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An "ab initio" Model for Quantum Theory and Relativity

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Abstract

The paper introduces a theoretical model aimed to show how the relativity can be made consistent with the non reality and non locality of the quantum physics. The concepts of quantization and superposition of states, usually regarded as distinctive properties of the quantum world, can be extended also to the relativity.

Keywords

Quantum Physics, Relativity, Cosmology

1. Introduction

The quantum theory and the relativity have stimulated influential ideas and experimental efforts to investigate and understand a huge number of natural phenomena from atomic to cosmic scale [1] [2]. However, with space ranges spreading from ~10^{-18} m to ~10^{26} m by about 44 orders of magnitude, is comprehensibly problematic the attempt to unify in the frame of a unique theory the whole variety of related natural phenomena. Yet, is symptomatic the fact that similar difficulties often arise even in formulating a more selective class of specific physical problems. In the case of the relativity, for example, something relevant should be still missing even at the mere cosmic scale; despite the great amount of its previsions and discoveries, remain problematic crucial topics like the progressive acceleration of universe expansion, the MOND (modified Newton), the dark matter and dark energy. A possible hint to overcome these difficulties is to identify an appropriate background of ideas that integrate or modify the preexisting ones; for example, at the quantum level, the major problem of the relativity is its link to the non-reality and non-locality of the quantum theory [3]. Also, the "handwritten" cosmological constant reluctantly introduced by Einstein after the Hubble experimental hint, is a further example of necessary revision of the general relativity even in its most representative cultural frame *i.e.* the cosmology. Is reasonable the suspect that focusing greater attention on the existing conceptual background is not an additional difficulty but a possible solution? To provide a contribution to this problem, the paper [4] has introduced an operative definition of space time

$$\frac{\hbar G}{c^2} \tag{1.1}$$

that implies as a corollary the statistical formulation of quantum uncertainty

$$\delta x \delta p_x = n\hbar = \delta\varepsilon\delta t, \tag{1.2}$$

being n an arbitrary integer. The purpose of this paper is to examine the physical information deductible by the definition (1.1): as it merges $\hbar$ and G/c^2, in principle, it seems reasonably valuable for quantum and relativistic implications. The most intuitive way to convert (1.1) into an effective equation allowing successive calculations is to introduce explicitly its physical dimensions

$$\frac{\hbar G}{c^2} = \frac{length^3}{time}, \tag{1.3}$$

which also calculates

$$\frac{\hbar^2 G}{c^2} = 8.2\times10^{-96}\ \mathrm{J\cdot m^3} \qquad \frac{\hbar}{time} = \epsilon. \tag{1.4}$$

The second (1.4) linked to (1.2) defines $\delta\varepsilon = \epsilon' - \epsilon''$, where ϵ' and ϵ'' are two arbitrary boundary energies, whereas the right hand side implies $\delta\varepsilon = n\hbar\omega$ with $\omega = \delta t^{-1}$ and $\varepsilon' \le n\hbar\omega \le \varepsilon''$; this means that $\hbar\omega$ is an energy in the range of allowed values $n\hbar\omega$ falling within $\delta\varepsilon$ with $n = 1, 2, \cdots$. The first equality (1.2) reads $2\pi\delta x = nh/\delta p = n\delta\left(\lambda^{-1}\right)$, having put $\delta x = x' - x''$ and $h\delta\left(p^{-1}\right) = h/p' - h/p'' = \lambda'^{-1} - \lambda''^{-1}$ coherently with $\delta\varepsilon$; then to any $x' \le x \le x''$ corresponds $h/\lambda' \le p \le h/\lambda''$ with $p = h/\lambda$ and that $2\pi x = n\lambda$ as well. Hence (1.2) summarizes contextually three fundamental statements of quantum physics. Now the crucial task of the present physical model is how to define specifically *length* and *time* of (1.3) to infer physical information. To this aim, (1.4) will be implemented via some fundamental parameters of the universe [5]

$$\begin{gathered} t_u^{estim} = 4.35\times10^{17}\ \mathrm{s}, \quad r_u^{estim} = 4.35\times10^{26}\ \mathrm{m}, \\ \Lambda = 1.9\times10^{-35}\ \mathrm{s^{-2}}, \quad H_u = 2.2\times10^{-18}\ \mathrm{s^{-1}}, \end{gathered} \tag{1.5}$$

i.e. the estimated age and radius of the Universe, from now on quoted with shortened notation t_u and r_u, the Einstein cosmological factor Λ and the Hubble constant H_u. Two quantities of interest are the mass m_{ob} detectable in the universe counting the stars only [5] and the critical density ρ_{cr} of Friedman equations [6]

$$m_{ob} = 3\times10^{52}\ \mathrm{kg}, \quad \rho_{cr} = \frac{3H_u^2}{8\pi G} = 8.6\times10^{-27}\ \mathrm{kg/m^3}. \tag{1.6}$$

Of course these numerical data must be intended as today's values. The mass m_{ob} is interesting, although approximated by defect for two reasons: because it refers to stars only and because it concerns by definition stars whose light has in fact reached us during the life time of the Universe.

The present theoretical model implements systematically uncertainty ranges to calculate quantum and relativistic quantities according to the logical step "local values $\rightarrow$ uncertainty ranges". From a formal point of view this statement can be acknowledged reminding the standard concept of measure errors: just as no one trusts the reliability of a single value measured in its experimental error bar, likewise (1.2) waive the signification of a local dynamical variable in its uncertainty range. Yet the true physical meaning of this replacement is one among the crucial points of the model, as it will be more thoroughly shown below; some examples of calculated results are also reported in the Section 6 to confirm the concepts exposed in the Sections 1 to 5. The uncertainty ranges are defined via the standard notation

$$\delta(any\ \ function\ \ f) = f'' - f', \tag{1.7}$$

being f'' and f' the range boundaries defined by two arbitrary values allowed to the concerned function; these values are arbitrary, unknown and unknowable by definition of quantum uncertainty. In general, both of them can be variables or constants. As δx implies x variable in an appropriate interval of values $x' \le x \le x''$, the dynamical variable x is assumed to take a range of random values between arbitrary boundaries x' and x''. Also, since the symbol δ introduces the meaning of change by definition, it also indicates the differential of f through the formal identity $f'' = f' + \delta f$; then the further identity

$$f'' = f' + \frac{\delta f}{\delta x}\delta x, \quad \delta x = x'' - x', \quad x' \le x \le x'' \tag{1.8}$$

introduces the ratio $\delta f/\delta x$ that in turn takes physical meaning under appropriate conditions. A further way to implement the ranges is that already highlighted about $\delta\varepsilon$, *i.e.* $n'\hbar\omega \le n\hbar\omega \le n''\hbar\omega$, which means

$$n' \le n \le n'', \tag{1.9}$$

being of course n' and n'' arbitrary and unknown integers. In the following the shortened notations δx^2 and $\delta(x)^2$ mean respectively $(\delta x)^2 = (x'' - x')^2$ and $(x'')^2 - (x')^2$. Eventually note that in principle both signs are allowed for any range; for example nothing hinders that $\delta x = x - x_0$ is defined by $x_0 \lesseqgtr 0$, so that $\delta x = x \pm x_0 = \pm|x \pm x_0|$ being both x and x_0 arbitrary. Sometimes in the following text a given result is obtained more than once in different contexts: this is not a redundant repetition, rather it must be intended as a check confirming that all conceptual steps progressively exposed are consistently linked each other. Despite the agnostic way to introduce (1.2), the remainder of this paper is able to formulate a self consistent theoretical physical model. The text is exposed in order to be as self contained as possible.

2. Preliminary Considerations

This section introduces some considerations having general character of straightforward corollaries of (1.1) to demonstrate that this definition of space time is physically sensible. All concepts introduced below are listed sequentially without calculations, while emphasizing their physical meaning; the validity of the various formulas inferred through the model will be concerned in the next section 6.

2.1. Energy and Energy Density in the Space Time

Implement first the dimensional analysis of (1.1), (1.3) and (1.2) defining $\hbar^2 G/c^2 = \epsilon\ell^3$ and $\eta = \epsilon/\ell^3$, where ϵ stands for energy and ℓ for length to introduce the concept of energy density η. Then multiplying and dividing side by side these two equations one finds

$$\frac{\hbar^2 G/c^2}{\eta} = \frac{\epsilon\ell^3}{\epsilon/\ell^3} = \ell^6, \quad \frac{\hbar^2 G}{c^2}\eta = \epsilon^2, \tag{2.1}$$

having identified ϵ pertinent to η with that defined by the second equation. Equation (2.1) yields two equations. Owing to (1.4) the first one is

$$\eta = \left(\frac{\epsilon c}{\hbar}\right)^2 G^{-1} = \left(\frac{c}{\tau}\right)^2 G^{-1}, \quad \tau = \frac{\hbar}{\epsilon}, \tag{2.2}$$

according which the value of η depends on that of ϵ via fundamental constants only: *i.e.* $\eta = \eta(t)$ if $\epsilon = \epsilon(t)$, whereas instead $\eta = const$ if $\varepsilon = const'$. It is necessary to make (2.2) consistent with (1.2), *i.e.* to regard ϵ as an uncertainty range $\epsilon - 0$; as the boundaries of any range are arbitrary, in this particular case it means considering the energies $0 \le \epsilon' \le \epsilon$ enclosed in the given boundaries. So $\eta = 0$ would be the deterministic value of energy density related to $\eta = 0$, whereas instead the actual value of $\eta = \eta - 0$ corresponds to the range size $\epsilon - 0$. Also, the dimensional relationships (2.1) yield the following equations

$$\delta\ell = \left(\frac{\hbar^2 G/c^2}{\eta}\right)^{1/6} = \left(\frac{\hbar G}{c^2}\delta t\right)^{1/3}, \quad \eta = \eta(\delta t), \quad \delta t \ne 0, \tag{2.3}$$

being δt an appropriate time range corresponding to the space range $\delta\ell$; the second equality is nothing else but the definition (1.3) of space time multiplied by δt. Let δt be equal in general to $\tau + t - t_0$, being the reference time t_0 an arbitrary constant and τ a further time constant. From a formal point of view, this is by definition the uncertainty time lapse $t - t_1$ with t variable and $t_1 = t_0 - \tau = const_t$. The corresponding space range $\delta\ell = \ell - \ell_1$, with $\ell_1 = const_\ell$, is such that dividing both sides by δt one finds

$$\frac{\delta\ell}{\delta t} = \delta\dot{\ell} = \left(\frac{\hbar G}{c^2}\right)^{1/3} \delta t^{-2/3} \tag{2.4}$$

and thus also

$$\delta\ell = \pm\sqrt{\frac{\hbar G}{\delta\dot{\ell} c^2}};\tag{2.5}$$

clearly this result merely rewrites (1.3) as $\delta\ell^3/\delta t$. Note that $\delta\dot{\ell}$ has physical dimensions of velocity; so, regarding $\delta\dot{\ell} \le c$, at any t (2.5) yields in particular

$$\delta\ell_{min} = \pm\sqrt{\frac{\hbar G}{c^3}} = \pm\ell_P.\tag{2.6}$$

The Planck length appears to be the smallest space range physically inferable through the definition (1.1) of space time, whereas the Planck time and energy can be nothing else but $t_P = \ell_P/c$ and $\epsilon_P = \hbar/t_P$ according to (1.4), whence the Planck mass $m_P = \epsilon_P/c^2$ too by dimensional reasons. Instead at $t = t_0$ by definition $\delta t \equiv \tau = const_t$ and thus $\delta\ell = const_\ell$; so (2.4) turns into

$$\left.\frac{\delta\ell}{\delta t}\right|_{t=\tau} = \frac{const_\ell}{const_\tau} = \left(\frac{\hbar G}{c^2}\right)^{1/3}\tau^{-2/3} = \left(\frac{\hbar G}{c^2\tau^2}\right)^{1/3}.\tag{2.7}$$

The left hand side introduces a velocity by definition constant. In particular, $\tau = t_P$ is consistent with the ratio of constants $const_\ell/const_t = c$, in analogy with $(\epsilon_0\mu_0)^{-1/2}$ of the classical electrodynamics.

2.2. Classical Newton law

Since $\epsilon c/\hbar$ in (2.2) has physical dimensions of acceleration, write thus

$$acceleration = \frac{\epsilon c}{\hbar} = 2.9\times 10^{42}\,\epsilon\ \mathrm{m/s^2};\tag{2.8}$$

then it must be possible to define via dimensional reasons also the related velocity, force and energy

$$\delta t\frac{\epsilon c}{\hbar} = velocity,\quad m\frac{\epsilon c}{\hbar} = force,\quad m\delta\ell\frac{\epsilon c}{\hbar} = energy;\tag{2.9}$$

both δt and $\delta\ell$ are arbitrary because they are introduced without hypothesis or conceptual constrain. Also,

$$\frac{\epsilon c}{\hbar} = \frac{\delta v}{\delta t} = \dot{v},\quad v = \frac{\delta\ell}{\delta t},\tag{2.10}$$

whereas holds the dimensional relationship $m\delta\ell\epsilon = mass^2 length^3/time^2$. As this latter defines in turn at the right hand side the physical dimensions of $G\times mass^3$, it is possible to write $mass\delta\ell\epsilon = Gmass^3$ *i.e.*

$$\epsilon = \pm G\frac{mass^3}{mass\times length} = \pm G\frac{m^2}{\delta\ell} = \pm G\frac{m_1 m_2}{\delta\ell};\tag{2.11}$$

the double sign being in agreement with (2.5). The energy ϵ is here identifiable with the analytical form of the classical Newton law, if $\delta\ell$ is regarded as the uncertainty range corresponding to the random distance between m_1 and m_1. First of all, with this available dimensional information only, it is impossible to identify the respective role of either mass; otherwise stated, as m_1 and m_2 were both formally inferred from a unique m^2, the quantum uncertainty requires that inertial and gravitational mass are physically equivalent. Moreover nothing is known in fact about $\delta\ell$, introduced in (2.1) simply as an arbitrary dimensional length. This suggests that actually (2.11), quoted here only as a preliminary check of (2.9), is the classical formula of a more general force, which in fact will be considered again in the next section 4. Yet it worth emphasizing since now that (2.11) is conceptually different from Gm_1m_2/r, seemingly analogous but actually wrong: indeed this latter implies an instantaneous action at distance *r*, whereas (2.11) implies the finite propagation time of the gravitational interaction through the range $\delta\ell$. This point will be concerned later, it is enough to anticipate here that the space range $\delta\ell$ contains inherently time information according to (1.2). The chance of defining $\dot{v}$ in (2.10) is justified in principle via (1.2) introducing first $v=|\mathbf{v}|$ as

$$\frac{\delta x}{\delta t}=v=\frac{\delta\varepsilon}{\delta p};\tag{2.12}$$

this link implies that in general the size of all uncertainty ranges defining v are time dependent themselves. Thus is sensible in principle $\dot{v}$ anticipated in (2.10). The fact that the boundary coordinates of δx are arbitrary does not exclude for example $\delta x=x_2(t)-x_1(t)$; so the uncertainty allows defining the possible time dependence of velocity modulus as change rate of $\delta x=\delta x(\delta t)$ as a function of time lapse δt. The way (2.12) of expressing (1.2) is significant for at least four reasons: (i) $\mathbf{v}$ is in fact four-vector because, being defined in (2.10) via uncertainty ranges, it must be regarded according to (1.2). (ii) It agrees in principle with the idea of replacing the concept of derivative with that of ratio of uncertainty ranges, as it will be more thoroughly confirmed in the next sections 2.6 and 4. (iii) Despite the quantum nature of (1.2), the gravity fits in a natural way the present model based on (1.1). (iv) Equation (1.2) requires that v must be upper bound. Indeed, consider any local momentum p included in a range of allowed values $p_0\le p\le p_1$; if v would tend to infinity then $\delta\varepsilon$ related to $v\delta p$ should be compatible with an infinite local energy, whereas instead $\epsilon_0\le\epsilon\le\epsilon_1$ correspond to a finite range of values allowed to the momentum. This absurd conclusion, *i.e.* finite $\delta\varepsilon$ and thus δp for infinite $v\delta p$ because of v, requires an upper finite value c of v in agreement with (2.7). This property of c is a corollary of (1.2), not a postulate; as such it must hold in any reference system.

2.3. Quantum Uncertainty and Space Time

The energy density (2.2) is inherent the concept of space time according to its own origin (1.3) and has several implications, first of all the existence of a pressure P_{st} internal to space time volume δx^3 previously symbolized as $length^3$; the subscript means *space time* to emphasize that it is inherently based on the definition (1.1) only. As sketched in the Appendix A [4], one finds

$$P_{st}=\xi\eta_{st},\quad \xi=\frac{1}{3},\frac{2}{3},1.\tag{2.13}$$

For example, assuming a spherical volume of space time of radius r_{st} crossed by a diametric light beam completely absorbed at its internal boundary, one calculates an internal outwards force

$$F_{st} = 4\pi r_{st}^2 \xi \eta_{st}, \quad \xi = \frac{1}{3}. \tag{2.14}$$

Moreover (2.2) reads according to the first (2.9)

$$\eta = \frac{velocity^2}{\delta t^2 G} = \frac{a^2}{G}, \quad a = \frac{velocity}{\delta t}, \tag{2.15}$$

so that $velocity \le c$ implies

$$\eta \le \frac{c^2}{\delta t^2 G}; \tag{2.16}$$

hence the first (2.9) reads

$$\epsilon = \frac{1}{\mathrm{n}} \frac{\hbar}{\delta t}, \quad \mathrm{n} = \frac{c}{velocity} \ge 1, \tag{2.17}$$

which requires in turn

$$\epsilon^* > \frac{\hbar}{\delta t}, \quad \epsilon^* = \mathrm{n}\epsilon. \tag{2.18}$$

This result compares the time range δt and the energy ϵ^* to it related via $\hbar$ only for any physical reason. Is remarkable the fact that this familiar inequality of quantum mechanics is here consequence of the other relativistic one $v \le c$, which actually reads $v < c$ for matter particles; so (2.18) supports the validity of the first (2.9). It is worth considering also the second and third dimensional equations (2.9) that read

$$force = \frac{energy}{\lambda\!\!\!^{-}_m}, \quad energy = \varepsilon \frac{\delta \ell}{\lambda\!\!\!^{-}_m}, \quad \lambda\!\!\!^{-}_m = \frac{\hbar}{mc}, \tag{2.19}$$

where $\lambda\!\!\!^{-}_m$ is the reduced Compton length of *m*. It appears reasonable to assume that the range size $\delta \ell$ is an integer number *n* of reduced Compton wavelengths, here regarded as the shortest wavelength relatable to one particle. If so, then putting

$$\delta \ell = n \lambda\!\!\!^{-}_m \tag{2.20}$$

the second (2.19) reads

$$energy = n\epsilon. \tag{2.21}$$

whereas the third (2.19) reads

$$2\pi \delta \ell = n \lambda_m, \quad \frac{h}{\lambda_m} = mc. \tag{2.22}$$

Appears here as a corollary an early postulate of the old quantum mechanics, *i.e.* the energy quantization due to the integer number *n* of steady waves allowed for a particle traveling in a closed path. This result is generalized via an arbitrary factor $\xi < 1$ in order that $\hbar/\lambda\!\!\!^{-}_m$ having physical dimensions of momentum reads

$$2\pi\delta\ell_{\xi} = n\lambda_{\xi}, \quad momentum = \frac{h}{\lambda_{\xi}} = mv_{\xi}, \quad v_{\xi} = \xi c, \quad \lambda_{\xi} = \xi\lambda_{m}. \tag{2.23}$$

All of this agrees with the results preliminarily obtained in section 1. A full paper [7] has been devoted to highlight the implications of (2.20), which in fact transfers n from its basic definition in (1.2) into the specific physical problem of a bound particle moving circularly around a central force field.

On the one hand (2.20) is justified by its direct corollaries (2.22) and (2.23), which however represent a particular case of boundary condition allowing steady wavelengths. While $\delta\ell_{\xi}$ is uniquely definable as the radius of a circumference, it must be replaced by a combination of minor and major semi-axes of an ellipse in order that (2.23) describes still the integer number of steady wavelengths along an elliptic perimeter. Are known in this respect various formulas, e.g. [8], that calculate this perimeter; the next subsection 5.7 will show how to infer through this reasoning the perihelion precession of orbiting planets. On the other hand the definition (2.23) of momentum is merely formal, being based on a dimensional assessment compliant with the condition $v < c$ via the arbitrary factor ξ. Nevertheless also in this reasoning energy quantization and De Broglie momentum are contextual. It is shown soon below that n of (2.17) is the refractive index of a dispersive medium. These considerations, crucial for the birth of the old quantum mechanics and here inferred as corollaries, suggest the necessity of defining more in detail the actual physical meaning of v.

2.4. Quantum Velocity and Space Time

The steps introduced by (2.9) are significant: whereas c in (1.1) is self-evident, it is a constant of nature, now rises the problem of clarifying further the physical meaning of velocity modulus v as a property of massive particles moving through the space time. To introduce v from a first principle, note that (2.12) yield

$$\delta\varepsilon = v\delta p = \frac{v}{c}\delta(pc), \quad v = \frac{\delta x}{\delta t};$$

since $mc \geq mv$, rewrite (2.23) according to (2.17); *i.e.* $mc = m\mathrm{n}v$ yields

$$p = mv = \frac{h}{\lambda}, \quad \mathrm{n} = \frac{c}{v} \geq 1, \quad \lambda = \mathrm{n}\lambda_{m}. \tag{2.24}$$

Introduce an arbitrary frequency ν of matter wave inherent the De Broglie momentum and write now

$$\delta\left(\frac{1}{\lambda}\right) = \delta\left(\frac{\nu}{v}\right) = \frac{1}{c}\delta(\mathrm{n}\nu) = \frac{\delta\nu}{c}\frac{\delta(\mathrm{n}\nu)}{\delta\nu}, \quad \nu = \frac{v}{\lambda},$$

so that

$$\frac{1}{\delta\nu}\delta\left(\frac{1}{\lambda}\right) = \frac{1}{c}\frac{\delta(\mathrm{n}\nu)}{\delta\nu} = \frac{1}{v_{g}}:$$

the second equality defining a new reciprocal velocity v_{g}^{-1} is justified by dimensional reasons according to the ratio at the left hand side. This yields the modulus of this velocity

$$v_g = \frac{c}{\delta(\mathrm{n}\nu)/\delta\nu}; \tag{2.25}$$

clearly v_g is the group velocity of a wave packet in a dispersive medium if $v = v(\nu)$ as a function of De Broglie wave frequency ν. Actually this holds for both $\delta \to 0$, *i.e.* the ratio of range sizes tends to the usual way to define the classical derivative for very small range sizes $\delta(\mathrm{n}\nu)$ and $\delta\nu$. Hence (2.12) is sensible.

Recall now the eq $2\pi\delta r = n\lambda$ found in (2.23), whose physical meaning introduces a crucial condition on any mass m_2 orbiting according to (2.11) around m_1 at constant distance having modulus δr: owing to the dual nature wave/corpuscle of matter, a steady m_2 wave is required to describe a stable orbiting system around m_1 at radial distance δr. Consider thus in this respect the De Broglie wave inferred in (2.24) and write

$$2\pi = \frac{n\lambda}{\delta r} = \frac{h}{\hbar}, \tag{2.26}$$

which brings, in agreement with (2.23), directly to

$$\frac{n\hbar}{\delta r} = \delta p_r = \frac{h}{\lambda} = p_\lambda. \tag{2.27}$$

At the right hand side appears the momentum p_λ of a De Broglie wave delocalized within $2\pi\delta r$. At the left hand side appears the radial momentum range δp_r of a corpuscle delocalized in δr: for example in the case of (2.11) it means that the space gap δr between the masses m_1 and m_2 implies a steady wavelength λ along the orbital path of the running mass around the rest mass, which confirms the indistinguishability of gravitational and inertial mass introduced in section 2.2. Rewrite now $\delta p_r = p_r - p_{r0} = p_\lambda - 0$ of (2.27) with vector notation, noting that δp_r is radial momentum range around the rest mass, whereas $\delta p_\lambda = p_\lambda - 0$ is momentum range size along the path of the orbiting mass in agreement with (1.9); then $\delta\mathbf{p}_r = \mathbf{p}_r - \mathbf{p}_{r0} = \mathbf{p}_\lambda$ implies $\delta\mathbf{p} = \mathbf{p}_r - \mathbf{p}_\lambda = \mathbf{p}_{r0}$. Regard first $\mathbf{p}_{r0} = 0$, as it is possible because the range sizes are arbitrary, in which case $\mathbf{p}_r = \mathbf{p}_\lambda$ reads more expressively $\mathbf{p}_\parallel = \mathbf{p}_\perp$, *i.e.* radial and tangential vectors; this result implies a circular orbit where a unique $|\delta\mathbf{r}|$ implies a unique $|\delta\mathbf{p}_r|$ and thus constant orbital $|\mathbf{p}_\lambda|$. Of course in general $p_{r0} \neq 0$, which suggests that not necessarily the orbit must be circular; even admitting δp_r constant, $p_{r0} = p_{r0}(t)$ implies p_λ variable. Even so, however, it is possible that $\delta p_r = \delta p_\lambda$ simply thinking $\delta\mathbf{p} = (\mathbf{p}_r - \mathbf{p}'_{r0}) - (\mathbf{p}_\lambda - \mathbf{p}''_{r0}) = 0$; *i.e.* $\mathbf{p}_{r0}$ has been split itself into its radial and tangential components, so that $\mathbf{p}_r$ and $\mathbf{p}_\lambda$ change in the respective ranges

$$|\delta\mathbf{p}_r| = |\delta\mathbf{p}_\lambda|: \tag{2.28}$$

this implies the momentum range conservation of the running mass regarded first via its radial delocalization momentum range δp_r and then also via its orbital delocalization momentum range δp_λ, normal to the radial distance in a circular path. Clearly this holds for the ranges $\delta \mathbf{p}_{\|}$ and $\delta \mathbf{p}_\perp$, whatever the local $\mathbf{p}_{\|}$ and $\mathbf{p}_\perp$ might be. Nonetheless two considerations support significantly the present way of reasoning. (i) From a relativistic point of view $p_{r0} = p_{r0}(\delta t)$ justifies the definition of the momentum vectors as 4-vectors owing to (1.2) and makes compatible the present conclusions even with another major effect: e.g. the orbit instability, sketched later in the subsection 5.3.4 as due to emission of gravitational waves. (ii) Examine a further consequence of (2.28) from a quantum point of view; as such this result should have a general validity, being it direct corollary of first principles. Consider a point source of particles, be they photons or matter corpuscles, and assume the tangential advancement of their wave front described by $\mathbf{p}_\lambda$ correspondingly to the radial advancement of the corpuscle beam $\mathbf{p}_r$. As the uncertainty requires considering the respective momenta consistently with the lack of deterministic trajectory, (2.28) suggests that to a wavelike radial propagation of the particle beam corresponds a wavelike propagation along a normal direction too. If a beam of corpuscles illuminates a solid plane with two slits, it is natural that the momentum $\delta \mathbf{p}_\lambda$ of running waves yields an interference pattern on a screen placed at distance $\delta \mathbf{r} = n\hbar/\delta \mathbf{p}_r$ behind the slit plane. More specifically, n controls the various distances on the interference plane with respect to the slits. In other words (2.28) anyway hold and account for the duality wave/corpuscle of matter revealed by n tangential waves that interfere on the screen simply because two slits generate two beams according to the Huygens principle. All of this is self consistent regardless of how are defined the relativistic $\mathbf{p}_{\|}$ and $\mathbf{p}_\perp$ of particles of the beam.

2.5. Relativistic Outcomes

The relativistic worth of these results follows straightforwardly, first of all because even the time is inherently involved by (1.2); multiplying side by side the first and third (2.9) one finds

$$\frac{\delta t \delta \ell \epsilon^2}{\hbar^2} m = \frac{energy \times velocity}{c^2}$$

and thus, owing to the first (2.17),

$$\frac{m}{\mathrm{n}^2}\frac{\delta \ell}{\delta t} = p_\ell = \frac{energy \times velocity}{c^2}, \quad v_\ell = \frac{1}{\mathrm{n}^2}\frac{\delta \ell}{\delta t}, \quad m v_\ell = p_\ell = \frac{v_\ell \epsilon_\ell}{c^2}. \tag{2.29}$$

Note that n^2 is here mere dimensionless multiplicative factor of v_ℓ, which however cannot be specifically calculated because both $\delta \ell$ and δt are uncertainty ranges unknown and conceptually unknowable; so n^2 contributes in determining the resulting v_ℓ, whatever its value might be. These dimensional equations introduce the velocity and relativistic momentum components v_ℓ and p_ℓ. Moreover write according to (2.12)

$$v_\ell = \frac{\delta \epsilon_\ell}{\delta p_\ell}, \tag{2.30}$$

Multiplying side by side this velocity and (2.29) one finds

$$\frac{\delta\epsilon_\ell}{\delta p_\ell}\epsilon_\ell = p_\ell c^2.$$

Then it is possible to write

$$\delta(\epsilon_\ell)^2 = \delta(p_\ell c)^2 \tag{2.31}$$

i.e. identically $\delta(\epsilon_\ell)^2 = \delta\left((p_\ell c)^2 + const\right)$, because of course $\delta(const) = 0$. Hence (2.31) is compatible with

$$\epsilon_\ell^2 = (p_\ell c)^2 + const. \tag{2.32}$$

Note that

$$\lim_{v\to 0}\frac{pc}{v} = m_0 c^2 \tag{2.33}$$

and then, being m_0 the rest mass of the particle, one finds

$$\epsilon_\ell^2 = (p_\ell c)^2 + \left(m_0 c^2\right)^2. \tag{2.34}$$

A simple reasoning shows therefore that the statistical formulation (1.2) of quantum uncertainty is immediate consequence of the space time definition (1.1) and that the energy and momentum equations of special relativity are actually quantum equations, likewise the Lorentz transformations; indeed merging (2.34), (2.27) and (2.29) one finds

$$\epsilon_\ell = \frac{m_0 c^2}{\sqrt{1 - v_\ell^2/c^2}}, \quad p_\ell = \frac{m_0 v_\ell}{\sqrt{1 - v_\ell^2/c^2}} = \frac{\epsilon_\ell v_\ell}{c^2} = \frac{h}{\lambda}. \tag{2.35}$$

Note that this definition of energy and momentum agrees with that of (2.23) simply putting

$$\epsilon_\ell = m_\ell c^2, \quad p_\ell = m_\ell v_\ell, \quad m_\ell = \frac{m_0}{\sqrt{1 - v_\ell^2/c^2}}. \tag{2.36}$$

The definition of m_ℓ is essential to overcome an evident difficulty about p_ℓ, which diverges for $v_\ell \to c$ when expressed via the Lorentz factor but remains finite when expressed via ϵ_ℓ. Clearly this is due to the fact that the former definition introduces explicitly the mass, the latter does not; so if is the mass that diverges because of the Lorentz factor, then both ϵ_ℓ and p_ℓ coherently increase till then their asymptotic ratio becomes $\epsilon_\ell / p_\ell = c$. However there is a further reason, more subtle, to explain this point even better; this reason will be concerned in the section 5.5.8.

