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On the Existence of Global Weak Solutions to 1D Sediment Transport Model

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Abstract

This paper is devoted to the study of the existence of weak solution in time with a periodic domain of sediment transport model. We consider a one-dimensional viscous sediment transport model which combines a viscous Shallow-Water system with a transport equation that describes the bottom evolution. The model studied does not take into account all the regularizing terms used by Roamba Brahima, Zongo Yacouba and Jean de Dieu Zabsonré (2017) and we use a better transport equation than that used by Zabsonré (2012).

Keywords

Shallow-Water, Sediment Transport, Viscous Model, Weak Solution

1. Introduction

In this paper, we study the existence of global weak solutions of one-dimensional sediment transport model. This work takes its inspiration from the work done in [1]. It should be noted that the author showed the stability of the weak solutions of a sediment transport model. The model presented is given on one dimension by the following equations:

$$\partial_t h + \partial_x (hu) = 0, \tag{1}$$

$$\partial_t (hu) + \partial_x (hu^2) - \nu \partial_x (h \partial_x u) + gh \partial_x (z_b + h) = 0, \tag{2}$$

$$\partial_t z_b + \partial_x (q_b(h,u)) - \frac{\nu}{2} \partial_x^2 z_b = 0. \tag{3}$$

where h, z_b, u represent respectively, the water, the sediment heights and the water velocity. The constant $g > 0$ is the gravity number and ν the kinematic viscovity. The discharge for the solid transport used by the author is:

$$q_b(h,u) = \beta hu\left(c_1|u|^2 + c_2\right)_+^m,$$

where β, c_1 and c_2 are the real numbers satisfying:

$$c_2 < 0, \quad 0 < c_1 < \frac{1}{4m\beta}, \quad 0 < m \le \frac{1}{2} \text{ with } u_+ = \max\{u, 0\}$$

In theory, the equation that describes sediment transport is a continuity equation. The expression of the conservation sediment volume equation is given by:

$$\partial_t z_b + \beta \partial_x q_b = 0, \tag{4}$$

where $\beta = \dfrac{1}{1-\rho_0}$ and ρ_0 is the porosity of the sediment layer, z_b is the movable bed thickness and q_b is the solid transport discharge (see [2]). In literature, many mathematical models describing the sediment transport with different expressions from solid transport discharge have been the subject of several studies (see [2] [3]). Among all these models, the best one to describe the solid discharge is given by the form:

$$q_b = z_b u \quad \text{(see [4])}.$$

Let us next recall some results on the existence of solutions of the one-dimensional Navier Stokes equations and viscous sedimentation models.

In [5], the author proved the existence of global weak solutions for 2D viscous Shallow Water equations and convergence to quasi-geotrophic model. In the paper, the authors shown the control of the vaccum thanks to an entropy named BD-entropy, which was introduced firstly in [6]. We note that the authors in [7] [8] have used this BD-entropy to get existence result of global weak solutions for Shallow-Water and viscous compressible Navier-Stokes equations. We have used this entropy in our work.

The authors in [9] have proved a result of global strong solution to the Navier-Stokes system with degenerate viscosity coefficient. This work has been developed in [10].

We integrate their ideas to limit the water height.

In [11], the authors show the existence of global weak solution for pollutant transport model; to have their result, the authors make a technical hypothesis on the pollutant layer and water layer in the form

$$h_2 \le h_1, \tag{5}$$

where h_1, h_2 represent respectively the pollutant and the water height.

The model studied in [11] comes from the form:

$$\begin{cases} \partial_t h_1 + \partial_x(h_1 u_1) = 0, \\ \partial_t(h_1 u_1) + \partial_x(h_1 u_1^2) + \dfrac{1}{2} g \partial_x h_1^2 - 4\nu_1 \partial_x(h_1 \partial_x u_1) + \dfrac{\alpha}{\rho_1}\gamma(h_1)u_1 - \dfrac{\delta_\xi}{\rho_1} h_1 \partial_x^3 h_1 \\ + r_1 h_1 |u_1|^2 u_1 + rgh_1 \partial_x h_2 + rgh_2 \partial_x(h_1 + h_2) = 0, \\ \partial_t h_2 + \partial_x(h_2 u_1) + \partial_x\left(-h_2^2 \dfrac{1}{\rho_2}\left(\dfrac{1}{c} + \dfrac{1}{3\nu_2} h_2\right)\partial_x p_2\right) = 0, \end{cases} \tag{6}$$

