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Endogenous Explanation for Random Fluctuation of Stock Price and Its Application: Based on the View of Repeated Game with Asymmetric Information

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Abstract

In this paper, we characterize the players' behavior in the stock market by the repeated game model with asymmetric information. We show that the discount price process of stock is a martingale driven by Brownian motion, and give an endogenous explanation for the random fluctuation of stock price: the randomizations in the market is due to the randomizations in the strategy of the informed player which hopes to avoid revealing his private information. On this basis, through studying the corresponding option pricing problem furtherly, we can give the expression of function φ.

Keywords

Repeated Game, Option Pricing, Martingale, Asymmetric Information

1. Introduction

In 2015, China's stock market had witnessed continuous drastic fluctuations, and the mystery of stock price fluctuations got social attention once again. One of the most important problems in financial mathematics is how to describe the fluctuation of stock price reasonably. On February 9, 2015, China officially launched the first option in the financial market—SSE 50ETF option. Under this background, it is of great theoretical significance and valuable application to research the stock price process and option pricing theory suitable for China's stock market.

At present, the common models to simulate stock price fluctuations include

Bachelier Model [1] [2], Black-Scholes Model [3], Diffusion Model (Merton [4] [5]), Random Volatility Model (Hull and White [6] [7], Heston [8]), Jump-Diffusion Model (Kou [9], Makate and Sattayatham [10], Boen and Hout [11], Bayraktar and Xing [12], Rodrigo [13]). Bachelier [1] initially put forward diffusion processes based on continuous-time processes in financial engineering and observed that the stock price movements are analogous to the motion of small particles suspended in liquids. With this assumption, he derived the underlying of motion and found the pricing formulas for put and call options on such stocks. Black and Scholes [3] derived a theoretical valuation formula for options, common stock, corporate bonds, and warrants by using the principle that creating portfolios of long and short positions in options and their underlying stocks will make no profit under the conditions of being priced correctly in the market. Recently, it has been demonstrated [14] that option pricing formulas of Bachelier and Black-Scholes coincide very well in the sense that Bachelier's model yields good short-time approximations of prices and volatilities. Moreover, Merton [4] presented an extension of these results for more general utility functions, price behavior assumptions, and income generated from noncapital gains sources. When asset prices are generated by a geometric Brownian motion, the two-asset case can be worked on without loss of generality. Aguilar *et al.* [15] further introduce a more general class of models based on the space-time-fractional diffusion equation in the framework of the risk-neutral approach. Heston [8] used a new approach to derive a closed-form solution for the price of a European call option on an asset with stochastic volatility. With proper choice of parameters, the stochastic volatility model appears to be a very flexible and promising description of option or stock prices. Taking into account the fact that the return distribution of assets may have a higher peak and two heavier tails than those of the normal distribution, in which this empirical phenomenon called volatility smile in option markets, Kou [9] proposed a double exponential jump-diffusion model for the purpose of random fluctuation of stock price. Makate and Sattayatham [10] proposed asset price dynamics to accommodate both jump-diffusion and jump stochastic volatility. Under this proposed model, an analytical solution is derived for a European call option via the characteristic function. This analytical solution can also be derived for American options [11], Asian options [12]. Rodrigo [13] used a Mellin transform approach to derive exact pricing formulas for barrier options with general payoffs and exponential barriers on underlying assets that have jump-diffusion dynamics.

Despite rapidly development of stock theory, the original Black-Scholes formula for a European call option remains the most successful and widely used application [16]. Black-Scholes formula is particularly useful because it relates the distribution of spot returns to the cross-sectional properties of option prices.

Black-Scholes model assumes that the stock price process $\{S_t\}$ satisfies the following stochastic differential equation:

$$\mathrm{d}S_t = S_t\left(\mu \mathrm{d}t + \sigma \mathrm{d}B_t\right) \tag{1}$$

where μ is the expected return rate of the stock, μ is the volatility of the stock, and $\{B_t\}$ is the standard Brownian motion. That's given by the Ito formula as follows.

$$S_t = \mathrm{e}^{\mu t - \frac{1}{2}\sigma^2 t + \sigma B_t} \tag{2}$$

It follows that the distribution of stock prices at any time is lognormal distribution in the Black-Scholes model.