2.6. The Uncertainty Equations

Merging the third (2.9) and (2.22) one finds

$$mc\delta\ell = n\hbar \tag{2.37}$$

and then also

$$mc^2\frac{\delta\ell}{c} = mc^2\delta t = n\hbar, \quad \delta t = \frac{\delta\ell}{c}. \tag{2.38}$$

Since m is arbitrary, consider now two masses m' and m'' such that $m'c^2(\delta\ell/c)=n'\hbar$ and $m''c^2(\delta\ell/c)=n''\hbar$. Subtracting side by side these equations the first (2.38) yields

$$(m'-m'')c^2\delta t=(n'-n'')\hbar,\quad \delta t=\delta\ell/c;$$

as of course $n'-n''$ is still an arbitrary integer, this result reads

$$\delta\epsilon\delta t=n'''\hbar,\quad \delta\epsilon=(m'-m'')c^2,\quad n'\le n'''\le n''. \tag{2.39}$$

So even n is defined in its own range of integer values, as stated in (1.9). Multiplying and dividing the left hand side of the first equation by an arbitrary velocity modulus v, by dimensional reasons one finds $\delta p=\delta\epsilon/v$ and $\delta x=v\delta t$ so that $\delta p\delta x=n\hbar$. Thus (2.37) implies via (2.22) and (2.19) the uncertainty equations

$$\delta\varepsilon_\ell\delta t_\ell=n\hbar,\quad \delta\ell\delta p_\ell=n\hbar, \tag{2.40}$$

i.e. just (1.2) merely with a different notation of the conjugate dynamical variables. These equations imply the indistinguishability of identical particles, because actually they concern the phase space rather than the particles themselves; in other words it is impossible to distinguish electron 1 from electron 2 delocalized in a region of space time if nothing in known about them. Indeed it has been shown in (2.29) to (2.35) that, for example, momentum and energy are directly related to the range sizes (2.40) regardless of any hypothesis about the particles themselves. On the one hand is remarkable the fact that the Newtonian definitions (2.9) imply the concept of uncertainty, thus confirming that actually even the classical gravity is rooted in the quantum equations (1.2). On the other hand the agnostic meaning of uncertainty, which implies lack of information about the boundaries of the ranges and about the local values of the dynamical variables allowed in their ranges, is not a postulate but a corollary of the way to introduce (2.40). The agnostic meaning of (1.2) follows from (2.9), in turn deductible themselves without need of further considerations besides the dimensional analysis of the physical definitions of time and energy. In fact is enough a general idea only: $\epsilon'\le\epsilon\le\epsilon''$ implied by $m'\le m\le m''$ follows from and corresponds to the number of quantum states $n'\le n\le n''$ allowed at any $t'\le t\le t''$. By consequence space and time coordinates lose their deterministic local meaning of classical physics, while however their uncertainty ranges fulfill the respective Lorentz transformations (2.35). Of course owing to (2.24) it also holds for the wave definitions of dynamical variables $\lambda'\le\lambda\le\lambda''$ of p and $v'\le v\le v''$ itself. In effect it follows because the velocity is defined as ratio of uncertainty ranges as shown in (2.29), (2.40) and (2.12); a deterministic value of v is pertinent only in classical physics and in Einstein relativity. Although the reasoning to infer (2.39) has quantum character, it involves the relativistic $\epsilon=mc^2$ and quantum $p=h/\lambda$ local values of dynamical variables randomly falling within the respective uncertainty ranges. It has been shown in [9] that n must be arbitrary integer in order that (1.2) are independent of any particular reference system. Summarizing shortly the reasoning therein carried out, let

$\delta x \delta p$ be defined by range sizes in a given reference system R and $\delta x' \delta p'$ in another R'. Since the respective products are equal to n and n' times $\hbar$, it is clear that (1.2) are actually indistinguishable in R and R', because n and n' are indistinguishable themselves; indeed n and n' are not specific values, rather they symbolize sets of arbitrary integers, so that any allowed n cannot be distinguished from any allowed n'. In other words the quantization makes indistinguishable the reference systems because the unique sequence $\hbar, 2\hbar, 3\hbar, \cdots$ is identically compatible with $\delta p \delta x$ in R and $\delta p' \delta x'$ in R'. Moreover all uncertainty ranges, e.g. $\delta x = x'' - x'$, contain themselves one boundary value, say x', that in principle could be referred to the origin of its own R, whereas the other boundary, say x'', determines the range size. However, being both boundary coordinates by definition unknown and conceptually unknowable, any link of δx to a specific R is conceptually missing as well. This conclusion is further confirmed in a more substantial way as shown in [9] and again sketched also here. Expressing the range sizes in the Planck units previously found, see (2.6) and following, (1.2) read in fact

$$n_x^* n_p^* = n = n_t^* n_\varepsilon^* \tag{2.41}$$

where n is the arbitrary integer of quantization whereas n_j^* are arbitrary real numbers defining the j-th range sizes as multiple of the respective Planck units, e.g. $\delta t = n_t^* t_P$ with $n_t'^* \le n_t^* \le n_t''^*$ as in (1.9); the primed and double primed notation of n_t^* indicates instead specific time values n_t^*, of course arbitrary. According to the link (2.41) between mere numbers it is in fact impossible even to introduce R itself, in agreement with the conclusion that by definition (1.2) hold in any reference system R inertial or not. Obviously an analogous conclusion holds for the ratios of uncertainty range sizes too, e.g. for $v = \delta x / \delta t$ of (2.12)

$$v = \frac{n_x^*}{n_t^*} \frac{\ell_P}{t_P} = \frac{n_x^*}{n_t^*} c: \tag{2.42}$$

owing to the arbitrariness of all range sizes, for example, this velocity can be intended in particular as an average value $\bar{v}$ of all ratios admissible for $x' \le x \le x''$ and $t' \le t \le t''$ or as the limit of both $\delta \to 0$, *i.e.* as $v \equiv v_{lim} = \partial x / \partial t$. Despite the notation, even ∂x and ∂t still have in principle the meaning of very small uncertainty range sizes whose ratio defines the modulus v as a new dynamical variable; yet the local value of v_{lim} is still numerically undefinable in the present quantum model because are not known the time and space coordinates n_t^* and n_x^* falling within ∂t and ∂x. However, this is true if the infinitesimal ∂x and ∂t are regarded in fact as independent ranges, as so far implicitly assumed for δx and δt. If both ranges are concurrently vanishing, as it happens in the usual concept of derivative, then their shared property of contextual vanishing adds supplementary information on the definition of v, which reads now v_{cl}: indeed it modifies the concept of quantum total uncertainty hitherto intuitively acknowledged for $\delta x / \delta t$. This is why v_{cl} is conceptually knowable in classical physics, but not in general in the present agnostic model based on (1.2) only where all n_j^* of (2.41) are random in their own $n_j'^* \le n_j^* \le n_j''^*$ and thus independent. This point deserves attention and is further explained in the next subsection. Here it is enough to remark the conceptual difference between the modulus of the four-vector $v_{rel} = |\mathbf{v}_{rel}|$ of special relativity and the quantum *v*, ratio of two independent and arbitrary uncertainty ranges defined in (2.29), (2.40) and (2.30). On the one hand the local moduli v_{cl} classical or v_{rel} relativistic are calculable, whereas the local *v* of (2.12) does not likewise any function whose local time and space coordinates are conceptually missing. On the other hand v_{rel} must be related to the reference system where is available deterministic information on local coordinates and time, whereas *v* waives "a priori" its own *R* because of (2.41). The fact that in principle *v* can be introduced without explicit link to a specific reference system has a further implication. Let *v* enter in the formula of a physical amount *f* written in terms of uncertainty ranges of dynamical variables only: if $f = f(\delta x, \delta p, \delta\varepsilon, \delta t)$ turns into $f = f(n_x^*, n_p^*, n_\varepsilon^*, n_t^*)$ while $v = v(n_x^*/n_t^*)$, then it is possible to introduce *f* itself without defining explicitly its *R*, whatever the specific physical meaning of *v* and *f* might be. Nevertheless the lack of a specific reference system does not imply in fact any ambiguity, as it will appear in the following section 6 where are calculated some numerical outcomes of the present model; rather the physical meaning of *v* results from that of δx and δt or $\delta\varepsilon$ and δp themselves according to (2.12). Although this physical model seems too agnostic to infer valuable information, note that these conceptual premises have been enough to infer the fundamental (1.2) from (1.1) and even preliminary relativistic results. The remainder of the paper aims to show that just this conceptual agnosticism allows to overcome the determinism of Einstein general relativity and plugs it into the elusive quantum world; the calculations will be carried only after having completed adequately the theoretical frame so far introduced.

2.7. Uncertainty, Covariance, Simultaneity

In general the choice of the reference system *R* is crucial in any classical physical model that implements deterministic local coordinates; in the Einstein relativity, the equations are required to be invariant with respect to the reference systems, including the non-inertial ones. Consider however a quantum problem formulated *only* via uncertainty ranges; in fact (2.41) shows that if all n_j^* and *n* are arbitrary, then there is no direct correlation between range sizes of dynamical variables and reference systems just because the former do not contain any information someway related to the latter. As it has been remarked for *n*, the only available information is that the product of two range sizes of conjugate dynamical variables must be quantized; *i.e.* both products $n_j^* n_{j'}^*$ of (2.41) must yield an arbitrary integer whatever the local values of the respective dynamical variables *j* and *j'* might be. So, if the local coordinates are replaced by ranges that fulfill (1.2) and (2.41), then is missing "a priori" the existence of privileged reference systems; moreover it is easy to show that, by consequence, the requirement of the different form of equations in *R* and *R'* becomes inessential. Is instructive in this respect the classical example reported in various textbooks, e.g. [10], of a point mass m tethered by a massless and inextensible wire, so that the mass moves circularly around a fixed coordinate. This example becomes significant noting that if the wire is broken, e.g. by the centrifugal force itself, thereafter the motion of the mass is rectilinear uniform along the tangent to the circumference in the breakdown point. This is true in *R* with origin fixed on the rotation center of the mass. In *R'* fixed on the moving mass, instead, the mass is at rest; when the wire is broken the mass deviates from its initial path, it follows a curved trajectory. The classical physics implements $F = ma$ and $F' = ma'$ respectively: elementary considerations show that $a' = a_r + a_{ce} + a_{co}$, where the subscripts stand for real, centrifugal and Coriolis terms. This is a typical example where the motion of the mass described in *R* and *R'* implies acceleration terms appearing in the non-inertial *R'* only. Einstein felt then the necessity of a covariant theory including the gravity. Classically it is possible to introduce *R* and *R'* along with any other *R"* arbitrarily chosen. Just for this reason the Einstein relativity aims to describe in general any physical system independently of a specific *R*, thus excluding the existence of such a "privileged" *R* but admitting however that in effect all these various *R* are actually definable. In any approach formulated via (1.2), instead, neither any deterministic coordinate of the tethered system nor its own *R* are actually definable owing to (2.41); in general it is possible only to say that exist arbitrary reference systems and that if two of them are inertial then hold (2.35). Consider indeed the dynamics of the rotating mass described by the Newtonian (2.9): is crucial the fact that the acceleration is defined as $a = (c/\hbar)\epsilon$, where the unique variable is ϵ. This latter however is a local energy that must be implemented solely via its uncertainty range $\delta\varepsilon = \epsilon_2 - \epsilon_1$: *i.e.* ϵ is to be regarded as $\epsilon_1 \le \epsilon \le \epsilon_2$, whereas a is a random value included within a range $a_1 \le a \le a_2$ of values. Also, in the present quantum model we should compare the ranges $\delta\varepsilon$ and $\delta\varepsilon' = \epsilon_2' - \epsilon_1'$ before and after the breakdown of the tethered system; in other words *a'* is now a different local value included within a different range $a_1' \le a' \le a_2'$ of values. But actually this comparison between inertial and non-inertial reference systems is conceptually meaningless in the present model: the range boundaries and the local values of dynamical variables are unknowable and irrelevant as concerns the physical description of any quantum problem, so it is irrelevant the local disagreement $a \ne a'$ and the fact that *a'* includes various additional terms with respect to a. The agnosticism implied by (1.2) compels considering these ranges before and after the wire breakdown regardless of how the corresponding local accelerations are made of; in other words the chance $\delta\varepsilon = \delta\varepsilon'$ of including the respective a and *a'*, in principle possible because a and ϵ differ by a proportionality constant factor, bypasses the necessity of discriminating *R* and *R'* to describe the tethered system with or without its breakdown. Since this holds for any uncertainty range by definition, in fact (2.41) waives the necessity of specifying either reference system to concern the dynamical variables. Accordingly it is possible to regard all ranges of (1.2) independently of their definition in a specific *R*: rather it is possible to introduce δx and δt independently of the local space and time coordinates $n_x'^*$ and $n_t'^*$ in Plank units, equivalent to $x = x(t)$ and $x' = x'(t')$, and regardless of their own R_x and R_t. It is clear now the last statement of the previous subsection: distinguishing $n_x^* = n_x^*(n_t^*)$ and $n_x'^* = n_x'^*(n_t'^*)$ to obtain the local velocity through the usual concept of derivative violates the concept of total uncertainty and implies turning the formulation of the physical problem into the deterministic definition of classical dynamical variables. This conclusion is reasonably extrapolable also to different times $n_t'^*$ and $n_t''^*$ in two different *R*: *i.e.* $n_t'^* = n_t''^*$, for example, would violate the principle of total lack of information about the local time variable, whereas the uncertainty leaves out any form of local determinism in any *R*. So $n_t'^*$ and $n_t''^*$ must fulfill the condition $n_t'^* \perp\!\!\!\perp n_t''^*$, where the symbol means "independent of": *i.e.* it states that $n_t'^*$ and $n_t''^*$ are independent local times in t_P units and introduces two crucial corollaries. On the one hand if $n_t'^*$ and $n_t''^*$ are independent local times in different reference systems, then $n_t'^* \perp\!\!\!\perp n_t''^*$ excludes the concept of simultaneity in *R* and *R'*; as any *t* in *R* is not numerically correlatable to *t'* in *R'* in a deterministic way, two events simultaneous in *R* are not automatically simultaneous in *R'*. On the other hand the local time $n_t'^*$ is illusory itself likewise the local coordinate $n_x'^*$; indeed during a given time lapse of length $n_t^* t_P$ it is meaningless to distinguish $n_t'^* > n_t''^*$ or $n_t'^* < n_t''^*$. The lack of the concept of simultaneity in the special relativity is obvious: if *c* is finite and invariant in all reference systems whereas instead space lengths and time lapses are subjected to (2.35) via δp and $\delta\varepsilon$ of (1.2), it is trivial to conclude that the time lapses δt and $\delta t'$ in different reference systems cannot imply the simultaneity of a given event for two observers in reciprocal motion. From a quantum point of view the same conclusion is due to the lack of single time coordinates to be compared in a deterministic way, being significant instead time ranges to be compared e.g. in two different reference systems. These statements, well known since the birth of the special relativity by consequence of the finite value of invariant *c*, appear here as straightforward corollaries of the quantum uncertainty. Nevertheless the present way to regard the quantum physics based on (1.2) has not only relativistic implications, partially already inferred in the subsection 2.5 and further considered in the next section 3, but also quantum implications. Here are two short examples of corollaries of (1.2) pertinent to the present reasoning.

(i) Quantum implication: the hydrogenlike atoms. Write

$$\delta p_r = p_r' - 0, \quad \delta\varepsilon = \varepsilon' - 0: \tag{2.43}$$

since the unprimed range sizes at the left hand sizes are arbitrary, the same must hold for the primed range sizes at the right hand side. Then it must be possible to implement identically both of them, regarding thus the lower boundary value

0 as a particular but not deterministic case. To infer physical information from these statements and check their validity, find the classical energy ϵ' of hydrogenlike atoms. Implementing (1.2) for p'_r, which actually owing to (2.43) is by definition radial range despite its notation, one finds

$$\epsilon' = \frac{p'^2_r}{2m} = \frac{(n\hbar)^2}{2mr'} = \frac{2(n\hbar)^2}{m(2r')^2} = \frac{2(n\hbar)^2}{m\delta r^2}, \quad \delta r = 2r'; \tag{2.44}$$

clearly r' has been introduced as a length by dimensional reasons and symbolizes the range $r' - 0$. Thus

$$\delta r = \pm\frac{2(n\hbar)^2}{m\epsilon'\delta r} = \pm\frac{2(n\hbar)^2}{m\left(-Ze^2/\delta r\right)\delta r} = \mp\frac{2(n\hbar)^2}{mZe^2}, \quad \epsilon' = -\frac{Ze^2}{\delta r}, \tag{2.45}$$

having introduced appropriate information, *i.e.* the specific electromagnetic interaction between nucleus and electron charges. One finds, with the second equation reasonably suggested by (2.44) and first (2.45) itself

$$\delta r = 2r_B, \quad r_B = r' = \frac{(n\hbar)^2}{mZe^2}$$

that yield Bohr radius and energy

$$\epsilon' = \varepsilon_B = -\frac{Ze^2}{\delta r} = -\frac{Ze^2}{2r'} = -\frac{\left(Ze^2\right)^2 m}{2(n\hbar)^2}. \tag{2.46}$$

Now it should be clear why range sizes and boundary coordinates are irrelevant as concerns the quantum problems, as in effect it has been demonstrated for various systems [11] [12]. In particular it is not necessary to specify R centered on the nucleus, it is enough to state that nucleus and electron are δr apart; the radial range size is then defined by the non deterministic Bohr radius via the integer $1 \le n \le \infty$. Is clear thus the meaning of the coefficient 2 in the second (2.44): an electron r' apart from the nucleus has total radial delocalization range $2r'$. In this model the quantum numbers are in fact numbers of quantum states.

(ii) Relativistic implication: the invariant equations. Write (1.2) as follows

$$\delta x\delta t = \delta x\frac{n\hbar}{\delta\varepsilon} = \delta x^2\frac{\delta p}{\delta\varepsilon}, \quad \delta x\delta t = \delta t\frac{n\hbar}{\delta p} = \delta t^2\frac{\delta\varepsilon}{\delta p}$$

so that, subtracting side by side,

$$c^2\delta t^2\frac{\delta\varepsilon/c}{\delta(pc)} - \delta x^2\frac{\delta p}{\delta\varepsilon} = 0.$$

Let be now, without loss of generality,

$$\frac{\delta\varepsilon/c}{\delta(pc)} = \frac{\delta p}{\delta\varepsilon} - Y,$$

being Y a function with physical dimensions of reciprocal velocity to be defined. Hence

$$\left(c^2\delta t^2 - \delta x^2\right)\frac{\delta p}{\delta \varepsilon} - Yc^2\delta t^2 = 0,$$

i.e.

$$c^2\delta t^2 - \delta x^2 = Yc^2\delta t^2\frac{\delta\varepsilon}{\delta p} = Yc^2\delta t\frac{n\hbar}{\delta p} = Yc^2\delta t\delta x. \tag{2.47}$$

The left hand side reads, again because of (1.2) and (2.34),

$$\begin{aligned} c^2\frac{n^2\hbar^2}{\delta\varepsilon^2} - \frac{n^2\hbar^2}{\delta p^2} &= \left(n\hbar c\right)^2\left(\frac{1}{\delta\varepsilon^2} - \frac{1}{\delta\left(p^2c^2\right)}\right) \\ &= \frac{\left(n\hbar c\right)^2}{\delta\varepsilon^2\delta\left(p^2c^2\right)}\left(\delta\left(p^2c^2\right) - \delta\varepsilon^2\right) \\ &= -\frac{\left(n\hbar c\right)^2\left(mc^2\right)^2}{\delta\varepsilon^2\delta\left(p^2c^2\right)}. \end{aligned} \tag{2.48}$$

If Y is a constant, *i.e.* $Y = c^{-1}$, (2.47) shows two invariant quantities of the special relativity correlated each other; $c^2\delta t^2 - \delta x^2$ is particularly important as it has been demonstrated in [13] to be conceptual foundation of the special relativity. Special attention deserves in this respect the operator formalism of quantum mechanics, which regards since the beginning the particles as waves; instead the last equations have concerned the corpuscular properties of matter. The next section shows how to introduce in this conceptual frame also the wave formalism, in agreement with the corpuscular/wave nature of the particles.

2.8. The Wave Formalism

Rewrite identically (2.32) as

$$\left(\epsilon_\ell + p_\ell c\right)\left(\epsilon_\ell - p_\ell c\right) = const, \tag{2.49}$$

which is trivially consistent with

$$\begin{aligned} &\epsilon'_\ell + p_\ell c = const', \quad \epsilon''_\ell - p_\ell c = const'', \quad \epsilon'_\ell = \epsilon''_\ell = \pm\epsilon_\ell, \\ &const' = const'' = \pm\sqrt{const}; \end{aligned} \tag{2.50}$$

it is trivially evident that multiplying side by side with the help of the third and fourth conditions (2.50) one obtains (2.49) and thus (2.32). Yet two more conditions make the first two (2.50) compatible with (2.32). Multiplying side by side $\epsilon'_\ell = const' - p_\ell c$ and $\epsilon''_\ell = const'' + p_\ell c$ one finds

$$\epsilon'_\ell\epsilon''_\ell = -p_\ell^2c^2 + \left(const' - const''\right)p_\ell c + const'const'', \tag{2.51}$$

whence the two chances allowed by $const' = const''$:

$$(1)\quad \epsilon'_\ell\epsilon''_\ell = \varepsilon^{*2}, \quad p_\ell = \pm ip^*, \quad const' = const'' = \pm\sqrt{const}, \tag{2.52}$$

which yields

$$\epsilon^{*2} = \left(p^*c\right)^2 + const^2,$$

and

$$(2)\quad \epsilon'_\ell \epsilon''_\ell = -\epsilon^{*2} = \left(i\epsilon^*\right)^2, \quad p_\ell = p^*, \quad const' = const'' = \pm i\sqrt{const}, \tag{2.53}$$

which yields

$$-\epsilon_\ell^{*2} = -\left(p^* c\right)^2 - const^2.$$

Both conditions (2.52) and (2.53) agree with (2.32) exactly likewise (2.50). The former implies imaginary momentum p_ℓ; also, since there is no reason to exclude $\epsilon'_\ell > 0$ and $\epsilon''_\ell > 0$, one must accept even imaginary energy ϵ_ℓ, which opens a new conceptual frame along with the imaginary momentum too. The relativistic (2.32) needs therefore a new interpretation to be consistent with complex dynamical variables, which in turn must be acknowledged themselves: it is evident that all of this implies in fact the corpuscle/wave behavior of matter to fit both quantum and relativistic results. While the compatibility of (2.49) with the initial (2.32) is trivial, that of (2.52) and (2.53) with (2.32) is still possible even defining complex quantities. Consider first (2.52) to calculate via $p^* = \pm p_\ell / i$ the corresponding complex range $\delta p^* = \pm \delta p_\ell / i = \pm n\hbar / i\delta x_\ell$ according to (1.2). Since any integer n can be expressed as a difference of two integers n' and n'' one finds

$$\delta p^* = p'^* - p''^* = \pm \frac{\hbar}{i}\frac{n}{\delta x_\ell} = \pm \frac{\hbar}{i}\frac{n'}{\delta x_\ell} \mp \frac{\hbar}{i}\frac{n''}{\delta x_\ell}, \quad n = n' - n'' \tag{2.54}$$

and thus also

$$p'^* \delta\psi'^*_\ell = \pm \frac{\hbar}{i} n' \frac{\delta\psi'^*_\ell}{\delta x_\ell}, \quad p''^* \delta\psi''^*_\ell = \pm \frac{\hbar}{i} n'' \frac{\delta\psi''^*_\ell}{\delta x_\ell}, \tag{2.55}$$

having multiplied both sides of these equations by $\delta\psi'^*_\ell$ and $\delta\psi''^*_\ell$ with the purpose of obtaining again via (2.55) a real value of momentum consistent with the relativistic p_ℓ of (2.49). Subtracting side by side (2.55) and writing explicitly $\delta\psi'^*_\ell = \psi'^*_\ell - \psi'_{\ell 0}$ and $\delta\psi''^*_\ell = \psi''^*_\ell - \psi''^*_{\ell 0}$, (2.54) yields

$$\begin{aligned} & p'^*\left(\psi'^*_\ell - \psi'_{\ell 0}\right) - p''^*\left(\psi''^*_\ell - \psi''^*_{\ell 0}\right) \\ & = \pm \frac{\hbar}{i} n' \frac{\delta}{\delta x_\ell}\left(\psi'^*_\ell - \psi'_{\ell 0}\right) \mp \frac{\hbar}{i} n'' \frac{\delta}{\delta x_\ell}\left(\psi''^*_\ell - \psi''_{\ell 0}\right). \end{aligned} \tag{2.56}$$

If the range boundaries $\psi'_{\ell 0}$ and $\psi''^*_{\ell 0}$, arbitrary in principle, are defined such that

$$p'^* \psi'_{\ell 0} = p''^* \psi''^*_{\ell 0},$$

then (2.56) reads

$$p'^* \psi'^*_\ell - p''^* \psi''^*_\ell = \pm \frac{\hbar}{i} n' \frac{\delta\psi'^*_\ell}{\delta x_\ell} \mp \frac{\hbar}{i} n'' \frac{\delta\psi''^*_\ell}{\delta x_\ell}$$

and thus

$$p'^* \psi'^*_\ell = \pm \frac{\hbar}{i} n' \frac{\delta\psi'^*_\ell}{\delta x_\ell}, \quad p''^* \psi''^*_\ell = \pm \frac{\hbar}{i} n'' \frac{\delta\psi''^*_\ell}{\delta x_\ell}. \tag{2.57}$$

It is easy to acknowledge that for $\psi'^*_\ell \to \psi'_{\ell 0}$ and $\psi''^*_\ell \to \psi''_{\ell 0}$, along with

$\delta x_\ell \to 0$, all δ of (2.57) turn into the classical ∂; so both equations are nothing else but the classical momentum wave equations with respective real eigenvalues p'^* and p''^* with an appropriate choice of $\psi'_{\ell 0}$ and $\psi''_{\ell 0}$. This result, which introduces the quantization required by (1.2), is clearly the wave formulation of momentum equation via the complex wave function ψ^*. Note that this result could have been obtained more shortly starting from (2.27) rewritten replacing $p_\lambda \equiv p_\ell$: regarding the De Broglie momentum as the imaginary momentum appearing in (2.52),

$$\frac{n\hbar}{\delta r} = p_\lambda$$

of (2.27) turns into

$$\frac{\hbar}{i}\frac{n}{\delta r} = \frac{p_\ell}{i} = \pm p^*,$$

whence, multiplying by the function $\delta\psi$ both sides,

$$\frac{\hbar}{i}\frac{\delta\psi}{\delta r} = \pm\frac{\delta\psi}{n}p^*.$$

Thus one finds

$$\frac{\hbar}{i}\frac{\delta\psi}{\delta r} = \pm\psi p^*, \quad \delta\psi = n\psi.$$

The last position expresses the range $\delta\psi$ as a function of n; *i.e.* it follows thinking the values of n in an arbitrary range $n' \le n \le n''$ and thus regarding $\delta\psi$ as a range of terms $n'\psi, (n'+1)\psi, \cdots, (n''-1)\psi, n''\psi$ through which one calculates the respective eigenvalues of momentum falling between $n'p^*$ and $n''p^*$. Consider now (2.53) with imaginary energy ε_ℓ^*. An identical reasoning holds of course here; trivial algebraic steps analogous to that from (2.55) to (2.57) yield

$$\epsilon'^*\psi_\ell'^* = \pm\frac{\hbar}{i}n'\frac{\delta\psi_\ell'^*}{\delta t_\ell}, \quad \epsilon''^*\psi_\ell''^* = \pm\frac{\hbar}{i}n''\frac{\delta\psi_\ell''^*}{\delta t_\ell} \tag{2.58}$$

compatibly with the existence of states of negative energy. Moreover $\psi_\ell'^*$ and $\psi_\ell''^*$ resulting from the uncertainty range formalism of (1.2) represent a combination of the n-th quantum states allowed for momentum and energy of particles. Once more the reason is that the uncertainty range boundaries are unknown and arbitrary; hence one could rewrite validly (2.54) replacing the upper boundary value p'^* with $\xi p'''^* + p'^*$ so that $\delta p^* = \xi p'''^* + p'^* - p''^*$, being ξ an arbitrary constant coefficient. If so, then (2.55) would result with $\delta\psi_\ell'^*$ and $\delta\psi_\ell''^*$ replaced by $\delta\left(\psi_\ell'^* + \xi\psi_\ell'''^*\right)$ and $\delta\left(\psi_\ell''^* + \xi\psi_\ell'''^*\right)$, whereas (2.57) would consist of two primed and double primed functions like this

$$p^{**}\psi_\ell^{**} = \pm\frac{\hbar}{i}\frac{\delta\psi^{**}}{\delta x_\ell}, \quad \psi_\ell^{**} = \psi_\ell'^* + \xi\psi_\ell'''^*. \tag{2.59}$$

From these considerations inferred as corollaries of (1.2) and (2.32) without need of postulates, was born the early wave mechanics and the modern quantum mechanics.

3. Relativistic Corollaries

This subsection examines four main implications of (2.2) noting that if $\eta = const$ then $\epsilon = const$ as well, whereas $\eta = \eta(\delta t)$ implies $\epsilon = \epsilon(\delta t)$ too. Owing to (1.4) and (2.9), the definition (1.1) of space time appears compliant with the idea of a dynamic system characterized by matter, energy and forces; also, the equivalence of mass and energy of the special relativity inferred in (2.33) agrees with the feature of space time characterized by the energy density η of (2.1) inherent to its definition (1.1). Without these results the space time would be an empty concept unavoidably abstract and unphysical. Instead, for reasons shown in the appendix B, does exist in principle an outwards pressure corresponding to the energy density η in (2.2), which in turn can be partially or totally counterbalanced by the attractive gravitational effect of matter/energy possibly present in a given volume of space time according to (2.32) and (2.33). The space time is therefore a dynamical system, in principle in equilibrium or non-equilibrium conditions, which evolves as a function of time. This point in particular, which anyway governs its dynamics, is now concerned to justify the possible presence of mass in a volume δx^3 of space time. Implement (1.1) to find a further result based again on a dimensional reasoning. Note the possible correlation

$$\frac{\hbar G}{c^2} \Leftrightarrow \frac{\hbar \delta\ell}{m}, \quad m \neq 0, \tag{3.1}$$

between quantities having the same physical dimensions; m is an arbitrary mass confined and delocalized within the arbitrary size $\delta\ell$ of an uncertainty space time range, thus without chance of information about its exact position. This section concerns just the physical conditions consistent with the delocalization of m in an uncertainty range, in agreement with (1.2).

3.1. Real and Virtual Mass

Are reasonably conceivable two conditions on the correlation (3.1), here expressed as follows

$$\frac{\hbar \delta\ell}{m} = \xi \frac{G}{c^2} \tag{3.2}$$

being ξ an appropriate proportionality factor.

(i) One concerns the Lorentz invariance of both definitions (3.1): for the first one this condition is self-evident because it is a constant, for the second one the condition must be purposely required. Write owing to (2.35)

$$\frac{\delta\ell}{m} \equiv \frac{\beta\delta\ell}{\beta m} = \frac{\delta\ell'}{m'}, \quad \beta = \sqrt{1 - v^2/c^2};$$

since both $\delta\ell'$ and m' are Lorentz transformations of $\delta\ell$ and m, it must be true that

$$m=\frac{m'}{\beta},\quad \delta\ell'=\beta\delta\ell. \tag{3.3}$$

With the given definition of β, for $v=0$ clearly $m\equiv m'$ and $\delta\ell'\equiv\delta\ell$; *i.e.* m is the v dependent dynamic mass corresponding to the rest mass m' defined in (2.33), whereas $\delta\ell'$ is the space contraction of the proper length $\delta\ell$. It is significant that (3.3) confirms the result (2.36) obtained via (2.35).

(ii) Consider now the limit of (3.2) for $m\to 0$; it is reasonable to expect that this limit is nothing else but the definition (1.1) of empty space time, *i.e.*

$$\hbar\lim_{m\to 0}\frac{\delta\ell}{m}=\frac{G}{c^2}\lim_{m\to 0}\xi=\frac{G}{c^2}\xi_0,\quad \xi_0=\hbar. \tag{3.4}$$

This limit ensures the consistency of the definitions (3.1) in agreement with the idea of m delocalized in $\delta\ell$: if no particle is delocalized, the range size is null. This suggests putting by dimensional reasons

$$\xi=\hbar+mv\delta\ell=v\delta\ell\left(m_o+m\right),\quad m_o=\frac{\hbar}{v\delta\ell}=\frac{\delta p_\ell}{nv},\quad m\neq 0 \tag{3.5}$$

so that the second (3.5) reads $m_o=\epsilon\delta p_\ell/\left(np_\ell c^2\right)$ thanks to (2.29) and thus

$$m_o c^2=\frac{\epsilon}{n}\frac{\delta p_\ell}{p_\ell}, \tag{3.6}$$

whereas (3.2) reads

$$\hbar\frac{\delta\ell}{m}=\left(m_o+m\right)v\delta\ell\frac{G}{c^2}. \tag{3.7}$$

Note that owing to (2.35), (2.26) and (2.27)

$$\frac{\hbar}{mv}=\frac{\hbar}{p\beta}=\frac{\lambda}{2\pi\beta}=\frac{\lambda'}{2\pi}=\frac{\delta r'}{n'},\quad \lambda'=\frac{\lambda}{\beta},\quad 2\pi\delta r'=n'\lambda'; \tag{3.8}$$

the last step of the chain means that whatever λ' might be, it is possible to define a corresponding $\delta r'$ that must identically fulfill the condition (2.26) being inessential the primed notation. Hence (3.7) yields

$$\frac{\delta r'}{n'}=\left(m_o+m\right)\frac{G}{c^2}. \tag{3.9}$$

On the one hand multiplying and dividing the right hand side by $m_o m$ (3.9) yields

$$\frac{\mu c^2}{n'}=G\frac{m_o m}{\delta r'},\quad \mu=\frac{1}{m_o}+\frac{1}{m}, \tag{3.10}$$

i.e. still holds a Newton-like law but with positive sign, yet in principle still consistent with (2.11). On the other hand, examine (3.9) that holds in general for any $m\neq m_o\neq 0$ and takes the meaning of quantum vacuum fluctuation consisting of the presence of two particles of masses m and m_o. Regard now (3.9) in the particular case especially important where these masses form a virtual pair particle/antiparticle. Specifying then with more expressive notation $m_o+m=m_v+m_v^*$, where the subscript v stands for *virtual*, write thus

$$\delta r' = n'\left(m_{\text{v}} + m_{\text{v}}^{*}\right)\frac{G}{c^2} = n'\frac{\epsilon_{\text{v}}}{F_P}, \quad \epsilon_{\text{v}} = \left(m_{\text{v}} + m_{\text{v}}^{*}\right)c^2, \quad F_P = \frac{c^4}{G} \tag{3.11}$$

via the Planck force F_P. Then

$$\delta p_r = \frac{n\hbar}{\delta r'} = \frac{n}{n'}\frac{h}{\lambda'} = \frac{n}{n'}p' = \frac{n}{n'}\frac{\hbar F_P}{\epsilon_{\text{v}}}, \quad \delta t_{\text{v}} = \frac{n\hbar}{\delta\epsilon_{\text{v}}} = n\frac{\hbar}{F_P\delta r'},$$
$$\delta\epsilon_{\text{v}} = n'\epsilon_{\text{v}}, \quad n' \neq n; \tag{3.12}$$

the last position holds because n' is due to the integer number of wavelengths consistent with $2\pi\delta r'$ according to (2.26), whereas n is clearly due to (1.2). In turn, $\delta\epsilon_{\text{v}} = n'\epsilon_{\text{v}}$ implies the rising of a random number n' of pairs of particles and antiparticles with total energy $n'\epsilon_{\text{v}}$ allowed to exist during the time lapse δt_{v}; it is significant that (3.1) implies n' fluctuation driven couples of particles/antiparticles. Examine now the chance of defining as a further particular case of (3.9) the lower limit value of $\delta r'$. Consider first to this purpose $n' = n = 1$, so that

$$\delta r = \delta r'_{n=1} = \left(m_o + m\right)\frac{G}{c^2}; \tag{3.13}$$

moreover it is possible to infer from the second (3.5)

$$\delta r_{\min} = \frac{\left(m + m_{oc}\right)G}{c^2}, \quad m_{oc} = \frac{\hbar}{c\delta\ell} = \frac{\hbar}{c\lambda\!\!\!^{-}_{mo}}, \quad n = 1, \quad v = c, \tag{3.14}$$

where the last position implies consequently $\delta\ell = \lambda\!\!\!^{-}_{mo}$ by definition whatever m_{oc} might be. Eventually, since m and m_o are arbitrary, m_o in particular has been defined by dimensional reasons only, it is also possible to consider $m_{oc} = m \neq 0$ for the simple reason that nothing hinders this position; in this last particular case

$$\delta r_{\min} \equiv \delta r_{bh} = 2m\frac{G}{c^2}, \quad n = 1, \quad v = c, \quad m_{oc} = m, \tag{3.15}$$

whereas the last position requires

$$m = m_{oc} = \frac{\hbar}{c\lambda\!\!\!^{-}_{mo}} = \frac{h}{c\lambda_{mo}} = \frac{p_{mo}}{c}. \tag{3.16}$$

Consider now the condition (3.5) to highlight when m can be assumed in fact confined in the range size $\delta\ell$ during an arbitrary time range δt: this requirement implies that $v_\ell\delta t \leq \delta\ell$, which must hold for both components $\pm v_\ell$ of the displacement velocity vector **v** in principle necessary to introduce the delocalization of m along $\delta\ell$; if δt is defined in R fixed on m, then the confinement condition requires actually $|v_\ell|\delta t \leq \delta\ell/2$ to allow at least one chance for m to remain really confined when displacing at rate $|v_\ell|$ towards either possible direction from the middle coordinate of $\delta\ell$. Thus under the condition

$$\frac{\hbar G}{c^2} = \frac{\hbar}{m}\frac{\delta\ell}{2} \tag{3.17}$$

the initial mass-free space time $\hbar G/c^2$ includes now m during the time range δt whatever its $\pm v_\ell$ might be. This explains the link between $\delta\ell \geq 2|v_\ell|\delta t$

defining $\delta r_{\min}$ and the factor 2 defining δr_{bh}. Hence the confinement condition of m in the given δr_{bh} implies (3.16), which in turn holds even for photons because m_{oc} is defined via c. Note eventually that (3.16) takes an interesting form writing

$$\frac{2mG}{c^2} = \frac{2mc^2G}{c^4} = \frac{2h\nu_{bh}}{F_P} = \delta r_{bh}, \quad m = \frac{h\nu_{bh}}{c^2} = \frac{h}{c\lambda_{mo}}, \quad h\nu_{bh} = \frac{1}{2}F_P\delta r_{bh}; \quad (3.18)$$

via the Planck force. So δr_{bh} takes the form of a zero point energy of a mass m oscillating with frequency ν_{bh} corresponding to (3.15) in its confinement range $\delta\ell$.

