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Revival of Order in the Chaotic Dynamics with Position and Time Dependent Perturbed System

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Abstract

Studying quantum properties of a system has been quite popular in quantum mechanics. One of the most important systems that are very crucial to the framework of quantum mechanics is the system of harmonic oscillator a system whose classical evolution is known to exhibit peculiar chaotic dynamics. We are motivated to investigate the behavior of quantum properties for a system with position and time dependent perturbed. Starting with Hamiltonian, we determined the equation of motion and obtained the wave function. The energy of the whole system using the operator ordering method was found. We show that the quantum mechanical picture alludes to a chaotic dynamics as expected. This is evidenced through the appearance of energy level crossings. An additional signature to this chaotic dynamics is observed in the transition of Eigen values from real to imaginary. We also show numerically that one can give the behavior of the system is Poincare section. By so doing we confirmed that increasing and decreasing the perturbation amplitude of the system becomes chaotic.

Keywords

Schrödinger Equation, Crossing, Chaotic Dynamics, Perturbed System, Harmonic Oscillator

1. Introduction

The quantum harmonic oscillator (QHO) is a problem of great interest in quantum mechanics. It is one of the very few, relatively non-trivial problems that has an exact analytic solution. The harmonic oscillator solution and algebraic formalism has applications throughout modern physics (e.g. many particle systems, quantum electrodynamics (QED), quantum field theory, approximations of real systems such as molecular bonds, etc.).

Harmonic oscillators are ubiquitous in physics. For example, the small vibrations of almost any mechanical system near the bottom of a potential well can usually be approximated by harmonic oscillators. This includes the case of small vibrations of a molecule about its equilibrium position or small amplitude lattice vibrations in a solid. The simple harmonic oscillator (SHO) can be used to understand a wide variety of physical phenomena ranging. However, for a real physical problem one has to introduce the perturbed harmonic oscillator. Nowadays, time and position dependent driving force present in the field of physics are most useful into the study of phenomena.

So for a long time, the study of time-dependent harmonic oscillator systems has attracted considerable attention in the literature [1] [2] [3]. The great interest in these systems is motived because they are exactly solvable quantum mechanically and can be used to successfully model many phenomena in different branches of physics, such as quantum optics [4] [5] quantum fluid dynamics [6] gravitation [7] and quantum chaos [8].

In particular, the time-dependent perturbed harmonic oscillator has been studied by various authors through different methods such as trial functions [9], path integral formulation [10] [11], Heisenberg picture approach [12], Wigner function [13], coherent and squeezed state approach [14] [15], canonical and unitary transformations [16] and invariant method combined with quadratic or linear invariant [17] [18].

The quantum-mechanical treatment of harmonic oscillator appears in the literature with [19] and without [20], a driving force. Furthermore, invariants in mechanical systems with explicitly time-dependent Hamiltonians are constants of motion of central importance in the study of dynamical systems [21]. Given a time-dependent Hamiltonian, one can obtain the corresponding time-dependent Schrödinger equation. Time dependent harmonic oscillator has several applications such as dissipative quantum tunneling effect in macroscopic system [22] and quantum motion of an ion in a Paul trap [23].

Over and above all this, in analytical study of Paul traps an ion confined within a Paul trap can be simulated, in a good approximation, with a harmonic oscillator [24]. Depending on trap parameters, the ions were found to equilibrate either as an apparently chaotic cloud or in an ordered structure.

The dynamics of a quantum system is always stable against small variations of the initial state described by the wave-function. This is due to the linearity and unitarity of the Schrödinger equation. Small variations in the Hamiltonian however can produce interesting and highly non-trivial effects on quantum time evolution such as quantum chaotic behavior. Various approaches to the quantum chaos signatures have been proposed. One notes in particular: The decay pattern of fidelity between a time evolved perturbed and unperturbed state of a system [25] [26], the correspondence between the statistics of eigenvalues and eigenvectors of quantum states and the canonical ensembles of the random matrix theory [27].