It is necessary that all the above models assume that the randomness of stock price fluctuations is due to external shocks, including the release of positive or negative information about listed companies, the adjustment of national macroeconomic policies, and so on. The randomness of stock price or stock return rate is usually described by Brownian motion, but there is no satisfactory explanation for how the Brownian motion is generated, for example, where does the Brownian motion in Equation (1) come from?

De Meyer and Saley [17] had provided an endogenous explanation of stock price fluctuations by establishing a simple repeated game model with asymmetric information, in which the stochastic strategies adopted by the two players in the market lead to random changes in stock prices due to information asymmetry. De Meyer [18] deduced that due to the randomization strategy of both sides of the game, the stock price $\{S_t\}$ follows the following process by establishing a repeated game model with more general asymmetric information:

$$S_t = f(B_t, t) \tag{3}$$

where the function $f(x,t): R \times [0,T] \to R$ is monotonically nondecreasing function with respect to x and makes the stock price $\{S_t\}$ a martingale process. But how this process manifests itself in actual financial markets, De Meyer [18] does not discuss.

This article will further promote De Meyer's repeated game model [18] with asymmetric information. We found that the discount price process of stock $\tilde{S}_t = \mathrm{e}^{-rt} S_t$ satisfies the Equation (3) under more general conditions. It holds:

$$\tilde{S}_t = f(B_t, t) \tag{4}$$

where r is the risk-free interest rate. In other words, the stock price process $\{S_t\}$ satisfies:

$$S_t = \mathrm{e}^{rt} f(B_t, t) \tag{5}$$

And we will further study the Pricing of European options in this process.

The remainder of this paper is organized as follows. In Section 2, we give an endogenous explanation for the random fluctuation of stock price by presenting an extension of a repeated game model with asymmetric information in the stock market, and put forward several hypotheses of natural trading mechanism. In addition, we give several notes on the game model. In Section 3, we estimate the distribution of stock prices by using option pricing formula with the acquisition of corresponding options market data. In Section 4, we draw a concise conclusion.

2. Game Model of Financial Transactions

2.1. Game Model

The financial transaction game model introduced by De Meyer [18] can be regarded as the extension of the classic Aumann-Maschler game model [19]. This model is a two-person zero repeated game model with incomplete information. We will take the stock market as an example to give a game model.

We assume that there are two kinds of participants in the stock market: the banker and the retail investors. They repeatedly trade a risky asset R (a stock) and a risk-free asset N (cash). At time $t = 1$, the clearing price of stock R is denoted as L, which is a random variable whose distribution is denoted as μ. The clearing price of the risk-free asset N is constantly assumed to be 1. The trade takes place in n rounds between time $t = 0$ and time $t = 1$. Each round of trading can be characterized by triples (I, J, T), where I and J respectively represent the action set of banker and retail investors, and $T : I \times J \to R^2$ is the trade transition function. If the banker and retail investors select strategy (i, j), then $T(i, j) = (A_{ij}, B_{ij})$ represents the transferred amount of assets from retail investors to banker: A_{ij} and B_{ij} respectively represent the amount of R and the amount of N that the banker gets from the retail investors. If $y_q = (y_q^R, y_q^N)$ and $z_q = (z_q^R, z_q^N)$ represent the respective portfolios of the banker and the retail investors at the end of round q, then

$$y_q = y_{q-1} + T(i_q, j_q); z_q = z_{q-1} - T(i_q, j_q) \tag{6}$$

is similar for the classic Aumann-Maschler model. The n round repeated game is carried out according to the following rules:

Round 0: the banker randomly selecting L based on probability measure μ, the banker knows the exact value of L and the retail investors only know that its probability distribution is μ, and the retail investors know that the banker knows the exact value of L, while the banker knows that the retail investors only know the distribution of L.

Round $(q = 1, 2, \cdots, n)$: the banker and retail investors adopt their strategies $i_q \in I$ and $j_q \in J$ independently according to their respective information and historical observations, and the strategies will be disclosed to both parties at the end of each round.