3.2. Invariant Equations of Special Relativity.

Rewrite Identically (1.3) as

$$\frac{\hbar G}{c^2} = \frac{v^2 length^3}{v^2 time}, \quad (3.19)$$

being $v \le c$ the modulus of an arbitrary velocity allowed in the space time containing mass, concerned in the previous subsection. In principle v could be the group velocity (2.25) of a wave packet propagating through space time volume filled with dispersive medium, or it could be the expansion rate of the boundary of space time volume compatible with (2.14), or eventually it could even be simply the velocity of a body of matter moving through the space time; it depends on how is defined v. To examine this point regard v as a possible velocity allowed in the space time, whatever it might represent in any reference system, and consider that (3.19) identically rewritten as

$$\frac{\hbar G}{v^2} = \frac{c^2}{v^2}\frac{length^3}{time} = \frac{c^2}{v^2}\frac{\delta x^3}{\delta t} \quad (3.20)$$

describes the swelling of the early space time volume introduced in (1.3), here indicated as δx^3 along with the factor c^2/v^2. This equation is justified by (2.13), (2.14), (2.2) and (2.9) and will be further implemented also in the next subsection 3.5. Rewriting explicitly (1.1) as a function of v one finds therefore

$$\frac{\hbar G}{v^2} = \frac{\Delta x^3}{\delta t}, \quad v^2 \le c^2, \quad (3.21)$$

where

$$\Delta x^3 = \frac{c^2}{v^2}\delta x^3; \quad (3.22)$$

accordingly the identity (3.19) becomes compliant with the space swelling rate during the time lapse δt, whereas (2.1) yields

$$\epsilon^2 = \hbar\eta\frac{v^2}{c^2}\frac{\hbar G}{v^2} = \hbar\eta\frac{v^2}{c^2}\frac{\Delta x^3}{\delta t}. \quad (3.23)$$

In principle this result is compatible with (2.4) and (2.14). A corollary of (3.22) follows starting again from (3.21) to write

$$\frac{\hbar G}{v^2}=\frac{\hbar G}{v_s^2+v_\ell^2}=\frac{\hbar G\delta t^2}{\delta s^2+\delta\ell^2},\quad v_s=\frac{\delta s}{\delta t},\quad v_\ell=\frac{\delta\ell}{\delta t},\tag{3.24}$$

being v_s and v_ℓ arbitrary velocities. Let be now $\delta s^2+\delta\ell^2$ such that by definition

$$\delta s_o^2+\delta\ell_o^2=c^2\delta t^2,\tag{3.25}$$

being δs_o and $\delta\ell_o$ specific values that for $\delta s\to\pm\delta s_o$ and $\delta\ell\to\pm\delta\ell_o$ verify (3.25). Hence

$$\frac{\hbar G}{v_o^2}=\frac{\hbar G\delta t^2}{c^2\delta t^2},\quad v_o^2\equiv c^2:\tag{3.26}$$

if (3.25) is true, then (3.26) is nothing else but the initial definition (1.1) of space time itself, already found in (2.47), whereas it appears that either δs_o^2 or $\delta\ell_o^2$ of (3.25) tend to the invariant interval of the special relativity. In other words, the step from (3.19) to (3.20) introducing the space time swelling implies the interval invariant rule. Consider indeed (3.22); Appendix B shows how to obtain from this equation the invariants

$$c^2\delta t^2-\delta\ell^2=c^2\delta t'^2-\delta\ell'^2,\quad \delta\ell\delta t=\delta\ell'\delta t'\tag{3.27}$$

according which trivial manipulations yield, as shown in (3.3),

$$\delta\ell^2=\delta\ell'^2\left(1-\frac{v^2}{c^2}\right),\quad \delta t^2=\frac{\delta t'^2}{1-v^2/c^2}.\tag{3.28}$$

The algebraic steps show that Lorentz transformations and invariant interval in inertial R and R', here introduced for simplicity via a one dimensional approach but immediately referable to a 4D formulation, are intrinsically inherent the space time definition (1.1) yield again space contraction and time dilation of special relativity. Equation (3.27) is particularly important because it is shown in [13] that the invariant interval is the conceptual basis of the special relativity, whence the chance of obtaining in particular (3.28).

3.3. Relativistic velocity

The results hitherto achieved compel explaining the concept of velocity. Multiplying both sides of (2.10) by v/c^3 one finds by consequence of (2.9) and according to (2.29)

$$\frac{v}{c^3}\frac{\delta v}{\delta t}=\frac{v}{c^3}\frac{2\pi\epsilon c}{h}=\frac{2\pi\epsilon v}{hc^2}=\frac{2\pi p}{h},\quad p=\frac{\epsilon v}{c^2};\tag{3.29}$$

it follows thus

$$\frac{2\pi}{\lambda}=\frac{v}{c^3}\frac{\delta v}{\delta t},\quad p=\frac{h}{\lambda},\tag{3.30}$$

whereas it is possible to define

$$\frac{n}{\delta r}=\frac{v}{c^3}\frac{\delta v}{\delta t},\quad \frac{2\pi}{\lambda}=\frac{n}{\delta r},\quad 2\pi\delta r=n\lambda\tag{3.31}$$

that allows the last equation. Once more it is worth emphasizing that the invariant (3.29) has been obtained along with the third crucial equation. Therefore, merely examining the definition (2.8) of acceleration one finds quickly results already obtained in the Equations (2.29) to (2.35) starting from the three (2.9). But now there is more. Rewrite the first (3.31) as

$$\frac{nc\delta t}{\delta r}=\frac{nc}{v_r}=\frac{v\delta v}{c^2}, \quad v_r=\frac{\delta r}{\delta t}, \tag{3.32}$$

which yields according to (2.22)

$$\frac{v_r v}{nc^2}=\frac{c}{\delta v} \tag{3.33}$$

that reads identically

$$1-\frac{v_r v}{nc^2}=1-\frac{c}{\delta v}=\frac{\delta v-c}{\delta v}$$

and thus, taking the reciprocals of both sides,

$$\frac{1}{1-v_r v/nc^2}=\frac{\delta v}{\delta v-c};$$

hence

$$\frac{\delta v-c}{1-v_r v/nc^2}=\delta v. \tag{3.34}$$

First of all eliminate *n*; it could be put equal to 1 by definition, yet it is easy to follow a general procedure valid for any *n*. With the positions

$$V_r=\frac{v_r}{\sqrt{n}}, \quad V=\frac{v}{\sqrt{n}}, \quad C=\frac{c}{\sqrt{n}}, \quad \delta V=\frac{\delta v}{\sqrt{n}} \tag{3.35}$$

(3.34) reads

$$\frac{\delta V-C}{1-V_r V/c^2}=\delta V=\delta V-C+C.$$

Trivial manipulations of this equation yield

$$(\delta V-C)\left(\frac{1}{1-V_r V/c^2}-1\right)=C$$

i.e.

$$\frac{(\delta V-C)V_r V/c^2}{1-V_r V/c^2}=C. \tag{3.36}$$

Let us elaborate further this result in order to obtain a significant equation; is useful in particular the position

$$(\delta V-C)\frac{V_r V}{c^2}=\xi(V_r-V), \tag{3.37}$$

where ξ is arbitrary proportionality factor. With δV in principle arbitrary as well because of the uncertainty, as previously stated, this position is allowed. This step appears important rewriting (3.36) via (3.37) as

$$\frac{(V_r - V)\xi}{1 - V_r V/c^2} = C,$$

which in turn yields

$$\frac{V_r - V}{1 - V_r V/c^2} = V^*, \quad V^* = \frac{C}{\xi} \tag{3.38}$$

where the resulting V^* is still an arbitrary velocity. This formula is actually well known, as it relates in special relativity $V_r - V$ to V^*; it is evident that in the particular cases where $V = c$ or $V_r = c$ then $V^* = c$, *i.e.* the sum of c plus any velocity returns always c.

This reasoning is not at all redundant repetition of a result already known: (3.30) and (3.31) are quantum properties obtained contextually to (3.29) that is the invariant definition of relativistic momentum. Hence the reasoning implies merging of quantum and relativistic results concurring to the definition of c as an invariant limit velocity: this crucial statement of relativity is here required by (1.2). Note that (3.38) has been obtained via V_r and V, which are arbitrary like the respective v_r and v but leave out n: *i.e.* the quantization is not essential to infer (3.38), as it has been emphasized while obtaining (3.36). On the one hand it explains why the relativity was formulated without suspecting the underlying quantization, which indeed appears hidden in (3.35) in the present model. On the other hand it means that the positions (3.35) are not merely formal, as it is evident rewriting (3.33) as

$$\frac{v_r v}{nc^2} = \frac{V_r V}{c^2} = \frac{C}{\delta V}: \tag{3.39}$$

the problem of (3.33) is that the left hand side vanishes for $n \to \infty$ incompatibly with the right hand side that never vanishes because $\delta v \le c$. Owing to (3.35), instead, at the right hand side of (3.33) appear just the velocities leading to the result (3.38) of actual interest without contradicting the arbitrariness of n.

So (3.38) completes the conclusion (2.18), where a well known quantum inequality was inferred just from a physical property of c; here also this property of c appears as a further corollary of (1.1) and (1.2). Once more, as already shown in further papers [4] [11] [12], relativistic and quantum principles appear in the present approach as harmonically coexisting concepts without "ad hoc" hypotheses.

4. Euler-Lagrange Equations and Gravitational Potential

Start from (1.2) that yields

$$\delta \dot{x} = -\frac{n\hbar}{\delta p^2}\delta \dot{p} = -\frac{\delta x}{\delta p}\delta \dot{p}; \tag{4.1}$$

also, recalling the considerations of subsection 2.6, let us define

$$\delta \dot{x} = -\frac{\delta}{\delta t}\delta x, \quad \delta \dot{p} = \frac{\delta}{\delta t}\delta p, \quad \delta \dot{\varepsilon} = \frac{\delta}{\delta t}\delta \varepsilon. \tag{4.2}$$

Note now that (4.1) can be rewritten as

$$\delta\dot{x} = -\frac{\delta x}{\delta p}\delta\dot{p} = -\frac{\delta\dot{p}}{\delta p}\delta x. \tag{4.3}$$

The step from (4.1) to (4.3) is not trivial. As anticipated in the subsections 2.6 and 2.7, the chance of exchanging the place of δx and $\delta\dot{p}$ fulfills the specific concept of derivative in the physical frame of the quantum uncertainty; in fact $\delta\dot{x}$ is mere ratio of arbitrary ranges finite by definition, to be regarded as independent differentials possibly but not necessarily tending both to zero. This subsection aims just to show that this way of intending the quantum derivative is physically sensible. Multiplying now both sides of the second equality (4.3) by δx, one finds

$$\delta\dot{x}\delta x = -\frac{\delta\dot{p}}{\delta p}\delta x^2. \tag{4.4}$$

Define now a function *f* consistent with this result, *i.e.* such that δf fulfills

$$\delta\dot{x}\delta x = \delta f = -\frac{\delta\dot{p}}{\delta p}\delta x^2, \quad f = f\left(x, \dot{x}, p, \dot{p}\right); \tag{4.5}$$

in turn (4.4) and (4.5) are consistent with the positions

$$\delta x = \frac{\delta f}{\delta\dot{x}}, \quad -\frac{\delta\dot{p}}{\delta p}\delta x = \frac{\delta f}{\delta x}. \tag{4.6}$$

As concerns the first equation, the first (4.2) yields

$$\delta\dot{x} = \frac{\delta}{\delta t}\frac{\delta f}{\delta\dot{x}},$$

whereas (4.4) reads with the help of the second (4.6)

$$\delta\dot{x} = -\frac{\delta\dot{p}}{\delta p}\delta x = \frac{\delta f}{\delta x}.$$

Hence, merging the last two results, one finds

$$\frac{\delta}{\delta t}\frac{\delta f}{\delta\dot{x}} = \frac{\delta f}{\delta x}. \tag{4.7}$$

According to (4.4) the function *f* has physical dimensions *length*^{2}/*time* and fulfills the same kind of equation of the Lagrangian $\mathcal{L}$ of a physical system; in fact f is proportional to $\mathcal{L}$ a multiplicative constant c^3/G apart. Since $f' = fc^3/G$ is an energy, this is in principle just the sought Lagrangian. Yet the way to obtain this equation via the proportionality constant does not require the condition $E_{kin} = E_{kin}\left(\dot{x}\right)$ and $E_{pot} = E_{pot}\left(x\right)$.

The Euler-Lagrange equations are well known; yet the non-trivial fact is that they have been inferred here as corollaries of (1.2) and (1.1), which are the conceptual root of both relativistic and quantum physics. Moreover this result supports the present way to regard the concept of derivative as a ratio of uncertainty ranges. Follow now two checks of the present way of reasoning.

(i) The classical Newton law inferred in the section 2.2 seemingly does not account for the finite propagation rate of any perturbation or interaction. Actually such information is explicitly available writing

$$\delta\dot{p} = -\frac{n\hbar}{\delta x^2}\delta\dot{x} = -\frac{\delta\varepsilon\delta t}{\delta x^2}\delta\dot{x} = -\frac{\delta\varepsilon}{\delta x^2}\delta t\delta\dot{x} = -\frac{\delta\varepsilon\delta\dot{x}}{\delta x^2 c}c\delta t = -\frac{\delta\varepsilon}{\delta x^2 c}\delta s\delta\dot{x}, \quad \delta s = c\delta t. \tag{4.8}$$

The force defined in this way is related to an energy $\epsilon' \le \epsilon \le \epsilon''$, vanishes with a δx^{-2} law, is proportional to the deformation rate $\delta\dot{x} \ne 0$ of the space time range δx, is positive or negative depending on whether δx swells or shrinks as a function of time and vanishes for $\delta s \to 0$; *i.e.* the force is defined within $\delta s \ne 0$. It reasonably means that a time range δt is necessary in order to allow its propagation at distance δs, outside which the force in null. The fact that δs has been defined via *c* means the carrier of the force must be a virtual photon or a graviton or anyway a massless particle propagating at speed *c*. Note that instead the classical Gm_1m_2/ℓ^2 has the form of a force propagating instantaneously because it is based only on (2.9) and (2.10) without implementing (1.2). Actually (2.11) avoids itself this error because it is expressed via the uncertainty range $\delta\ell$, not via the deterministic ℓ; since (1.2) involve inherently δt, (2.11) could have been written itself as $Gm_1m_2\delta p_\ell/(\delta\epsilon_\ell\delta t_\ell)$ thus involving anyway the time range δt_ℓ governing its propagation. This holds of course for any force. As concerns the gravity note that also now it is possible to repeat for (4.8) the considerations introduced for (2.11): $\delta s\delta\varepsilon = mass \times length^3 \times time^{-2}$, *i.e.* from a dimensional point of view $\delta\varepsilon\delta s\delta\dot{x}c^{-1} = mass^2 \times G$. Hence, whatever the actual form of the function $\delta\varepsilon\delta s\delta\dot{x}c^{-1}$ might be, it is reasonable to regard its series expansion whose first order term is a constant; if so, then neglecting for the moment the higher order terms, it is possible to write

$$\delta\dot{p} = \pm G\frac{m^2}{\delta x^2} + \cdots = \pm G\frac{m_1m_2}{\delta x^2} + \cdots. \tag{4.9}$$

This formula is formally similar to (2.11), yet it incorporates the idea of a non instantaneous long term force that worried Newton himself. The form of the higher order terms will be concerned later, see next (5.114).

(ii) Consider eventually that (2.12) yields with the help of (1.2) and (4.2)

$$\dot{v} = \frac{\delta\dot{\varepsilon}\delta p - \delta\varepsilon\delta\dot{p}}{\delta p^2} = \frac{\delta\dot{\varepsilon}}{\delta p} - \frac{vF}{\delta p} = -\frac{n\hbar}{\delta p\delta t^2} - \frac{vF}{\delta p} = -\frac{\delta x}{\delta t^2}\left(1 + \frac{vF\delta t^2}{n\hbar}\right)$$
$$= -\frac{\delta x}{\delta t^2}\left(1 + \frac{F\delta\ell}{\delta\varepsilon}\right) = -\frac{\delta x}{\delta t^2}\left(1 + \frac{\delta\varepsilon_\ell}{\delta\varepsilon}\right)$$

where

$$F = \delta\dot{p}, \quad \delta\ell = v\delta t, \quad \delta\varepsilon_\ell = F\delta\ell, \quad \delta\varepsilon = \frac{n\hbar}{\delta t}. \tag{4.10}$$

In summary it is possible to write this result as

$$\dot{v} = -\frac{\delta\varphi}{\delta x}, \quad \delta\varphi = \frac{\delta x^2}{\delta t^2}\left(1 + \frac{\delta\varepsilon_\ell}{\delta\varepsilon}\right), \tag{4.11}$$

where φ is a new function having physical dimensions *velocity*^{2}; with vector notation the first equation reads

$$\dot{\mathbf{v}} = -\nabla\varphi. \tag{4.12}$$

This definition, inferred here as a corollary, was taken in [13] as a basis to infer special and general relativity; the sign of $\delta\varphi$ depends on whether $\delta\varepsilon_\ell/\delta\varepsilon \gtreqless -1$. Moreover (4.11) also reads according to (2.8)

$$\dot{v} = \frac{\varepsilon c}{\hbar} = -a_x\left(1+\frac{\delta\varepsilon_\ell}{\delta\varepsilon}\right), \quad a_x = \frac{\delta x}{\delta t^2}, \tag{4.13}$$

where a_x has physical dimensions of acceleration. It is immediate to acknowledge that φ of (4.11) is the definition gravitational potential [13], which will be more specifically concerned in the next subsection 5.5. Also, $\dot{v}$ is not simply a_x but includes a further addend $-a_x\delta\varepsilon_\ell/\delta\varepsilon$.

These concepts, more systematically examined in the next sections, have been preliminarily introduced to show the validity of the definition (1.1) of space time, which will be implemented next according to (1.3).

Space Time Curvature

Consider (3.20) under the particular condition where the velocity v^2 can be expressed as follows

$$v^2 = a\delta\ell, \quad v = v(\delta\ell,\delta t), \quad \delta\ell = \delta\ell(\delta t), \tag{4.14}$$

being *a* acceleration by dimensional reasons. Hence (3.20) yields

$$\delta\left(\frac{\hbar G}{v^2}\right) = \frac{\hbar G}{a}\delta\left(\frac{1}{\delta\ell}\right) - \frac{\hbar G}{a^2\delta\ell}\delta a = \frac{\hbar G}{a}\delta\left(\frac{1}{\delta\ell}\right) - \frac{\hbar G}{v^2}\frac{\delta a}{a} = \frac{\hbar G}{a}\left(\delta\left(\frac{1}{\delta\ell}\right) - \frac{\delta a}{v^2}\right).$$

Summarizing therefore this result as

$$\delta\left(\frac{\hbar G}{v^2}\right) = \frac{\hbar G}{a}\left(\frac{1}{\delta\ell_1} - \frac{1}{\delta\ell_2} - \frac{\delta a}{v^2}\right), \quad \delta\left(\frac{1}{\delta\ell}\right) = \frac{1}{\delta\ell_1} - \frac{1}{\delta\ell_2}, \tag{4.15}$$

by definition of uncertainty range $(\delta\ell)^{-1}$, there are in principle three chances. The first one is that

$$\delta\left(\frac{\hbar G}{v^2}\right) = \frac{\hbar G}{a_0}\left(\frac{1}{\delta\ell_1} - \frac{1}{\delta\ell_2}\right), \quad a \equiv a_0, \quad \delta a = 0, \tag{4.16}$$

with notation emphasizing that a is a constant. Moreover are also possible for (4.15)

$$\delta\left(\frac{\hbar G}{v^2}\right) = -\frac{\hbar G}{a}\frac{1}{\delta\ell_2}, \quad \frac{1}{\delta\ell_1} = \frac{\delta a}{v^2}$$

and

$$\delta\left(\frac{\hbar G}{v^2}\right) = \frac{\hbar G}{a}\frac{1}{\delta\ell_1}, \quad \frac{1}{\delta\ell_2} = -\frac{\delta a}{v^2}.$$

Clearly a is the acceleration describing the change of space time swelling rate as a function of time; since the time is inherent the physical dimension of

$\hbar G$ appear natural the positions (4.14). In the particular case (4.16) where a is constant, it is possible to write

$$v^* = \frac{a_0}{v^*}\delta\ell, \quad v^* = \frac{\delta\ell}{\delta t^*}, \quad \delta t^* = \frac{v^*}{a_0}, \tag{4.17}$$

whereas (4.15) becomes

$$\delta\left(\frac{\hbar G}{v^2}\right) = \frac{\hbar G}{a_0}\left(\frac{1}{\delta\ell_1} - \frac{1}{\delta\ell_2}\right). \tag{4.18}$$

Both $\delta\ell_1$ and $\delta\ell_2$ are arbitrary, in particular these boundaries of the range $(\delta\ell)^{-1}$ can be positive or negative; the resulting sign of mean that for any physical reason $\delta\ell$ expands or shrinks with constant acceleration a_0 as a function of time. Implementing again the arbitrariness of range boundaries, the last equation reads

$$\delta\left(\frac{1}{\delta\ell}\right) = \pm\mathcal{R}, \quad \mathcal{R} = \frac{1}{\delta\ell_2} \pm \frac{1}{\delta\ell_1}, \tag{4.19}$$

being $\mathcal{R}$ the Laplace-like curvature radius of space time with principal curvature radii $\pm\delta\ell_1$ and $\pm\delta\ell_2$. In general the signs of these radii depend on the specific problem [14], e.g.: for a liquid droplet in a gaseous environment are both positive, for a gas bubble in a liquid environment both negative, for a liquid meniscus between solid cylinders with saddle-like geometry one positive and one negative. So it is not surprising that in principle all chances have been found in the present general approach starting from first principles as concerns the space time swelling.

It is significant anyway that the concept of space time curvature is definable in a natural way even in the present quantum/relativistic context through the concept of uncertainty range. It is instructive in this respect the crucial role of (1.2) in linking quantum and relativistic points of view. Consider two remarks.

(i) Consider (1.2) to express in particular radial range size δr and conjugate radial momentum range size

$$\delta r = \frac{n\hbar}{\delta p_r}. \tag{4.20}$$

Specify this equation as done in (2.24) and (2.25) *i.e.* implementing the De Broglie definition $p_r = h/\lambda_r$ of radial momentum, corollary itself of (1.2) [15]. Accordingly write

$$2\pi\delta r = \frac{nh}{\delta p_r} = \frac{n}{\delta(p_r/h)} = \frac{n}{\delta(1/\lambda_r)}; \tag{4.21}$$

then also consider that by definition $\delta\lambda_r^{-1}$ is nothing else but $\lambda_r'^{-1} - \lambda_r''^{-1}$, whatever the range boundaries λ_r' and λ_r'' might be. Hence (4.21) turns into

$$2\pi\delta r = \frac{nh}{\delta p_r} = \frac{n}{1/\lambda_r' - 1/\lambda_r''} = \frac{n}{\mathcal{R}}, \quad \mathcal{R} = \frac{1}{\lambda_r'} - \frac{1}{\lambda_r''}, \tag{4.22}$$

with notation emphasizing again that $\mathcal{R}$ is clearly the Laplace curvature radius according to the reasoning carried out to infer (4.19); this in turn supports the condition (4.14).

(ii) In general the space time size change rate $\delta\dot{x}$, as defined in (4.2), is related to the rising of a force field $\delta\dot{p}$ within δx itself; so write via (4.10)

$$\delta\dot{x} \lesseqgtr 0, \quad \delta\dot{p} = \frac{\delta}{\delta t}\delta p \gtreqless 0: \tag{4.23}$$

the sign of the force field $\delta\dot{p}$ depends on whether δx swells or shrinks for any physical reason, whereas the force field is null if $\delta\dot{x} = const$. Then, combining this result with the second (2.9) one finds

$$\frac{\delta}{\delta t}\delta x = -\frac{n\hbar}{\delta p^2}\frac{m\epsilon c}{\hbar} = -\frac{nm\epsilon c}{\delta p^2} = -\frac{m\epsilon c}{n\hbar^2}\delta x^2 = -force\frac{\delta x^2}{n\hbar} \tag{4.24}$$

that in turn reads

$$force = -\frac{n\hbar}{\delta t}\frac{\delta(\delta x)}{\delta x^2} = -\delta\varepsilon\left(\frac{\delta x_2}{\delta x^2} - \frac{\delta x_1}{\delta x^2}\right) = -\delta\varepsilon\left(\frac{1}{\delta x'} - \frac{1}{\delta x''}\right) = -\delta\varepsilon\mathcal{R}, \quad \delta\varepsilon = \frac{n\hbar}{\delta t}; \tag{4.25}$$

the last equality is legitimated in analogy with (4.19) once having defined by dimensional reasons $\delta x'^{-1} = \delta x_2/\delta x^2$ and $\delta x''^{-1} = \delta x_1/\delta x^2$ whatever δx_1 and δx_2 might be. The interesting fact is that in the second (2.9) the concept of force was directly related to that of acceleration $\epsilon c/\hbar$, here the same force is related to the concept of curvature via the space time ranges $\delta x'$ and $\delta x''$ replacing the acceleration. The only possible conclusion is that Newtonian concept of force and relativistic concept of space time curvature are equivalent in describing the concept of *force*. The Einstein intuition becomes corollary of the quantum uncertainty.

5. The General Relativity as a Corollary

Some relevant concepts of general relativity are quoted in this section to show how to generalize the approach hitherto followed for the special relativity. Are examined in particular further significant implications of the quantum uncertainty ranges, to show how both special and general relativity contextually merge in a unique non-local and non-real conceptual frame. Some hints in this respect have been early examined in [16]; further topics are here reminded along with new considerations just to point out what have to do these typical concepts of quantum theory with the gravitational field. Indeed the problem of quantum gravity involves non only the quantization of this field according to the distinctive concept of superposition of quantum states, but also the inherent concepts non-reality and non-locality.

5.1. The So-Called "EPR Paradox"

It has been shown in the subsection 2.8 that the wave formalism is a corollary of (1.2) together with the relativistic properties inferred in subsections 2.5, 3.1 and

3.2; these results make in principle the present model compatible with the standard answer of wave quantum mechanics to the paradox. Yet, although it would be legitimate to skip additional comments to the ample literature already existing on this topic, it is instructive to emphasize the distinctive contribution provided to the paradox by the present model based on the quantum uncertainty. Deserve attention the following crucial points of this theoretical framework: (i) the concept of uncertainty ranges replacing the local dynamical variables is in principle compatible with the concept of entanglement; (ii) the difficulty of superluminal distance is bypassed, because the deterministic concept of distance between physical objects is unphysical; (iii) the concept of non-locality reduces to that of unpredictable randomness of particles confined and delocalized in quantum uncertainty ranges and excludes any kind of local information; (iv) by consequence of (iii), the concept of "non-locality" is strictly related to that of "non-reality".

Consider two particles, whose delocalization is in principle possible either in their own independent uncertainty ranges or in one shared uncertainty range. In the first case the particles in δx_1 and δx_2 are in general non-interacting, e.g. any physical reason that deforms δx_1 like in (4.23) does not necessarily affect δx_2, so that the force field $\delta \dot{p}_1$ in δx_1 does not imply $\delta \dot{p}_2$ in δx_2 too: the particle in δx_1 experiences a force, whereas that in δx_2 does not, *i.e.* the particles do not interact. In the second case a unique delocalization range $\delta \dot{x}$ also results from the way the particles interact, even if no external perturbation causes or affects $\delta \dot{p}$: this is the typical case of (3.12) where pairs of virtual particles with opposite charges and spins are generated by vacuum energy fluctuation. First of all, the modulus v of velocity of any particle in a given point of space time cannot be specifically local velocity because are missing by definition both the space coordinates x_1 and x_2 from which to which a particle moves and the time coordinates t_1 and t_2 defining the displacement time lapse; it follows that it is unphysical to define velocity and distance and thus superluminal distances. In fact two particles confined in δx are neither far away nor close each other, they simply *are* in δx. This agrees with the Aharonov-Bohm effect [17] simply acknowledging that one particle is neither "here" or "there", rather it is simply everywhere. This holds even though the particles are delocalized in different δx_1 and δx_2: as the boundary coordinates of uncertainty ranges are arbitrary, certainly the impossibility of determining distances and velocities holds identically also for two particles in their own uncertainty ranges. Hence do not exist "spooking actions at a distance" but rather "actions at a spooky distance": once having renounced to the classical determinism and accepted (1.2) there is no way to distinguish the behavior of particles far apart or close each other confined in a given delocalization range, whatever their interaction mechanism might be. Particularly interesting is the former case of two entangled particles born within a unique uncertainty range where, for example when one γ photon decays by interacting with a nucleus or via vacuum fluctuation, e.g.