Though non linear Schrödinger equations have been investigated extensively within the quantum chaos domain [28] [29] [30], the chaotic dynamics of position dependent systems have received less attention. In [31], a limit cycle-chaos transition is demonstrated numerically in the forced classical PDEM duffing-type oscillator.

We seek in this work to investigate the behavior of a quantum system under the influence of external driving force with the aid of Lewis and Reisenfeld approach.This effect will be done at the level of energy, and the level of Poincare section.This paper is organized as follows. In Section 2 we present a general theory of harmonic oscillator with time and position dependent driving force, the model is presented and the solution is given. In Section 3 we analyze the chaotic effects produced in mechanical oscillators due to driving force amplitude. Finally, in Section 4 we summarize our results and conclusions.

2. Problem Statement

Harmonic oscillator is an important model system pervading many areas in classical physics; it is likewise ubiquitous in quantum mechanics. The non-relativistic Schrödinger equation with a harmonic oscillator potential is readily solve with standard analytic methods, whether in one or three dimensions. The harmonic oscillator at constant mass has the following parameters:

The Hamiltonian

$$\hat{H}=\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2 \tag{1}$$

The energy

$$E_n=\hbar\omega\left(n+\frac{1}{2}\right) \tag{2}$$

And the wave function

$$\psi_n(x)=\frac{1}{\pi^{\frac{1}{4}}\sqrt{2^n n!}}H_n(x)\exp\left(-\frac{x^2}{2}\right) \tag{3}$$

However, we will take a different tack in this work.

In quantum mechanics, perturbation theory is a set of approximation schemes directly related to mathematical perturbation for describing a complicated quantum system in terms of a simple on. The idea is to start with a simple system for which a mathematical solution is known, and add an additional perturbing Hamiltonian representing a weak disturbance to the system [32].

The quantum Hamiltonian for a time and position dependent driving force harmonic oscillator system are given by

$$\hat{H}=\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2+F(x,t) \tag{4}$$

where $\hat{x}$ and $\hat{p}$ are the position and momentum of the mass, respectively, and $F(x,t)$ is the external driving force.

With

$$F(x,t) = A\cos(bx + \Omega t) \tag{5}$$

For which one can write the following time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left[\frac{\hat{p}^2}{2m_0} + \frac{1}{2}m_0\omega^2\hat{x}^2 + A\cos(bx + \Omega t)\right]\psi(x,t) \tag{6}$$

Assuming the system is initially prepared in an eigenstate of the free harmonic oscillator, we include the external driving force using the mean field approximation such that the potential energy of the system can be expressed as:

$$\begin{aligned} V(x,t) &= \frac{1}{2}m_0\tilde{\omega}^2\hat{x}^2\left[1 + \frac{2A}{m_0\tilde{\omega}^2}\cos(\Omega t)\left\langle x^{-2}\sum_{j=0}^{\infty}(-1)^j b^{2j}\frac{x^{2j}}{(2j)!}\right\rangle\right] \\ &= \frac{1}{2}m_0\tilde{\omega}^2\hat{x}^2\left[1 + \beta\cos(\Omega t)\right] \end{aligned} \tag{7}$$

where n is the quantal index the Harmonic Oscillator and

$$\beta = \frac{2A}{m_0\omega^2(2n+1)}\mathrm{e}^{-\frac{b^2}{2\omega}}L_n\left(\frac{b^2}{2\omega}\right) \tag{8}$$

By defining a new frequency as

$$\tilde{\omega}^2 = \tilde{\omega}(t)^2 = \omega^2\left[1 - \beta\cos(\Omega t)\right] \tag{9}$$

one obtains the following effective harmonic oscillator:

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left[\frac{\hat{p}^2}{2m_0} + m_0\tilde{\omega}^2\hat{x}^2\right]\psi(x,t) \tag{10}$$