Specifically, the banker's behavioral strategy $\sigma = (\sigma_1, \cdots, \sigma_n)$ is a list of transition probabilities that depend on his private information and historical observations:

$$\sigma_q : R \times (I \times J)^{q-1} \to \Delta(I) \tag{7}$$

where $\sigma_q(L, i_1, j_1, \cdots, i_{q-1}, j_{q-1})$, in the q round, is the probability distribution of i_q selected by the banker under the condition that the liquidation price of risk asset R is L and the historical observations of the game are $(i_1, j_1, \cdots, i_{q-1}, j_{q-1})$. Also, the behavioral strategy set of the retail investors $\tau = (\tau_1, \cdots, \tau_n)$ is a list of transition probabilities dependent on his historical observations:

$$\tau_q : (I \times J)^{q-1} \to \Delta(J) \tag{8}$$

where $\tau_q(i_1, j_1, \cdots, i_{q-1}, j_{q-1})$, in the q round, is the probability distribution of i_q selected by the retail investors under the condition that the historical observations of retail investors are $(i_1, j_1, \cdots, i_{q-1}, j_{q-1})$.

The behavioral strategy sets of banker and retail investors are respectively recorded as Σ_n and T_n. Triple (μ, σ, τ) induces a unique probability measure $\pi(\mu, \sigma, \tau) \in \Delta(R \times I^n \times J^n)$, where $\Delta(A)$ represents the totality of probability measures that exist for any moment estimation defined over set A. Since the initial portfolios y_0 and z_0 have been given in advance, the liquidation price of the initial asset portfolio is constant, and its value does not affect their respective behavior strategies. Without loss of generality, we're going to assume $y_0 = z_0 = (0,0)$. Therefore, the game is a zero game, and the banker's return function is:

$$g_n(\mu, \sigma, \tau) = E_{\pi(\mu,\sigma,\tau)}\left[y_n^R L + y_n^N\right] \tag{9}$$

Banker's maximum return $\underline{V}_n(\mu)$ and retail investor's minimum return $\overline{V}_n(\mu)$ are respectively:

$$\underline{V}_n(\mu) = \sup\nolimits_{\sigma \in \Sigma_n} \inf\nolimits_{\tau \in T_n} E_{\pi(\mu,\sigma,\tau)}\left[g_n(\mu, \sigma, \tau)\right] \tag{10}$$

$$\overline{V}_n(\mu) = \inf\nolimits_{\tau \in T_n} \sup\nolimits_{\sigma \in \Sigma_n} E_{\pi(\mu,\sigma,\tau)}\left[g_n(\mu, \sigma, \tau)\right] \tag{11}$$

Obviously, $\underline{V}_n(\mu) \le \overline{V}_n(\mu)$ is always true. When the equal sign holds, the value of the above game exists:

$$V_n(\mu) := \underline{V}_n(\mu) = \overline{V}_n(\mu) \tag{12}$$

In fact, given the game rules, the above game can be completely determined by two parameters: the distribution μ of risk asset L and the number of rounds n. For convenience, we abbreviate the above game as $G_n(\mu)$.

2.2. Natural Trading Mechanism

The hypotheses of the natural trading mechanism are as follows:

(H1) Existence of game value:

$\forall \mu \in \Delta(R)$, values of the single period game exist, namely, $V_1(\mu) = \underline{V}_1(\mu) = \overline{V}_1(\mu)$.

(H2) Boundedness of transactions:

There exists a constant C, making $\forall i, j : |A_{ij}| \le C$.

(H3) Positive homogeneous:

$$\forall \alpha > 0, \forall L : V_1([\alpha L]) = \alpha V_1([L]) \tag{13}$$

(H4) The translational invariance of the riskless part of a risky asset:

$$\forall \beta \in R : V_1([L + \beta]) = V_1([L]) + V_1([\beta]) \tag{14}$$

(H5) Positive value of information:

There exist a L, making $V_1([L]) > 0$, where the symbol $[X]$ represents the distribution of the random variable X.

(H6) Lipschitz continuity:

There exists $p \in [1,2)$ and $K \in R$, making all $X, Y \in \Delta(R)$:

$$\left| M[X] - M[Y] \right| \le K \left\| X - Y \right\|_{l^p} \tag{15}$$

For fixed n, we assume that (σ^*, τ^*) is the equilibrium solution to $G_n(\mu)$. We further assume that the price process of the stock $S^{(n)}_{q\ q=1,\cdots,n}$ during the n rounds of the game is

$$S^{(n)}_q = E_{\pi(\mu,\sigma^*,\tau^*)}\left[L \mid i_1,\cdots,i_{q-1}, j_1,\cdots, j_{q-1} \right] \tag{16}$$