[18] [19]; in fact the latter chance found in (3.9) reasonably agrees with (2.2). Accordingly, in the conceptual frame based on (1.1) and (1.2) the EPR paradox shouldn't even be formulated: the present model is inherently non-local by definition. Moreover the agnosticism of (1.2), not purposely invoked here but assumed since the beginning as the unique leading idea of the present physical model, implies a conceptual gap in (3.1) between elusiveness of (1.1) and reality of (3.9); the former is mere dimensional definition of the framework allowed for latent events, the latter made feasible by the measure process breaks the latency of possible events. Since nothing is "a priori" known about m_{v} and m_{v}^{*}, e.g. number of pairs or energy and lifetime of pairs and so on, the present model is inherently also non-real by definition. In other words the physical agnosticism implied by the concept of uncertainty as hitherto exposed, corresponds to the non real essence of the quantum world before the experiment; hence one must accept the idea that also the relativistic properties hitherto inferred are subjected to the same non-weird but logical consequences of (1.2) without need of postulating any "collapse" of wave function into a well defined quantum state. It means that n introduced in (1.2) and next appearing in (3.9) remains arbitrary and undefinable until when the measurement converts it into a specific n_{obs}; in turn, the wave formalism allows calculating the probabilities inherent the superposition of allowed states. Consider now the orientation of the possible spins of the particles with respect to an arbitrary direction. When measured, their spin orientation must yield a total angular momentum equal to zero like that of the empty space time (1.1) before the vacuum fluctuation (3.11). Physical information in this respect is provided only by the angular momentum conservation law, which however presupposes a measurement process. In general this is a perturbation action that affects the quantum state of any particle. In particular, being both particles in the same δx, the measure process perturbs the system of entangled particles wherever they might be, not either particle only. If for any physical reason the shared δx is modified, then the consequent $\delta\dot{x}$ implies $\delta\dot{p}$ and thus a force field in δx that in turn perturbs the couple of particles. Is clear at this point the connection of the present reasoning with the possible spins of m_{v} and m_{v}^{*} introduced in (3.9); in a certain sense the concept of entanglement is here stronger than usually intended once having removed the idea of "superluminal" or relativistic "luminal" distance, no longer conceived as separate and mutually excluding distinguishable situations. With the language of wave formalism, the quantum state of two entangled particles is a superposition of luminal and superluminal states. Nonetheless the conclusion is the same: the most controversial premise of the entanglement, the simultaneous perturbation linking particles infinitely apart via spooky action, is here automatically removed. The answer provided by the total agnosticism of (1.2) can bypass also the simultaneity inherent inherent entangled pairs, as proposed by the mere wave formalism, and suggests a further implication. The EPR paradox, conceived to demonstrate the inadequacy of wave mechanics, demonstrates instead the in-

adequacy of the deterministic metrics that fail explaining via tensor calculus the correlations between entangled particles; the experimental data show indeed that the relativity needs the "external" contribution of the wave formalism introducing the concept of entanglement. The present model explains both relativistic results and wave formalism while removing in principle the paradox itself. This is because the total agnosticism of (1.2) makes the relativity non-local and non-real itself. On the one hand the concept of non-locality, unpredictable randomness of particles delocalized in quantum uncertainty ranges excluding any kind of local information, is in turn related to that of "non-reality": in fact this idea does not violate any relativistic principle, apart from its out of place local determinism. On the other hand it is not crucial whether the spins of entangled particles are actually aligned or counter-aligned inside δx before the correlation experiment, rather it is only required that they are in fact measured counter-aligned after the correlation experiment. In this sense the experiment creates the reality fulfilling the angular momentum conservation although starting from any undefined and undefinable state, be it wave/corpuscle duality or dead/alive states of Schrödinger's cat or luminal/superluminal distance. Eventually appears clear the task of the present section: to find relativistic results without starting from a deterministic metrics, whatever it might be. To confirm that all of these considerations hold also for the relativity, the next subsections concern a few selected topics purposely chosen to emphasize the role of the quantum uncertainty in the general relativity: the latter is in fact a corollary of the former. The most important point in this respect is the equivalence principle, which is soon examined first in the section below.

5.2. The Equivalence Principle

Two relevant results previously obtained, Equations (2.9) to (4.25), address directly to Einstein's equivalence principle, as it has been explained through the simple reasoning early concerned in [9]: the reasoning is so crucial and short to deserve being sketched here for completeness. Think a space time uncertainty range $\delta x = x_2 - x_1$ with time dependent size, and two observers sitting on the boundary coordinates of this range. Let for example the lower boundary $x_1 = x_1(t)$ be defined with respect to the origin of an arbitrary reference system *R*, *i.e.* it defines the "position" of δx in *R* at a given time, whereas x_2 is a fixed coordinate that defines the "size" of δx. Although neither information is actually definable and accessible, it appears in principle that if $x_1(t)$ is subjected to change as a function of time for any physical reason, then the size change rate $\delta \dot{x}$ of δx is related to the rising of a force acting on a particle possibly delocalized in δx; indeed (4.23) and (4.10) predict a local force field $\dot{p}$ whose strength falls within a range of forces $\delta \dot{p} = \dot{p}_2 - \dot{p}_1$. More specifically, is interesting the following chain of equations inferred with the help of (4.2) and (2.9)

$$\delta \dot{x} = -\frac{n\hbar}{\delta p^2}\delta \dot{p} = -\frac{\delta x}{\delta p}\frac{m\varepsilon c}{\hbar} = -\frac{\delta x}{\delta p}ma = -\frac{\delta x^2}{n\hbar^2}m\epsilon c = -\frac{\delta x^2}{n\hbar}\,force: \qquad (5.1)$$

hence, owing to (4.2),

$$force = n\hbar\frac{\delta\dot{x}}{\delta x^2} = -n\hbar\frac{\delta}{\delta t}\left(\frac{1}{\delta x}\right) = -\frac{\delta}{\delta t}\delta p, \tag{5.2}$$

i.e. the force field $\delta\dot{p} = m\epsilon c/\hbar$ due to the space time deformation rate $\delta\dot{x}$ is repulsive or null or attractive depending on $\delta\dot{x} \lesseqgtr 0$ respectively. The chances of $\delta x(t)$ are swelling, shrinking, remaining constant. Also, as

$$m\frac{\epsilon c}{\hbar} = -n\hbar\frac{\delta}{\delta t}\left(\frac{1}{\delta x}\right), \tag{5.3}$$

(5.1) yields with the help of (1.2)

$$-\hbar\delta\left(\frac{1}{\delta x}\right) = \frac{m\epsilon c\delta t}{n\hbar} = \frac{m\epsilon c}{\delta\epsilon} = \frac{\epsilon\delta t}{n\lambda_m} = \frac{\epsilon}{\lambda}\delta t, \quad \lambda = n\lambda_m;$$

the last position has been explained about (2.22) when commenting the eq (4.23). The fact that *force* of (5.2) results equal to $n\hbar$ times the left hand side of the chain shows the quantum nature of this force. Eventually the right hand side of (5.3) shows the geometrical essence of an attractive force, indeed

$$\frac{\hbar}{\delta t}\mathcal{R} = \frac{-\epsilon}{\lambda}, \quad \mathcal{R} = \delta\left(\frac{1}{\delta x}\right) = \frac{1}{\delta x_2} - \frac{1}{\delta x_1}$$

in agreement with the second (2.19). Clearly $\mathcal{R}$ is the space time Laplace-like curvature radius corresponding to the attractive gravity force $-\epsilon$ of (2.11), as explained in (4.19).

Once having expressed the deformation of space time in terms of range size change rate $\delta\dot{x}$, return now to the Einstein equivalence principle considering for simplicity the change of $x_1(t)$ only with constant x_2; this is enough to account for the rising of a force field inside δx and highlight the reasonable conclusions of two independent Newtonian observers sitting on either boundary of $\delta x = x_1(t) - x_2$. The key points are: (i) the observer 1 sitting on x_1 experiences an acceleration since his variable coordinate is defined with respect to the origin of *R*, *i.e.* this observer moves far from or towards to the origin of *R* during the deformation of δx; (ii) the observer 2 feels anyway a force field inside δx although he is at rest in *R*.

Therefore the observer 2 concludes that an external field is acting on δx, whereas the observer 1 acknowledges an acceleration as if his position in *R* would be perturbed by the force field in δx. Once more the consistent conclusion is that in fact the space time deformation rate $\delta\dot{x}$ causes itself the rising of a force field and that an accelerated reference frame is equivalent to such a force field. Only for $\delta x \to 0$ the force field appears as a local classical force. It is immediately evident the role of the quantum uncertainty in this explanation of the concept of force, required by the physical equivalence of the boundary coordinates in lack of any discriminating information about their behavior: indeed $\delta\dot{p}$ is nothing else but a corollary of $\delta\dot{x}$ via (1.2), whereas the conclusions of the two observers are equally valid. External gravity force and space time deformation driven force are indistinguishable because the properties of space boundaries x_1 and x_2 of δx are conceptually arbitrary and unknowable by definition. Note that other forces of nature are directly related the interactions between particles, e.g. the electromagnetic interaction; the gravity force is instead inferred as a property of the space time that manifests under deformation of the uncertainty range sizes. This short discussion allows explaining what have to do the Equations (1.3) and (1.4) via (1.2) with relativistic and quantum physics: the Einstein intuition and thought experiment are now corollaries of the quantum uncertainty.

5.3. Quantum Angular Momentum

This topic has been concerned in [11] [16] [20]. Here are sketched for completeness some selected reminds only, useful later. By definition the component of angular momentum along an arbitrary direction defined by the unit vector $\mathbf{z}$ is $\mathrm{M}_z = \mathbf{r} \times \mathbf{p} \cdot \mathbf{z}$, which reads in the present conceptual frame as $\mathrm{M}_z = \delta \mathbf{r} \times \delta \mathbf{p} \cdot \mathbf{z}$ *i.e.* $\mathrm{M}_z = \delta \mathbf{p} \cdot (\mathbf{z} \times \delta \mathbf{r}) = \delta \mathbf{p} \cdot \delta \mathbf{s}$ where $\delta \mathbf{s} = \mathbf{z} \times \delta \mathbf{r}$. Hence owing to (1.2) the unit vector $\delta \mathbf{s}_z = \delta \mathbf{s} / |\delta \mathbf{s}|$ yields

$$\mathrm{M}_z = \delta \mathbf{p} \cdot \delta \mathbf{s} = \delta p \cdot \frac{\delta \mathbf{s}}{|\delta \mathbf{s}|} |\delta \mathbf{s}| = \delta \mathbf{p} \cdot \delta \mathbf{s}_z |\delta \mathbf{s}| = \delta p_s \delta s = l\hbar; \tag{5.4}$$

here *n* of (1.2) has been replaced by *l* according to the usual notation $l\hbar$ to express the component M_z of ***M***. The problem is now that the direction of $\mathbf{z}$ is arbitrary and unknown; so repeating the reasoning with a different $\mathbf{z}'$ would be physically insignificant, as it does not add anything conceptually new to the given result. The only information available is that *l* is an integer number $\gtreqless 0$ depending on the scalar $\delta \mathbf{p} \cdot \delta \mathbf{s}_z$.

Let us sketch some properties of quantum angular momentum, which will be useful in the next subsection, assuming that $-L \le l \le L$; *i.e.* *l* ranges between two allows values $-L$ and *L*, of course arbitrary, whereas (5.4) holds for any *L*, exactly as done in (1.9). The following considerations emphasize the reasoning carried out in [11], although here the steps to calculate M^2 differ slightly from that therein exposed: consider here that if M_z is the only component knowable, then M^2 must be somehow related to M_z only. Note that

$$\sum_{l=-L}^{L} l = 0, \quad \sum_{l=-L}^{L} |l| = 2\sum_{0}^{L} l = L(L+1), \quad -L \le l \le L. \tag{5.5}$$

The first equality follows by symmetry between the given limits of *l*, while the second equality is straightforward consequence of the first one; this explains why the second equation computes all *l*-th states of the given component as twice the sum from $l = 0$ to $l = L$. Since the angular quantum number *l* is actually a number of allowed quantum states likewise *n* of (1.2), the idea is now that M^2 should be defined by its own quantum angular number of *l* states and that in turn this latter is related to the sum of all *l*-th states allowed to its unique definable component M_z. In other words (5.4) suggests counting all quantum states of M_z included in the range $-L \to L$, *i.e.* summing all positive quantum numbers $|l|$ of allowed states as done in (5.5); this result should be reasonably related to M². To verify this reasoning consider the *j*-th component M_j of **M**, which owing to (5.4) and first (5.5) reads

$$\frac{\mathrm{M}_j^2}{\hbar^2}=\sum_{l_j=-L_j}^{L_j} l_j^2=\left(2L_j+1\right)\frac{L_j\left(L_j+1\right)}{3}. \tag{5.6}$$

Since the number of states from $-L_j$ to L_j is $2L_j+1$, it is possible to calculate the average $\left\langle \mathrm{M}_j^2\right\rangle$ as

$$\frac{\left\langle \mathrm{M}_j^2\right\rangle}{\hbar^2}=\left(2L_j+1\right)^{-1}\frac{L_j\left(L_j+1\right)}{3} \tag{5.7}$$

Although one component only of **M** is calculable, assume reasonably that in a isotropic space time

$$\left\langle \mathrm{M}_x^2\right\rangle=\left\langle \mathrm{M}_y^2\right\rangle=\left\langle \mathrm{M}_z^2\right\rangle; \tag{5.8}$$

taking these averages, (5.7) with equal L_j for all three components yields, as done in [11],

$$\sum_{j=1}^{3}\frac{\left\langle \mathrm{M}_j^2\right\rangle}{\hbar^2}=3\frac{\left\langle \mathrm{M}_j^2\right\rangle}{\hbar^2}=\frac{\mathrm{M}^2}{\hbar^2}=L\left(L+1\right),\quad L_j\equiv L. \tag{5.9}$$

In effect, once having written $\mathrm{M}^2=3\left\langle \mathrm{M}_j^2\right\rangle$, the knowledge of the three components (5.6) reduces in fact to that of one component only; hence it is natural that this result coincides with that of (5.5) expressed in $\hbar^2$ units and confirms (5.8). Follow now three important corollaries.

(i) Implement now these quantum results considering arbitrary numbers *n* of states to describe also properties of orbiting systems. Rewrite thus (5.7) as

$$\frac{\mathrm{M}^2}{\hbar^2}=3\frac{\left\langle \mathrm{M}_j^2\right\rangle}{\hbar^2}=\left(2n+1\right)n\left(n+1\right),\quad 2n+1=\frac{\mathrm{M}^2/\hbar^2}{n(n+1)}; \tag{5.10}$$

i.e. $\mathrm{M}^2/\hbar^2$ consists of $2n+1$ states, from $-L$ to L. An immediate corollary of the third (2.9) reads

$$\frac{m\varepsilon c\delta\ell}{\hbar}=ma\delta\ell=energy$$

and thus

$$\frac{\varepsilon\delta\ell}{\hbar c}=\frac{energy}{mc^2}\ge 1; \tag{5.11}$$

indeed mc^2 is rest energy, whereas *energy* denotes in general the dynamic energy of a physical system. Thus, since according to (2.11)

$$\varepsilon^2=\varepsilon_G^2=\left(G\frac{m^2}{2\delta\ell}\right)^2,\quad \hbar^2=\frac{\mathrm{M}^2}{L\left(L+1\right)},$$

(5.11) squared reads

$$\frac{\left(m^2G/2\right)^2 L\left(L+1\right)}{\mathrm{M}^2c^2}\ge 1$$

so that

$$m^2 \ge \left(\frac{|\mathbf{M}|c}{mG}\right)^2 \frac{4}{L(L+1)}. \tag{5.12}$$

As the minimum non-zero value of $L(L+1)$ is 2, it is possible to infer

$$m^2 > \left(\frac{|\mathbf{M}|c}{mG}\right)^2. \tag{5.13}$$

Eventually, since it is possible to regard $\delta\ell = \delta\ell_m + \delta\ell_q$ as done in (3.25), being q is electric charge, one identifies (5.13) as a spin effect; in an analogous way $\delta\ell_q$ as a charge effect, indeed the analytical form of the Coulomb law is analogous to that of the Newton law. So $\epsilon = \epsilon_m + \epsilon_q$ yields a new addend $\epsilon_q \delta\ell_q = q^2$. Replace thus $\mathrm{M}_z c$ with q^2, as both have physical dimensions $energy \times length$; the same reasoning yields now

$$\left(\frac{|\mathbf{M}|c}{mG}\right)^2 + \left(\frac{q^2}{mG}\right)^2 \le m^2. \tag{5.14}$$

(ii) A further corollary concerns the spin of particles and the Pauli principle. Since in $\hbar^2$ units $L(L+1) \equiv (L+1/2)^2 - 1/4$, it is possible to write

$$\mathrm{M}^2 + \left(\frac{1}{2}\hbar\right)^2 + \left(L+\frac{1}{2}\right)\hbar^2 = \left(L+\frac{1}{2}\right)\left(L+\frac{1}{2}+1\right)\hbar^2 \tag{5.15}$$

i.e.

$$\mathcal{M}^2 = J(J+1)\hbar^2, \quad J = L+\frac{1}{2}, \quad \mathcal{M}^2 = \mathrm{M}^2 + \left(\frac{1}{2}\hbar\right)^2 + \left(L+\frac{1}{2}\right)\hbar^2. \tag{5.16}$$

Note that the left hand side of (5.15) defines an angular momentum in fact allowed, so $\mathcal{M}$ in (5.16) is a half integer angular momentum due to $\hbar/2$, which is clearly by analogy with $l\hbar$ the component along an arbitrary direction of a new half integer angular momentum. In [9] is concerned the spin of particles more in detail starting from (5.15). Here this topic is not further concerned for brevity, e.g. to show why actually $J = L \pm S$; it is interesting instead to remark that the Pauli principle follows as a corollary of (5.16) [20]. This interpretation of the Pauli principle is a crucial consequence of the fact that l and L are not mere quantum numbers, but numbers of allowed quantum states likewise n of (1.2).

(iii) Consider the following definition of M_z, which reads

$$\hbar = \zeta m v \delta\ell = \zeta m_0 c \delta\ell_0 : \tag{5.17}$$

both equalities express actually the component of angular momentum with ζ accounting for the sin and cos factors. Then owing to (2.35) and (3.3) the second equality reads

$$\hbar = \zeta m_0 c \delta\ell_0 = \zeta \frac{m_0 c^2}{\beta} \frac{\beta \delta\ell_0}{c} = \zeta\varepsilon \frac{\delta\ell'}{c}, \quad \delta\ell' = \beta\delta\ell_0, \quad \varepsilon = \frac{m_0 c^2}{\beta},$$

in agreement with (3.3). Hence

$$\varepsilon = \frac{\hbar c/\zeta}{\delta\ell'} = \frac{e^2}{\delta\ell'} = \frac{e^2}{\beta\delta\ell}, \quad \zeta = \frac{\hbar c}{e^2} = \alpha^{-1} \tag{5.18}$$

and thus

$$\zeta\varepsilon = \frac{\hbar c}{\delta\ell'} = \frac{e^2\zeta}{\delta\ell'} = \frac{e_P^2}{\delta\ell'}, \quad e_P = \frac{e}{\sqrt{\alpha}} \tag{5.19}$$

by definition of Planck charge. The first equality (5.17) yields

$$\hbar = \zeta mv\delta\ell = \hbar\frac{mv^2}{e}\frac{c\delta\ell}{ev}$$

so that

$$\mathcal{A} = \frac{ev}{c\delta\ell''} = \frac{mv^2}{e}, \quad \delta\ell'' = \zeta\delta\ell, \tag{5.20}$$

having merged ζ with the arbitrary range size of $\delta\ell$. This simple reasoning has defined via M_z the fine structure constant, the Coulomb law, the magnetic potential $\mathcal{A}$ and the definition of Planck charge.

5.4. Black Hole

Consider (3.2) rewritten according to (2.20) as

$$\frac{Gm}{\delta\ell} = v^2, \quad \delta\ell = n\lambda\!\!\!^{-}_m = \frac{n\hbar}{mc}, \quad v^2 = \frac{c^2\hbar}{\xi}, \tag{5.21}$$

where v is velocity by dimensional reasons. Let us define now a dimensionless parameter ζ such that

$$\frac{v^2}{c^2} = 1 - \zeta\frac{v^2}{c^2}; \tag{5.22}$$

then, dividing both sides of (5.21) by c^2 one finds

$$\frac{Gm}{c^2\delta\ell} = \frac{v^2}{c^2}, \quad \frac{v^2}{c^2} = \frac{1}{\zeta+1}, \quad v = v(\zeta), \quad 0 < \zeta < \infty. \tag{5.23}$$

To show the physical meaning of (5.22), consider first the particular case $v^* = v(\zeta = 1)$; so (5.21) rewritten with pertinent notation $m = m^*$ and $\delta\ell = \delta\ell^*$ reads

$$\frac{Gm^*}{c^2\delta\ell^*} = 1 - \frac{v^{*2}}{c^2} = \frac{v^{*2}}{c^2} = \frac{1}{2}, \quad v_\pm^* = \pm\frac{c}{\sqrt{2}}, \quad v_-^{*2} + v_+^{*2} = c^2, \quad \zeta = 1, \tag{5.24}$$

and yields

$$\delta\ell^* = \frac{2m^*G}{c^2} \tag{5.25}$$

in agreement with (3.15) and (3.16). Note that (5.24) corresponds to (3.24) already introduced to infer (3.25); in particular one implements here $v_s^2 = v_\ell^2$, so the implications of (5.22) are in fact related to and confirm (3.27). In fact (5.24) allows defining as a function of a unique reference length $\delta\ell_0$ two internal and external surfaces corresponding to $v_\pm^*$ that fulfill (5.25); indeed δt^* implies

$$\delta\ell_{\pm}^{*}=\delta\ell_{0}\pm\frac{c\delta t^{*}}{\sqrt{2}},$$

i.e. two characteristic lengths $\delta\ell_{+}^{*}$ and $\delta\ell_{-}^{*}$ are definable as

$$\delta\ell_{+}^{*}=\delta\ell_{0}+\frac{c}{\sqrt{2}}\delta t^{*},\quad \delta\ell_{-}^{*}=\delta\ell_{0}-\frac{c}{\sqrt{2}}\delta t^{*}. \tag{5.26}$$

Assuming that $\delta\ell_{+}^{*}\equiv\delta\ell^{*}$ of (5.25), then it is possible to calculate $\delta\ell_{0}$; hence (5.26) yields

$$\delta\ell_{+}^{*}=\frac{2m^{*}G}{c^{2}},\quad \delta\ell_{-}^{*}=\frac{2m^{*}G}{c^{2}}-\sqrt{2}c\delta t^{*},\quad \delta t^{*}\le\frac{\sqrt{2}m^{*}G}{c^{3}}, \tag{5.27}$$

the third result being due to $\delta\ell_{-}^{*}\ge 0$. The result (5.25) agrees with that already found in (3.15), which shows that the particular v^{*} defined by (5.24) has really a specific physical meaning. Moreover (5.27) shows the existence an inner event horizon at distance $\delta\ell_{-}^{*}$ with respect to the gravity center of m^{*} compatible with the outer one of (5.25). The event horizon appears here as an outer shell of the black hole of finite thickness, rather than an ideal two dimensional boundary surface. The fact of having implemented uncertainty ranges instead of deterministic metrics excludes in the present model the rising of divergence at the gravity center of the black hole, which is local inner boundary coordinate of the radii $\delta\ell_{+}$ and $\delta\ell_{-}$ and then unphysical.

Encouraged by the particular result (5.27), let us generalize the condition (5.24) by considering instead $\zeta\ne 1$ in (5.23): two different chances of generalization concern reasonably $0<\zeta<1$ and $\zeta>1$.

On the one hand, considering $\zeta<1$ and putting now $v^{*2}=\zeta v^{2}$, (5.22) and (5.23) yield

$$\frac{Gm^{*}}{\left(1-v^{*2}/c^{2}\right)\delta\ell^{*}}=\frac{Gm_{o}}{\delta\ell_{o}}=c^{2},\quad m_{o}=\frac{m^{*}}{\sqrt{1-v^{*2}/c^{2}}},\quad \delta\ell_{o}=\delta\ell^{*}\sqrt{1-v^{*2}/c^{2}}: \tag{5.28}$$

these equations have well known meaning consistent with (2.36) regarding

$$m^{*}=m_{rest},\quad \delta\ell^{*}=\delta\ell_{rest}, \tag{5.29}$$

i.e. m_{o} and $\delta\ell_{o}$ are defined in a reference system R_{o} moving at constant rate v_{o} with respect to R where m^{*} and $\delta\ell^{*}$ are at rest. These results agree with and confirm the different reasoning exposed to infer (3.3).

On the other hand, also follows from (5.28) the chance of defining via $\delta\ell^{*}$ the time range δt_{o} such that

$$\frac{\delta\ell_{o}}{c}=\frac{m_{o}G}{c^{3}}=\frac{Gm^{*}}{c^{3}\sqrt{1-v^{*2}/c^{2}}}=\frac{\delta t^{*}}{\sqrt{1-v^{*2}/c^{2}}},\quad \delta t^{*}=\frac{Gm^{*}}{c^{3}}:$$

as expected owing to (3.28), $\delta\ell_{o}/c=\delta t_{o}$ agrees with Lorentz transformation of proper time lapse δt *i.e.*

$$\delta t_{o}=\frac{\delta t^{*}}{\sqrt{1-v^{*2}/c^{2}}} \tag{5.30}$$

with the same physical meaning of (5.28). So the second (5.28) is the well known velocity dependent dynamic mass law with respect to the rest mass, the third one yields the Lorentz contraction of the proper length $\delta\ell_{rest}$. Although (5.24) concerns in particular $v = v_{\pm}^*$ only, the velocity $v_{\pm}^*$ shows the condition under which hold (5.28) and the following (5.25) that describe a law of nature; the same holds for (3.11) implied by (3.3).

Follows now an interesting implication of (5.25). Consider

$$\delta\ell^{*2} = \frac{4m^{*2}G^2}{c^4} = 4m^*G\frac{m^*G}{c^4},$$

which dividing both sides by c^2 yields

$$\frac{\delta\ell^*}{c^2} = \frac{4m^*G}{c^2\delta\ell^*}\frac{m^*}{F_P}, \quad F_P = \frac{c^4}{G} \tag{5.31}$$

and thus

$$\frac{\delta\ell^*}{c^2} = \frac{1}{a^*} = \delta\phi\frac{m^*}{F_P}, \quad \delta\phi = \frac{4m^*G}{c^2\delta\ell^*}, \tag{5.32}$$

being by dimensional reasons a^* acceleration. This way of rearranging (5.31) is significant as, owing to (2.8),

$$\frac{\epsilon^* c}{\hbar} = \frac{c^2}{\delta\ell^*}. \tag{5.33}$$

On the one hand the first equality (5.32) does not conflict with (2.8) as this definition yields

$$\epsilon^* = \frac{\hbar c}{\delta\ell^*} = \frac{\hbar c^3}{2m^*G} = \frac{p_P^2}{2m^*}, \tag{5.34}$$

being p_P the Planck momentum, whereas the second equality reads $F_P = m^*a^*\delta\phi$; hence $\delta\phi$ is consistent with the concept of angle inherent more in general $\mathbf{F}_P = m^*\mathbf{a}^*$. On the other hand (5.32) yields via (2.9)

$$F_P = \frac{m^*c^2}{\delta\ell^*}\delta\phi = \frac{m^*c^2}{\delta\ell^*}\frac{\delta s}{\delta s}\delta\phi,$$

being δs an arbitrary length range; hence

$$F_P\delta s = m^*c^2\delta\phi'\delta\phi, \quad \delta\phi' = \frac{\delta s}{\delta\ell^*}.$$

Regard δs as the length of arc of circumference of the circle osculating the true path that defines the angular deviation $\delta\phi$; so $F_P\delta s$ calculates the work done by the force component F along a curved trajectory. Then

$$F\delta s = m^*c^2\delta\phi = m^*\frac{\epsilon^* c}{\hbar}\delta s = \epsilon^*\frac{\delta s}{\lambda_m^*}, \quad \delta\phi = \frac{4m^*G}{c^2\delta\ell^*} = \frac{F\delta s}{m^*c^2},$$
$$F = \frac{F_P}{\delta\phi'}, \quad \lambda_m^* = \frac{\hbar}{m^*c}, \tag{5.35}$$

having specified $\delta\ell = \delta s$ in the third (2.9) for obvious reasons. The second equation is the Einstein result of light beam bending in the gravity field of m^*; the deviation angle $\delta\phi$ is also equal the ratio between the work $F\delta s$ done by the field along the photon path δs and the rest energy m^*c^2 of the field source. Moreover it appears that $\epsilon^* = F\lambda\!\!\!^{-}{}_m^*$, *i.e.* the energy ϵ^* corresponding to F involves the Compton length of m^*. The mathematical approximation inherent the Einstein result corresponds here to having assumed δs as a circumference arc instead of a more general curved arc of the actual photon trajectory.

Now note eventually an interesting corollary of (5.33) that reads $\epsilon^* = \hbar c/\delta\ell^*$ and is identically rewritten as

$$\epsilon^* = \frac{\hbar c}{\delta\ell^*} = \frac{\hbar c z^2}{\delta\ell^* z^2} = \frac{\hbar c}{z^2}\frac{z^2}{\delta\ell^*} = \frac{1}{\alpha}\frac{z^2}{\delta\ell^*}, \quad \alpha = \frac{z^2}{\hbar c}, \tag{5.36}$$

where z is a new arbitrary parameter not yet introduced in the present model to be appropriately defined in agreement with (5.18). Simply renaming z as $z = \pm e$, where both signs are compatible with $z^2 = e^2$, it follows that $\pm e$ is the electric charge, whereas the proportionality factor α linking the Coulomb law hidden in ϵ^* via $\hbar c$ is actually the fine structure constant. This last result is closely related to the results from (5.17) to (5.20) previously found. Emphasize now that the particular condition (5.24) is sensible, although it has been introduced preliminarily just in order for (5.23) to match (3.17) and not as a consequence of a fundamental requirement; yet (5.24) can be generalized while regarding (5.28) as mere particular case. The key point is to replace (5.23) via a function $\mathcal{F}$ of ζ and δr_{bh} defined as follows

$$\left(\delta r_{bh} - \frac{m^*G}{c^2}\right)^2 - \left(\frac{m^*G}{c^2}\right)^2 = \mathcal{F}(\zeta) = \mathcal{F}(\zeta') + \mathcal{F}(\zeta''), \quad \mathcal{F}(\zeta = 1) = 0; \tag{5.37}$$

The condition on $\zeta = 1$ corresponds to (5.24), because then this equation admits solutions $\delta r_{bh} = 0$ and again $\delta r_{bh} = 2m^*G/c^2$. Discarding the null solution, δr_{bh} is just that expected according to (MJM) pertinent to $\zeta = 1$, in agreement with (5.25). To generalize (5.24) it is enough to consider (5.37) with $\mathcal{F}(\zeta') = -\mathcal{F}(\zeta'')$ for $\zeta = 1$ and with any $\mathcal{F}(\zeta') \neq -\mathcal{F}(\zeta'')$ for $\zeta \neq 1$, which in principle it is possible with both $0 < \zeta' < 1$ and $1 < \zeta'' < \infty$. This generalized equation can be rewritten putting respectively

$$\mathcal{F}(\zeta') = \left(\delta r'_{bh} - \frac{m^*G}{c^2}\right)^2 - \left(\frac{m^*G}{c^2}\right)^2, \quad \mathcal{F}(\zeta'') = \left(\delta r''_{bh} - \frac{m^*G}{c^2}\right)^2 - \left(\frac{m^*G}{c^2}\right)^2. \tag{5.38}$$

On the one hand it is reasonable to assume that these equations concern two different properties of δr_{bh}; on the other hand it is also reasonable to guess that two properties of the mass m^* can be charge and angular momentum due to its possible angular spinning or to its possible spin or both. Anyway, since the uncertainty ranges at left hand side represent square lengths, it is immediate to conclude that the same holds for the right hand size terms; in other words, to include the charge terms it is enough to express the space range sizes that appear in the Coulomb law of (5.36). So, in Planck units, $Q^2/\ell^2 = c^4/G$ yields $\mathcal{F}(\zeta') = \ell_Q^2 = Q^2G/c^4$ according to (5.36). An analogous reasoning for the rotation of m^* via the angular momentum $\mathbf{J}$ reads $\mathcal{F}(\zeta'') = \ell_J^2 = J^2/(m^*c)^2$.

Replacing both contributions (5.38) into (5.37), whose left hand side is <0 because $\left(m^* G/c^2\right)^2 > \left(m^* G/c^2 - \delta r_{bh}\right)^2$, one finds thus

$$\left(\delta r_{bh} - \frac{m^* G}{c^2}\right)^2 - \left(\frac{m^* G}{c^2}\right)^2 = -\frac{Q^2 G}{c^4} - \frac{J^2}{\left(m^* c\right)^2},$$

with appropriate signs at the right hand side. This yields the well known result consistent with (5.14)

$$\delta r_{bh} = \frac{G}{c^2}\left(m^* + \sqrt{m^{*2} - \frac{Q^2}{G} - \frac{J^2 c^2}{\left(m^* G\right)^2}}\right). \tag{5.39}$$

5.5. From Special to General Relativity

Rewrite the first (3.32) as

$$2nc\frac{\delta t}{\delta r} = \delta\left(\frac{v^2}{c^2}\right) = \frac{v_1^2}{c^2} - \frac{v_0^2}{c^2} \tag{5.40}$$

having simply implemented at the right hand side the definition of $range = \delta\left(any\ \ function\right)$. Note that the possible chance $n \to \infty$ requires $\delta t \to 0$ or $\delta r \to \infty$ or both, because the right hand side of (5.40) is anyway finite being $v_1 \le c$ and $v_0 \le c$. This problem requires defining appropriately δr and δt as done with (3.33) to infer the relativistic property (3.38) of *c*. Consider that any range size tending to zero implies in fact the classical determinism for the concerned variable: *i.e.* $t_0 \le t \le t_1$ means that $t \to specific\ local\ value$ for $t_1 \to t_0$, whatever t_0 and t_1 might be. This conclusion holds for all quantum variables and ensures here that (5.40) is definable for any *n* implementing the positions (3.35), *i.e.* replacing δt and δr as follows

$$2c\frac{\delta t_n}{\delta r_n} = \delta\left(\frac{v^2}{c^2}\right), \quad \delta r_n = \frac{\delta r}{\sqrt{n}}, \quad \delta t_n = \delta t\sqrt{n}; \tag{5.41}$$

as of course δt_n and δr_n are still arbitrary, it is possible to implement likewise (5.41) or (5.40) examining some possible cases where v_0 and/or v_1 are in particular constant or more in general any functions of *n*. Anyway note that (5.41) fulfill the invariant condition of special relativity already concerned in (2.47)

$$\delta r_n \delta t_n = \delta r \delta t: \tag{5.42}$$

in fact, being *n* arbitrary likewise δr and δt, it is possible to regard the left and right hand sides as if they would refer to different inertial reference systems. This point will be further clarified below.