With this manipulation, the original classical equation of motion

$$\ddot{x}(t) + \omega^2 x(t) = \frac{Ab}{m_0}\sin(bx(t) + \Omega t) \tag{11}$$

Now reads

$$\ddot{x}(t) + \tilde{\omega}^2 x(t) = 0 \tag{12}$$

Its solution is obtained in terms of the sine elliptic (S_e) and cosine elliptic (C_e) functions as

$$x(t) = K_1 C_e\left[\frac{4\omega^2}{\Omega^2}, -\frac{2\beta\omega^2}{\Omega^2}, \frac{\Omega}{2}t\right] + K_2 S_e\left[\frac{4\omega^2}{\Omega^2}, -\frac{2\beta\omega^2}{\Omega^2}, \frac{\Omega}{2}t\right] \tag{13}$$

where $K_{1,2}$ are integration constants. In **Figure 1** and **Figure 2**, we compare the analytic solution (solid curve) to the numerical solution (dotted curve) for small driving force amplitude and large driving force amplitude respectively. For small amplitude (A) **Figure 1**, both solutions closely match while for larger amplitude (A) **Figure 2**, the solutions show distinct behaviours.

Without loss of generality, we can set $K_2 = iK_1$, then the solution can be

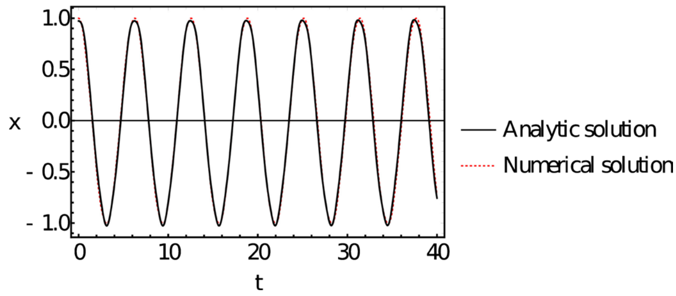

Figure 1. Comparison of the numerical solution (dotted red curve) and the analytic solution (solid curve) for large small amplitude ($A = 0.5$).

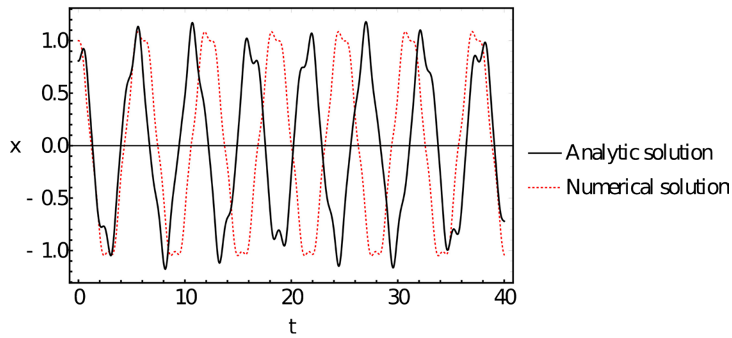

Figure 2. Comparison of the numerical solution (dotted curve) and the analytic solution (solid curve) for large perturbation amplitude ($A = 10$).

expressed in polar form as

$$u = K_1 r e^{i\theta} \tag{14}$$

where r and θ are time-dependent functions given by

$$r = \left(C_e^2 + S_e^2\right)^{1/2}; \quad \theta = \arctan\left(\frac{S_e}{C_e}\right) \tag{15}$$

In what follows, we employ the Lewis and Reisenfeld approach to resolution of the time-dependent quantum harmonic oscillator. This consists in using the complex solution Equation (14) to construct the operators

$$\begin{aligned} \hat{A} &= i\left(u^* p - \hbar m_0 \dot{u}^* x\right), \\ \hat{A}^\dagger &= -i\left(up - \hbar m_0 \dot{u} x\right), \end{aligned} \tag{16}$$