with

$$\partial_x p_2 = \rho_2 g \partial_x (h_1 + h_2) \text{ and } \gamma(h_1) = \left(1 + \frac{\alpha}{3\nu_1} h_1\right)^{-1}. \quad (7)$$

where h_1, h_2 represent respectively, the water and the pollutant heights, u_1 is the water velocity, ρ_1 and ρ_2 are the densities of each layer of fluid; the ratio of densities is denoted $r = \frac{\rho_2}{\rho_1}$, ν_i is the kinematic viscosity, p_2 the pressure of the pollutant layer and g is the constant gravity. The coefficients δ_ξ, α, r_1, c, are respectively the coefficients of the intrefaz tension, friction at the bottom, quadratic friction and friction at the interfaz.

From a mathematical point of view, the momentum equation of the studied model is similar to this model except that does not take into account the laminar and quadratic friction terms ($\frac{\alpha}{\rho_1}\gamma(h_1)u_1$ and $r_1 h_1 |u_1|^2 u_1$) and the intefaz tension term ($\frac{\delta_\xi}{\rho_1} h_1 \partial_x^3 h_1$). And these terms have been instrumental in showing the strong convergence of $\sqrt{h}u$ in [11].

In this paper, we investigate the following regularized model with better transport equation:

$$\partial_t h + \partial_x (hu) = 0, \quad (8)$$

$$\partial_t (hu) + \partial_x (hu^2) - 2\nu\partial_x (h\partial_x u) + gh\partial_x (z_b + h) + gz_b \partial_x (h + \varepsilon z_b) = 0, \quad (9)$$

$$\partial_t z_b + \partial_x (z_b u) - \nu\partial_x^2 z_b = 0. \quad (10)$$

where $(t, x) \in (0, T) \times]0,1[$.

We introduce the ratio of densities by *r*, we note $\varepsilon = \frac{1}{r}$ and we assume that

$$1 < \varepsilon < 2 \text{ and } \forall \varepsilon > 1, \text{there exist } \alpha \text{ such that } 0 < \alpha < \varepsilon - 1. \quad (11)$$

The two Equations ((8), (9)) form the hydrodynamic component that is modeled by the equations of shallow water that are used to study fluid movement. The last two terms in (9) are terms of exchange that are obtained by derivation (see [12]). The morphodynamic component is shaped by a sediment transport Equation (10).

In our paper, we make a similar hypothesis assuming that

$$z_b < \frac{2}{\varepsilon}(\varepsilon - \alpha - 1)h. \quad (12)$$

This condition implies that the thickness of the sediment layer is small compared to that of the fluid. It will allow us to recover the BD entropy.

In this work, our contribution from the mathematical point of view is on the one hand to show the existence of weak solutions of a sediment transport model without regularizing terms (friction terms and tension terms), which the authors in [11] have resorted to have a result of the existence of weak global solutions of pollutant transport model. On the other hand this paper means the result of stability in [1] or we consider one-dimensional regularized sediment transport model using a better transport equation than the one used in [1].

Our paper is organized as follows. First of all, we give in the Section 2 the definition of global weak solutions. Secondly, we establish a classical energy associated to our model, which give some regularities on the unknowns. Thirdly, we give some results allowing us to limit inferiorly the height of water which is very fundamental for the continuation since this limit study gives us additional regularities on the unknowns. Fourthly, we give an existence theorem of global weak solutions. And finally, we give the proof of the energy associated to the model and existence theorem including the limits passage in the Section 3.