1) v_0 and v_1 both constants.

In this case write (5.40) as

$$2nc\frac{\delta t}{\delta r} = const - \frac{v_0^2}{c^2} = const\left(1 - \frac{v_0^2/const}{c^2}\right) = const\left(1 - \frac{v_0^{*2}}{c^2}\right) = const\,\beta^{*2} \tag{5.43}$$

where

$$const = \frac{v_1^2}{c^2}, \quad v_0^* = \frac{v_0}{\sqrt{const}}, \quad \beta^* = \sqrt{1 - \frac{v_0^{*2}}{c^2}}, \tag{5.44}$$

so that (5.43) reads

$$\left(2n\frac{c}{const}\right)\frac{\delta t/\beta^*}{\beta^*\delta r} = 1 = \left(2n\frac{c}{const}\right)\frac{\delta t'}{\delta r'}, \quad \delta t' = \frac{\delta t}{\beta^*}, \quad \delta r' = \beta^*\delta r; \tag{5.45}$$

thus, apart from the factor in parenthesis appearing at both sizes and thus irrelevant, (5.45) implies again

$$\delta r'\delta t' = \delta r\delta t, \tag{5.46}$$

in agreement with (5.42) and the result of the Appendix B. Regarding the primed and unprimed quantities in the respective inertial reference systems R and R' reciprocally moving at constant rate v_0^*, appear again the Lorentz space contraction and time dilation together with the invariant behavior of the product $\delta(space)\times\delta(time)$ of uncertainty ranges. This conclusion is not trivial: (5.46) holds regardless of n in parenthesis of (5.45) and justifies the chance of regarding the uncertainty ranges likewise the local coordinates of relativity, while remarking however that the former only and not the latter fulfill the Heisenberg principle.

2) $v_0 = v_0(n)$ and $v_1 = v_1(n)$.

In this case it is possible to start from (5.41) for further considerations about (5.40). Rewrite (5.41) as

$$2\frac{\delta\ell_n}{\delta r_n} = \delta\left(\frac{v^2}{c^2}\right) = \delta\left(1-\beta^2\right) = -\delta\left(\beta^2\right), \quad \delta\ell_n = c\delta t_n;$$

merging the equalities at the right hand side one finds

$$4\frac{\delta\ell_n^2}{\delta r_n^2} = -2\delta\left(1-\beta^2\right)\beta\delta\beta = -2\delta\left(1-\beta^2\right)\beta^2\frac{\delta\beta}{\beta}$$

$$= -2\beta\delta\left(1-\beta^2\right)\beta\delta\left(\log\beta\right) = -2\beta\beta_0\delta\left(1-\beta^2\right)\frac{\beta}{\beta_0}\delta\left(\log\frac{\beta}{\beta_0}\right),$$

being β_0 an arbitrary constant, and thus eventually

$$4\frac{\delta\ell_n^2}{\delta r_n^2} = -2\beta\beta_0\delta\left(1-\beta^2\right)w\delta\log w, \quad w = \frac{\beta}{\beta_0}.$$

This result is interesting because it takes the final form

$$\frac{2}{Q}\frac{\delta\ell_n^2}{\delta r_n^2} = -w\delta\log w = -w\left(\log w - \log w_0\right), \quad Q = Q(\beta) = \beta\beta_0\delta\left(1-\beta^2\right),$$

having simply replaced $\delta\log(w)$ with the usual notation of uncertainty range $\log(w) - \log(w_0)$ as in (1.7). Since the range boundaries are arbitrary and is reasonable to expect that w_0 is defined by $\beta = \beta_0$, then $w_0 = 1$ yields

$$\frac{\delta\ell_n^{o2}}{\delta r_n^{o2}} = -w\log w, \quad \frac{\delta\ell_n^{o2}}{\delta r_n^{o2}} = \frac{2}{Q}\frac{\delta\ell_n^2}{\delta r_n^2}: \tag{5.47}$$

the interesting fact is that at the left hand side appears the ratio of the square lengths $\delta\ell_n^{o2}$ and δr_n^{o2} whatever the value of the dimensionless factor Q might be, at the right hand side appears the entropy like term $-w\log w$. Since $\beta<1$ by definition, w can be regarded as probabilities via an appropriate choice of the arbitrary constant β_0. It is evident that the arbitrariness of the quantum ranges plays a fundamental role in this respect; indeed, being the range sizes arbitrary by definition, the left hand size ratio is indistinguishable from and thus identically readable as the initial $\delta\ell_n$ and δr_n in turn reducible to the respective lengths of (5.40). This result is helpful for its implications in the next section 5.7

3) Virial theorem.

It is possible to assess now (5.40), which is useful to obtain information whenever the quantization condition of the dynamical variables is explicitly required. Since the boundaries of any uncertainty range are arbitrary by definition, examine the two possible cases where v_0 is in particular a constant or it is in general any function of n. Write (5.40) with the help of (2.35) as

$$2nc\frac{\delta t}{\delta r}=2nc\frac{\delta t}{n\hbar/\delta p_r}=2\delta(p_r c)\frac{\delta t}{\hbar}=\frac{\delta(p_r c)}{\epsilon_r}\omega\delta t$$
$$=\frac{v_r}{c}\frac{\delta(p_r c)}{p_r c}\omega\delta t=\delta\left(\frac{v^2}{c^2}\right)=\frac{v_1^2-v_0^2}{c^2},$$
$$\epsilon_r=\frac{\hbar\omega}{2}$$

having regarded ϵ_r as a zero point energy. So, multiplying by m the fourth equality, one finds

$$\frac{mv_r}{c^2}\frac{\delta(p_r c)}{p_r}\omega\delta t=\frac{\beta_r}{c^2}\delta(p_r c)\omega\delta t=m\delta\left(\frac{v^2}{c^2}\right),\quad mv_r=p_r\beta_r,$$

whereas the second equality reads

$$\beta_r\delta(p_r c)\omega\delta t=m\delta(v^2). \tag{5.48}$$

Here is useful the classical approximation $\beta_r\approx1$ assuming for simplicity $v_r\ll c$; (5.48) simplifies to

$$\delta(p_r c)\omega\delta t=m\delta(v^2)=\delta(mv^2). \tag{5.49}$$

Even (5.49) contains the same classical approximation of (5.48), because (3.3) shows that the dynamic mass becomes in fact a constant. Both steps clarify the physical meaning of δv^2 and $p_r c$, as (5.49) reads

$$\omega\delta t=-\frac{2\delta\mathrm{T}}{\delta\mathrm{U}},\quad \mathrm{T}=\frac{mv^2}{2},\quad \mathrm{U}=-p_r c, \tag{5.50}$$

so that the formal position $p_r c=-\mathrm{U}$ yields

$$\omega\delta t=-\frac{2\delta\mathrm{T}}{\delta\mathrm{U}}=1,\quad 2\mathrm{T}=-\mathrm{U}. \tag{5.51}$$

The positions (5.50) allow to find the expected result $\omega=\delta t^{-1}$, which fits the classical virial theorem $2\mathrm{T}=-\mathrm{U}$ consistent with the second equation; the classical limit of (5.40) has sensible physical meaning.

4) Quantum gravity.

Multiply now both sides of (5.40) by $\hbar/\delta t$ in order to find

$$\frac{n\hbar c}{\delta r}+\frac{\hbar v_0^2}{2c^2\delta t}=\frac{\hbar v_1^2}{2c^2\delta t};\tag{5.52}$$

a further result is now available implementing the definitions

$$\omega=\frac{c}{\delta r},\quad \omega=\frac{v_0^2}{c^2\delta t}\tag{5.53}$$

that yield

$$n\hbar\omega+\frac{1}{2}\hbar\omega=\frac{1}{2}\frac{\hbar v_1^2}{c^2\delta t}=\frac{1}{2}\frac{\hbar v_1^2}{c\delta\ell},\quad \delta\ell=c\delta t\tag{5.54}$$

and merge themselves into

$$\delta r=\frac{c^3}{v_0^2}\delta t=\frac{c^2}{v_0^2}\delta\ell.\tag{5.55}$$

The left hand side of (5.54) diverges for $n\to\infty$; however this is not a problem, being allowed by $\delta t\to 0$.

The first (5.53) is sensible, as it reads $2\pi\nu\delta r=c$, *i.e.*

$$2\pi\delta r=\frac{c}{\nu}=\frac{nv_r}{\nu}=n\lambda_r,\quad v_r=\frac{c}{n},\quad \lambda_r=\frac{v_r}{\nu};\tag{5.56}$$

the given definition of v_r, quantized likewise the number n of respective λ_r, is compatible with $v_r\le c$ and with (5.41). Thus one finds once again the key quantum equation (2.22) and accordingly also

$$\begin{aligned}&\omega=\frac{nv_r}{\delta r},\quad energy=\hbar\omega=n\epsilon,\quad \epsilon=\hbar\frac{v_r}{\delta r}=\hbar\frac{v_r}{v_r\delta t}=\frac{\hbar}{\delta t},\\&\delta r=v_r\delta t,\quad \frac{h}{\lambda_r}=\frac{n\hbar}{\delta r}=\delta p_r\end{aligned}\tag{5.57}$$

and then, owing to the second (5.53), also

$$\frac{nv_r}{\delta r}=\frac{v_0^2}{c^2\delta t}.$$

Since according to (5.41)

$$\frac{\delta r}{\delta t}=n\frac{\delta r_n}{\delta t_n},$$

then

$$v_r=\frac{v_0^2}{c^2}\frac{\delta r_n}{\delta t_n},\quad \omega=\frac{v_{rn}}{\delta r_n},\quad v_{rn}=\frac{v_r}{\sqrt{n}};$$

the last position is coherent with that of (3.35). Eventually (5.41) yields

$$\frac{v_1^2}{c^2}=\frac{v_0^2}{c^2}+2c\frac{\delta t_n}{\delta r_n}=\left(v_r+2c\right)\frac{\delta t_n}{\delta r_n};$$

in this way both positions (5.53) are consistent with (5.41): *i.e.* defining appropriately $\delta r/\delta t$ via $\delta r_n/\delta t_n$ there is no divergence of velocity in (5.40) and (5.54) for $n\to\infty$. A further corollary of the positions (5.53) is found merging them by eliminating *c*: replacing $c=\omega\delta r$ in the second one, two interesting results are

$$\omega^3\delta r^2=\frac{v_0^2}{\delta t},\quad \frac{\omega^3\delta r^2}{c}=\frac{v_0^2}{c\delta t}=a. \tag{5.58}$$

The familiar form at left hand side of the first equation will be useful below, see next (5.71) and (5.72); the second equation provides a further definition of acceleration. Both equations support the results just obtained. Consider now (5.54) noting that the physical dimensions at the right hand side are

$$\frac{\hbar v_1^2}{2c}=mass\times length^3\times time^{-2}, \tag{5.59}$$

whereas at the left hand side appears instead the expression of vibrational harmonic system along with its zero point energy; this suggests that $\hbar v_1^2/c$ should be itself proportional to m^2G by dimensional reasons, *i.e.* $\hbar v_1^2/c=\xi m^2G$, being ξ the proportionality constant. Defining therefore without loss of generality the dimensional square mass as $m^2=m_1m_2$ because *m* is in fact arbitrary likewise as m_1 and m_2 themselves, write for this distinctive reading of the energy defined by (5.53) in agreement with (2.11)

$$n\hbar\omega+\frac{1}{2}\hbar\omega=\frac{\xi}{2}G\frac{m_1m_2}{\delta\ell}=\xi\left(G\frac{m_1m_2}{\delta\ell}-\frac{1}{2}G\frac{m_1m_2}{\delta\ell}\right), \tag{5.60}$$

where it is possible to identify at the right hand side

$$G\frac{m_1m_2}{\delta\ell}=-U,\quad -\frac{1}{2}G\frac{m_1m_2}{\delta\ell}=\frac{U}{2}=-T \tag{5.61}$$

with the help of the quantum virial theorem inferred in (5.51). Then

$$n\hbar\omega+\frac{1}{2}\hbar\omega=\xi\left(-U-T\right),\quad 2T=-U,$$

so that, putting $\xi=1$, the result is

$$n\hbar\omega+\frac{1}{2}\hbar\omega=-H,\quad \xi=1,\quad H=U+T<0,\quad \omega^2=\frac{k_f}{m}. \tag{5.62}$$

The right hand side is the Hamiltonian of the orbiting system, *i.e.* Newtonian binding energy $-\epsilon_G$, which agrees with the idea of harmonic oscillator as a bound system itself. The minus sign of ϵ_G means that the force constant k_f defining the quantum oscillator frequency implies an attractive energy between two orbiting masses; a repulsive energy would be instead inconsistent with steady quantum oscillations. As expected, whatever m_1 and m_2 might be, an appropriate *n* shows the actual quantization of orbital motion: for large masses, *n* is so large that the quantization is hidden by the values $n\approx n+1$, being $\delta\ell$ the circular orbit radius. So the steady harmonic oscillations are the quantum equivalent of the steady orbital motion of a Newtonian system via the circular frequency ω: the force constant k_f governing the one dimensional linear oscillation of the former turns into the behavior of m in a central force field governed by G and seemingly non-quantized. The last equation (5.62) is direct consequence of (5.53) because

$$\omega^2 = \frac{c^2}{\delta r^2} = \frac{mc^2}{m\delta r^2} = \frac{\epsilon/\delta r^2}{m} = \frac{k_f}{m}: \tag{5.63}$$

i.e. classically ω^2 is proportional to m^{-1} via the factor k_f. Moreover, the link between a one dimensional oscillation and a two dimensional system orbiting on an arbitrary plane implies 2 degenerate states, as the clockwise and counter clockwise rotation are both allowed and in principle indistinguishable; this also holds in the quantum world, indeed l takes all values $-n \le l \le n$ identically to $-l$. So the macroscopic measurable orbiting energy (2.11) is twice that $m_1 m_2 G/2\delta\ell$ of (5.60). This degeneracy can be also regarded as a statement of equivalence between inertial and gravitational mass: the degenerate energies concern now the systems where m_1 moves around m_2 or m_2 around m_1, depending on either reference system R_1 or R_2 where the respective mass is at rest. Without the equivalence principle, R_1 and R_2 would not interchangeable, as instead it is true according to the quantum (2.41). The macroscopic Newton law inferred from an oscillating quantum system reveals and requires the sought equivalence.

5) Implications of harmonic oscillations.

As a closing remark consider now the following dimensional definitions

$$m_\omega = \xi \frac{c^3}{\omega G}, \quad \rho_\omega = \xi \frac{\omega^2}{G}, \tag{5.64}$$

being ξ an appropriate proportionality factor plausible in any dimensional equations and thus introduced here; the subscript emphasizes that these definitions implement explicitly the frequency ω. These positions aim to calculate the force constant k_f of the harmonic oscillator implied by (5.54)

$$k_f = m_\omega \omega^2 = \xi \frac{c^3 \omega}{G}. \tag{5.65}$$

In fact (5.64) are alternative to (2.1) and (2.2) in defining via ω mass, density and energy density in an arbitrary volume $\delta\ell^3$. The connection with these equations is given by

$$\omega^2 = \frac{\rho_\omega G}{\xi} = \frac{m_\omega}{\delta\ell^3}\frac{G}{\xi}, \quad \rho_\omega = \frac{m_\omega}{\delta\ell^3} \tag{5.66}$$

and yields

$$k_f = \frac{m_\omega^2 G}{\xi\delta\ell^3} = \frac{\omega^4 \delta\ell^3 \xi}{G} = -\frac{\epsilon_G}{\xi\delta\ell^2} \tag{5.67}$$

so that

$$\epsilon_G = -G\frac{m_\omega^2}{\delta\ell}. \tag{5.68}$$

Moreover (5.64) defines

$$\eta_\omega = \xi \frac{(c\omega)^2}{G}. \tag{5.69}$$

The physical meaning of these results will be highlighted by calculating their numerical values in section 6.

6) Gravitational waves.

This subsection aims to sketch that the gravitational waves are actually quantized and fit the result inferred in (5.60); details on the physical model and results have been already concerned in a paper [16] on this topic. To add further considerations in this respect and highlight this point, let us start from the Einstein formula

$$-\frac{dE}{dt} = \frac{32G}{5c^5}\left(\frac{m_1 m_2}{m_1 + m_2}\right)^2 r^4\omega^6, \quad \omega^2 r^3 = (m_1 + m_2)G \tag{5.70}$$

where r and ω are the deterministic radius and angular frequency of an orbiting body in a circular orbit; in this case holds the second equation, which is direct consequence of the third Kepler law. The explanation of these formulas is reported in [13]; here the priority is remarking how to inspect the energy loss dE by emission of gravitational wave energy. Rewrite (5.70) in order to replace the deterministic orbit parameters r and ω to highlight their quantum meaning hidden in the given formulas and replace the integration factor 32/5 = 6.4 with 2π, which differ by about 1.8% only; this numerical replacement allows highlighting conveniently the following considerations. Regard then the original Einstein result as

$$-\frac{\delta E}{\delta t} = \frac{2\pi}{W_p}\left(\frac{m_1 m_2}{m_1 + m_2}\right)^2 \delta r^4\omega^6, \quad \omega = \frac{2\pi}{\delta t}, \quad W_P = \frac{c^5}{G}; \tag{5.71}$$

all quantities with notation δ are now uncertainty ranges, δt is the time lapse to complete one orbit.

First of all rewrite identically the first equation with the help of the second (5.70) and (5.60) itself as follows

$$-\frac{\delta E}{\delta t} = \frac{2\pi}{W_p}\left(\frac{\epsilon_G \delta r}{\omega^2 \delta r^3}\right)^2 \delta r^4\omega^6 = \frac{2\pi}{W_p}(\epsilon_G\omega)^2; \tag{5.72}$$

moreover replace once more $\delta t = n\hbar/\delta\varepsilon$ according to (1.2), so that

$$-\delta E\delta\varepsilon = n\hbar\frac{2\pi}{W_p}(\varepsilon_G\omega)^2 = nhW_P(t_G\omega)^2, \quad t_G = \frac{\varepsilon_G}{W_P}$$

with notation emphasizing that $energy/W_P$ has physical dimensions of time. Also, this equation becomes

$$-\delta E\frac{\delta\epsilon}{W_P} = -\delta E\delta t_G = nh(t_G\omega)^2, \quad \delta t_G = \frac{\delta\epsilon}{W_P}$$

and eventually

$$-\delta E = nh\nu_G \left(t_G \omega\right)^2, \quad \nu_G = \frac{1}{\delta t_G}, \tag{5.73}$$

with notation emphasizing that the energy loss $-dE$ can be nothing else but loss of some quanta $h\nu_G$. Now it is possible to introduce the last step and compare this result to (5.60), according which $-\epsilon_G = nh\nu + h\nu/2$; therefore $dE = \delta\epsilon_G = h\nu\delta n$ yields

$$-h\nu\delta n = nh\nu_G \left(t_G \omega\right)^2.$$

Just this conclusion is the key to guess the dimensionless $t_G \omega$ that appears to be just a correction factor: being δn integer, $-\delta E$ can be nothing else but something like $n' h\nu$ with n' integer in order to fit (5.60) [21]. Also, as t_G is introduced via $\delta\varepsilon$ and thus arbitrary, put then $\left(t_G \omega\right)^2 = n'/n$ so that

$$-\delta E = n' h\nu_G. \tag{5.74}$$

Otherwise stated, $t_G \omega$ has been defined in order that (5.74) is consistent with (5.62). In synthesis, the initial Einstein formula, deterministic, becomes here a very simple quantum result, showing at the right hand side the number n' of gravitational energy quanta lost. Also here $-\delta E$ expresses the fact that n' must be intended as $n'' \le n' \le n'''$, with n'' and n''' of course arbitrary, once more according to (1.9).

Although for brevity this result has been introduced here as mere elaboration of Einstein's early achievement, reversing the steps from (5.74) to (5.70) one could find the initial δE whose quantization is however hidden. The paper [9] concerns instead an "ab initio" model, where are also described further implications of this result. The Einstein formula is actually a quantum of gravitational energy dissipated by an orbiting system. In this quoted paper, published before the experimental evidence of the gravitational waves, it is remarked that not necessarily the gravitational system must collapse; rather both signs possible for δn describe the exchange of gravitational quanta between orbiting systems, possibly the so called gravitons, could be regarded in principle in analogy with electromagnetic excitation and decay of atoms by exchange of photons. This supports the idea of gravitons inherent the gravity propagation rate (4.8).

7) Quantum remarks on the Newton equation.

At this point some remarks on (5.40) and (5.60) deserve attention.

(i) According to (5.59), (5.60) and (5.62), write

$$\frac{\hbar}{2c} = G\frac{m_1 m_2}{v_1^2}, \tag{5.75}$$

which yields

$$\lambda_{m_1} = \frac{\hbar}{m_1 c} = \frac{2m_2 G}{v_1^2}, \quad \lambda_{m_2} = \frac{\hbar}{m_2 c} = \frac{2m_1 G}{v_1^2}; \tag{5.76}$$

so, taking the limit $v_1 \to c$, one finds respectively

$$\lambda_{m_1} = \frac{2m_2 G}{c^2}, \quad \lambda_{m_2} = \frac{2m_1 G}{c^2}.$$

The black hole radii are thus the limit of (5.40) for $v_1 \to c$ suggest an interesting feature of a bound gravitational system where either mass is a black hole; merging of their masses occurs when the event horizon of the latter approaches the Compton length of the former.

(ii) It is usual to say that at the center of a black hole there is a space time singularity. Emphasizing that no singularity is explicitly required by or directly implied in the present conceptual frame, such a singularity is actually unknowable and thus unphysical: according to (1.2) and (2.41), by definition non-deterministic, no information is accessible about what happens inside an uncertainty range. Thus the concept of local singularity is merely an arbitrary extrapolation allowed in the classical world only; here instead the relativity is conceived in the quantum frame of (1.2).

(iii) Via (5.59) $\hbar v^2/2c = m^2 G$ yields an expression for the force constant k_f of harmonic oscillations (5.62)

$$k_f = \omega^2 m = \frac{\omega^2 \hbar v^2}{2Gmc} = \frac{1}{2}\frac{\lambda\!\!\!^{-}_m}{G} a^2, \quad a = \omega v \tag{5.77}$$

Where a is a further definition of acceleration being $v\omega = \delta x/\delta t^2$. This result, which clearly plugs the force constant of the quantum oscillator into the frame of the general relativity, will be further considered in the next subsection.

(iv) The reduced Compton length $\lambda\!\!\!^{-}_{m1}$ of m_1 defines in (5.76) the Schwarzschild radius of the interacting mass m_2 as a limit case. This result follows the link between the energy of a two body gravitational system of masses and the energy of a quantum oscillator with binding force constant k_f; in turn this link suggests that the quantum relationship between m_1 and m_2 makes their event horizons equal to the respective Compton lengths; these latter correspond to the minimum approaching distance below which the masses merge into a unique black hole of total mass $m_1 + m_2$ with event horizon $2(m_1+m_2)G/c^2$. Indeed

$$\lambda\!\!\!^{-}_{m1} + \lambda\!\!\!^{-}_{m2} = \lambda\!\!\!^{-}_{\mu} = \frac{\hbar}{\mu c}, \quad \mu = \frac{m_1 m_2}{m_{tot}}, \quad \frac{1}{2}\mu c^2 = G\frac{m_1 m_2}{\lambda\!\!\!^{-}_{\mu}} = -U, \quad U < 0. \tag{5.78}$$

(v) The fact that (5.60) is related to an arbitrary number n of quantum states allowed to a harmonic oscillator shows that even the gravity equation at the right hand side can be expressed as a superposition of states corresponding to and defined by the respective n. Start from (5.60) written according to (5.53) as

$$\frac{Gm_1 m_2}{2\delta\ell} - \frac{1}{2}\hbar\omega = n\hbar\omega = (n' - n'')\hbar\omega, \quad \omega = \frac{v_0^2}{c^2 \delta t}, \quad n = n' - n''; \tag{5.79}$$

then, with the last position where n' and n'' are arbitrary integers too, one finds via (5.61) and (1.9)

$$\delta\varepsilon_G = \delta\varepsilon_\psi, \quad \delta\varepsilon_G = T - \frac{1}{2}\hbar\omega, \quad \delta\varepsilon_\psi = n'\hbar\omega - n''\hbar\omega \tag{5.80}$$

In this way the average kinetic energy of Newton orbital motion identified in (5.61) defines the range size $\delta\varepsilon_G$ equal to that $\delta\varepsilon_\psi$ of the quantized energies corresponding to the wave functions $\psi_\ell'^*$ and $\psi_\ell''^*$ defined by (2.58). As concerns *U*, write according to (5.61)

$$-U=\frac{Gm_1m_2}{\delta\ell}=(2n+1)\hbar\omega=\frac{\mathrm{M}_j^2/\hbar^2}{n(n+1)}\hbar\omega.$$

So the average potential energy of the gravitational system is equal to $\hbar\omega$ times the number $2n+1$ of allowed states of angular momentum; in other words the left hand side consists of $2n+1$ degenerate states $\hbar\omega$. At the left hand side still appears the gravitational energy of the same masses now $\delta\ell$ apart, at the right hand side the energy $\hbar\omega_{\ell,n}$ with the same coefficient $2n+1$ due to the degeneracy of l_j states consistent with *n*. Thus Newton's equation is equivalent to a superposition of *l* states of energy $\hbar\omega_{\ell,n}$ having quantum origin.

(vii) Consider (5.25) and write

$$\frac{2m_{bh}G}{c^2}=\frac{2m_{bh}c^2G}{c^4}=\frac{2h\nu_{bh}}{F_P}=\delta\ell_{bh},\quad m_{bh}=\frac{h\nu_{bh}}{c^2},\quad h\nu_{bh}=\frac{1}{2}F_P\delta\ell_{bh};$$

via the Planck force $F_P=c^4/G$. Recalling (5.54), it is possible to regard the last term as a zero point energy and thus to define by analogy a more general energy given by

$$E_n=nh\nu_{bh}+\frac{1}{2}h\nu_{bh}=(2n+1)F_P\delta\ell_{bh}.$$

To interpreter this result, think a set of *n* non-interacting harmonic oscillators consisting of point masses m_{bh} that displace with frequency ν_{bh} by a length $\delta\ell$ with respect to their equilibrium positions: the energy of such a system results averaging the mechanical work of F_P to displace all point masses and their oscillation energy $h\nu_{bh}$. This average $E_n/n(n+1)$ concerns clearly the corpuscle/wave behavior of m_{bh}.

8) The invariance in quantum special relativity.

The starting key equations are now (3.29), which yield

$$p=\frac{\varepsilon v}{c^2},\quad \delta p=\frac{\delta(\varepsilon v)}{c^2};$$

multiplying side by side these equations

$$p\delta p=\frac{(\varepsilon v)\delta(\varepsilon v)}{c^4}$$

one finds

$$\frac{1}{2}\delta(pc)^2=\frac{1}{2}\frac{\delta(\varepsilon v)^2}{c^2} \tag{5.81}$$

and thus

$$\delta(\varepsilon v/c)^2-\delta(pc)^2=0, \tag{5.82}$$

being of course by definition

$$\delta(\varepsilon v/c)^2=(\varepsilon_1 v_1/c)^2-(\varepsilon_2 v_2/c)^2,\quad \delta(pc)^2=(p_1c)^2-(p_2c)^2. \tag{5.83}$$

Clearly (5.82) reads identically $(\varepsilon v/c)^2-(pc)^2=0$, which of course is fulfilled by $p=\epsilon v/c^2$; yet, considering the ranges (5.83) of dynamical variables instead of their local values (2.32), (5.82) is also fulfilled by

$$(\varepsilon v/c)^2-(pc)^2=const \tag{5.84}$$

because $\delta(const)=0$. This equation yields thus

$$(\varepsilon v/c)^2=(pc)^2+const. \tag{5.85}$$

Owing to (3.29), if in particular $const=0$ and $v\neq c$ then (5.85) reduces trivially to an identity. The fact that (3.29) holds even for $v=c$, whereas (2.35) do not, is the key to understand and verify the next step: although both p and ε diverge for $v=c$ according to (2.35), their ranges δp and $\delta\varepsilon$ defined in (5.82) do not. Even for $v=c$ they take the form $\infty-\infty$ that in principle could admit finite limits uniquely defined by (5.84). In effect it has been already remarked that have physical meaning the uncertainty ranges and not the local dynamical variables, random unknown and unknowable according to (1.2). Then, even implementing the particular position $v=c$, (5.85) resulting from (5.82) yields for $const\neq 0$ the well known result

$$\varepsilon^2=const+(pc)^2,\quad const=\left(m_0c^2\right)^2,\quad v=c, \tag{5.86}$$

being m_0 a rest mass according to (2.33). This well known result of special relativity together with (WHF) defines the energy and the Lorentz transformations already found in (2.35) and (3.28); the fact of having replaced the local values with the respective uncertainty ranges makes plausible the step from (5.85) to (5.86). The quantum uncertainty is thus essential to generalize (5.85): eq (5.86) has been concerned here although already obtained in (2.34) just to emphasize the link between uncertainty and special relativity and the physical importance of the uncertainty ranges. This is the reason of having repeated the result (2.34).

Let us show further that however (5.86) is not itself the most useful result for the next purposes just because its deterministic local values. Start therefore directly from (1.2); squaring both sides of $\delta\varepsilon=v\delta p$, being of course $v=\delta x/\delta t$, one finds owing to (2.35)

$$(\delta\varepsilon)^2=(v/c)^2\left(\delta(pc)\right)^2=\left(\delta(pc)\right)^2-\beta^2\left(\delta(pc)\right)^2,\quad \beta=\sqrt{1-v^2/c^2}. \tag{5.87}$$

Next, calculating δp as $\delta p=p_2-p_1=m_{02}v/\beta-m_{01}v/\beta$ via (2.35), one finds

$$\beta\delta(pc)=\beta\left(m_{02}vc/\beta\right)-\beta\left(m_{01}vc/\beta\right)=\left(m_{02}-m_{01}\right)vc; \tag{5.88}$$

as expected from previous considerations, there is no difficulty to calculate this result of (5.87) for $v=c$. So

$$(\delta\varepsilon)^2-\left(\delta(pc)\right)^2=-\left(\delta\left(m_0c^2\right)\right)^2,\quad \delta m_0=m_{02}-m_{01}, \tag{5.89}$$

whence it is possible to infer

$$
\begin{aligned}
(\delta\varepsilon)^2-(\delta(pc))^2 &= (n\hbar)^2\left(\frac{1}{(\delta t)^2}-\frac{c^2}{(\delta x)^2}\right) \\
&= (n\hbar)^2\frac{(\delta x)^2-c^2(\delta t)^2}{(\delta t\delta x)^2}=-\left(\delta m_0c^2\right)^2 .
\end{aligned}
\tag{5.90}
$$

Clearly the right hand side is a constant, thus invariant by definition; hence the ratio at the left hand side is an invariant as well. It is known indeed that this ratio is defined by two invariant quantities. Since the first (5.46) demonstrates that $\delta x\delta t$ is a relativistic invariant, it follows that the numerator is also invariant as well itself. The ranges at the left hand side of the last equality correspond thus to the $\delta m_0 = m_{02} - m_{01}$ at the right hand side. The invariant interval $\delta x^2 - c^2\delta t^2$, in particular, has been stated in [13] as the conceptual foundation of the special relativity; just for this reason it is remarkable the fact that in the present model i(5.46) and (5.90) are actually straightforward corollaries of the quantum uncertainty. The crucial difference between (5.90) and (5.86) is that now ε and pc appear through their uncertainty ranges and not as deterministic values. This result not only demonstrates the link between special relativity and quantum physics, but also allows further important steps concerning directly the general relativity. Although this point has been examined in several previous papers, see e.g. [4] [11] [12], the next section reports some relevant considerations just on this topic. Consider once more (5.83) for $v_1 = v_2 = c$, as already done to infer (5.86); the reasoning is still that already highlighted, but now extended to find a further interesting result. Write explicitly (5.83) with the help of (5.34) as follows

$$
\delta\left(\varepsilon^2\right)=\varepsilon^2-\left(m_0c^2\right)^2,\quad \delta(pc)^2=(pc)^2-\frac{p^2}{2m_1}\varepsilon, \tag{5.91}
$$

being m_1 a new arbitrary mass; here we have simply expressed also the right hand side in the form of a range of square energies. Then (5.82) for $v = c$ reads

$$
\varepsilon^2-\left(m_0c^2\right)^2=(pc)^2-\frac{p^2}{2m_1}\varepsilon. \tag{5.92}
$$

Clearly this equation reduces for $m_1 \to \infty$ to the standard form (5.86) of Einstein's special relativity. In fact the additional term in (5.92), more general than (5.86), is a known result of quantum gravity that helps solve three cosmological paradoxes [22]. More details about (5.92) are reported in [12].

