* for all stella octangula numbers is

$$\frac{x(x^2 + 10x + 1)}{(1-x)^4} = x + 14x^2 + 51x^3 + 1124x^4 + \cdots.$$

- From (2) and (168) it follows that

$$\sum_{k=1}^{n} (SO)_k = \frac{1}{2} n(n+1)(n^2 + n - 1). \quad (212)$$

- The sum of reciprocals of all stella octangula numbers appears as

$$\sum_{k=1}^{\infty} \frac{1}{(SO)_k} = -\frac{1}{2}\left(2\gamma + \Psi\left(-\frac{1}{\sqrt{2}}\right) + \Psi\left(\frac{1}{\sqrt{2}}\right)\right) \approx 1.1114472084. \quad (213)$$

- Similarly, we find

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(SO)_k}=\frac{1}{2}\left(\Phi\left(-1,1,1+\frac{1}{\sqrt{2}}\right)+\Phi\left(-1,1,1-\frac{1}{\sqrt{2}}\right)-2\ln 2\right)$$
$$\approx 0.942739143439. \tag{214}$$

34. Biquadratic Numbers $(BC)_n$

A biquadratic number can be written as a product of four equal factors of natural numbers. Thus, 1, 16, 81, 256, 625, 1296, 2401, 4096 are first few biquadratic numbers.

- Last digit of a biquadratic number can only be 0 (in fact 0000), 1, 5 (in fact 0625), or 6.
- The n-th biquadratic number is the sum is the sum of the first n Haüy rhombic dodecahedral numbers. Indeed, from (2), (11), and (168) it follows that

$$\sum_{k=1}^{n}(2k-1)\left(2k^2-2k+1\right)=n^4.$$

- Fermat's Last Theorem confirms that a fourth power cannot be the sum of two other fourth powers, for details see Agarwal [14]. In 1770, Edward Waring (1736-1798, England) conjectured (known as Waring's problem) that every positive integer can be expressed as the sum of at most 19 fourth powers, and every integer larger than 13,792 can be expressed as the sum of at most 16 fourth powers. In 1769, Euler conjectured that a fourth power cannot be written as the sum of three fourth powers, but in 1988 [34], Noam David Elkies (born 1966, USA) disproved Euler's conjecture. Out of infinite number of possible counterexamples the following of Elkies is notable

$$20615673^4=18796760^4+15365639^4+2682440^4.$$

- The *generating function* for all biquadratic numbers is

$$\frac{x\left(x^3+11x^2+11x+1\right)}{(1-x)^5}=x+16x^2+81x^3+256x^4+\cdots.$$

- From (2), (11), (168), and the identity

$$(n+1)^5-1=\sum_{k=1}^{n}\left[(k+1)^5-k^5\right]=5\sum_{k=1}^{n}k^4+\sum_{k=1}^{n}\left(10k^3+10k^2+5k+1\right)$$

it follows that

$$n^5+5n^4+10n^3+10n^2+5n=5\sum_{k=1}^{n}k^4+\frac{5}{2}n^4+\frac{25}{3}n^3+10n^2+\frac{31}{6}n$$

and hence

$$\sum_{k=1}^{n}k^4=\frac{1}{30}n(n+1)(2n+1)\left(3n^2+3n-1\right)=\frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n. \tag{215}$$

- The following identity due to Abu-Ali al-Hassan ibn al-Hasan ibn al-Haitham (965-1039, Iraq) combines the sum of numbers raised to the power of four with different sums of numbers raised to the power of three

$$\sum_{k=1}^{n} k^4 = \left(\sum_{k=1}^{n} k^3\right)(n+1) - \sum_{k=1}^{n}\left(\sum_{j=1}^{k} j^3\right). \quad (216)$$

- From (11), (15), (168), and (215) it follows that

$$\sum_{k=1}^{n} t_k^2 = \frac{1}{30} t_n \left(3n^3 + 12n^2 + 13n + 2\right) = \frac{1}{10}\left(\sum_{k=1}^{n} t_k\right)\left(3n^2 + 6n + 1\right). \quad (217)$$

- To find the sum of reciprocals of all biquadratic numbers, we shall use the derivation of (39). First in the two expansions of $(\sin x)/x$, we compare the coefficients of x^4, to get

$$\frac{\pi^4}{5!} = \sum_{\substack{p,q\in\mathbb{N}\\ p\neq q}} \frac{1}{p^2 q^2}. \quad (218)$$