Obviously, $S^{(n)}_{q\ q=1,\cdots,n}$ is a discrete time process. Define:

$$S^{(n)}_t = S^{(n)}_{[\![nt]\!]}, 0 \le t \le 1 \tag{17}$$

where $[\![x]\!]$ represents the largest integer without exceeding x. At this point, $\left\{ S^{(n)}_t \right\}_{0 \le t \le 1}$ is a continuous time process induced by $S^{(n)}_{q\ q=1,\cdots,n}$

Theorem 1. If (H1)-(H5) holds, then for all $\mu \in \Delta(R)$, we have

1) $\lim_{n\to\infty} \frac{1}{n} V_n(\mu) = V_1([E(\mu)])$, where $E(\mu)$ is the expectation of μ.

2) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} \left(V_n(\mu) - nV_1([E(\mu)]) \right) = \rho E\left[f_\mu(Z) Z \right]$, where $Z \sim N(0,1)$ is the standard normal distribution, $f_\mu(Z) \sim \mu$ and ρ is a constant.

3) As n tends to positive infinity, $\left\{ S^{(n)}_t \right\}_{0 \le t \le 1}$ converges to $S_t = f(t, B_t)$ in a finite-dimensional distribution, and $f(1,x) = f_\mu(x)$.

Proof of Theorem 1. On the premise that $V(\mu)$ satisfies (H3). (H4) and (H6), we can get

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n}} \right) V(\mu) = \rho E\left[f_\mu(Z) Z \right] \tag{18}$$

For all $(\mathcal{F}, X) \in V_n(\mu)$, we have

$$\sum_{q=0}^{n-1} E\left[M\left[X_q \mid \mathcal{F}_q \right] \right] = nM\left[E(\mu) \right] \tag{19}$$

where $\mathcal{F} = (\mathcal{F}_q)_{q=1,\cdots,n}$ is the group of information on the probability space $(\Omega, B(\Omega), P)$, and $X = (X_q)_{q=1,\cdots,n}$ is the martingale of $\mathcal{F}$.

$$V_n(\mu) - nM\left[E(\mu) \right] = \bar{V}_n(\mu) \tag{20}$$

If there exists $(\mathcal{F}^*, X^*) \in V_n(\mu)$ satisfying $V_n(\mathcal{F}^*, X^*) = V_n(\mu)$, then we have

$$\bar{V}_n(\mathcal{F}^*, X^*) = \bar{V}_n(\mu) \tag{21}$$

We can easily prove conclusion (1) in the Theorem 1 from (19). From (18) and (20), we have

$$\lim_{n\to\infty} \frac{V_n(\mu) - nV_1\left[E(\mu) \right]}{\sqrt{n}} = \lim_{n\to\infty} \frac{\bar{V}_n(\mu)}{\sqrt{n}} = \rho E\left[f_\mu(Z) Z \right] \tag{22}$$

So conclusion (2) in the Theorem 1 is proved. From (18) and (21), obviously, conclusion (3) in the Theorem 1 is true.

The proof process of the above theorems is extremely complex, and readers interested in the detailed proof process can refer to Li [20].

2.3. Several Notes on the Game Model

1) The natural trading mechanism was first proposed in article [18]. This paper made some changes to the assumptions in article [18], among which the biggest difference from the initial assumption is (H4). In article [18], the corresponding hypothesis is

$$\forall \beta \in R : V_1\left(\left[L+\beta\right]\right) = V_1\left(\left[L\right]\right) \tag{23}$$

and then, combined with H3 and H2, we can get

$$\forall \beta \in R : V_1\left(\left[\beta\right]\right) = V_1\left(\left[0+\beta\right]\right) = V_1\left(\left[0\right]\right) = \lim_{\alpha \downarrow 0} V_1\left(\left[\alpha\right]\right) = \lim_{\alpha \downarrow 0} \alpha V_1\left(\left[1\right]\right) = 0 \tag{24}$$

Equation (18) means that no profits will be made by trading risk-free assets. But in some cases, for example, when transaction costs are taken into account, Equation (18) is no longer true. In Li [20], a game model with transaction costs is presented, which does not satisfy the risk trading mechanism in article [18], but satisfies the hypothesis (H1)-(H5) in this paper.