5.6. Red Shift and Time Dilation

Starting from (2.10) and (2.9) consider $\dot{v} = \epsilon c/\hbar = -\nabla\varphi$ [13], being $-\nabla\varphi$ a force per unit mass related to $\dot{v}$ due to the gravitational potential φ, and write in the present one-dimensional approach

$$
\frac{\epsilon c}{\hbar}=-\frac{\delta\varphi}{\delta x}
$$

whence

$$\frac{\epsilon\delta x}{\hbar c}=-\frac{\delta\varphi}{c^2}.$$

Let us show now that

$$-\frac{\delta\varphi}{c^2}=\frac{\epsilon\delta x}{\hbar c}=\frac{\delta\omega}{\omega}.$$

Being by definition $\delta x=x_2-x_1$, regard the second equality considering that $\epsilon x_1/\hbar c=\omega_1/\omega$ *i.e.* $\epsilon x_1\omega/c=\hbar\omega_1$: indeed this result reads $\epsilon v_1/c=\hbar\omega_1$ because $\omega x_1=v_1$ by dimensional reasons. Owing to (3.29) in turn this means $p_1c=\hbar\omega_1$ at the coordinate x_1 where the gravitational potential φ reads $\varphi=\varphi_1$; as consistently to δx corresponds $\delta\varphi=\varphi_2-\varphi_1$, while an analogous reasoning holds of course for $\varepsilon x_2/\hbar c=\omega_2/\omega$, one finds

$$\frac{\delta\omega}{\omega}=\frac{\delta(-\varphi)}{c^2},\quad \delta\omega=\omega_2-\omega_1,\quad \delta(-\varphi)=-\varphi_2-(-\varphi_1); \tag{5.93}$$

this result is the red shift of a photon moving radially in the attractive gravitational potential $\varphi<0$ because $\omega_2>\omega_1$ implies $|\varphi_2|>|\varphi_1|$, *i.e.* lower frequency at the coordinate where the central gravitational potential is weaker. A corollary of this result is found replacing reasonably $\omega\rightarrow t^{-1}$; (5.93) yields

$$\frac{\delta\omega}{\omega}=\frac{\delta(1/t)}{1/t}=-\frac{\delta t}{t}=\frac{\delta(-\varphi)}{c^2}=-\frac{|\delta\varphi|}{c^2}$$

i.e. summarizing

$$\frac{t-t_0}{t}=\frac{|\delta\varphi|}{c^2},\quad \delta t=t-t_0. \tag{5.94}$$

It is sensible to regard t_0 as a proper time, with respect to which is defined t determining δt; as the right hand side describes gravitational potential rising from 0 to φ with $\varphi-0<0$ (5.94) reads $1-t_0/t=|\varphi|/c^2$ *i.e.*

$$t_0=\left(1-\frac{|\varphi|}{c^2}\right)t. \tag{5.95}$$

Owing to proper time $t_0<t$, this result yields time dilation δt due to gravity field with respect to field null.

5.7. Black Hole Entropy

Define the ratio of Planck length and mass, $\ell_P=\sqrt{\hbar G/c^3}$ and $m_P=\sqrt{\hbar c/G}$, which reads $\ell_P/m_P=G/c^2$; as (5.25) reads $\delta\ell^*/2m^*=G/c^2$, the starting point of this section is

$$\frac{\delta\ell^*}{m^*}=\frac{2G}{c^2}=\frac{2\ell_P}{m_P}. \tag{5.96}$$

This section is based on the ideas exposed about (5.47).

(i) Surface entropy.

Assume that the surface entropy is an extensive property that increases proportionally to the black hole surface. Squaring both sides of (5.96), it is possible to define a function S^* as follows

$$S^* = \left(\frac{m^*}{m_P}\right)^2 = \frac{\delta\ell^{*2}}{4\ell_P^2} = \frac{1}{4\pi}\frac{4\pi\delta\ell^{*2}}{4\ell_P^2} = S_0\frac{A^*}{4\ell_P^2}, \quad A^* = 4\pi\delta\ell^{*2}, \quad S_0 = \frac{1}{4\pi} \tag{5.97}$$

i.e. such that

$$S_{sur}^* = \frac{S^*}{S_0} = \frac{A^*}{4\ell_P^2}. \tag{5.98}$$

In effect the given ratio of squared masses in (5.97) is equal to the ratio of square lengths that, owing to (5.47), has physical meaning of surface entropy; so S_{sur}^* is the sought function that increases linearly with the surface A^* defined by the black body length $\delta\ell^*$ via the proportionality constant S_0. Since accordingly

$$S_{sur}^* = 4\pi S^* = 4\pi\left(\frac{m^*}{m_P}\right)^2 = 4\pi\frac{m^{*2}G}{\hbar c},$$

the factor 4π reminds the Gauss theorem and suggests its link to the flux $\mathbf{\Phi}^*$ of an appropriate function $\mathbf{F}^*$ through a surface element δA^* of A^*. So write

$$\delta\Phi^* = \mathbf{F}^*\cdot\delta\mathbf{A}^*, \quad \mathbf{F}^* = F^*\frac{\mathbf{r}}{r^3}, \quad F^* = -m^{*2}G, \quad \delta\mathbf{A}^* = \mathbf{n}\delta A^*, \tag{5.99}$$

where $\mathbf{r}$ is unit vector directed inside the flux surface δA^* whereas $\mathbf{n}$ is a unit vector oriented outside the surface δA^*. Hence

$$\Phi = \int\left(F^*\frac{\mathbf{r}\cdot\mathbf{n}}{\delta r^3}\delta A^*\right) = \int\left(m^{*2}G\frac{\delta A^*}{\delta r^2}\right) = \int\left(m^{*2}G\delta\Omega\right) = 4\pi m^{*2}G \tag{5.100}$$

and then the last (5.97) yields

$$S_0 = \frac{m^{*2}G}{\hbar c} = \frac{\hbar G/c^2}{c\lambda\!\!\!^{-}{}_{m^*}^2} = \frac{\ell_P^2}{\lambda\!\!\!^{-}{}_{m^*}^2}, \quad S^* = \frac{A^*}{4\lambda\!\!\!^{-}{}_{m^*}^2}. \tag{5.101}$$

With $\mathbf{F}^*$ related to the classical Newton law, the given definition (5.97) of S_0 makes S_{sur}^* proportional to the incoming flux $\mathbf{\Phi}$ of gravity force at the black hole surface expressed in $\hbar c$ units; so S_{sur}^* does not depend explicitly on m^* but on $|\mathbf{\Phi}|$ it generates. The presence of $\hbar$ and c in (5.101) shows the link between black body matter and usual matter inherent the standard definition (5.47) of entropy, so merging quantum and relativistic concepts. This definition of S_0 is sensible, as it results to be a ratio of square lengths too, whose physical meaning of entropy agrees with that of S_{sur}^*.

Equation (5.98) is the famous Hawking-Bekenstein surface entropy of a spherical non-rotating black body.

(ii) Volume entropy

In an analogous way it is possible to calculate the volume entropy. Once knowing that $\left(m^*/m_P\right)^2$ is related to $\left(\delta\ell^*/\ell_P\right)^2$, which has physical meaning of entropy, it is reasonable to guess that now the expected $\left(\delta\ell^*/\ell_P\right)^3$ related to

$\left(m^*/m_P\right)^3$ should yield a result m^*/m_P times that previously obtained. Write thus

$$S'^* = \left(\frac{m^*}{m_P}\right)^3 = \frac{4\pi\delta\ell^{*3}/3}{4\pi\left(2\ell_P\right)^3/3} = S'_0\frac{V^*}{8\ell_P^3}, \quad S'_0 = \left(\frac{4\pi}{3}\right)^{-1}, \quad V^* = \frac{4\pi}{3}\delta\ell^{*3} \quad (5.102)$$

and then, as before,

$$S_{vol}^* = \frac{S'^*}{S_0'^*} = \frac{V^*}{8\ell_P^3}; \quad (5.103)$$

this result corresponds to (5.98). Moreover (5.101) becomes

$$S_{vol}^* = \frac{4\pi}{3}\left(\frac{m^*}{m_P}\right)^3 = \frac{1}{3}\frac{m^*}{m_P}S_{sur}^*, \quad (5.104)$$

where now the mas m^* appears explicitly; also, owing to (5.101), (5.104) should reasonably yield the volume entropy of a spherical black body with an analogous meaning of $S'_0 = 3S_0$. Also, since

$$\frac{V^*}{8\ell_P^3} = \frac{m^*}{m_P}\frac{m^{*2}G}{\hbar c},$$

the flux ratio $|\Phi|/\hbar c$ multiplied by the huge ratio m^*/m_P corresponds to the expected $V^*/\ell_P^3 \gg A^*/\ell_P^2$.

5.8. Perihelion Precession

Consider the square ranges $(\delta\varepsilon)^2$ and $(\delta(pc))^2$ of (5.89), now defined explicitly as

$$\left(\varepsilon_{sr} - \varepsilon_\zeta\right)^2 = \left(cp_{sr} - cp_\zeta\right)^2 + \left(m_0c^2\right)^2, \quad m_0 = m_{sr} - m_\zeta; \quad (5.105)$$

the subscript *sr* stands for "*special relativity*, whereas ε_ζ and cp_ζ are appropriate boundary energies to be defined. This choice is possible because the boundaries of uncertainty ranges are arbitrary. Once having defined ε_{sr} consistently with the special relativity energy (5.86), cp_ζ and ε_ζ represent instead the actual momentum and energy in a gravitational system congruently to the respective quantities of the special relativity. In other words the boundary energies and momenta of $(\delta\varepsilon)^2$ and $(\delta(pc))^2$ of (5.89) are chosen in (5.105) in order to generalize the corresponding values of special relativity to the case of a gravitational system just taking advantage of the arbitrariness and agnosticism inherent the uncertainty. Write therefore

$$\varepsilon_{sr}^2 + \varepsilon_\zeta^2 - 2\varepsilon_{sr}\varepsilon_\zeta = \left(cp_{sr}\right)^2 + \left(cp_\zeta\right)^2 - 2c^2p_{sr}p_\zeta + \left(m_0c^2\right)^2$$

that fulfills by definition

$$\varepsilon_{sr}^2 = \left(cp_{sr}\right)^2 + \left(m_{sr}c^2\right)^2, \quad (5.106)$$

so that

$$\varepsilon_\zeta^2 - 2\varepsilon_{sr}\varepsilon_\zeta = \left(cp_\zeta\right)^2 - 2c^2p_{sr}p_\zeta + \left(m_\zeta c^2\right)^2 - 2m_{sr}m_\zeta c^4; \quad (5.107)$$

then, putting

$$\varepsilon_{\xi}^{2}=2\left(\varepsilon_{sr}\varepsilon_{\zeta}-c^{2}p_{sr}p_{\zeta}-m_{sr}m_{\zeta}c^{4}\right), \tag{5.108}$$

the right hand side reads

$$\varepsilon_{\zeta}^{2}=\left(cp_{\zeta}\right)^{2}+\left(m_{\zeta}c^{2}\right)^{2}+\varepsilon_{\xi}^{2} \tag{5.109}$$

where

$$\varepsilon_{\xi}^{2}=2\varepsilon_{sr}\varepsilon_{\zeta}\left(1-\frac{c^{2}p_{sr}p_{\zeta}}{\varepsilon_{sr}\varepsilon_{\zeta}}-\frac{m_{sr}m_{\zeta}c^{4}}{\varepsilon_{sr}\varepsilon_{\zeta}}\right)=2\epsilon_{\xi}^{2}\left(1-\frac{v_{sr}p_{\zeta}}{\varepsilon_{\zeta}}-\frac{m_{sr}m_{\zeta}c^{4}}{\varepsilon_{sr}\varepsilon_{\zeta}}\right),\quad \epsilon_{\xi}^{2}=\varepsilon_{sr}\varepsilon_{\zeta}.$$

Note that now are used notations like p_{ζ}, and not δp_{ζ}, because are implemented boundaries of uncertainty ranges that of course are not deterministic values. From a formal point of view the Equation (5.109) is similar to (2.34), inferred through a quantum approach, apart from the additional term ε_{ξ}^{2}; this is clearly consistent with the fact that by definition the subscript ζ refers to dynamical variables of general relativity introduced in (5.105). On the one hand (5.109) confirms the quantum gravity result (5.92), where in effect the corrective term $\epsilon p^{2}/m$ also appears with respect to the Einstein energy equation of the standard special relativity; this comparison suggests that even ε_{ξ}^{2} should be someway reducible to that in (5.92), which in fact has been also inferred reasoning on the boundaries of energy uncertainty ranges. On the other hand, ϵ_{ξ}^{2} represents the sought generalization of (2.34) or (5.86). Eventually rewrite (5.109) as

$$\frac{\varepsilon_{\zeta}^{2}-\left(m_{\zeta}c^{2}\right)^{2}}{\left(cp_{\zeta}\right)^{2}}=1+\frac{\varepsilon_{\xi}^{2}}{\left(cp_{\zeta}\right)^{2}}. \tag{5.110}$$

This equation is interesting; it depends upon how is defined ε_{ξ} at the right hand side. For example specify in particular ε_{ξ}, in principle arbitrary itself, according to (2.11); in this way it is possible to finalize (5.110) to the purpose of generalizing energy and momentum of the special relativity to the corresponding dynamical variables of the general relativity. In this specific case, possible and reasonable, regard ε_{ξ} and p_{ζ} defining

$$\left(\frac{\varepsilon_{\xi}}{cp_{\zeta}}\right)^{2}=\left(-\frac{Gm_{1}m_{2}}{\chi c\mathrm{M}}\right)^{2},\quad \varepsilon_{\xi}=-\frac{Gm_{1}m_{2}}{\ell_{\zeta}},\quad \ell_{\zeta}p_{\zeta}=\chi\mathrm{M},$$

where $\mathrm{M}=|\mathbf{M}|$ has physical dimensions of modulus of angular momentum and χ is regarded as a proportionality factor linking $\ell_{\zeta}p_{\zeta}$ to M to be defined. Since the last equation has mere dimensional basis, χ can be defined conveniently as the factor that converts $\ell_{\zeta}p_{\zeta}$ into an average value $\mathrm{M}=\langle|\mathbf{M}|\rangle$ of the modulus of the orbiting angular momentum $\mathbf{M}$. This result turns next (5.110) into

$$\frac{\varepsilon_{\zeta}-m_{\zeta}c^{2}}{cp_{\zeta}}\frac{\varepsilon_{\zeta}+m_{\zeta}c^{2}}{cp_{\zeta}}=1+\left(\frac{Gm_{1}m_{2}}{\chi c\mathrm{M}}\right)^{2},$$

which reads eventually

$$\delta\phi = \frac{1}{k} + \frac{1}{k\chi^2}\left(\frac{Gm_1m_2}{c\mathrm{M}}\right)^2, \quad \delta\phi = \frac{\varepsilon_\zeta - m_\zeta c^2}{cp_\zeta}, \quad k = \frac{\varepsilon_\zeta + m_\zeta c^2}{cp_\zeta}. \qquad (5.111)$$

Note that if $G = 0$ then $\delta\phi = k^{-1}$; *i.e.* actually the value of k is not essential as concerns $\delta\phi$, it is simply a reference value with respect to which is defined the change of ϕ with respect to a value ϕ_0 in the absence of field. The essential quantity is instead the range size $\delta\phi$, *i.e.* the shift of ε_ζ with respect to the rest energy $m_\zeta c^2$. Hence it is sensible to introduce $\Delta\phi = \delta\phi - k^{-1}$ in order to account for the change $\phi - \phi_0$ the gravitational effect in parenthesis only. Rewrite then (5.111) as

$$\Delta\phi = const\left(\frac{Gm_1m_2}{c\mathrm{M}}\right)^2, \quad const = \frac{1}{k\chi^2}: \qquad (5.112)$$

the second position regards $k\chi^2$ as mere proportionality constant of the gravitational term in parenthesis, assumed to be the one physically relevant because in fact it concerns the parameters G and M that govern the orbital behavior of m_1 and m_2. Remind now (2.26) that refers to a circular orbit of m_2 in the gravity field of m_1; in the case of an elliptic orbit one expects that the early steady condition consistent with $\lambda = \lambda_{circ}$ is reasonably to be replaced by a different wavelength λ_{ell}. Let be $\lambda_{ell} > \lambda_{circ}$, although still being $n\lambda_{ell}/\delta r_{ell} = const$ in order to generalize (2.26) $n\lambda_{circ}/\delta r_{circ} = 2\pi$ while fulfilling the same kind of equation; with this assumption (2.26) is simply a particular case for $\delta r_{ell} \to \delta r_{circ}$. Clearly δr_{ell} is now an "effective" radius, taking into account that the perimeter C of ellipse is actually a function of its semi axes a and b; an approximate formula is for example $C \approx \pi\left[3(a+b) - \sqrt{(3a+b)(a+3b)}\right]$, which is reliable for the present purposes because for $a = b$ it reduces to $C = 2\pi a$. If $b > a$ then $C \approx \pi\left[6a + 3(b-a) - \sqrt{(3a+b)(a+3b)}\right]$ reduces to $C \approx 6a\pi$ if the difference $3(b-a) - \sqrt{(3a+b)(a+3b)}$ of positive numbers mutually self eliding becomes negligible with respect to $6a$; so $\delta r_{ell} \approx 6a$. With this numerical approximation, clearly $2\pi\delta r_{circ} = n\lambda_{circ}$ is to be replaced by $6\pi\delta r_{ell} = n\lambda_{ell}$ with identical physical meaning; so, merging (5.112) with the quantum condition $const = n\lambda_{ell}/\delta r_{ell} = 6\pi$ yields eventually

$$\Delta\phi \approx 6\pi\left(\frac{Gm_1m_2}{c\mathrm{M}}\right)^2, \quad const = 6\pi. \qquad (5.113)$$

Are worth noticing in this respect two remarks on the "Kepler problem" exposed in the textbooks [13] [21].

(i) The first one introduces the condition $2\pi j/i$ in a non-relativistic approach, where i and j are integer numbers to get a steady closed trajectory via a rational fraction of 2π; in effect even in a classical model the perihelion precession is still possible, although insufficient to explain the astronomical observations. Here this condition is replaced by the quantum condition of an integer number of wavelengths in an elliptic orbit.

(ii) The second one lucidly shows step by step how to infer classically this famous Einstein formula of Mercury perihelion simply comparing two forms of potential energy of orbiting system; the mathematical formulation introduces first the mere Newtonian potential $U_N = \beta_1/r$ and then also assumes an extended potential form $U_\zeta = \beta_2/r^2$, where β_1 and β_2 are appropriate constants. Of course the plain Newton law does not justify U_ζ, so that the approach shown in the book had mere speculative/didactic character. Nevertheless the procedure therein reported, very instructive and significant in principle, has actual interest here because the present theoretical frame does admit in fact higher order potential terms like U_ζ besides U_N once replacing the deterministic r with δr. Indeed (4.23) and (4.3) yield according to (1.2) and (2.12)

$$\begin{aligned} force &= n\hbar\frac{\delta\dot{x}}{\delta x^2} = -n\hbar\frac{\delta}{\delta t}\left(\frac{1}{\delta x}\right) = -\frac{n\hbar}{\delta t}\delta\left(\frac{1}{\delta x}\right) \\ &= -\delta\varepsilon\delta\left(\frac{1}{\delta x}\right) = -\frac{n\hbar v}{\delta x}\delta\left(\frac{1}{\delta x}\right) = \frac{n\hbar v}{\delta x^3}\delta(\delta x): \end{aligned} \tag{5.114}$$

the notation emphasizes that the time derivative $\delta\dot{x}$ is actually regarded as ratio of ranges $\delta(x^{-1})$ and δt as previously explained in the sections 2.6 and 2.7. The former is in particular relevant: it reminds the curvature of space time, as explained in (4.15) and (4.16). Since the last equality of the chain yields in turn

$$\begin{aligned} &force = \frac{n\hbar v}{\delta x^3}\delta(\delta x) = \frac{n\hbar v\delta t'}{\delta x^3}\frac{\delta}{\delta t'}\delta x = \frac{n\hbar\delta x'}{\delta x^3}\frac{\delta}{\delta t'}\delta x = \frac{n\hbar\delta x'}{\delta x^3}\delta\dot{x}', \\ &\delta x' = v\delta t' = \frac{\delta\varepsilon\delta t'}{n\hbar}, \end{aligned} \tag{5.115}$$

is clear the implication here: requiring that $\delta x' = const' = \delta x_0$ with an appropriate choice of $\delta t'$, the result due to the space time curvature reads

$$force = \frac{n\hbar\delta x_0}{\delta x}\frac{\delta\dot{x}''}{\delta x^2}, \quad \delta\dot{x}'' = \frac{\delta}{\delta t'}\delta x. \tag{5.116}$$

It is evident that the reasoning from (5.114) to (5.116) just shown can be repeated, thus obtaining higher power potential terms. So the simple fact of having justified via (1.2) the potential (5.116) allows obtaining with elementary methods of classical mechanics the sought result, the perihelion precession tentatively exemplified classically in the quoted textbook. Here has been proposed an alternative derivation just to show that (5.105) is enough to obtain a crucial result of general relativity. Now, after having introduced the conceptual frame outlined by (1.1) and (1.2), it is sensible to proceed with calculations implementing (1.5).

6. Cosmological Calculations

$$\frac{\hbar G}{c^2} = 7.8\times 10^{-62}\ \mathrm{m}^3\cdot\mathrm{s}^{-1} \tag{6.1}$$

In this section are calculated the numerical values of some relevant formulas inferred in the previous sections; after having outlined the theoretical frame, the aim to show how the values (1.5) at today's time fit the concepts hitherto introduced. The strategy of the calculation scheme is that of implementing the values (1.5) more than once in various equations, whose global self-consistency supports the validity of the single results and outlines a unique conceptual frame. The fact of having inferred relativistic concepts via (1.2), see for example (5.93) (5.95) and (5.113) and the results in [16], shows that in fact even cosmological information should be sensibly accessible despite the agnosticism of the quantum uncertainty. The following calculations are carried out assuming the value of π approximately equal to that of the flat Euclidean space. The paper sketches also a few results already published in order to be as self-contained as possible. For sake of clarity the calculations are listed one by one in the various points below.

1) Examining the numerical values quoted in (1.5), it appears that $\Lambda \approx 4H_u^2$. This numerical evidence suggests that actually the true relationship between the literature estimates of H_u^2 and Λ is reasonably

$$\Lambda = \frac{4}{3}\pi H_u^2. \tag{6.2}$$

This fact is interesting because an energy density η is calculable via (2.2) as $(c/\tau)^2 G^{-1}$ according to (1.4); in particular the *time*$^{-2}$ dependence inherent Λ is $4\pi/3$ times that calculated via H_u^2 because, despite the same physical dimensions, the former contains geometrical information with respect to the latter. Since η is anyway *energy/volume*, (6.19) calculated as a function of H_u^2 and Λ read

$$\frac{c^2 H_u^2}{G} = \frac{c^2\Lambda}{G4\pi/3} = \frac{\epsilon}{(4\pi/3)\delta\ell^3}: \tag{6.3}$$

hence according to the first equality the energy density ϵ/V_η involving H_u^2 implies the total energy ϵ calculated in the total volume $V_\eta = (4\pi/3)\delta\ell^3$, whereas according to the second equality involving Λ the volume pertinent ϵ is in fact expressed via $V = \delta\ell^3$. In other words H_u requires explicitly the space time curvature implied by spherical geometry of mass containing universe, Λ does not. For this reason $\sqrt{\Lambda}$ and H_u are numerically interchangeable via a coefficient ~2 apart. Note now that

$$2\pi\frac{c}{r_u} = \omega_u = 4.33\times10^{-18}\ \text{s}^{-1}, \quad \sqrt{\Lambda} = 4.36\times10^{-18}\ \text{s}^{-1}, \quad \frac{c}{r_u} = \nu_u, \tag{6.4}$$

which shows the link between the cosmological constant and the estimated universe radius. This link is further confirmed considering that $\sqrt{c^2/\Lambda}$ and c/H_u are lengths; then calculate

$$\ell_\Lambda = \frac{c}{\sqrt{\Lambda}} = 6.88\times10^{25}\ \text{m}, \quad \ell_H = \frac{c}{H_u} = 1.36\times10^{26}\ \text{m} \tag{6.5}$$

and implement the quantum definition $2\pi\delta r = n\lambda$ of (2.26), whose right hand side reads $n/\mathcal{R}$ in (4.22) after having replaced λ with the reciprocal curvature radius $\mathcal{R} = \lambda_r'^{-1} - \lambda_r''^{-1}$ via the De Broglie momentum wavelengths. Taking appropriately δr as the universe radius in the estimates (1.5), the numerical result is

$$\begin{array}{ccc} 2\pi\delta r & = & n\mathcal{R}^{-1} \\ \Updownarrow & & \Updownarrow \\ 2\pi\sqrt{\dfrac{c^2}{\Lambda}} & = & nr_u \\ \Updownarrow & & \Updownarrow \\ 4.32\times10^{26}\ \text{m} & = & n4.35\times10^{26}\ \text{m} \end{array} \qquad \delta r = \sqrt{c^2\Lambda^{-1}} \quad r_u = \mathcal{R}^{-1} \tag{6.6}$$

With $n=1$ the agreement of r_u with the estimates (1.5) is surprisingly decent. On the one hand this result highlights the link of Einstein cosmological constant Λ and today's radius of Universe via the early quantum condition (2.23) in agreement with (6.4): as expected the estimated value r_u is related to the reciprocal curvature radius $\mathcal{R}$ of the Universe, here defined by the range of wavelengths $\delta\lambda^{-1}$ corresponding to δr.

On the other hand it is possible to match the dimensional definition of space time by defining the functions $\left(c/\sqrt{\Lambda}\right)^3 H_u$ and $\left(c/H_u\right)^3\sqrt{\Lambda}$ along with r_u^3/t_u that share the dimensional property $length^3\times time^{-1}$. A reasonable chance to merge these definitions in a self consistent way is in fact owing to (6.2)

$$\sqrt{H_u\left(\frac{c}{\sqrt{\Lambda}}\right)^3\frac{r_u^3}{t_u}} = 1.16\times10^{61}\ \text{m}^3\cdot\text{s}^{-1}, \quad \sqrt{\Lambda}\left(\frac{c}{H_u}\right)^3 = 1.10\times10^{61}\ \text{m}^{33}\cdot\text{s}^{-1}. \tag{6.7}$$

It is worth noticing that

$$\frac{\hbar G}{c^2}\sqrt{\Lambda}\left(\frac{c}{H_u}\right)^3 \approx 1\left(\text{m}^3\cdot\text{s}^{-1}\right)^2. \tag{6.8}$$

Combining together these equations thanks to their numerical values nicely coincident, one finds via Λ the link between the fundamental cosmological parameters

$$\sqrt{H_u\left(\frac{c}{\sqrt{\Lambda}}\right)^3\frac{r_u^3}{t_u}} = \sqrt{\Lambda}\left(\frac{c}{H_u}\right)^3, \quad i.e.\ \ \frac{r_u^3}{t_u} = \frac{c^3\Lambda^{5/2}}{H_u^7} = 1.70\times10^{62}\ \text{m}^3\cdot\text{s}^{-1}; \tag{6.9}$$

it appears in particular the link between the estimated volume of of universe, proportional to r_u^3, and its estimated age t_u. Owing to the physical dimensions of Λ and H_u and their time dependence, one expects

$$\frac{r_u^3}{t_u} \propto \textit{function of}\left(time^2\right). \tag{6.10}$$

Whatever this time function $X = X\left(\delta t^2\right)$ might be, the fact that the dependence of the ratio at the left hand side involves δt^2 means that δt can be identically positive or negative, *i.e.* the time can run in principle away from or back towards the initial big bang. It is worth noticing a numerical accident, *i.e.* both today's values (6.7) are reciprocal of $7.83\times10^{-62}\ \text{m}^3/\text{s}$ calculable directly with the fundamental constants of (1.1).

2) In (6.4) Λ was related to ω_u pertinent to r_u; now, to extend further the link between theoretical results and (1.5), implement H_u too. Are relevant in this respect the Equations (5.64) to (5.65) that define mass, density and energy density as a function of the frequency ω appearing in (5.62) and (5.63). Indeed via k_f it is possible to identify further mass density and mass directly related to the gravitational effect of matter in the universe, which clarify the meaning of such ω; it has also to do with the geometrical implication of matter on the space time curvature, introduced in (6.2) and emphasized in (6.6). So ω_u is a property of the space time, as it results in (6.4) regardless of any mass, instead ω is related to the gravitational effect of matter in the universe. To clarify this point, let us introduce the following definitions

$$r_u = \xi \frac{2c}{\omega}, \quad \omega = \frac{H_u}{2} \tag{6.11}$$

The first definition is nothing else but a way to express $length = c/frequency$ via the proportionality constant 2ξ, the second expresses ω as a function of H_u via the coefficient 1/2; in this way $\hbar H_u/2$ takes the meaning of zero point energy of the universe, which of course is sensible once having introduced (5.60), (5.77) and (5.79). These positions are useful to obtain via the first and second (5.64)

$$\xi = \frac{r_u H_u}{4c} = 0.8, \quad \rho_\omega = \xi \frac{\omega^2}{G} = \frac{r_u H_u^3}{16cG} = 14.4 \times 10^{-27} \frac{\text{kg}}{\text{m}^3}; \tag{6.12}$$

as reasonably expected $\xi \sim 1$, whereas instead a proportionality constant very different from 1 would have suggested that something important was neglected when formulating the pertinent dimensional definition. However the value of ξ is not the only reason to justify the definitions (6.11). Indeed note that

$$m_\omega = \xi \frac{c^3}{\omega G} = \frac{r_u H_u}{4c} \frac{2c^3}{H_u G} = \frac{r_u c^2}{2G} = 2.93 \times 10^{53}\ \text{kg}: \tag{6.13}$$

this result is significant because thanks to (6.11) it acknowledges the familiar (5.25), *i.e.* the link between $m_\omega = m^*$ and the universe radius $r_u = \delta\ell^*$ of that equation. Eventually it is easy to calculate also

$$\begin{aligned} \eta_\omega &= \xi \frac{(c\omega)^2}{G} = \frac{r_u H_u^3 c}{16G} = 1.30 \times 10^{-9} \frac{\text{J}}{\text{m}^3}, \\ k_f &= m_\omega \omega^2 = \frac{r_u (H_u c)^2}{8G} = \frac{r_u \eta_{vac}}{8} = 3.54 \times 10^{17} \frac{\text{kg}}{\text{s}^2}. \end{aligned} \tag{6.14}$$

It has been shown in (5.79) and (5.77) that the Newton gravity energy is expressible via a harmonic oscillator whose energy is governed by the force constant k_f. Here is a confirm of this idea calculating k_f via H_u while involving also the estimated size of the universe; indeed, defining by dimensional reasons

$$\rho_{kf} = \sqrt{\frac{k_f}{r_u^3 G}} = \frac{H_u c}{r_u G \sqrt{8}} = 8.03 \times 10^{-27} \frac{\text{kg}}{\text{m}^3}, \quad M_{kf} = \rho_{kf} r_u^3 = 6.61 \times 10^{53}\ \text{kg}. \tag{6.15}$$

This gravitational contribution due to the mass density ρ_{k_f} results thus decently comparable with the literature critical value $\rho_{cr} = 8.6\times10^{-27}\ \text{kg}\cdot\text{m}^{-3}$ [6] quoted in (1.6) and with the value $9.9\times10^{-27}\ \text{kg/m}^3$ reported in [5]. It is worth emphasizing again in this respect that even small deviations of r_u and H_u from the estimates (1.5) can alter the conclusions here affordable, e.g. the comparison of ρ_ω and ρ_{cr} of the Friedman equations; the comparison here proposed aims to check that the present calculations yield, at least in principle, sensible numbers compatible with other literature outputs. As a further check about the link between (6.13) and (5.25), with $\delta\ell^* = r_u$ the value of m^* is greater than m_{ob} in (1.6), as it must be, and fulfills (5.27). In fact, $\delta t^* = t_u$ and $m^* = m_\omega$ just calculated fulfill $t_u \le \sqrt{2} m_\omega G/c^3$; indeed

$$4.35\times10^{17} < 1.1\times10^{18}\ \text{s}.$$

3) Note that $\hbar H_u^2/c$ and $\hbar\Lambda/c$ have both physical dimensions of force; introduce thus an arbitrary length ℓ according which holds the following correlation

$$|m\ell\Lambda| = \left|G\frac{m^2}{\ell^2}\right| \quad\Rightarrow\quad |M_{tot} r_u \Lambda| = \left|G\frac{M_{tot}^2}{r_u^2}\right| : \tag{6.16}$$

the first equation is a mere dimensional definition of force, in the second equation the dimensional mass and length are identified with the respective actual properties (1.5) of the universe. Considering the latter for the calculation, write then owing to (6.2)

$$\Lambda = \frac{4}{3}\pi H_u^2 = G\frac{M_{tot}}{r_u^3},\quad M_{tot} = 2.34\times10^{55}\ \text{kg}, \tag{6.17}$$

i.e. $M_{tot} G = V_u H_u^2$ involves through H_u the geometrical volume V_u of the universe.