It may be easily verified that these operators satisfy the commutation relation below:

$$\left[\hat{A}, \hat{A}^\dagger\right] = \hat{I} \tag{17}$$

which form a pair of first order invariants that satisfy the quantum Liouville-Neumann equation

$$\begin{aligned} i\frac{\partial \hat{A}}{\partial t} + \left[\hat{A}, H\right] = 0 \\ i\frac{\partial \hat{A}^\dagger}{\partial t} + \left[\hat{A}^\dagger, H\right] = 0 \end{aligned} \tag{18}$$

and can in turn be used to construct the Lewis quadratic invariant

$$\hat{I} = \hat{A}\hat{A}^{\dagger} + \frac{\hbar\delta}{2} \tag{19}$$

whose eigenstates ϕ_n^L are related to those of $H - \phi_n$ by

$$\phi_n = \mathrm{e}^{\xi}\phi_n^L \tag{20}$$

where ξ are the Lewis phases. Substituting Equation (14) in Equation (12), one obtains

$$i\left[\ddot{\theta} + 2\frac{\dot{r}}{r}\dot{\theta}\right] + \frac{\ddot{r}}{r} + \omega^2 - \dot{\theta}^2 = 0 \tag{21}$$

whose imaginary part shows that $\delta = r^2\dot{\theta}$ is a time-invariant quantity. By requiring Equation (14) to satisfy the Wronskian condition

$$\hbar m_0\left(\dot{u}^* u - \dot{u}u^*\right) = i \tag{22}$$

the integral constant K_1 is obtained as

$$K_1 = \frac{1}{\sqrt{2\hbar m_0\delta}} \tag{23}$$

Once the ground state ψ_0^L, is obtained by solving $A\psi_0^L = 0$, then the n^{th} eigenstate of $\hat{I}$ follows from $\psi_n^L = \left(\frac{1}{\sqrt{n!}}\right)\left(\hat{A}^{\dagger}\right)^n \psi_0^L$. The n^{th} eigenstate of H reads:

$$\psi_{0,n} = \left(\frac{1}{\sqrt{2^n n!}}\right)\left[\frac{m_0\delta}{\pi\hbar r^2}\right]^{1/4} H_n\left[\sqrt{\frac{m_0\delta}{\hbar r^2}}x\right]\exp\left[-\frac{m_0}{2\hbar}\left(\frac{\delta}{r^2} - i\frac{\dot{r}}{r}\right)x^2\right]\mathrm{e}^{i\xi} \tag{24}$$

The Lewis phases are given by

$$\xi = \left\langle -\int\left[\frac{1}{\hbar}H\mathrm{d}t\right]\right\rangle = -\int\frac{1}{\hbar}E_n\mathrm{d}t \tag{25}$$

The dispersion relations

$$\begin{aligned}\left\langle x^2\right\rangle &= \hbar^2 u^* u = \frac{\hbar^2 r^2}{2m_0\delta}(2n+1)\\ \left\langle p^2\right\rangle &= \hbar^2 m_0^2 \dot{u}^* \dot{u}^* = \frac{\hbar m_0}{2\delta}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)(2n+1)\end{aligned} \tag{26}$$

are readily obtained, from where follows the uncertainty relation

$$\Delta x\Delta p = \frac{\hbar}{2} = \left[1 + \frac{(\dot{r}r)}{\delta^2}\right]^{1/2}(2n+1) \geq \frac{\hbar}{2} \tag{27}$$

$$E_{n,n} = \left(n + \frac{1}{2}\right)\frac{\hbar}{2}\left[\frac{\omega^2}{\dot{\theta}} + \frac{\dot{r}}{\delta}\dot{\theta}\right] \tag{28}$$

3. Results and Discussions

Taking note of the identities

$$C_e(a,0,x)=\cos(\sqrt{a}x);\quad S_e(a,0,x)=\frac{1}{\sqrt{a}}\sin(\sqrt{a}x) \tag{29}$$

we find that when $A = 0$, the solution

$$x(t)=K_1C_e\left[\frac{4\omega^2}{\Omega^2},0,\frac{\Omega}{2}t\right]+K_2S_e\left[\frac{4\omega^2}{\Omega^2},0,\frac{\Omega}{2}t\right] \tag{30}$$

reduces to the solution of the classical equation of motion of the free harmonic oscillator

$$x_{fho}(t)=K_1\cos(\omega t)+\frac{K_2\Omega}{2\omega}\sin(\omega t) \tag{31}$$

The equations in (31) for $K_2=\frac{2\omega iK_1}{\Omega}$, therefore become

$$r=K_1;\quad \theta=\omega t \tag{32}$$

substituting these quantities in Equation (28), we obtain the expected spectrum $E_n=\hbar\omega\left(n+\frac{1}{2}\right)$, of the free quantum harmonic oscillator.