2. Mains Results

We start this section by giving the initial conditions of model and the definition of weak solutions to (8)-(10).

$$h(0,x)=h_0(x),\quad z_b(0,x)=z_{b_0}(x),$$
$$u(0,x)=u_0(x),\quad (hu)(0,x)=m_0(x)\quad \text{in }]0,1[. \tag{13}$$
$$h_0\in L^2(0,1),\quad z_{b_0}\in L^2(0,1),\quad \partial_x(h_0)\in L^2(0,1),$$
$$\partial_x m_0\in L^1(0,1),\quad m_0=0\quad \text{if } h_0=0,\quad \frac{|m_0|^2}{h_0}\in L^1(0,1). \tag{14}$$

Definition 2.1. *We say that* (h,z_b,u) *is a weak solution of* (8)-(10), *with initial data* (13), (14) *verifying the entropy inequality* (18); *if for all smooth test functions* $\phi=\phi(t,x)$ *with* $\phi(T,)=0$, *we have*:

$$h_0\phi(0,.)-\int_0^T\int_0^1 h\partial_t\phi-\int_0^T\int_0^1 hu\partial_x\phi=0, \tag{15}$$
$$-z_{b_0}\phi(0,.)-\int_0^T\int_0^1 z_b\partial_t\phi-\int_0^T\int_0^1 z_bu\partial_x\phi+\nu\int_0^T\int_0^1\partial_x {z_b}^2\partial_x\phi=0, \tag{16}$$
$$\begin{aligned}&h_0u_0\phi(0,.)-\int_0^T\int_0^1 hu\partial_t\phi-\int_0^T\int_0^1 hu^2\partial_x\phi+2\nu\int_0^T\int_0^1 h\partial_xu\partial_x\phi\\&+g\int_0^T\int_0^1(h\partial_xz_b)\phi+g\int_0^T\int_0^1(z_b\partial_xh)\phi\\&+g\int_0^T\int_0^1(h\partial_xh)\phi+g\varepsilon\int_0^T\int_0^1(z_b\partial_xz_b)\phi=0.\end{aligned} \tag{17}$$

Proposition 2.1. *For* (h,u,z_b) *smooth solution of the system* (8)-(10) *with boundary conditions* (13)-(14), *we show the following relation*:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 h|u+\nu\partial_x\log h|^2+\frac{1}{2}(1-\nu)\int_0^1 h|u|^2+\frac{1}{2}g\int_0^1\left[|h+z_b|^2+(\varepsilon-1)|z_b|^2\right]\\&+\nu g\int_0^1|\partial_xh+\partial_xz_b|^2+\nu\int_0^1 h|\partial_xu|^2+\alpha\nu g\varepsilon\int_0^1|\partial_xz_b|^2\\&+\nu g\int_0^1\left[-\frac{\varepsilon}{2}\frac{z_b}{h}+\varepsilon-\alpha-1\right]|\partial_xz_b|^2+\nu g\left(1-\frac{\varepsilon}{2}\right)\int_0^1|\partial_xh|^2\le 0.\end{aligned} \tag{18}$$

Corollary 2.1 *Let* (h,z_b,u) *be a solution of model* (8)-(9). *Then, thanks to* ***Proposition*** 2.1 *we have*:

$\sqrt{h}u$ is bounded in $L^\infty\left(0,T;L^2\left(0,1\right)\right)$, $\sqrt{h}\partial_x u$ is bounded in $L^2\left(0,T;L^2\left(0,1\right)\right)$,

h is bounded in $L^\infty\left(0,T;L^2\left(0,1\right)\right)$, z_b is bounded in $L^\infty\left(0,T;L^2\left(0,1\right)\right)$,

$\partial_x h$ is bounded in $L^2\left(0,T;L^2\left(0,1\right)\right)$, $\partial_x z_b$ is bounded in $L^2\left(0,T;L^2\left(0,1\right)\right)$,

$\partial_x\sqrt{h}$ is bounded in $L^\infty\left(0,T;L^2\left(0,1\right)\right)$,

$$\left(\sqrt{1-\frac{\varepsilon}{2}}\right)\partial_x h \text{ is bounded in } L^2\left(0,T;L^2\left(0,1\right)\right),$$

$$\left(-\frac{\varepsilon}{2}\frac{z_b}{h}+\varepsilon-\alpha-1\right)\partial_x z_b \text{ is bounded in } L^2\left(0,T;L^2\left(0,1\right)\right).$$

Remark 2.1 *Thanks to the Corollary* 2.1, *we have*

h is bounded in $L^\infty\left(0,T;L^2\left(0,1\right)\right)$, $\partial_x h$ is bounded in $L^2\left(0,T;L^2\left(0,1\right)\right)$,

which implies that h is bounded in $L^\infty\left(0,T;H^1\left(0,1\right)\right)$. According the Sobelev embeddings, we have:

h is bounded in $L^\infty\left(0,T;L^\infty\left(0,1\right)\right)$.