Now squaring (39), to obtain

$$\sum_{k=1}^{\infty} \frac{1}{k^4} + 2\sum_{\substack{p,q\in\mathbb{N}\\ p\neq q}} \frac{1}{p^2 q^2} = \frac{\pi^4}{36},$$

which in view of (218) gives

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{36} - 2\frac{\pi^4}{5!} = \frac{\pi^4}{90}. \quad (219)$$

- From (219) it immediately follows that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^4} = \frac{7\pi^4}{720}. \quad (220)$$

35. Pentatope Numbers $(PTOP)_n$

The fifth cell of any row of Pascal's triangle starting with the 5-term row 1 4 6 4 1, either from left to right or from right to left are defined as pentatope numbers. First few these numbers are 1, 5, 15, 35, 70, 126, 210, 330, 495. Thus, the n-th pentatope number is defined as

$$(PTOP)_n = \binom{n+3}{4} = \frac{1}{24} n(n+1)(n+2)(n+3) = \frac{1}{6} t_n t_{n+2} = \frac{1}{4}(n+3)T_n. \quad (221)$$

These numbers can be represented as regular discrete geometric patterns, see Deza [3]. In biochemistry, the pentatope numbers represent the number of possible arrangements of n different polypeptide subunits in a tetrameric (tetrahedral) protein.

- Two of every three pentatope numbers are also pentagonal numbers. In fact, the following relations hold

$$(PTOP)_{3n-2} = P_{(3n^2-n)/2} \quad \text{and} \quad (PTOP)_{3n-1} = P_{(3n^2+n)/2}. \quad (222)$$

- The *generating function* for all pentatope numbers is

$$\frac{x}{(1-x)^5} = x + 5x^2 + 15x^3 + 35x^4 + \cdots.$$

- From (2), (11), (168), and (215) it follows that

$$\sum_{k=1}^{n}(PTOP)_k=\frac{1}{120}n(n+1)(n+2)(n+3)(n+4)=\frac{1}{5}(n+4)(PTOP)_n. \quad (223)$$

- As in (17), we have

$$\sum_{k=1}^{\infty}\frac{1}{(PTOP)_k}=\sum_{k=1}^{\infty}\left(\frac{8}{k(k+1)(k+2)}-\frac{8}{(k+1)(k+2)(k+3)}\right)=\frac{4}{3}. \quad (224)$$

- We also have

$$\sum_{k=1}^{\infty}\frac{(-1)^{n-1}}{(PTOP)_k}=32\ln 2-\frac{64}{3}\approx 0.8473764446. \quad (225)$$

36. Partitions by Polygonal Numbers

Recall that *general polygonal number* can be written as $p_n^r=n[(r-2)n-(r-4)]/2$, where p_n^r is the *n*th *r*-gonal number. For example, for $r=3$ it gives triangular number, and for $r=4$ gives a square number. Fermat in 1638 claimed that every positive integer is expressible as at most *k* *k*-gonal numbers (Fermat's Polygonal Number Theorem). Fermat claimed to have a proof of this result, however his proof has never been found. In 1750, Euler conjectured that every odd integer can be written as a sum of four squares in such a way that $n=a^2+b^2+c^2+d^2$ and $a+b+c+d=1$. In 1770. Lagrange proved that every positive integer can be represented as a sum of four squares, known as *four-square theorem*. For example, the number $1638=4^2+6^2+25^2+31^2=1^2+1^2+6^2+40^2$ has several different patricians, whereas for the number $1536=0^2+16^2+16^2+32^2$ this is the only partition. In 1797-8, Legendre extended the theorem in with his *three-square theorem*, by proving that a positive integer can be expressed as the sum of three squares if and only if it is not of the form $4^k(8m+7)$ for integers *k* and *m*. Later, in 1834, Jacobi gave a formula for the number of ways that a given positive integer *n* can be represented as the sum of four squares. In 1796, Gauss proved the difficult triangular case (every positive integer is the sum of three or fewer triangular numbers, which is equivalent to the statement that every number of the form $8m+3$ is a sum of three odd squares, see Duke [35]), commemorating the occasion by writing in his diary the line ΕΓΡΗΚΑ! num $=\Delta+\Delta+\Delta$, and published a proof in his book *Disquisitiones Arithmeticae* of 1798. For this reason, Gauss's result is sometimes known as the Eureka theorem. For example, $16=6+10$, $25=1+3+21$, $39=3+15+21$, $150=6+66+78$. The full polygonal number theorem was resolved finally in 1813 by Cauchy. In 1872, Henri Léon Lebesgue (1875-1941, France) proved that every positive integer is the sum of a square number (possibly 0^2) and two triangular numbers, and every positive integer is the sum of two square numbers and a triangular number. For further details, see Grosswald [36], Ewell [37] [38], and Guy [39].