2) About the asymmetry of information, in round 0, the banker has already known the exact value of the risky asset L, while the retail investors only have known its distribution. This leads to the fact that in the two-party strategy of each round, the banker's strategy (7) can depend on L, while the retail investor's strategy (8) is independent of L. But retail investors can infer the specific value of L through the banker's strategy, while the banker intentionally conceals the specific information of L through the randomization strategy, which leads to the random fluctuation of the stock price in the process of mutual game.

3) About hypothesis (H1), different from the Aumann-Maschler model, since the strategy sets of both players in our repeated game model are not finite set, in some cases, the minimum and maximum operators in Equations (10) and (11) may not be interchangeable (Mertens *et al.* [21]), so the corresponding single-period game values may not exist.

4) In conclusion (1) of theorem 1, it is noted that $E(\mu)$ is a constant, so $V_1\left(\left[E(\mu)\right]\right)$ corresponds to the value of the single-period game with complete information. Therefore, conclusion (1) indicates that with the increase of the number of games n, the average value of n round games tends to the value of the single-stage games with complete information. This is because with the increase of the number of games n, retail investors will gradually guess the information known by the banker according to the banker's strategy, at this time, the incomplete information repeated game will gradually become the complete information repeated game.

5) Conclusion (3) in the Equation (1) is consistent with the conclusion of risk-neutral theory. The risk-neutral theory shows that the discounting process of risky assets is a martingale under the equivalent martingale measure (Harri-

son and Pliska [22] [23], Dalang *et al.* [24]). In the above repeated game model, in order to simplify the problem, we do not consider the discount problem of risk-free assets, and thus the stock price is, actually, the discount price of the stock in conclusion (3). At this point, the discount price process of stock is a martingale driven by standard Brownian motion under wiener measure, in other words, Equation (4) holds.

6) The model actually implies that the banker and retail investors are engaged in high-frequency trading: when the number of rounds n is sufficiently large, the stock price process converges to the Martingale model (3). However, China's stock market implements the T + 1 trading rule, and thus high-frequency trading seems to be unworkable in China. We treating the banker and the retail investors as two groups, high-frequency trading will occur in the short term, so that the stock price process can still converge to the martingale model.

7) The game behaviors between the banker and the retail investors in the actual stock market are much more complex than those described by theoretical models. Though, we get an interesting result: the discount price process of stock is a martingale driven by Brownian motion. Noting that there is no external randomness hypothesis in the game model, it is strange that the result is a "random result", so we can give an endogenous explanation of random fluctuation in stock prices, in which the random fluctuation of stock price comes from the random trading strategy of the banker and the retail investors: the banker adopts the random strategy to interfere the retail trader's judgment of the known information in order to get the maximum profit. Of course, the impact of external shocks on stock prices can play a critical role in some situations, but given the complexity of the model, we do not consider the impact of external shocks on stock prices here.

3. Option Pricing Formula

In the previous section, we deduced the discount prices process $\{\tilde{S}_t\}$ of stock through the repeated game model with asymmetric information. It has the following form:

$$\tilde{S}_t = f(B_t, t) \tag{25}$$

and $\tilde{S}_t$ is a martingale process.

Further, from the Itô formula we can get:

$$\mathrm{d}\tilde{S}_t = \partial_x f(B_t, t)\,\mathrm{d}B_t + \left(\partial_t f(B_t, t) + \frac{1}{2}\partial_{xx}^2 f(B_t, t)\right)\mathrm{d}t \tag{26}$$

where x is a random variable, $\partial_x f(B_t, t)$ is the first partial derivative of $f(B_t, t)$, $\partial_{xx}^2 f(B_t, t)$ is the second-order partial derivative of $f(B_t, t)$. If $\tilde{S}_t$ is a martingale process, and if and only if the drift term in the above process is zero, then $f(x, t)$ satisfies the heat equation:

$$\partial_t f(x, t) + \frac{1}{2}\partial_{xx}^2 f(x, t) = 0 \tag{27}$$

Thus, the discount price process of the stock can be uniquely determined by the function *f* which satisfies Equation (21). In particular, if we take $f(x,t) = \mathrm{e}^{\sigma x - \frac{1}{2}\sigma^2 t}$, then

$$\tilde{S}_t = \mathrm{e}^{\sigma B_t - \frac{1}{2}\sigma^2 t} \tag{28}$$

At this point, it is the famous Black-Scholes option pricing model, and it can be seen from Equation (22) that $\tilde{S}_t$ obeys lognormal distribution.