We deduce when there exists constant $\beta<\infty$ such as:

$$h\le\beta$$

Corollary 2.2. (*see* [9])

There exists a constant $C>0$ such as

$$h\ge C. \tag{19}$$

Thanks to the Remark 2.1 and the Corollary 2.1, we have the following additional regularities:

Corollary 2.3. *Let* $\left(h,z_b,u\right)$ *be a solution of model* (8)-(10).

Then, thanks to Corollary 2.1, Corollary 2.2 and sobelev injections, we have:

h is bounded in $L^\infty\left(0,T;L^\infty\left(0,1\right)\right)$, z_b is bounded in $L^\infty\left(0,T;L^\infty\left(0,1\right)\right)$,

u is bounded in $L^\infty\left(0,T;L^\infty\left(0,1\right)\right)$, $\sqrt{h}$ is bounded in $L^\infty\left(0,T;L^\infty\left(0,1\right)\right)$,

hu is bounded in $L^\infty\left(0,T;L^2\left(0,1\right)\right)$, $\partial_t h$ is bounded in $L^\infty\left(0,T;W^{-1,2}\left(0,1\right)\right)$.

We need the following three lemma for proof of the above Proposition 2.1.

Lemma 2.1. (*Energy equality*) *Let* $\left(h,z_b,u\right)$ *be a smooth solution of* (8)-(10). *Then the following equality holds*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left[h|u|^2+g\left|z_b+h\right|^2+g\left(\varepsilon-1\right)\left|z_b\right|^2\right]+\nu\varepsilon g\int_0^1\left|\partial_x z_b\right|^2\\&+2\nu\int_0^1 h\left(\partial_x u\right)^2+\nu g\int_0^1\partial_x h\partial_x z_b=0.\end{aligned} \tag{20}$$

Lemma 2.2. *If* $\left(h,z_b,u\right)$ *is a smooth solution of* (8)-(10), *the following equality holds*:

$$\begin{aligned}&\nu\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 h\left|\partial_x\log h\right|^2+\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u\partial_x h+g\int_0^1\left(1+\frac{z_b}{h}\right)\left|\partial_x h\right|^2\\&=\int_0^1 h\left(\partial_x u\right)^2-g\int_0^1\left(1+\varepsilon\frac{z_b}{h}\right)\partial_x h\partial_x z_b.\end{aligned} \tag{21}$$

Lemma 2.3. (*BD-entropy*) *For smooth solutions* $\left(h,z_b,u\right)$ *of* (8)-(10), *we have the following equality*:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 h\left|u+\nu\partial_x \log h\right|^2 + g\nu\int_0^1\left(1+\frac{z_b}{h}\right)\left|\partial_x h\right|^2$$
$$= \nu\int_0^1 h\left(\partial_x u\right)^2 - g\nu\int_0^1\left(1+\varepsilon\frac{z_b}{h}\right)\partial_x h\partial_x z_b + \frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 h\left|u\right|^2 \quad (22)$$

Theorem 2.1. *There exists global weak solutions to the system* (8)-(10) *with initial data* (13), (14) *and satisfying energy inequality* (18).

3. Proof of the Energies Inequalities and Theorem 2.1

3.1. Proof of Proposition 2.1

Proof. **Lemma 2.1**

First, we multiply the momentum equation by u and we integrate from 0 to 1.

$$\int_0^1 u\left(\partial_t\left(hu\right)+\partial_x\left(hu^2\right)\right)+\frac{1}{2}\int_0^1 gu\partial_x h^2 - 2\nu\int_0^1\partial_x\left(\nu h\partial_x u\right)u$$
$$\int_0^1 hu\partial_x\left(z_b+h\right)+g\int_0^1 z_b u\partial_x\left(h+\varepsilon z_b\right)=0. \quad (23)$$

We use the mass conservation equation of (8) for simplification. Then, we obtain:

- $\int_0^1 u\partial_t\left(hu\right)+\partial_x\left(hu^2\right)=\int_0^1\frac{1}{2}\partial_t\left(hu^2\right)$
- $-2\nu\int_0^1\partial_x\left(h\partial_x u\right)u = 2\nu\int_0^1 h\left(\partial_x u\right)^2$
- $g\int_0^1 h\partial_x\left(h+z_b\right)u = -g\int_0^1\left(h+z_b\right)\partial_x hu = \frac{1}{2}g\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left|h\right|^2 + g\int_0^1 z_b\partial_t h$
- $g\int_0^1 z_b u\partial_x\left(h+\varepsilon z_b\right) = -g\int_0^1\left(h+\varepsilon z_b\right)\left(-\partial_t z_b+\nu\partial_x^2 z_b\right)$ $= g\int_0^1 h\partial_t z_b + \frac{1}{2}\varepsilon g\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left|z_b\right|^2 + \nu g\int_0^1\partial_x h\partial_x z_b + \nu\varepsilon g\int_0^1\left|\partial_x z_b\right|^2$

Substituting all these terms we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left[h\left|u\right|^2+g\left|h\right|^2\right]+2\nu\int_0^1 h\left(\partial_x u\right)^2+g\int_0^1\partial_t\left(z_b h\right)+\frac{1}{2}\varepsilon g\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left|z_b\right|^2$$
$$+\nu g\int_0^1\partial_x h\partial_x z_b+\nu\varepsilon g\int_0^1\left|\partial_x z_b\right|^2=0. \quad (24)$$

Next we simplify to have the proclaimed equality.

Proof. **Lemma 2.2**

Using again the equation (8) to find

$$\partial_t\partial_x h+\partial_x\left(h\partial_x u\right)+\partial_x\left(u\partial_x h\right)=0.$$

Replacing $\partial_x h$ by $h\partial_x \log h$, we have:

$$\partial_t\left(h\partial_x\log h\right)+\partial_x\left(h\partial_x u\right)+\partial_x\left(hu\partial_x\log h\right)=0$$

We multiply the previous equation by $\partial_x \log h$ to have:

$$\frac{1}{2}\partial_t\left(h\left(\log h\right)^2\right)-\frac{1}{2}\frac{\left(\partial_x h\right)^2}{h^2}\partial_x\left(hu\right)+\partial_x^2 u\partial_x h+2\partial_x u\frac{\left(\partial_x h\right)^2}{h}+u\partial_x^2 h\frac{\partial_x h}{h}=0. \quad (25)$$

Let us multiply the momentum equation by $\partial_x \log h$ and simplify to have:

$$\left(\partial_t u+u\partial_x h\right)\partial_x h-2\nu\partial_x u\frac{\left(\partial_x h\right)^2}{h}-2\nu\partial_x^2 u\partial_x h+g\left(\partial_x h\right)^2 \tag{26}$$
$$+g\partial_x z_b\partial_x h+g\frac{z_b}{h}\partial_x h\partial_x z_b=0.$$

We multiply the Equation (25) by ν add to the Equation (26) and integrate to have

$$\nu\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 h\left|\partial_x\log h\right|^2+g\int_0^1\left|\partial_x h\right|^2+g\int_0^1\frac{z_b}{h}\left|\partial_x h\right|^2$$
$$=-\int_0^1\left(\partial_t u+u\partial_x u\right)\partial_x h-g\int_0^1\partial_x h\partial_x z_b-g\varepsilon\int_0^1\frac{z_b}{h}\partial_x z_b\partial_x h.$$

We use the mass equation to rewrite: $\left(\partial_t u+u\partial_x u\right)\partial_x h$ as

$$\left(\partial_t u+u\partial_x u\right)\partial_x h=\partial_t u\partial_x h+\left(-\partial_t h-h\partial_x u\right)\partial_x u$$
$$=\partial_t\left(u\partial_x h\right)+\partial_x\left(u\partial_t h\right)-u\partial_t\left(\partial_x h\right)+u\partial_x\left(\partial_t h\right)-h\left(\partial_x u\right)^2.$$