Figure 3 shows that for increasing values of A each energy level admits a critical value of the perturbation amplitude $A_{c,n}$ at which it undergoes a transition from real to complex. We found the root of the function $E_n(0,A,b,c)=5.01443$, in the interval {A, 0.172} and obtained $A_{c,0}$. For the ground state, and for the named parametrization, $A_{c,0}=0.173$ where $E_0=5.01$. Beyond this value of $A_{c,0}$, the ground state is vanishes from the spectrum.

Figure 4 shows that there are also critical values of the wavevector $b_{c,n}$ at which the eigenvalues undergo transition from complex to real.

The more excited levels however always remain real.

Small perturbation strengths influence the ground state significantly and have little effect on the more excited states as shown in **Figure 5**. As the perturbation strength gets very large, the ground state becomes inaccessible while the excited states are markedly altered as shown in **Figure 6**.

The solution $x(t)$ in the configuration:

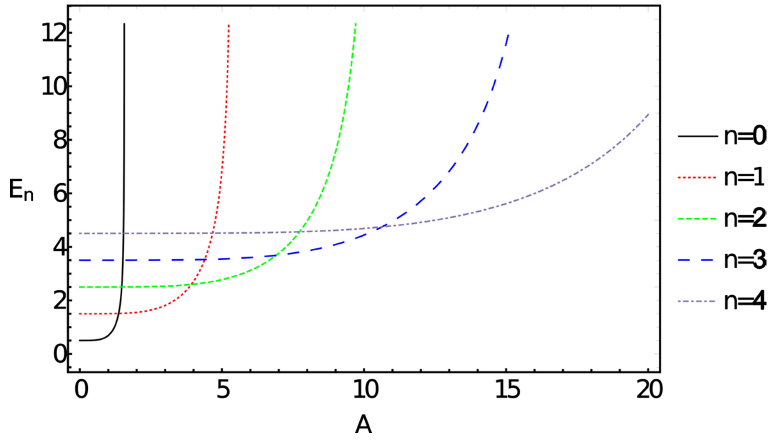

Figure 3. Plot of energy against the perturbation amplitude.

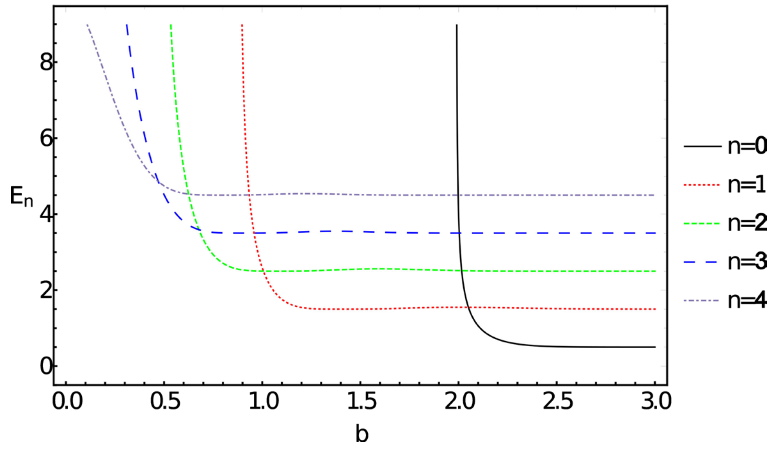

Figure 4. Plot of energy against the perturbation amplitude.