4) Consider first (2.3) in the particular case where δt, in principle arbitrary likewise any uncertainty range, is taken equal to the reciprocal Hubble parameter measured today

$$\left(\frac{\hbar^2 G/c^2}{\eta}\right)^{1/6} = \left(\frac{\hbar G}{c^2} H_u^{-1}\right)^{1/3},\quad \delta t = H_u^{-1}. \tag{6.18}$$

Then one finds

$$\eta_{vac} = \frac{(cH_u)^2}{G} = 6.52\times10^{-9}\ \text{J/m}^3 : \tag{6.19}$$

since no mass appears explicitly in this calculation, the energy density calculated in this way is reasonably regarded as, and in fact its order of magnitude agrees with, the acknowledged vacuum energy density in the Universe [23]. So η_{vac} is related to the swelling rate H_u of the universe; also, right hand side of (6.18) yields

$$\left(\frac{\hbar G}{c^2 H_u}\right)^{1/3} = 3.2\times10^{-15}\ \text{m}. \tag{6.20}$$

The value of this length fits well the order of magnitude of four neutron radii;

indeed

$$\frac{1}{4}\left(\frac{\hbar G}{c^2}H_u^{-1}\right)^{1/3}=r_{neut}^{calc}=8.10432\times10^{-16}\ \text{m},\quad r_{neut}^{estim}=8.0\times10^{-16}\ \text{m}\ [24],\quad (6.21)$$

which suggests that η_{vac} is due to barionic matter consisting of two virtual neutrons. This indeed reminds the concepts of virtual particles introduced in (3.9) and (3.12); this estimate does not require that two corpuscles of neutral matter are actually present in V_η, indeed no mass appears explicitly in (6.19) despite its dimensional meaning of mass is due to *G*; rather the reasoning simply establishes an energy/volume ratio equivalent to that of two virtual barions, *i.e.* energy fluctuation driven time transients in V_η. The fact that $\eta_{vac}=\epsilon_\eta/V_\eta$ of (6.19) is defined by an energy ϵ_η corresponding to two diametric sizes of neutron, suggests that V_η should define one quantum state of vacuum with energy ϵ_η equivalent to that of two virtual barions; in principle two neutrons with opposite spins could fit one allowed quantum state. As η_{vac} corresponds to two virtual particles supposed non interacting in a volume V_η, about $\epsilon_\eta\sim0.93\times2\ \text{GeV}$, the volume size of one quantum state of vacuum should be of the order of $1.86\times10^9\times1.6\times10^{-19}/\eta_{vac}$, *i.e.*

$$V_\eta\sim0.047\ \text{m}^3.\quad (6.22)$$

It is worth emphasizing that the energy density (6.19) is likely an upper limit: indeed no more than two neutrons with opposite spins can occupy one quantum state defined by (6.22), so that a lower energy density is in principle still possible e.g. with one virtual neutron only and an unoccupied state in V_η. This could happen for example when a black hole traps either *m* or m_o of (3.12) only; indeed even m_v and m_v^* are δr apart in (3.9), according to the well known Hawking mechanism [25]. It appears that in fact (6.19), greater than η_ω of (6.14) although having the same order of magnitude, is upper limit of vacuum energy density.

In summary, the universe is definable as a space time environment characterized by its own size, energy and energy density, which in turn implies straightforwardly pressure as shown in the appendix A. These features introduce in a natural way another thermodynamic property, the temperature; this topic is concerned in 6).

5) Since $\hbar^2G/c^2$ has physical dimensions of $energy\times volume$, define via η_{vac} and η_ω the energies

$$\epsilon_\eta=\sqrt{\hbar\frac{\hbar G}{c^2}\eta_{vac}}=\hbar H_u,\quad \epsilon_\omega=\sqrt{\hbar\frac{\hbar G}{c^2}\eta_\omega}=\frac{\hbar H_u}{4}\sqrt{\frac{r_uH_u}{c}}\quad (6.23)$$

calculated implementing respectively the maximum and actual vacuum energy densities (6.19) and (6.14). This is interesting because it is possible to calculate

$$\epsilon_\eta=2.32\times10^{-52}\ \text{J},\quad \epsilon_\omega=1.04\times10^{-52}\ \text{J},\quad \delta\varepsilon_{vac}=\epsilon_\eta-\epsilon_\omega=1.28\times10^{-52}\ \text{J};\quad (6.24)$$

these figures compare well with

$$\frac{1}{2}\hbar H_u = 1.15\times10^{-52}\ \text{J}, \quad \frac{\delta\varepsilon_{vac}}{\hbar} = 1.22\times10^{-18}\ \text{s}^{-1} = \frac{H_u}{2}, \tag{6.25}$$

in agreement with the definition (6.11). It appears that the quantum energy ε_η directly implied by the Hubble constant is just that given by (2.2). This calculation is interesting as it emphasizes the direct link between Hubble constant and definition of space time (1.1) through the vacuum energy density, while also confirming (2.2) and thus the Newtonian (2.11) via the concept of acceleration (2.8).

6) Implement (3.23) that yields

$$\epsilon^2 = \frac{\epsilon'\hbar}{\delta t}, \quad \epsilon' = \frac{v^2}{c^2}\left(\eta\Delta x^3\right) \tag{6.26}$$

where ϵ and ϵ' are in general time dependent energies related to the initial energy density η. The physical meaning of these energies results particularly significant rewriting the first (6.26) as follows

$$T = \frac{\mathcal{T}}{\sqrt{\delta t}}, \quad T = \frac{\epsilon}{k_B}, \quad \mathcal{T} = \sqrt{\frac{\epsilon'\hbar}{k_B^2}}, \quad \epsilon' = \frac{\left(k_B\mathcal{T}\right)^2}{\hbar}. \tag{6.27}$$

The first equation is tested with the help of the universe timeline temperature vs time published by the Fermilab and reported in [26]. This point has been already concerned in [20], where it is shown through the plot of temperature vs time implementing the timeline data; it appears that actually $\mathcal{T}$ is a constant, it is the best fit coefficient of T vs $\delta t^{1/2}$. In principle the result $\epsilon' = const$ can be understood thinking a series expansion of ϵ' as a function of time with constant zero order term and neglecting the possible time dependent higher order terms. To infer ϵ' of (6.27) and justify the linear plot calculable via (6.27), note that merging the Stefan Boltzmann black body constant σ and the Boltzmann constant k_B it is possible to define a new constant whose physical dimensions are

$$\frac{k_B^2}{\sigma} = energy \times time \times length^2 \times temperature^2,$$

whence the chance of defining in turn also

$$\sigma^* = \frac{k_B^2}{\sigma \times length^2 \times energy} = temperature^2 \times time.$$

This suggests in turn implementing

$$T^2 = \frac{const}{\delta t}, \quad const = \zeta\sqrt{\frac{k_B^2}{\sigma\delta\ell^2\varepsilon}}. \tag{6.28}$$

Here T fulfills an equation similar to (6.27) despite the different definition of *const* replacing $\mathcal{T}$; the proportionality factor ζ is justified when converting any dimensional relationship into an equation containing the corresponding dynamical variables. Of course if this equation is physically sensible one expects $\zeta \approx 1$. To calculate (6.28) it is necessary to identify the physical meaning of ε and $\delta\ell$.

(i) As concerns ϵ, merge (6.28) and (6.27) to obtain

$$\sqrt{\frac{\epsilon'\hbar}{k_B^2}} = T\sqrt{\delta t} = \zeta\sqrt{\frac{k_B^2}{\sigma\delta\ell^2\epsilon}}$$

so that

$$\epsilon'\epsilon\hbar = \zeta^2\frac{k_B^4}{\delta\ell^2\sigma}, \quad \frac{k_B^4}{\sigma} = \frac{60\hbar^3c^2}{\pi^2}; \tag{6.29}$$

the second equation is the well known definition of σ. Hence

$$\frac{\epsilon'}{\hbar}\frac{\epsilon}{\hbar} = \frac{60\zeta^2}{2\pi^2}\frac{c^2}{\delta\ell^2} = \frac{60\zeta^2}{\pi^2}\nu''^2, \quad \nu''^2 = \frac{c^2}{\delta\ell^2}$$

and eventually

$$\frac{\nu'}{\nu''}\frac{\nu}{\nu''} = \frac{60\zeta^2}{\pi^2}, \quad \nu = \frac{\epsilon}{\hbar}, \quad \nu' = \frac{\epsilon'}{\hbar}. \tag{6.30}$$

(ii) As concerns $\delta\ell$, note that defining $\delta\ell^2 = \ell_P^2 = \hbar G/c^3$ and replacing in (6.29), trivial algebraic steps yield

$$\epsilon'\epsilon = 60\zeta^2\hbar W_P, \quad W_P = \frac{c^5}{G}.$$

At this point, assuming

$$\nu'\nu = \nu''^2, \quad \epsilon' = \epsilon, \quad \delta\ell = \ell_P = 1.6\times10^{-35}\ \text{m}$$

(6.29) yields

$$\zeta = \sqrt{\frac{\pi^2}{60}} = 0.4, \quad \epsilon^2 = 0.16\frac{k_B^4}{\ell_P^2\hbar\sigma}, \quad \epsilon = 1.9\times10^9\ \text{J}. \tag{6.31}$$

The numerical value of ζ is acceptable. It is possible now to calculate (6.27), which reads

$$T = \frac{1.9\times10^9}{\sqrt{\delta t}}; \tag{6.32}$$

two significant particular cases of photon energies at $t = t_P$ and $t = t_u$ are

$$T(t_P) = 8\times10^{30}\ \text{K}, \quad T(t_u) = 2.73\ \text{K}. \tag{6.33}$$

This result has been already concerned in [20], where the coefficient *const* has been described in more detail.

(iii) It is possible to confirm (6.32) considering now that the Stefan Boltzmann constant σ/c that has physical dimensions $energy\ density \times temperature^{-4}$; since the energy density has physical dimensions $mass/(length \times time^2)$ a simple dimensional analysis brings to

$$T = \left(\frac{mass \times c}{\sigma \times length}\right)^{1/2}\delta t^{-1/2} \tag{6.34}$$

i.e. also now appears the dependence of T upon $\delta t^{-1/2}$ via the ratio $m/length$; so, equating the dimensional definition (6.34) and (6.27) via an appropriate proportionality constant ζ, it is possible to write

$$\zeta\left(\frac{mc}{\sigma\delta\ell}\right)^{1/4}=T\sqrt{\delta t}=\sqrt{\frac{\varepsilon'\hbar}{k_B^2}}$$

and therefore

$$\epsilon'=\left(\frac{k_B^2}{\hbar\sqrt{\sigma/c}}\right)\zeta^2\sqrt{\frac{m}{\delta\ell}}\quad i.e.\quad \epsilon'=1.31\times10^{-4}\zeta^2\sqrt{\frac{m}{\delta\ell}}\ \text{J}. \tag{6.35}$$

This equation contains two unknown ε' and $m/\delta\ell$ and reads also, according to (6.14),

$$\epsilon'=\left(\frac{k_B^2}{\hbar}\sqrt{\frac{k_f}{\sigma/c}}\right)\frac{\zeta^2}{\sqrt{\omega^2\delta\ell}}=\frac{7.8\times10^4}{\sqrt{a}},\quad a=\frac{\omega^2\delta\ell}{\zeta^4}=\frac{H_u^2}{4\zeta^4}\sqrt{\frac{\hbar G}{c^3}}. \tag{6.36}$$

The definition of acceleration appearing here is analogous to that of (4.13) a constant ζ^{-4} apart.

7) The dimensional analysis, proven useful to infer (5.25) by implementing (3.17), allows once more to obtain valuable information via (3.1). Replacing now in this equation $\delta\ell=2r_u$, the diametric size of the Universe, one finds the interesting numerical correspondence

$$\frac{2r_u\hbar}{M_{st}}=\frac{\hbar G}{c^2},\quad M_{st}=1.17\times10^{54}\ \text{kg},\quad M_{ob}=m_{ob}\left(\frac{r_u}{ct_u}\right)^3=1.11\times10^{54}\ \text{kg}, \tag{6.37}$$

being m_{ob} the estimated value reported in (1.6) of light emitting mass by observable stars only, whereas the subscript *st* stands for "space time"; so, owing to $r_u>ct_u$ and assuming a uniform dissemination of stars and galaxies in the whole Universe, the second (6.37) scales up m_{ob} to the total light emitting mass M_{ob} including all stars really existing, all observable if the light speed would be infinite: in other words it means to extrapolate the features of the observable universe even to that non observable from our standpoint. Therefore, owing to the physical meaning of $\hbar G/c^2$, the left hand side of the first (6.37) concerns the ability of space time to generate light emitting mass $M_{st}=M_{ob}$ throughout its volume per unit time, *i.e.* it yields the total mass of "ordinary" observable matter existing in the Universe regardless of the ability of its emitted light to reach any particular observation point. If so, then

$$\frac{2r_u}{G/c^2}=m_{ob}\left(\frac{r_u}{ct_u}\right)^3=M_{ob}, \tag{6.38}$$

i.e.

$$r_u=\frac{1}{2}\frac{M_{ob}G}{c^2}=\sqrt{\frac{2c^5t_u^3}{m_{ob}G}}. \tag{6.39}$$

The first equality relates r_u to the total amount M_{ob} of ordinary matter known to us. It is attracting at this point the chance of comparing this partial value of mass to that M_{tot} effectively existing in the universe. So, whatever the numerical values of M_{tot} might be, nothing hinders to think

$$r_u = \frac{1}{2}\frac{(M_{tot} - M_x)G}{c^2}, \quad M_{ob} = M_{tot} - M_x, \tag{6.40}$$

being M_x an extra amount of mass additional to that observable, to be defined according to the idea that it collects other possible mass contributions additional to that of all observable stars. Note that this conclusion is not due to experimental evidences on motion of celestial bodies, but to the numerical definition of $M_{tot} > M_{ob}$. So, regarding M_x as a sum of possible extra-masses not yet considered, let us write in general

$$M_{tot} = M_{ob} + \sum_i m_i. \tag{6.41}$$

8) At this point, comparing (6.13) and (6.37) one finds

$$4m_\omega = 1.17 \times 10^{54} \text{ kg}, \quad M_{ob} = 1.1 \times 10^{54} \text{ kg}. \tag{6.42}$$

and then (6.40) reads

$$M_{tot} - M_x = M_{ob} = 4m_\omega. \tag{6.43}$$

On one hand $M_{ob} = 4m_\omega$ supports the idea of regarding $M_{tot} = M_x + 4m_\omega$ with the coefficient 4 of m_ω agreeing the idea of sum of masses in (6.41), while being $M_{ob} \approx (\Lambda/H_u^2) m_\omega$ according to (6.2). On the other hand let us calculate and compare now the vacuum mass calculated through the vacuum energy density

$$M_{vac} = \frac{4}{3}\pi r_u^3 \frac{\eta_{vac}}{c^2} = \frac{4}{3}\pi r_u^3 \frac{H_u^2}{G} = \frac{\Lambda}{G} r_u^3 = M_\Lambda. \tag{6.44}$$

Clearly both sides are an identity owing to (6.2), *i.e.* $M_{vac} = M_\Lambda$ the factor $4\pi/3$ has been included in agreement with (6.3) and (6.17) owing to (6.19): the fact that η_{vac} has been calculated as a function of H_u^2 whereas here M_{tot} has been identified as a property related to Λ justifies the necessity of introducing the geometrical coefficient. Clearly M_{vac} concerns the total mass of the universe related to $m_\text{v} + m_\text{v}^*$ inferred in (3.11); in other words the total virtual mass originated by the vacuum cannot overcome the actual visible mass existing in the universe, otherwise energy and energy density of the virtual and real universe could not be in equilibrium. First of all it appears that total real mass and vacuum mass fluctuation corresponding to the vacuum energy density are systems in equilibrium, *i.e.* according to (6.17),

$$M_\Lambda = M_{vac} = M_{tot}; \quad M_{tot} = 2.34 \times 10^{55} \text{ kg}. \tag{6.45}$$

As M_Λ is gravitational mass, it follows that $\eta_{vac} = \eta_\Lambda$, *i.e.* vacuum and gravitational matter with the same energy density are a system in equilibrium; also, both densities are representative of the mass and vacuum energy of the whole universe. As an immediate corollary of this result it is possible to calculate

$$\frac{M_{ob}}{M_{tot}} = 0.048, \tag{6.46}$$

i.e. the whole observable mass in the universe, the one we know, is about 5% only of the whole actual mass. This acknowledged result is interesting because

M_{ob} has been calculated in (6.43) via m_{ω}, which in turn shows that r_u fulfills the black hole condition (5.25). Moreover it is possible to write $\eta_{vac} = \eta_{tot}$ and thus $\eta_{vac} V_u / c^2 = \eta_{tot} V_u / c^2$, while being $\eta_{tot} V_u / c^2 = M_{tot}$ by definition the total in the volume V_u of the universe.

9) Concern now the residual mass M_x prospected by (6.43) and (6.41). With reference to (3.11) and (3.12):

-one of these terms, call it M_{ob}^*, should concern the antimatter virtual mass m_{v}^* created along with the matter virtual mass m_{v} with the condition $m_{\mathrm{v}}^* = m_{\mathrm{v}}$;

-a second term, call it $m_{\delta\varepsilon}$, should concern the energy gap $\delta\varepsilon_{\mathrm{v}} / c^2$ due to fluctuation frequency $\delta t_{\mathrm{v}}^{-1}$ of (3.12);

-a third term, call it m_{m}, should be directly related to ϵ_{v} accounting for the fluctuation energy driven formation of the virtual masses themselves.

Without excluding in principle possible additional terms, let us examine these terms only; write then

$$M_{tot} = \left(M_{ob} + M_{ob}^* \right) + m_{\mathrm{m}} + m_{\delta\varepsilon}.$$

First of all, let be reasonably $M_{ob} = M_{ob}^*$, *i.e.* the amount of visible matter is equal to the amount of visible antimatter; without concerning now for the present purposes where the antimatter should actually be, a glance to (3.12) suggests that this last equation reads

$$M_{tot} = 2M_{ob} + m_{\mathrm{m}} + m_{\delta\varepsilon}$$

and yields

$$2\frac{M_{ob}}{M_{tot}} + \frac{m_{\mathrm{m}}}{M_{tot}} + \frac{m_{\delta\varepsilon}}{M_{tot}} \approx 1;$$

The notation emphasizes that are considered in this estimate two terms m_{m} and $m_{\delta\varepsilon}$ of the sum (6.41) only. Owing to (6.46) this result reads

$$\frac{m_{\mathrm{m}}}{M_{tot}} + \frac{m_{\delta\varepsilon}}{M_{tot}} \approx 0.9. \tag{6.47}$$

Next it is reasonable to assume

$$\frac{m_{\delta\varepsilon}}{M_{tot}} \approx 2\frac{m_{\mathrm{m}}}{M_{tot}}:$$

indeed $m_{\delta\varepsilon} = 2m_{\mathrm{m}}$ because the mass equivalent of the fluctuation energy ϵ_{v} in (3.11) is twice that of $m_{\mathrm{v}} c^2 = m_{\mathrm{v}}^* c^2$. This last position inserted in (6.47) is enough to conclude

$$\frac{m_{\mathrm{m}}}{M_{tot}} \approx 0.3, \quad \frac{m_{\delta\varepsilon}}{M_{tot}} \approx 0.6, \quad \frac{M_{ob}}{M_{tot}} \approx 0.1. \tag{6.48}$$

These terms estimate quite well the known ratios between the amounts of dark matter and dark energy with respect to the amount of familiar visible matter in the universe.

10) This last point concerns the black holes. Consider $\delta\ell$ and m of (5.25)

omitting the asterisks for brevity; then define the following feature of the black hole

$$\rho = \frac{m}{(4\pi/3)\delta\ell^3} = \frac{c^6}{(32\pi/3)m^2G^3}. \tag{6.49}$$

To examine the physical implications of ρ multiply both sides of ρ^{-1} by an arbitrary mass m', reciprocal volume δr^{-3} and c via a proportionality factor ζ; noting that $m'c/(\rho\delta r^3)$ has physical dimensions *length*/*time*, one finds according to (4.2)

$$\delta\dot{r}' = \zeta\frac{m'c}{\delta r^3\rho} = \frac{32\pi\zeta}{3}\frac{G^3m^2m'}{\delta r^3c^5}, \quad \delta\dot{r}' = \frac{\delta}{\delta t}\delta r'. \tag{6.50}$$

The ratio at the left hand side has been indicated as $\delta\dot{r}'$ to emphasize that it is in general different from δr and its sign depends on ζ. The physical reason of these steps appear recalling (3.9); rewriting the right hand side as follows

$$\delta\dot{r}' = \frac{32\pi\zeta}{3}\frac{Gm'}{\delta r}\frac{Gm^2}{\delta r^2}\frac{G}{c^5} \tag{6.51}$$

it is possible to implement (3.9) and the position $m^2 = m_o m$, as already explained in (2.11). Hence

$$-\delta\dot{r}' = 4\pi\frac{Gm'}{\delta r}\frac{Gm_o m}{\delta r^2}\frac{G}{c^5}, \quad \zeta = -\frac{12}{32}; \tag{6.52}$$

even in this case the proportionality constant is of the order of unity. At this point it is also legitimate to regard the arbitrary m' as $m' = m_o + m$ so that

$$\delta\dot{r}' = 4\pi\frac{1}{n'}F\frac{G}{c^3} \quad \frac{Gm'}{c^2\delta r} = \frac{1}{n'}, \quad m' = m_o + m, \quad F = -\frac{Gm_o m}{\delta r^2} \tag{6.53}$$

and then

$$F = \frac{n'}{4\pi}\delta\dot{r}'\frac{\hbar}{\ell_P^2} = \frac{n'}{4\pi}\frac{\delta}{\delta t}\left(\frac{\delta r'}{\hbar}\frac{\hbar^2}{\ell_P^2}\right) = \frac{n'^2}{4\pi}\frac{\delta}{\delta t}\left(\frac{\delta r'}{n'\hbar}\frac{\hbar^2}{\ell_P^2}\right)$$
$$= \frac{n'^2}{4\pi}\frac{\delta}{\delta t}\left(\frac{p_P^2}{\delta p_r'}\right) = -\frac{n'^2}{4\pi}\delta\dot{p}_r'\left(\frac{p_P}{\delta p_r'}\right)^2,$$

being p_P the Planck momentum. Note eventually that it is possible to put $\delta p_r' = n'p_P$, in which case the last equality of the chain reads $4\pi F = -\delta p_r'$, so that the last (6.53) yields

$$4\pi F = -4\pi G\rho m\delta r = -\delta\dot{p}_r' = -\frac{\delta}{\delta t}\delta p_r', \quad \rho = \frac{m_o}{\delta r^3}$$

i. e. F is proportional to $\delta\dot{p}_r$ as it must be. Then, owing to (4.2) and regarding m as a constant,

$$-4\pi G\rho = -\frac{1}{m\delta r}\frac{\delta}{\delta t}\delta p_r' = -\frac{1}{m\delta r}\frac{\delta}{\delta t}\frac{\delta\varepsilon\delta t}{\delta r} = -\frac{1}{m}\frac{\delta(\delta\varepsilon)}{\delta r^2}$$
$$= -\frac{\delta(\delta\varepsilon/m)}{\delta r^2} = -\frac{\delta^2(\varepsilon/m)}{\delta r^2} = -\frac{\delta^2\phi}{\delta r^2}, \quad \phi = \frac{\varepsilon}{m}.$$

This result is nothing else but the gravitational Poisson's equation whereas contextually ϕ yields energy per unit mass *i.e.* the gravitational potential, which shows that the definition (6.52) of ζ is sensible. As stated in (5.70) and (5.71) the factor 4π fits well the numerical coefficient 64/5, the deviation being a few % only; so with the definitions of ζ and m' (6.51) reads

$$\delta\dot{r} = n'\delta\dot{r}' = -4\pi \frac{G(m+m_o)}{\delta r} \frac{Gm_o m}{\delta r^2} \frac{G}{c^5}, \tag{6.54}$$

which is is nothing else by the Einstein collapse rate of two orbiting masses: $\delta\dot{r}$ has the correct relativistic form expected for the orbit radius contraction of a gravitational system due to its energy dissipation rate via gravitational waves, but now it is quantized in agreement with (5.74). Hence (6.49) is sensible starting point to calculate the dynamics of a gravitational system compliant at least in principle with the general relativity. Of course here the reasoning has been simplified and shortened for sake of brevity only; the aim of this last point is to justify the validity of (6.49) and its ability of defining the black hole energy density

$$\eta = \rho c^2 = \frac{c^8}{(32\pi/3) m^2 G^3} \tag{6.55}$$

To examine (6.49) and (6.55), rewrite first as usual m as w times the solar mass M_S that is

$$M_S = 1.989 \times 10^{30} \text{ kg};$$

so one obtains

$$\rho_w = \frac{1.9 \times 10^{19}}{w^2} \text{ kg/m}^3, \quad \eta_w = \frac{1.7 \times 10^{36}}{w^2} \text{ J/m}^3, \quad 0 < w < \frac{M_{tot}}{M_S}. \tag{6.56}$$

Also, noting that $density \times G = time^{-2}$, it is possible to relate a time range δt_w to the first (6.56) given by

$$\delta t_w = \frac{M_S G \sqrt{32\pi/3}}{c^3} w = 2.8 w \times 10^{-5} \text{ s}. \tag{6.57}$$

These values have been expressed via the arbitrary factor w as a function of the Sun mass, which as such has no particular physical meaning; this position simply helps to express the results of the following calculations in term of solar masses, as it is usual in astrophysics. A first interesting corollary implied by (6.55) is that if it is correct, as in effect it seems sensible owing to its reasonable implication (6.54), then it happens that η diverges for $m \to 0$: this means that there must be a lower limit of m to allow in practice the rising of the expected black hole behavior. In other words, the smaller m, the smaller the probability of forming a black hole for example by end life collapse of a low w star; the probability of forming small black holes should likely decrease with $\sim w^{-2}$.

Note that (6.56) and (6.57) have been inferred implementing exclusively (5.25), *i.e.* in (6.49) $m/\delta\ell^3$ has been calculated with a unique m fulfilling also $\delta\ell$ of (5.25). So (6.56) and (6.57) refer to the black holes according to (6.49) and evidence the chance of a huge range of mass and energy densities in principle possible, regardless of the specific explanation about their actual formation mechanism/process.

The black hole density introduced in (6.49) can be rewritten according to the following equivalent forms

$$\rho = \frac{m}{(4\pi/3)\delta\ell^3} = \frac{c^6}{(32\pi/3)m^2G^3} = \frac{c^2}{(8\pi/3)G\delta\ell^2} = \frac{1}{(8\pi/3)G\delta t^2}. \tag{6.58}$$

As expected, in the first equality the density is proportional to this mass contained in a space time volume $\delta\ell^3$. Yet (5.25) allows introducing the second expression where ρ is proportional to m^{-2} and even other forms where the mass does not appear at all explicitly; in the last equality does not appear even the size of the black hole, rather ρ depends on δt^2 via *G*, thanks to the physical dimensions of this latter. These terms are not trivial duplicates of the first standard definition: the third equality depends on $\delta\ell^{-2}$, and thus is particularly interesting to describe the black hole evaporation rate, which is clearly a surface phenomenon, whereas the last equality introduces explicitly the time. The physical interest of (6.58) is confirmed calculating ρ with $\delta t = t_u$ of (1.5); the result is

$$\rho_u = \frac{3}{8\pi G t_u^2} = 9.45\times10^{-27}\ \text{kg/m}^3 \tag{6.59}$$

in agreement with (6.15) and [5]. Also note in particular

$$\rho = \frac{c^2}{\hbar G}\frac{\hbar}{(8\pi/3)\delta\ell^2} = \frac{3}{2}\frac{c^2}{\hbar G}\frac{\hbar}{4\pi\delta\ell^2} = \frac{3\hbar/2A}{\hbar G/c^2}, \quad A = 4\pi\delta\ell^2: \tag{6.60}$$

this result is significant as it introduces the space time definition (1.1) and the black hole surface *A*. It is evident that all ways to define ρ can be legitimately implemented; since these ways are of course self consistent, the choice of the most appropriate form depends on its ability to describe the evaporation time δt_{ev} as a function of the initial black hole volume V_{bh}. The ability of a black hole to evaporate is now assumed to be a property of the space time defined in (1.1); write then the last equation according to (1.3) as

$$\rho\frac{A}{\hbar} = \frac{t}{V} = \frac{3/2}{\hbar G/c^2}, \quad \rho = \frac{3}{2}\frac{\hbar/A}{\hbar G/c^2}, \quad \delta\rho = -\frac{3}{2}\frac{\hbar/A^2}{\hbar G/c^2}\delta A \tag{6.61}$$

where the first equality has been written by dimensional reasons. Once again therefore it is possible to specify this equation to the case of interest via a proportionality constant ζ; writing $\zeta t/V = \delta t_{ev}/V_{bh}$ yields

$$\delta t_{ev} = \frac{3\zeta}{2}\frac{V_{bh}}{\hbar G/c^2}, \quad V_{bh} = \frac{4\pi}{3}\frac{8m_{bh}^3G^3}{c^6} = \frac{4\pi}{3}\frac{8M_S^3G^3}{c^6}w^3,$$

whence the result

$$\delta t_{ev} = \zeta\frac{V_S}{\hbar G/c^2}w^3, \quad V_S = \frac{32\pi M_S^3G^3}{c^6} \tag{6.62}$$

whose value calculated with (6.1) is

$$\delta t_{ev} = 4.2\zeta\times10^{72}w^3\ \text{s}.$$

The problem is now to determine ζ, which is not mere numerical proportionality factor; rather it is a multiplicative factor of V introduced in (6.61). Indeed during evaporation V cannot simply be the initial volume V_{bh} of the evaporating black hole, whereas after evaporation the region of space time previously occupied by the black hole turns into vacuum; this implies a change of local space time curvature, initially equal to that of the black hole surface. Eventually also consider that any evaporation process is a surface driven event; thus one expects that the black hole surface swells progressively and diffuses into vacuum with respect to the initial volume of black hole matter, whatever it consists of. So one expects reasonably $\zeta > 1$, which explains while the time calculated here is smaller than that inferred by Hawking, about 6×10^{74} s.

Admitting an effective volume V'_{bh} of evaporating black hole, which reasonably implies a reduced average density due to a decreased local density at and below the surface, it is possible to rewrite (6.62) as

$$\delta t_{ev} = \frac{V'_{bh}}{\hbar G/c^2} w^3, \quad V'_{bh} = \frac{4\pi}{3}\left(\zeta'\delta\ell\right)^3 = \frac{4\pi}{3}\left(\frac{2\zeta' mG}{c^2}\right)^3, \quad \zeta = \zeta'^3,$$

being ζ' the swelling parameter such that by definition $\delta\ell' = \zeta'\delta\ell$. To account for the loss of curvature after evaporation it is reasonable to implement (6.2) that relates H_u to Λ; so, putting $\zeta' = \Lambda/H_u^2$, the last equation reads

$$V'_{bh} = \frac{4\pi}{3}\left(\frac{2mG}{c^2}\frac{\Lambda}{H_u^2}\right)^3$$

so that the order of magnitude of this value $\delta t_{ev} = 2.7\times10^{74} w^3$ s fits that calculated by Hawking.

7. Discussion

The present theoretical model has shown the chance to infer quantum and relativistic outcomes starting neither from the deterministic metrics of special and general relativity nor from the operator formalism of the wave quantum theory, which is actually a byproduct of (1.2) as shown in the section 2.8.

This preliminary idea was suggested by the textbook [13] where, starting from the metrics to infer the Lorentz transformations, are determined in particular the transformation properties of three components of angular momentum. On the one hand this result is unavoidable, because the reasoning underlying the algebraic steps is correctly exposed in the quoted book; unfortunately however the quantum theory admits one component of angular momentum only, through which is calculable even M^2 itself as shown in the section 5.3. Clearly the wrong point is not the theoretical reasoning implied while implementing the deterministic metrics, but just the fact of starting from such a metrics in a self contained way regardless of its quantum compliance. On the other hand an analogous difficulty also rises when relativistic features are sought starting from the fundamental postulates of the wave quantum mechanics: it is difficult to acknowledge what have to do the Lorentz transformations with the indistinguishability of identical particles and their reference to Bose or Fermi statistics.

In other words, merging quantum theory and relativity into a unique conceptual frame is problematic because the two-way correspondence "deterministic metrics $\leftrightarrows$ wave quantum theory" doesn't work.

This conceptual gap is in turn due to the initial purposes of either theory: the wave quantum theory was in fact born to explain why the electron does not fall into the nucleus, the relativity to formulate a covariant approach to the nature laws. The right direction to follow is thus to merge not the whole theories themselves, but rather their fundamental roots from which everything follows. It is intuitive that the physical frame able to account for the conceptual pillars of both theories consequently will also be able to account for their specific topics; in effect it has been easy to show throughout the exposition of the present model that relativistic and quantum outcomes are contextually inferred in a straightforward and simple way. Thus the strategy of the present model follows the idea of waiving the standard premises of both theories, not because they are wrong per se but because they are incompatible, at least in the usual form currently implemented: instead of thinking an advanced relativistic formulation of problems into which to include successively also the quantum requirements, or vice versa, seems more practicable the idea of identifying a common conceptual root to start with, in order to infer as a natural corollary the fundamental axioms of both theories. In principle it seems hard the idea of abandoning the deterministic metric able to formulate covariant laws of physics, although it conflicts with the Heisenberg principle and the non locality/non reality; likewise it seems equally hard to give up the corpuscle wave dualism capable of explaining the tunnel effect, although it has seemingly nothing to do with the perihelion precession.

In addition this preliminary intent is still not enough to outline adequately the physical problem, there is a further conceptual difficulty.