So we have:

$$\nu\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 h\left|\partial_x\log h\right|^2+\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u\partial_x h+g\int_0^1\left|\partial_x h\right|^2+g\int_0^1\frac{z_b}{h}\left|\partial_x h\right|^2$$
$$=\int_0^1 h\left(\partial_x u\right)^2-g\int_0^1\left(1+\varepsilon\frac{z_b}{h}\right)\partial_x h\partial_x z_b.$$

Proof. **Lemma 2.3**

We multiply the Equation (21) by ν, to have

$$\nu^2\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 h\left|\partial_x\log h\right|^2+\nu\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u\partial_x h+g\nu\int_0^1\left(1+\frac{z_b}{h}\right)\left|\partial_x h\right|^2 \tag{27}$$
$$=\nu\int_0^1 h\left(\partial_x u\right)^2-g\nu\int_0^1\left(1+\varepsilon\frac{z_b}{h}\right)\partial_x h\partial_x z_b.$$

We add to the right and left of the Equation (27) the term $\nu\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 h|u|^2$ to have the proclaimed inequality.

To complete the proof of the Proposition 2.1, we sum up the tow energies of the Lemma 2.1 and Lemma 2.3 and we use the inequality $ab\le\frac{1}{2}\left(a^2+b^2\right)$ to have the proclaimed result.

3.2. Proof of Theorem 2.1

This section is devoted to the prove of Theorem 2.1. Let $\left(h^k,z_b^k,u^k\right)$ be a sequence of weak solutions with initial data

$$h^k\big|_{t=0}=h_0^k,\quad z_b^k\big|_{t=0}=z_{b_0}^k,\quad \left(h^k u^k\right)\big|_{t=0}=m_0^k$$

such that

$$h_0^k\to h_0\ \text{ in } L^1(\Omega),\quad z_{b_0}^k\to z_{b_0}\ \text{ in } L^1(\Omega),\quad m_0^k\to m_0\ \text{ in }\left(L^1(\Omega)\right)^2,$$

and satisfies

$$\int_0^1 \left[h_0^k \left| u_0^k + \nu \partial_x \log\left(h_0^k\right) \right|^2 + \nu^2 h_0^k \left| \partial_x \log h_0^k \right|^2 + h_0^k \left| u_0^k \right|^2 + 2g \left| h_0^k + z_{b_0}^k \right|^2 + 2g(\varepsilon - 1) \left| z_{b_0}^k \right|^2 \right] \leq C.$$

Such approximate solutions can be built by a regularization of capillary effect.

3.2.1. Strong Convergence of $\sqrt{h^k}$, h^k and z_b^k

We first give the spaces in which $\sqrt{h^k}$ is bounded.

Integrating the mass equation, we obtain directly $\sqrt{h^k}$ in $L^\infty\left(0,T;L^2(0,1)\right)$.

As **Corollary 2.3** gives us $\partial_x \sqrt{h^k}$ in $L^\infty\left(0,T;L^2(0,1)\right)$, so

$$\sqrt{h^k} \text{ is bounded in } L^\infty\left(0,T;L^\infty(0,1)\right). \tag{28}$$

Moreover, always using the mass equation, we obtain the following equality:

$$\begin{aligned} \partial_t \sqrt{h^k} &= \frac{1}{2}\sqrt{h^k}\partial_x u^k - \partial_x\left(\sqrt{h^k}u^k\right) \\ &= \frac{1}{2}\sqrt{h^k}\partial_x u^k - u^k \partial_x \sqrt{h^k} - \sqrt{h^k}\partial_x u^k \end{aligned}$$

which gives that $\partial_t \sqrt{h^k}$ is bounded in $L^2\left(0,T;L^2(0,1)\right)$.

Applying Aubin-Simon lemma ([13] [14]), we can extract a subsequence, still denoted $\left(h^k\right)_{1\leq k}$, such that

$$\sqrt{h^k} \text{ strongly converges to } \sqrt{h} \text{ in } L^2\left(0,T;L^2(0,1)\right).$$

According to the Corollary 2.2, we show that

$$\left|h^k - h\right| \leq \sqrt{c_2}\left|\sqrt{h^k} - \sqrt{h}\right| \Rightarrow \left|h^k - h\right|^2 \leq c_2 \left|\sqrt{h^k} - \sqrt{h}\right|^2.$$

This ensures that

$$h^k \text{ strongly converges to } h \text{ in } L^2\left(0,T;L^2(0,1)\right).$$

We have $z_b^k \in L^2\left(0,T;L^\infty(0,1)\right)$. Moreover, we have $\partial_t z_b^k = -\partial_x\left(z_b^h u^k\right) + \nu\partial_x^2 z_b$.