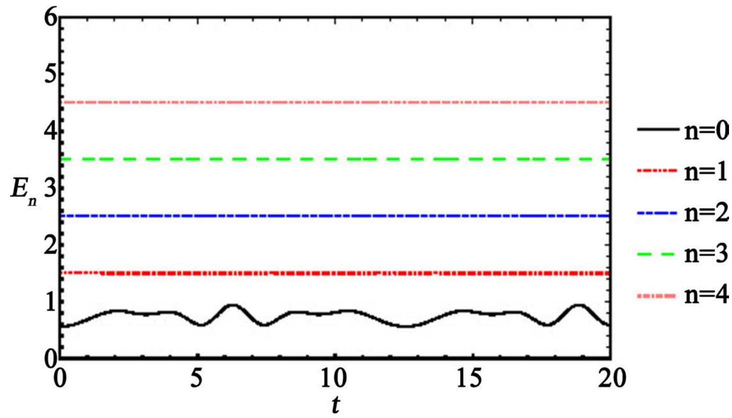

Figure 5. Plot of energy against the perturbation amplitude with small perturbation strengths.

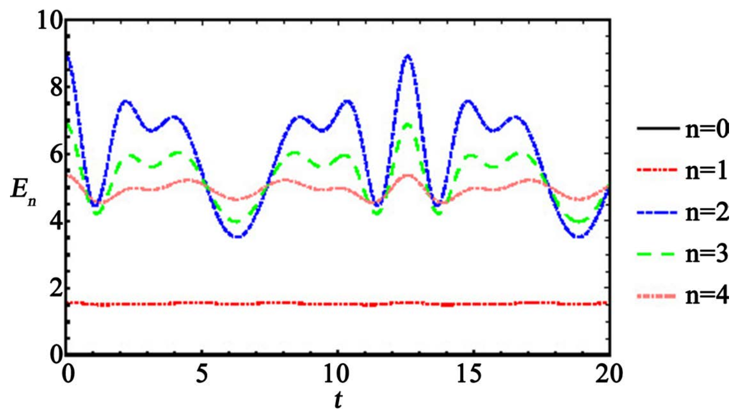

Figure 6. Plot of energy against the perturbation amplitude with large amplitude strengths.

$$x(t) = K_1 C_e \left[1, -\frac{2\beta\omega^2}{\Omega^2}, \frac{\Omega}{2}t\right] + K_2 S_e \left[1, -\frac{2\beta\omega^2}{\Omega^2}, \frac{\Omega}{2}t\right] \tag{33}$$

and

$$x(t) = K_1 C_e \left[4, -\frac{2\beta\omega^2}{\Omega^2}, \frac{\Omega}{2}t\right] + K_2 S_e \left[4, -\frac{2\beta\omega^2}{\Omega^2}, \frac{\Omega}{2}t\right] \tag{34}$$

as per the behaviour of Mathieu functions are complex valued. Equation (33) and Equation (34) translate respectively into the conditions: $\omega = \frac{\Omega}{2}$ and $\omega = \Omega$. The real and imaginary parts of $x(t)$ at these configurations oscillate with increasing amplitude which is reminiscent of parametric resonance. In these cases, the entire spectrum of the quantum mechanical problem is Imaginary.

The functions $C_e(a,b,z)$ and $S_e(a,b,z)$ are complex valued for $a > \frac{b}{2}$. This translates into the fact that we must have $\beta < 1$. One is thus tempted to link the onset of an imaginary spectrum to a chaotic behavior in the classical domain.

To confirm the assertion that the system exhibits a quantum chaotic behavior, we performed a numerical experiment, the result of which shows that the system is indeed oscillatory see **Figure 1** and **Figure 2**. An efficient pictorial tool that helps to monitor the chaotic or orderly behavior of a system is the Poincare section. The dots in such a phase space represent the state of the system at a particular instant of time. In an ordered regime, the dots get arrayed in circular tracks called KAM tori, named after Kolmogorov, Arnold and Moser who first introduced such a map.

One interesting behavior we have observed in a plane-wave perturbed harmonic oscillator with time-dependent mass is its ability to pass from an ordered regime see **Figure 7(a)** through chaos **Figure 7(b)** then back to order **Figure 7(c)** as the perturbation amplitude increases. The system however never returns to its original state.