Usually, the idea of quantum relativity reminds concepts like quantization or gravitational interaction between particles moving at speed near c or even superposition of gravitational states. In this respect nothing hinders in principle to conceive the actual corpuscles as waves: then, since $F=\dot{p}$, it is anyway possible to introduce $p=h/\lambda$ and next to define $F=-h\dot{\lambda}/\lambda^2$. Eventually, introducing the uncertainty (2.41), it is possible to proceed towards a gravity field valid in all reference systems.

This outline of alternative approach shortly sketched as a corollary of De Broglie momentum would certainly allow an innovative relativity without insurmountable efforts. But unfortunately this is not the true crucial point: the classical mechanics or standard relativity could not fit the conceptual character of the quantum world without accounting for two points that no mathematical code could ever introduce: the non locality and non reality, without which phenomena like the entanglement could never be explained or even conceived. Without these distinctive quantum features, would be out of our mind the Bell inequality, which instead is a fingerprint of the gap between relativistic and quantum theories as shortly sketched in Appendix C.

Actually the most important problem is to obtain a non-local and non-real general relativity.

These features seem oxymora when concerning real corpuscles that someway must be referred sooner or later to the Newton law, may be as a particular case or limit condition. In other words the crucial point is either to make relativity non local and non real or to demonstrate that quantum physics is local and real. Yet the experimental data show that the second alternative is unphysical; so the only attempt to formulate a successful connection between the theories is the first chance, which however requires a new conceptual reformulation well beyond the mere mathematical strategies.

In this respect the results of the present model indicate that (1.2) are a simple and reliable candidate to account for both theories. The standard quantum mechanics implements the operator formalism that by definition is related to the wave behavior of particles; yet to match relativity it is more sensible to implement the corpuscular behavior of particles according to the uncertainty, which inherently imply both delocalized mass and wave behavior. This intuitive statement summarizes the basic idea on which has been conceived the strategy of this paper. Clearly the mathematical formulation of the theoretical model must be consistent with these premises.

The universe implies the uncertainty. Indeed the mere definition (1.1) of space time takes implementable physical meaning when written first as in (1.3), which in turn provides physical information when rewritten further as in (2.1) and then as in (2.2) and next as in (2.29).

Often the algebraic steps have been inspired by and based on initial dimensional relationships, rather than on mathematical equations: the former are actually conceptual similarities, the latter prospect specific local values. This is for example the case of Equations (2.9) or (3.1) or (2.29); yet (1.3) and (3.20) are examples of how the dimensional premises turn into a physical formulation to be compared with the experience. But just this comparison rises a further crucial point: the concept of measurement.

As in fact the strategy of the present paper has followed these ideas, anyway the resulting (2.40) of the section 2.6 can be nothing else but agnostic relationships between ranges of dynamical variables preliminarily introduced in (1.2); strictly speaking their agnostic essence is a corollary of the initial abstract considerations, in turn based on the physical dimensions of the fundamental constants of nature. The physical kernel of these constants contains however all ingredients necessary to "materialize" their dimensional implications: as a matter of fact the uncertainty ranges of dynamical variables inferred in this way, see e.g. (2.29) and (2.35) or (5.45) and (5.46), fulfill not only the same relativistic transformation properties of the local dynamical variables but also the Heisenberg requirement. It is then evident the more general character of the present approach and thus its comprehensiveness even of the relativity: (1.2) ensure an approach more general than that of the standard wave mechanics, because they include this latter: the results evidence that one thing is the wave mechanics, other thing is the quantum theory based on (1.2) that appears compliant even with the relativity.

In particular, the remarkable Einstein intuitions of regarding the force as space time curvature and the equivalence principle that unifies the concepts of force and accelerated reference system, are here corollaries of the heuristic concept (1.1) of space time that bypasses the deterministic tensor formalism to describe a curved space geometry. Actually even the curvature (4.22) and (4.25) is the particular case of a more general concept of space time deformation (4.23), in turn due to the agnostic idea of replacing local coordinates, regarded as random, unknown and unknowable and thus unphysical, with uncertainty ranges. Is crucial the fact that these uncertainty ranges surrogate not only the relativistic equations defined by local coordinates of the standard relativity but also the deterministic metrics itself, thus demonstrating their fundamental physical meaning.

It has been emphasized that (1.2) remind standard concepts of statistical measure errors: just as no one trusts the reliability of a single measure in its experimental error bar, likewise (1.2) waive the signification of a local dynamical variable in its uncertainty range. Regarded in this way, the uncertainty has nothing weird or puzzling: yet its conceptual requirement makes the local dynamical variables elusive and in fact non existing, just as do not exist in principle measurements absolutely exempt of any errors. Yet this simple idea becomes in the quantum world a conceptual limitation of the human knowledge, thus demonstrating that the reality we see is elusive like that resulting by unavoidable measurement errors affecting the "true" ideal value. Nonetheless it would be wrong to regard the uncertainty with mere negative and reductive meaning: considering uncertainty ranges instead of deterministic local values is crucial to infer the equivalence principle of the general relativity, section 5.2, and even the Newton law, section 2.2.

First of all, it is worth noticing that even combinations of uncertainty ranges additional to (1.2) have physical meaning and allow defining characteristic features of quantum matter even without necessarily pairing conjugate dynamical variables. A short example is sketched just below with reference to Equations (2.43) to (2.46):

$$\delta\varepsilon\delta p_r = \frac{(n\hbar)^2}{\delta t\delta r} = \epsilon' p_r' = -\frac{Ze^2}{r'} m \frac{r'}{t'} = -\frac{Ze^2 m}{t'},$$
$$\epsilon = \epsilon' - 0, \quad p_r' = p_r' - 0, \quad t' = t' - 0,$$

whence

$$\delta r = -\frac{(n\hbar)^2}{Ze^2 m}\frac{t'}{\delta t}. \tag{7.1}$$

Let be by definition

$$t'-0 \geq \delta t = t'-t'', \quad \frac{t'-0}{t'-t''} = \frac{n'}{n''}\frac{\delta\epsilon''}{\delta\epsilon'}, \quad \delta r = \delta r\left(n,n',n''\right), \tag{7.2}$$

consistently with the arbitrariness of ranges in (1.2). On the one hand with the equality sign, possible in (7.2) in the particular case $t''=0$, (7.2) reduces to Bohr's radius; it answers the early question about why the electron does not fall into the nucleus. On the other hand, excluding $\delta r=0$ incompatible with (1.2), it also appears that electron transitions between states with different energy are also allowed as a function of time, along with the necessity of quantum numbers additional to n to account for the related variety of spectral lines.

Thus appears appropriate to start with is the definition (1.1) of space time, which in addition to the concept of uncertainty brings to invariants of special relativity in (2.47) and appendix B, contextually to the dual wave corpuscle behavior of matter, section 2.4, the quantization and the indistinguishably of identical particles, section 2.6, along with the rationale of the non local and non real properties of the quantum world, section 2.7.

The arbitrariness of uncertainty ranges hitherto invoked is thus not purposeful; it is a fundamental concept that pervades systematically any algebraic step of the present model, including even the relativistic formulas. Accordingly both theories become non real and non local because of the conceptual lack of deterministic dynamical variables. On the one hand this lack prevents knowing the intrinsic physical properties of a quantum corpuscle, being instead accessible its status perturbed by the measure process; indeed this perturbation driven range of values is just that appearing in (1.2). On the other hand this lack also prevents defining deterministic space distances and time lapses and regarding separately luminal or superluminal states: the corpuscle is not "here" or "there" but simply *is* in its space time uncertainty range. These concepts, previously considered distinctive of the quantum world only, become shared in the present model even with the relativistic world.

Coherently the Schrödinger cat simultaneously dead and alive teaches that the quantum world admits the superposition of states each one of which corresponds to a real chance for the usual classical world. This way of thinking, widely accepted as unavoidable weirdness, appears instead legitimated now even for a gravitational system; the failure of the concept of trajectory in the quantum world results here through an orbit circular but simultaneously elliptic whose geometrical parameters, like major and minor semi-axes, are someway hidden in the unknowable parameter p_{r0} of (2.28). Put in this way, the superposition of states appears coherent the idea that a planet orbit is neither circular not elliptic; it is a superposition of chances "per se" legitimate and thus not singularly refutable without a valid reason, which actually does not exist.

On the one hand the outline just proposed supports the idea that the space time of (1.1) is a physical entity characterized by its own properties like energy density, pressure, energy and force. In other words, the volume δx^3 expressing the dimensional definition $length^3$ of (1.3) is a physical entity that can be further implemented according to its own properties in the various sections.

On the other hand the uncertainty is regarded without having in mind only its original quantum implications, for this reason has been emphasized its immediate derivation from the operative definition of space time proposed in (1.1) as early exposed in [4]. In other words have physical meaning the uncertainty ranges, and not the random local dynamical variables.

To link quantum and relativistic physics implies a conceptual cost; for example the Lorentz transformation of x and x' does not read $(x-Vt)/\beta \rightarrow (x'+Vt')/\beta$ because the local time and space coordinates are unknown, it is only possible to consider $\delta x \rightarrow \delta x'$ but only the origins of the inertial reference systems. As a first remark about the quantum theory in this respect, note that the previous considerations are enough to bypass a wave based quantum approach only, as it is currently done, to start instead from the (1.2) totally agnostic but just for this reason more general; nevertheless the wave formalism and all its well known implications are in effect a straightforward corollary of the quantum uncertainty. As a further remark, is worth emphasizing that in this model are missing equations of motion to be solved; yet it is natural because (1.2) skip even the probability of local position. So also concepts like "comoving coordinates" are useless because is missing the concept itself of local coordinate, systematically replaced throughout this model by the physical concept of coordinate ranges; nothing is assumed known about these latter, while the same holds for any other dynamical variable. Nevertheless, just this agnostic approach allowed to obtain in a straightforward way relevant outcomes of general relativity and numerical results of the section 6 skipping crucial concepts like distances between objects, classically defined. Although the present model waives concepts definable in the frame of a deterministic metric, the conceptual limit put by the uncertainty selects the allowed knowledge actually accessible to the observer; e.g. by this reason one component of angular momentum is physically definable. Without being aware of this conceptual limit, relativistic and quantum theories would remain incompatible with each other.

Also note that aim and formulation of the present model are in principle different from that of Dirac in describing the relativistic hydrogenlike atoms: in His model, Dirac implements known relativistic concepts to infer a wave equation consistent with the ideas already formulated by Einstein. Here instead the fundamental principles of both theories are consistently cogenerated "ab initio" in a self contained way. The only common premise is that both (1.1) and in turn (1.2) merge together space and time, which therefore are meaningless separately: the former implicitly by dimensional reasons, the latter explicitly. Thus it is evident that (1.2) cannot imply any metrics, i.e the chance of defining lengths, angles and so on, just because size and orientation in the space time of all uncertainty ranges are completely unknown by definition; nevertheless the conceptual physical formulation of vectors follows by extrapolating physical concepts, *i.e.* simply guessing the meaning of dynamical variables corresponding to the range sizes. An example is (4.12) inferred from (4.11), whereas a Dirac-like equation including also the Lamb term has been inferred in [9].

In this context, special attention deserves the quantization, which is not mere mathematical feature of the quantum reality but has a general valence in that involves even the relativity for the reasons already emphasized in the sections 2.6 and 2.7 and further sketched here thinking in particular to (2.41). It has been shown that n arbitrary integer makes indistinguishable $\delta x\delta p$ from $\delta x'\delta p'$ and thus $\delta\varepsilon\delta t$ from $\delta\varepsilon'\delta t'$ in the respective R and R'; accordingly is also lost, by quantum reasons, the concept of simultaneity initially proposed by Einstein through the invariance of c. Not only is missing "a priori" the existence of privileged reference systems, but also becomes inessential the requirement of the different form of equations in R and R'. Since this holds for any uncertainty range by definition, in fact (2.41) bypasses the necessity of specifying inertial or non-inertial reference systems of the dynamical variables; accordingly it is possible to regard the ranges of (1.2) independently of how is defined their own R.

The last remark to be emphasized at this point is the kind of mathematical approach compliant with the premises of the present model, *i.e.* the dimensional analysis. To exemplify shortly how an elementary dimensional equation carries effectively physical information, consider $\hbar = mass\times velocity\times length$ and turn in into $\hbar = \zeta\delta\ell mv$; as previously explained, the dimensionless proportionality constant ζ aims to convert abstract dimensional concepts into operative dynamical variables of physical interest. At the right hand side appear four quantities that do not need being constant themselves, must be constant their product. Putting

$$\zeta = \frac{\beta_0}{\beta},$$

where β_0 is a constant, one finds

$$\hbar = \beta_0\delta\ell\frac{mv}{\beta} = \delta\ell' p, \quad \delta\ell' = \beta_0\delta\ell, \quad p = \frac{mv}{\beta}.$$

Hence it is also possible to write

$$\hbar = \beta_0\left(\frac{v\delta\ell}{c^2}\right)\frac{mc^2}{\beta} = \varepsilon\beta_0\delta t', \quad \delta t' = \frac{mv}{c^2}, \quad \varepsilon = \frac{mc^2}{\beta},$$

which yield

$$\hbar = \varepsilon\delta t, \quad \delta t' = \frac{\delta t}{\beta_0}$$

and then also

$$\delta\ell'\delta t' = \delta\ell\delta t.$$

Specify now

$$\frac{\beta_0}{\beta} = \frac{\sqrt{1-v_0^2/c^2}}{\sqrt{1-v^2/c^2}}, \quad \zeta = \zeta(v):$$

this last position defines the momentum p, the energy ϵ, the Lorentz transformations of $\delta\ell$ and δt and the space time invariant of the special relativity; $\delta\ell$ and δt are proper length and time. The second statement has introduced a condition on ζ additional to $\zeta m v \delta\ell = const$, which in fact reads $f(v) m\delta\ell = const$; *i.e.* three arbitrary and independent variables concur to define *const* to obtain the given results. This actually leaves undetermined the numerical specification of $\hbar$, whose unique value holds for any $\delta\ell$ and m whatever $f(v)$ might be. It is interesting a remark: the consequence of having introduced β_0 and β is that the mere definition of $\hbar$ introduces physical definitions of new values of dynamical variables tanks to the Lorentz factor, which not only codes their transformation between inertial reference systems but also concurs to a dynamic physical reality coming up to the static initial one. It should be clear now why the dimensional equations previously introduced effectively enable a theoretical model based on arbitrary quantities that, as such, can be nothing else but non-real and non local by definition. At this point is attracting the idea of verifying the effectiveness of such a dimensional analysis for (1.1) too, the starting point of this paper. Consider thus the following chain of equalities with the help of (1.2)

$$\frac{\hbar G}{c^2} = \frac{\hbar}{c^2}\frac{\delta\ell_0^3}{m_0\delta t_0^2} = \frac{\hbar}{c^2}\frac{n_0\hbar\delta\varepsilon_0^2}{m_0\delta p_0^3} = \frac{\hbar}{m_0c^2}\frac{n_0\hbar\delta v_0^2}{\delta p_0} = \tau_0\delta x_0\delta v_0^2, \qquad v_0 = \frac{\delta\varepsilon_0}{\delta p_0}, \quad \tau_0 = \frac{\hbar}{m_0c^2}; \tag{7.3}$$

then write

$$\frac{\hbar G}{c^2} = \tau_0\delta x_0\delta v_0^2 = \frac{\delta x_0\delta\ell_0^2}{\tau_0}, \quad \delta\ell_0 = \delta(v_0\tau_0), \quad \delta\ell_0 = \tau_0\delta v_0,$$

which yields

$$c^2\tau_0^2 - \delta\ell_0^2 = c^2\tau_0^2 - \frac{\hbar G/c^2}{\delta x_0/\tau_0} = c^2\tau_0^2 - \frac{\tau_0^2\hbar G/c^2}{\delta x_0\tau_0} = c^2\tau_0^2\left(1 - \frac{\hbar G/c^4}{\delta x_0\tau_0}\right).$$

Hence

$$c^2\tau_0^2 - \delta\ell_0^2 = c^2\tau_0^2\left(1 - \frac{v_0^2}{c^2}\right), \quad v_0 = \frac{\hbar G/c^2}{\delta x_0\tau_0},$$

which reads thus

$$\Delta\ell_0^2 = c^2\tau_0^2 - \delta\ell_0^2 = \Delta\ell'^2, \quad \Delta\ell'^2 = c^2\tau_0^2\left(1 - \frac{v_0^2}{c^2}\right).$$

It means: $c^2\tau_0^2$ less something yields Lorentz contraction of $c^2\tau_0^2$ itself in a different inertial reference system moving at relative constant rate v_0.

8. Conclusions

The basic assumption of the present physical model, surprisingly simple and intuitive despite the complexity of the concerned task, implies a huge amount of further considerations; a more systematic examination has been carried out in [16]. This paper does not aim to introduce new equations to be solved to infer information: yet, owing to (1.2), quantum systems for which a simple analytical solution exists are described in complete agreement with the wave formalism without solving any equation. The same holds for relativistic quantities, e.g. (2.35), (5.113) and (5.79).

The implications of the approach hitherto introduced consist of at least four open points. (i) According to (3.10) the gravitational interaction between matter and antimatter is repulsive, being found with a positive sign. On the one hand it has been found in other papers [9] that both signs are compatible with the Newton law inferred in the present conceptual frame, as it results also in (2.11) and (4.9); this follows from $\delta\dot{p} = -n\hbar(\delta x)^{-2}\delta\dot{x}$, examined in section 5.2, whose sign depends uniquely on the size change rate $\delta\dot{x}$ of δx that in principle can swell or shrink. If this reasoning would find experimental confirm, then would be validated the idea prospected in [4] *i.e.* the antimatter originated contextually to the ordinary matter as in (3.11) could be repelled at the boundary of the universe, which should consist of an outer shell of antimatter.

(ii) Various definitions of acceleration have been found in this paper, e.g. $\epsilon c/\hbar$ in (2.9) and ωv in (5.77), to which in principle correspond respective forces. It is sensible to ask at this point whether these definitions are redundant repetitions of a unique force, while being however formally different only, or in fact these definitions concern forces of different physical nature. Seemingly (2.9) and (5.77) can be regarded as analogous results, because they merge into $\epsilon = \hbar\omega$ a factor v/c apart. However this is not generally true, because ϵ is not necessarily replaceable with $\hbar\omega$ only. For example a tidal force does not involve the Planck constant. In effect $\delta x/\delta t^2$ of (4.13), $v^2/\delta\ell$ of (4.14) and $c^2/\delta\ell$ of (5.32) appear profoundly different; the arbitrary ranges defining these expressions share only the physical dimensions $length/time^2$. So it is sensible to examine the problem of establishing if and when the various forces in principle implied are effectively reducible to a unique concept or instead to the fundamental forces of nature, e.g. typical of subnuclear interactions.

(iii) The fundamental structure of Nature of physical interest is actually enclosed in the three constants of (1.1); the existence of a fourth essential concept, the charge, is revealed by (5.36) and also appears in connection to *G*, as shown by the numerical evidence involving the fine structure constant

$$e = \kappa G\alpha, \quad e = 4.80320\times 10^{-10}\ \text{u.e.s.}$$
$$G = 6.67408\times 10^{-8}\ \text{cm}^3\cdot\text{g}^{-1}\cdot\text{s}^{-2}, \quad \alpha = 0.007297,$$

which holds with a value of the dimensional constant $\kappa = 0.98628\ \text{g}^{3/2}\cdot\text{cm}^{-3/2}\cdot\text{s}$ implementing of course c.g.s.e system where the charge is directly defined through the usual dynamical variables. The deviation of e from αG is 1.37% only; $\kappa \approx 1$ and α suggest non accidental link between e and G that deserves being investigated.

(iv) Assume the presence of a massive object of mass $\mathcal{M}$, e.g. supermassive black holes or galaxies, in an arbitrary region of space time and consider (5.95) in a surrounding region of space time arbitrarily apart with respect to the local position of the object; it is nevertheless reasonable the fact that according to (5.30) and (5.94) in proximity of the massive object the local time run δt due to the gravitational field is slowed down with respect the proper time range δt_0 in an empty space time. The time delay $\delta\mathcal{T} = \delta t_0 - \delta t$ between two points $\delta\mathcal{L}$ apart of these space time regions implies the rising of an energy gap $\delta\mathcal{E} = n\hbar/\delta\mathcal{T}$ between the respective regions; this gap in turn implies by consequence the existence of a momentum $\delta\mathcal{P} = n\hbar/\delta\mathcal{L}$ in the intermediate region $\delta\mathcal{L}$ of space time. Also now $\delta\mathcal{E}$ and $\delta\mathcal{P}$ are defined by an appropriate interval of allowed values $n_1 \le n \le n_2$ of *n*, of course arbitrary. This implies that someway massive objects and surrounding space time interact. The question arises: are $\delta\mathcal{E}$ and $\delta\mathcal{P}$ anyhow related to dark energy and dark matter, as prospected in Equations (6.24) to (6.27) of [9]?

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix A

In principle the quantum energy density $\eta = \epsilon/V$ in a given three dimensional space time volume V can be expressed as $\epsilon/\delta x^3$ or $\epsilon/\delta x^2\delta x_0$ or $\epsilon/\delta x\delta x_0^2$: any value allowed to the first definition of η is identically allowed to the second and third definitions as well for $\delta x_0 = \delta x$. Yet it does not hold when considering the change $\delta\eta$ if $\delta x_0 = const$ whereas δx is allowed to change; clearly the three definitions imply different $\delta\eta$. Define

$$\eta_k = \frac{\epsilon}{\delta x^k \delta x_0^{3-k}}, \quad \eta = \frac{\epsilon}{\delta x^3}, \quad k = 1,2,3$$

so that, implementing the dimensional analysis likewise as in (2.9),

$$\delta\eta_k = -k\frac{\epsilon}{\delta x^{k+1}\delta x_0^{3-k}}\Delta x_k = -k\eta_k\frac{\Delta x_k}{\delta x} = -k\frac{\epsilon/\delta x}{V/\Delta x_k} = -k\frac{F}{S_k} = P_k$$

and for $k = 3$ in particular

$$\delta\eta = -3\eta\frac{\Delta x}{\delta x} = -3\frac{\epsilon/\delta x}{V/\Delta x} = -3\frac{F}{S} = P;$$

by definition F is force while Δx_k and Δx are arbitrary displacements of the boundaries defining the initial volume defining η, so that P_k and P are pressures because S and S_k are surfaces. In other words the energy density inside V is assumed to change at constant ϵ merely because of the change of space time volume. Hence, taking the ratios side by side to eliminate $F = \epsilon/\delta x$, the first and last terms of the two chains yield

$$\frac{\delta\eta_k}{\delta\eta} = \frac{P_k}{P} = \frac{k}{3}\frac{\eta_k}{\eta}r_k, \quad r_k = \frac{\Delta x_k}{\Delta x}.$$

Split now this result in order that

$$P_k = \frac{k}{3}\eta_k, \quad P = \frac{\eta}{r_k}. \tag{2.13}$$

Introduce at this point the concept of uncertainty, which implies lack of any information about the actual sizes not only of δx and δx_k but also of Δx and Δx_k. This implies in turn that P corresponding to δx must fulfill the same rule of the three P_k allowed. This is in fact possible defining

$$r_k = \frac{\Delta x_k}{\Delta x} = \frac{3}{k},$$

owing to the boundary condition $\Delta x = \Delta x_k$ for $k = 3$. So P is one of the values resulting by merging (2.13)

$$P = \frac{k}{3}\eta, \quad k = 1,2,3 \tag{A.1}$$

as it is known. Indeed this result is just that of the first (2.13), which merely specifies P_k as a function of the respective η_k, whereas in the second (2.13) P coincides with the first one thanks to the definition of k_k.

Appendix B

The starting point is the space time swelling Equation (3.22)

$$c^2 = \frac{\Delta x^3}{\delta x^3} v^2$$

rewritten as

$$c^2 = (q_1 + q_2) v^2, \quad \frac{\Delta x^3}{\delta x^3} = q_1 + q_2, \tag{B.1}$$

where q_1 and q_2 are appropriate factors expressing the swelling ratio. Hence, multiplying by δt^2 both sides of the first equation, $c^2 \delta t^2 = (q_1 + q_2) v^2 \delta t^2$ yields

$$c^2 \delta t^2 = (q_1 + q_2) \delta \ell^2, \quad \delta \ell^2 = \mathrm{v}^2 \delta t^2, \tag{B.2}$$

from which one obtains

$$c^2 \delta t^2 \left(1 - \frac{q_1 \delta \ell^2 / \delta t^2}{c^2}\right) = q_2 \delta \ell^2$$

i.e.

$$c^2 \delta t^2 \left(1 - \frac{\mathrm{v}^2}{c^2}\right) = q_2 \delta \ell^2, \quad \mathrm{v}^2 = q_1 \frac{\delta \ell^2}{\delta t^2}. \tag{B.3}$$

From this result one finds

$$c^2 \delta t^2 \beta^2 = q_2 \delta \ell^2, \quad \beta^2 = 1 - r, \quad r = \frac{\mathrm{v}^2}{c^2}$$

that in turn yields two chances via the positions

$$\frac{c^2 \delta t^2}{q_2} = \delta \ell_p^2, \quad \frac{q_2 \delta \ell^2}{c^2} = \delta t_p^2 \tag{B.4}$$

where

$$\beta^2 \delta \ell_p^2 = \delta \ell^2, \quad \delta t^2 = \frac{\delta t_p^2}{\beta^2}; \tag{B.5}$$

these equations imply also

$$\delta \ell^2 \delta t^2 = \delta \ell_p^2 \delta t_p^2. \tag{B.6}$$

Let the subscript *p* mean *proper*, so that $\delta \ell^2 \delta t^2$ refers to proper length and time; *i.e.* $\delta \ell_p^2 \delta t_p^2$ is defined in an inertial reference system R_p where the particle is at rest, moving at constant rate v with respect to *R* defining the left hand side. Let us show that the space time invariant (B.6), inferred contextually to the time dilation and space contraction via the Lorentz factor β, implies the interval rule

$$c^2 \delta t^2 - \delta \ell^2 = c^2 \delta t_p^2 - \delta \ell_p^2. \tag{B.7}$$

The key equations are that already introduced, *i.e.* owing to (B.5)

$$\delta t_p = \frac{\delta \ell \delta t}{\delta \ell_p} = \beta \delta t, \quad \delta \ell^2 = \delta \ell_p^2 \beta^2 = \delta \ell_p^2 - r \delta \ell_p^2.$$

To show that (B.7) reduces to an identity owing to (B.4), write

$$c^2\delta t^2 - \delta\ell^2 = q_2\delta\ell^2 - \frac{c^2\delta t^2}{q_2}; \tag{B.8}$$

in effect this is an identity for $q_2 = -1$. As expected (B.4) requires then $\delta\ell_p = \pm ic\delta t$. So (B.1) reads

$$c^2 = (q_1 - 1)v^2, \quad \frac{\Delta x^3}{\delta x^3} = q_1 - 1,$$

whence

$$\frac{v^2}{c^2} = \frac{1}{q_1 - 1}, \quad 2 < q_1 < \infty.$$

If $\Delta x^3 = \delta x^3$ for $q_1 = 2$ the universe would be steady. As in general $q_1 \neq 2$, (B.7) and (B.8) hold even for an expanding universe.

Appendix C

Consider three terms

$$N(A,B_n), \quad N(B,C_n), \quad N(A,C_n) \tag{C.1}$$

that represent the occurrence/non-occurrence numbers $N(\ldots,\ldots)$ of three values A,B,C of arbitrary physical events corresponding to the physical properties P_A, P_B, P_C; the subscript n stands here for *not*, *i.e.* the given physical property does not hold. Regard separately the chance of equality

$$N(A,B_n) + N(B,C_n) = N(A,C_n) \tag{C.2}$$

and then also the chances of inequality

$$N(A,B_n) + N(B,C'_n) \neq N(A,C'_n): \tag{C.3}$$

(C.2) means that a value C exists for a given physical property P_C such that its corresponding C_n is compatible with the equality sign. Moreover, to account also for the inequality chance (C.3), it is necessary that a different value $C' \neq C$ of the given P_C also exists in relation to the same P_A and P_B represented by the values A and B along with the corresponding A_n and B_n.

Let P_C take first the values C and C_n, and consider (C.2). Summing $N(A_n,C_n)$ at both sides of (C.2) yields

$$N(A,B_n) + N(B,C_n) + N(A_n,C_n) = N(A,C_n) + N(A_n,C_n)$$

equal to

$$N(A,B_n) + N(B,C_n) + N(A_n,C_n) = N(C_n); \tag{C.4}$$

indeed, summing both occurrences and non-occurrences of the property A at the right hand side means considering the occurrence of C_n only. Adding also $N(B_n,C_n)$ at both sides of this last equation, one finds

$$N(A,B_n) + N(B,C_n) + N(A_n,C_n) + N(B_n,C_n) = N(C_n) + N(B_n,C_n)$$

that is equal to

$$N(A,B_n)+N(A_n,C_n)=N(B_n,C_n).$$

Subtracting now side by side (C.2) from this last equation one finds

$$N(A_n,C_n)-N(B,C_n)=N(B_n,C_n)-N(A,C_n)$$

and thus

$$N(A_n,C_n)+N(A,C_n)=N(B_n,C_n)+N(B,C_n).$$

$$N(C_n)=N(C_n). \tag{C.5}$$

This identity is inferred from and thus agrees with the initial (C.2). It is clear that reverting the order of these steps starting from this identity, one finds again (C.2). Let now P_C take different values C' and C'_n, in which case the equality does no longer hold with the same values pertinent to P_A and P_B. In general, with the same reasoning repeated for initial (A,C') and (A,C'_n), one expects that (C.3) yields $N(C')\neq N(C'_n)$: an initial discrepancy cannot result in a final identity through the same steps that have converted an initial identity into a final identity. Repeat now the reversed steps (C.5) to (C.2) starting from this inequality; it is clear that now one finds the corresponding inequality (C.3), rewritten in general

$$N(A,B_n)+N(B,C'_n)\gtrless N(A,C'_n) \tag{C.7}$$

via two inequalities. To understand whether (C.7) is consistent with (C.5) it is necessary to specify the properties P_A,P_B,P_C: the problem is comparing "determinism vs non-determinism". In this paper determinism has been referred to either existence or not of local space time coordinates; now this concept is extended to the properties of the values (C.1) and concern non locality and non reality of the quantum world. The problem of interest is thus to establish whether or not the physical properties of a system of particles do exist "a priori" or are created by the interaction with the experimental apparatus that perturbs the initial unknown system. (C.1) have been written in order that the property P_C appears with *not* value C_n only and the property P_A with occurrence value A only, whereas B appears in both forms; to infer information about non local and non real systems in comparison with real and local systems, consider a possible scheme where B is a property like particle spin or photon polarization. In the case of spin let B and B_n represent the respective chances of spins paired or not. The scheme implements the following attributions of values

$$A=non\ real,\ A_n=real,\quad B=spin\uparrow\downarrow,\ B_n=spin\uparrow\nearrow,$$
$$C=local,\ C_n=non\ local,$$

of the properties P_A,P_B,P_C that imply by consequence the following interpretation

$$\underset{non\ real,\uparrow\nearrow}{(A,B_n)}\quad+\quad\underset{non\ local,\uparrow\downarrow}{(B,C_n)}\quad\geq\quad\underset{non\ real,\ non\ local}{(A,C_n)}. \tag{C.8}$$

At the right hand side of the inequality is concerned the quantum theory, which is *both* non real *and* non local; indeed the sections 2.6 and 2.7 have shown that the uncertainty requires contextually these features. At the left hand side is concerned any non-quantum theory, which is *either* non local *or* non real only. The symbol ≥ is understandable regarding $N(\ldots,\ldots)$ as probabilities, which is possible simply introducing a normalization factor to unity; obviously the sum of probabilities of either property verified is higher or at least equal to that describing both probabilities contextually verified. The first column represents therefore the values A and B of the properties P_A and P_B in a non-quantum local theory, because it is non real only, and thus without spin correlation; this correlation requires indeed a non local spooky action to occur. In the second column the values B and C of the properties of P_B and P_C represents again a non-quantum theory, because of its non locality only, but now with spin correlation just due to its non locality. The third column represents the quantum theory, which is *both* non local and non real. The symbol ≥ identifies the Bell inequality. Predetermined physical properties, typical of non-quantum physical theories, classical and even relativistic as well, fulfill the inequality: the determinism of relativistic metrics belongs to a classical vision of universe, although enriched by covariance of physical laws, four dimensional premise, invariant light speed introduced by Einstein. The violation of the inequality does not require the existence of "hidden variables" to bypass the difficulty of a superluminal action between particles. In fact, hidden variables are excluded in the present conceptual frame based on (1.2) and bypassing the wave functions where these hypothetical variables could be somehow encoded. A further remark on the Bell inequality is that it reads

$$\left(N\left(A,B_n\right)-\xi_1 N\left(A,C_n\right)\right)+\left(N\left(B,C_n\right)-\xi_2 N\left(A,C_n\right)\right)\geq 0,\quad \xi_1+\xi_2=1, \quad \text{(C.9)}$$

having moved to the left side and split $N\left(A,C_n\right)$; *i.e.*, since (C.1) are in fact numbers, $\delta N_1 \geq \delta N_2 \leftrightarrow c^2 \geq v^2$ show that the Bell inequality consists of arbitrary ranges constrained similarly to c^2 with respect to v^2.