We have $z_b^k \in L^\infty(0,1)$ and $u^k \in L^2(0,1)$, so $h^k u^k \in L^2(0,1)$, according to the Sobolev embeddings, we show that the first term is in $W^{-1,2}(0,1)$. By analogy we prove that the last term is in the same space and we also get $\partial_t z_b^k$ in this space. Thanks to the Aubin-Simon lemma, we find:

$$z_b^k \text{ strongly converges to } z_b \text{ in } L^2\left(0,T;W^{-1,2}(0,1)\right).$$

3.2.2. Strong Convergence of $h^k u^k$ and u^k

We have h^k that's bounded in $L^\infty\left(0,T;L^\infty(0,1)\right)$ and u^k that's bounded in $L^2\left(0,T;L^\infty(0,1)\right)$, What gives us $h^k u^k$ is bounded in $L^2\left(0,T;L^\infty(0,1)\right)$.

Let's look now $\partial_x\left(h^k u^k\right)$. We have:

$$\partial_x\left(h^k u^k\right) = h^k \partial_x u^k + u^k \partial_x h^k,$$

basing on the estimates obtained on h^k in Corollary 2.3, we get:

$$h^k u_k^k \text{ bounded in } L^2\left(0,T;W^{1,2}(0,1)\right).$$

Moreover, the momentum Equation (9) enables us to write the time derivation of the water discharge:

$$\partial_t\left(h^k u^k\right) = -\partial_x\left(h^k\left(u^k\right)^2\right) - \frac{1}{2}g\partial_x h^{k2} + 2\nu_1\partial_x\left(h^k\partial_x u\right)$$
$$- gh^k\partial_x z_b^k - gz_b^k\partial_x\left(h^k + \varepsilon z_b^k\right)$$

we then study each term:

- $\partial_x\left(h^k\left(u^k\right)^2\right) = \partial_x\left(h^k u^k\left(u^k\right)\right)$ which is in $L^2\left(0,T;W^{-1,2}(0,1)\right)$.
- as h^k is in $L^\infty\left(0,T;L^\infty(0,1)\right)$, and $\partial_x h^k$ is in $L^2\left(0,T;L^2(0,1)\right)$ and we can write the following relation:

$$\partial_x\left[\left(h^k\right)^2\right] \text{ is bounded in } L^2\left(0,T;L^2(0,1)\right).$$

- $\partial_x\left(h^k\partial_x u^k\right)$ is bounded in $L^2\left(0,T;W^{-1,2}(0,1)\right)$.
- The last three terms are bounded in $L^\infty\left(0,T;W^{-1,2}(0,1)\right)$.

Then, applying Aubin-Simon lemma, we obtain,

$$\left(h^k u^k\right)_k \text{ stongly converges to } h_1 u \text{ in } C^0\left(0,T;W^{-1,2}(0,1)\right).$$

3.2.3. Strong Convergence of u^k, $h^k\partial_x u^k$ and $\sqrt{h^k}u^k$

Thanks to Corollary 2.3 and Corollary 2.2, we have u^k, in bounded in $L^\infty\left(0,T,L^\infty(0,1)\right)$ and $\partial_x u^k$ is bounded in $L^2\left(0,T,L^2(0,1)\right)$. In order to obtain new estimates on u^k, we are going to control the right hand side of the following equation:

$$\partial_t u^k = -u^k\partial_x u^k + 2\nu\partial_x \log h^k\partial_x u^k - g\partial_x h^k - g\partial_x z_b^k$$
$$- gz_b^k\partial_x \log h^k - g\varepsilon\frac{z_b}{h}\partial_x z_b^k + 2\nu\partial_x^2 u^k.$$

Thanks to the estimates obtained on h^k, z_b^k and u^k, all the terms to the right of equality except the last term are in $L^2\left(0,T,L^2(0,1)\right)$.