It can be found from Equation (19) that the unknown variable in the martingale model is a binary function $f(x,t)$ satisfying Equation (21). A significantly practical problem is how to estimate the function *f*.

It can be seen from Equation (21) that we need to attach certain boundary conditions to obtain the explicit expression or numerical solution of *f*. To do this, we assume that there is $\varphi(x) = f(x,T)$ at the end time *T*, then the explicit expression of $f(x,t)$ can be obtained.

$$f(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi\left(x + \sqrt{t}\, y\right) \mathrm{e}^{-\frac{y^2}{2}} \mathrm{d}y \tag{29}$$

Therefore, we only need to know the terminal function $\varphi(x)$, and then we can get the expression of the function $f(x,t)$ from Equation (23). Obviously we have got the $\tilde{S}_T = \varphi(B_T)$. Furthermore, if we can get the distribution $F(x) = P(\tilde{S}_T \le x)$ of the discounted price of a stock at a fixed future time *T*, then $\varphi(x)$ can be obtained by transforming *F* appropriately.

As a matter of fact, it is very difficult to use data from the stock market to calculate the distribution of the discounted price of the stock at a certain fixed moment T in the future. But with the acquisition of corresponding options market data, Breeden and Litzenberger [25] proposed a method to estimate the distribution of stock prices based on option quotes at a certain fixed moment T in the future, and called the estimated distribution of stock prices as implied probability distribution. Using this method, we will give a way to estimate $\varphi(x)$.

Option Pricing

We consider the corresponding option pricing problem first. A European call option with a strike price of K and an expiration date of T is priced at time t $(0 \le t \le T)$ as $C(f,K;t)$, and we have the following option pricing formula:

Theorem 2:

$$C(f,K,t) = E\left[\mathrm{e}^{rt}\left(f\left(B_{T-t} + x, T\right) - K\mathrm{e}^{-rT}\right)^+\right]\Big|_{x=f^{-1}(t,S_t)} \tag{30}$$

Proof of Theorem 2.

Based on the principle of risk-neutral pricing,

$$\begin{aligned} C(f,K,t) &= E\left[\mathrm{e}^{-r(T-t)}\left(S_T - K\right)^+ \mid B_{s,s<t}\right] \\ &= E\left[\mathrm{e}^{rt}\left(\tilde{S}_T - K\mathrm{e}^{-rT}\right)^+ \mid B_{s,s<t}\right] \\ &= E\left[\mathrm{e}^{rt}\left(f\left(B_T - B_t + B_t, T\right) - K\mathrm{e}^{-rT}\right)^+ \mid B_{s,s<t}\right] \end{aligned} \tag{31}$$

From the properties of Brownian motion $\{B_t\}$, $B_T - B_t$ is independent of $B_{s,s<t}$, then Equation (24) can be reduced to

$$C(f,K,t) = E\left[\mathrm{e}^{rt}\left(f\left(B_T - B_t + x, T\right) - K\mathrm{e}^{-rT}\right)^+\right]\Big|_{x=B_t} \tag{32}$$

Since $S_t = f(B_t,t)$ and f is monotonic with respect to x, we have

$$B_t = f^{-1}(S_t,t) \tag{33}$$

It is noted that B_{T-t} and $B_T - B_t$ are identically distributed, so Equation (24) holds.

In particular, when $f(x,t) = \mathrm{e}^{\sigma x - \frac{1}{2}\sigma^2 t}$, in other words, the stock price process satisfies the Black-Scholes model, we substitute it into Equation (24), then we can get

$$C_{BS}(f,K,t) = S_t N(d_1) - K\mathrm{e}^{-r(T-t)} N(d_2) \tag{34}$$

where $d_1 = \dfrac{\ln\left(\dfrac{S_t}{K}\right) + \left(r + \dfrac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{(T-t)}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$,

$N(x) = \dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-\frac{y^2}{2}}\mathrm{d}y$ are the standard normal distribution cumulative distribution functions. Equation (28) is the famous Black-Scholes option pricing formula.