37. Conclusions

Triangular numbers which are believed to have been introduced by Pythagoras himself play a dominant role in all types of figurative numbers we have addressed in this article. In fact, Equation (1) says natural number n is the difference of t_n and t_{n-1}, whereas Gauss's Eureka theorem stipulates that n can be written as the sum of three triangular numbers. Equation (32) shows that square number S_n is the sum of t_n and t_{n-1}. Equation (44) shows that square number R_n is 2 times of t_n. Relation (51) says pentagonal number P_n is $(1/3)t_{3n-1}$, whereas (53) gives $P_n = t_n + 2t_{n-1} = t_{2n-1} - t_{n-1}$. Equation (60) informs that hexagonal number H_n is the same as t_{2n-1}. Relation (67) says generalized pentagonal number $(GP)_n$ is the same as $t_1 + 6t_{n-1} = t_n + 4t_{n-1} + t_{n-2}$. Equation (77) informs that heptagonal number $(HEP)_n$ is the same as $t_n + 4t_{n-1}$. Relation (87) declares that octagonal number O_n is equal to $t_n + 5t_{n-1}$. Equation (98) implies that nonagonal number N_n is equal to $t_n + 6t_{n-1}$. Relation (110) says decagonal number D_n is the same as $t_n + 7t_{n-1}$. Equation (121) informs that tetrakaidecagonal number $(TET)_n$ is the same as $t_n + 11t_{n-1}$. Relation (133) shows that centered triangular number $(ct)_n$ is the same as $t_n + t_{n-1} + t_{n-2}$, whereas relation (137) confirms that centered square number $(cS)_n$ is equal to $t_n + 2t_{n-1} + t_{n-2}$. Equation (141) says centered pentagonal number $(cP)_n$ is equal to $t_n + 3t_{n-1} + t_{n-2}$, whereas Equation (145) tells centered heptagonal number $(cHEP)_n$ is the same as $t_n + 5t_{n-1} + t_{n-2}$. Relation (149) informs that centered octagonal number $(cO)_n$ is the same as $t_n + 6t_{n-1} + t_{n-2}$, whereas (153) shows centered nonagonal number $(cN)_n$ is the same as $t_n + 7t_{n-1} + t_{n-2}$, and relation (157) tells centered decagonal number $(cD)_n$ is the same as $t_n + 8t_{n-1} + t_{n-2}$. Equations (161) shows that star number $(ST)_n$ is the same as $t_n + 10t_{n-1} + t_{n-2}$, whereas Equation (165) shows that centered tetrakaidecagonal number $(cTET)_n$ is the same as $t_n + 12t_{n-1} + t_{n-2}$. Relation (168) shows that the sum of the first n cubic numbers is the same as t_n^2. Equation (171) shows that tetrahedral number T_n is the same as $(1/3)(n+2)t_n$. Relation (175) says square pyramidal number $(SP)_n$ is the same as $(1/6)(n+1)t_{2n}$. From the definition of octahedral numbers and (175) it follows that $(OH)_n = (1/6)(nt_{2n-2} + (n+1)t_{2n})$. From the relation (182) it follows that pentagonal pyramidal number $(PP)_n$ is the same as nt_n. Equation (186) says hexagonal pyramidal number $(HP)_n$ is the same as $(1/3)(4n-1)t_n$. From Equation (191) it follows that heptagonal pyramidal number $(HEPP)_n$ is the same as $(1/3)(5n-2)t_n$. Equation (195) suggests that octagonal pyramidal number $(OP)_n$ is the same as $(2n-1)t_n$. Relation (199) tells nonagonal pyramidal number $(NP)_n$ is the same as $(1/3)(7n-4)t_n$. Equation (203) informs that decagonal pyramidal number $(DP)_n$ is the same as $(1/3)(8n-5)t_n$, whereas relation (207) indicates that tetrakaidecagonal pyramidal number $(TETP)_n$ is the same as $(4n-3)t_n$. Thus, in conclusion almost all figurative numbers we have studied are directly related with triangular numbers.

Acknowledgements

The author is grateful to Professor Chan Heng Huat, National University of

Singapore, for his generous guidance and encouragement in the preparation of this article. He is also thankful to Professor Chao Wang and referees for their comments.

Authors Contribution

Not applicable.

Funding

This work received no external funding.

Institutional Review Board Statement

Not applicable.

Informed Consent Statement

Not applicable.

Data Availability Statement

Not applicable.

Conflicts of Interest

There is no conflict of interest.

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