On the other hand, $\partial_x u^k$ is bounded in $L^2\left(0,T,L^2(0,1)\right)$, this leads us to $\partial_x^2 u^k$ is bounded $L^2\left(0,T,W^{-1,2}(0,1)\right)$ Aubin Simon's lemma leads us to the following result:

$$\left(u^k\right)_k \text{ strongly converges to } u \text{ in } L^2\left(0,T;W^{-1,2}(0,1)\right).$$

However, the function $\left(h^k,\partial_x u^k\right)\mapsto h^k\partial_x u^k$ is a continuous in $L^\infty\left(0,T;L^\infty(0,1)\right)\times L^2\left(0,T;L^2(0,1)\right)$ to $L^2\left(0,T;L^2(0,1)\right)$.

So,

$$h^k\partial_x u^k \text{ weakly converges to } h\partial_x u \text{ in } L^2\left(0,T;L^2(0,1)\right).$$

Thanks to the Corollary 2.2, we say that it exists constants $0<\alpha$ and $\beta<+\infty$ such as $\alpha\le h^k\le\beta$.

For all constant $\kappa>\beta$, we have the following norm:

$$\int_0^T\int_0^1\left|\sqrt{h^k}u^k-\sqrt{h}u\right|^2\leq\kappa\int_0^T\int_0^1\left|u^k-u\right|^2\to 0$$

So,

$$\sqrt{h^k}u^k \text{ strongly converges to } \sqrt{h}u \text{ in } L^2\left(0,T;\left(L^2(\Omega)\right)^2\right).$$

3.2.4. Strong Convergence of $z_b^k u^k$, $\partial_x h^k$, $\partial_x z_b^k$ $z_b^k\partial_x h^k$, $z_b^k\partial_x h^k$ $h^k\partial_x z_b$ and $z_b^k\partial_x z_b^k$

- We have $\partial_x h^k$ bounded in $L^2\left(0,T;H^1(0,1)\right)$ and $\partial_t\partial_x h^k$ is bounded in $L^\infty\left(0,T;H^{-2}(0,1)\right)$ since $\partial_t h^k$ is bounded in $L^\infty\left(0,T;H^{-1}(0,1)\right)$. Thanks to compact injection of $H^1(0,1)$ in $L^2(0,1)$ in one dimension, we have:

$$\partial_x h^k \text{ strongly converges to } \partial_x h \text{ in } L^2\left(0,T;L^2(0,1)\right)$$

- The bound of $\partial_x z_b^k$ in $L^2\left(0,T;L^2(0,1)\right)$ gives us:

$$\partial_x z_b^k \text{ weakly converges to } \partial_x z_b \text{ in } L^2\left(0,T;L^2(0,1)\right).$$

- Thanks to the strong convergence of h^k, z_b^k, $\partial_x h^k$, we have:

$$h^k\partial_x h^k \text{ strongly converges to } h\partial_x h \text{ in } L^1\left(0,T;L^1(0,1)\right),$$

$$z_b^k\partial_x h^k \text{ strongly converges to } z_b\partial_x h \text{ in } L^1\left(0,T;L^1(0,1)\right),$$

$$h^k\partial_x z_b^k \text{ strongly converges to } h\partial_x z_b \text{ in } L^1\left(0,T;L^1(0,1)\right),$$

$$z_b^k\partial_x z_b^k \text{ strongly converges to } z_b\partial_x z_b \text{ in } L^1\left(0,T;L^1(0,1)\right).$$

To end we have u^k weakly converges to u in $L^2\left(0,T;L^2(0,1)\right)$ and the strong convergence of z_b^k to z_b, gives us:

$$z_b^k u^k \text{ weakly converges to } h_2 u \text{ in } L^1\left(0,T;L^1(0,1)\right).$$

4. Conclusion

This paper extends from a mathematical point of view the work done in [11], or the authors showed a result of the existence of global weak solutions of a pollutant transport model using regularizing terms such as (the friction terms, tension term). Moreover, the work done in this article takes inspiration from the work done in [1], or we show the existence of global weak solutions of a one-dimensional sediment transport model with regularizing terms (exchange terms) and physical hypothesis, without the regularizing terms used in [11] considering a better transport equation than that used in [1]. In our future work, we will work to remove the physical hypothesis, regularizing terms and focus on results of the existence of strong solutions.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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