4. Conclusion

It should be noted that for any of the regimes:

$$b \gg 1; \ \Omega \gg 1; \ A \ll 1 \tag{35}$$

the condition $\beta < 1$ is satisfied and the system approaches regular dynamics with $x(t) \approx x_{fho}(t)$. For Large amplitude (A), this condition is no longer satisfied and one observes not only level crossing (which is a quantum mechanical signature of chaos) but also a transition of one eigenvalue after another from real to complex. After each transition, there is a parameter domain in which the rest of the levels above are normally ordered before the next energy level "whites out" of the spectrum. This may explain Marco Frasca's [33] observation of the system Equation (35) occasionally passing from fully developed chaos to regular motion with increasing perturbation strength. The "whiting out of levels" in the quantum mechanical system explains why in the classical counterpart, the reforming KAM tori never resemble the original. In the present observation, the appearance of complex eigen values in the quantum mechanical system is synonymous to suppression of regular dynamics in the classical system.

It has been known that on the one hand, parametric resonance can induce

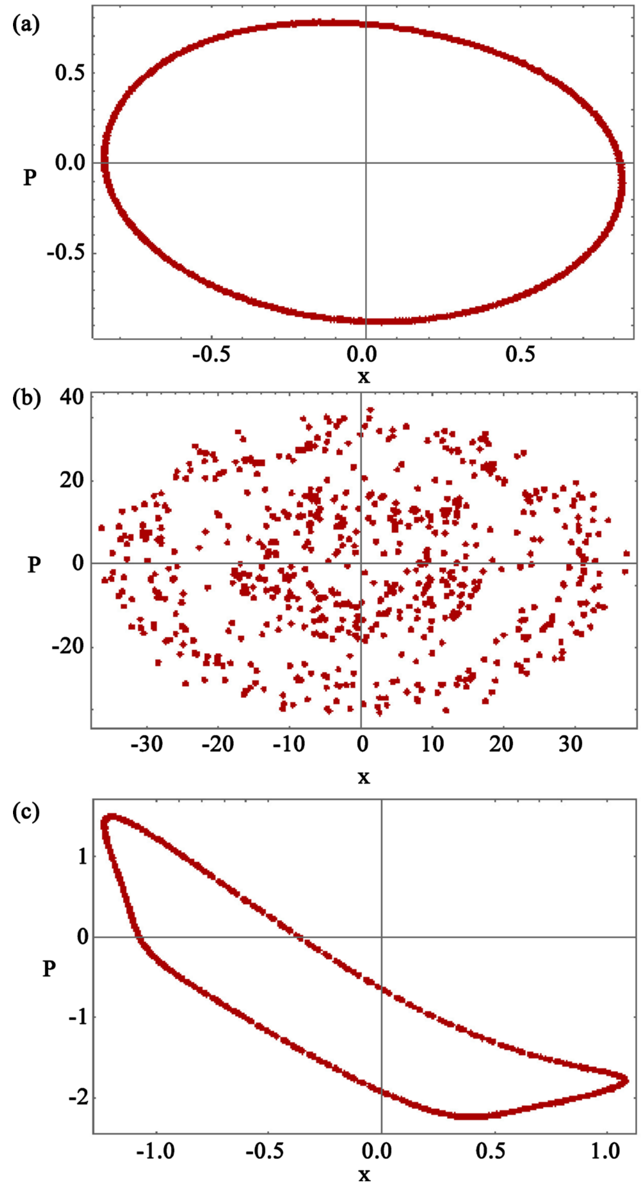

Figure 7. Poincare section for different values of amplitude (A); (a) Poincare section with amplitude (A = 0.05); (b) Poincare section with amplitude (A = 5); (c) Poincare section with amplitude (A = 7).

chaos [34] and on the other hand it can suppress it [35] [36]. At parametric resonance, level crossing vanishes at all times for all perturbation strengths. On the other hand, whenever $\omega = \Omega/2$ it is not possible to observe fully developed chaos in the classical system as invariant tori would always be traced in the phase space. The physical realization of such a system would have a far reaching impact in the field of Chaos control.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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