If we define the price of the corresponding European put option at time t as $P(f,K,t)$, it is easy to verify the following option parity formula based on the no-arbitrage principle:

$$C(f,K,t) + K\mathrm{e}^{-r(T-t)} = P(f,K,t) + S_t \tag{35}$$

Theoretically, with the acquisition of the option parity Formula (29), the properties of the European put option can be derived from the properties of the European call option, so this paper only studies the European call option.

At time $t = 0$, we can observe the price $C(K)$ of European call options with a strike price K and an expiration date of T from the options market. If our martingale model (18) matches the actual market well, then $C(K) = C(f,K,0)$. We have the following theorem.

Theorem 3:

$$\varphi\left(\sqrt{T}N^{-1}\left(1 - \mathrm{e}^{rT}\frac{\partial C(x)}{\partial x}\bigg|_{x=K}\right)\right) = K\mathrm{e}^{-rT} \tag{36}$$

Proof of Theorem 3.

Firstly, according to the option pricing Formula (24),

$$C(K) = E\left[\left(\tilde{S}_T - \mathrm{e}^{-rT}K\right)^+\right] \tag{37}$$

similar to the method in article [25], by taking the first order partial derivative of K in Equation (31), we can obtain

$$F\left(\tilde{S}_T \le K\mathrm{e}^{-rT}\right) = 1 - \mathrm{e}^{rT} \left.\frac{\partial C(x)}{\partial x}\right|_{x=K} \tag{38}$$

Substitute $\tilde{S}_T = \varphi(B_T)$ into the above formula and we get

$$F\left(\tilde{S}_T \le K\mathrm{e}^{-rT}\right) = F\left(\varphi(B_T) \le K\mathrm{e}^{-rT}\right) = F\left(B_T \le \varphi^{-1}\left(K\mathrm{e}^{-rT}\right)\right) \tag{39}$$

And since B_T and $\sqrt{T}B_1$ are identically distributed, we have

$$\begin{aligned} F\left(B_T \le \varphi^{-1}\left(K\mathrm{e}^{-rT}\right)\right) &= F\left(\sqrt{T}B_1 \le \varphi^{-1}\left(K\mathrm{e}^{-rT}\right)\right) \\ &= F\left(B_1 \le \frac{1}{\sqrt{T}}\varphi^{-1}\left(K\mathrm{e}^{-rT}\right)\right) \end{aligned} \tag{40}$$

Notice that B_1 is a standard normal distribution, so we have

$$N\left(\frac{1}{\sqrt{T}}\varphi^{-1}\left(K\mathrm{e}^{-rT}\right)\right) = 1 - \mathrm{e}^{rT} \left.\frac{\partial C(x)}{\partial x}\right|_{x=K} \tag{41}$$

After sorting out, Equation (30) holds.

In the actual option market, for the same maturity T, there are many options with different strike prices $K_1 < K_2 < \cdots < K_M$, so we can get the corresponding option price $C(K_i)$. We can also use the following formula to estimate $C(K_i)$.

$$\left.\frac{\partial C(x)}{\partial x}\right|_{x=\frac{K_i+K_{i+1}}{2}} = \frac{C(K_{i+1}) - C(K_i)}{K_{i+1} - K_i} \tag{42}$$

And then we can get

$$x_i = \sqrt{T}N^{-1}\left(1 - \mathrm{e}^{rT}\frac{C(K_{i+1}) - C(K_i)}{K_{i+1} - K_i}\right);\ y_i = \frac{K_i + K_{i+1}}{2}\mathrm{e}^{-rT} \tag{43}$$

Theoretically, we can get $\varphi(x_i) = y_i$ from Theorem 3. Thus, the specific form of the function φ can be estimated by using parameter or non-parameter statistical methods.

After obtaining the expression of φ, the estimation formula of function $f(x,t)$ can be obtained by using Equation (23), and then the European option can be priced by using Equation (24).

4. Conclusions

In this paper, a repeated game model with asymmetric information is established to describe the game behavior between the banker and the retail investors in the stock market. This paper also deduces that the discount price process of stock follows a martingale model. Through this model, we can give an endogenous explanation of the random fluctuation of stock price: the banker conceals the information he knows by adopting the stochastic strategy, which leads to the random fluctuation of the stock price.

Furthermore, we study the European option pricing problem based on martingale model, and give the corresponding option pricing formula and the estimation method of unknown function. This paper mainly carries on the theoretical research, the further empirical analysis and the performance of this model in

China's options market will be our future research direction.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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