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Building Wave Functions of the Outer Electron for Alkaline Atoms Theory and Wave Functions Representation

Amaury de Kertanguy

LERMA, Observatoire de Paris-Meudon Place Jules Janssen, Meudon, France
Email: amaury.dekertanguy@gmail.com

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Abstract

The effects of the polarization potential serve to model spectra of alkaline atoms. These effects have been known for a long time and notably explained by the physicist Max Born (1926). The experimental knowledge of these alkaline spectra enables us to specify the values of these quantum defects. A simple code is used to calculate two quantum defects for which δ_l can be distinguished as: δ_s $l=0$ and δ_p $l=1$. On the theoretical part, it is possible to have an analytical expression for these quantum defects δ_l. A second code gives the correct wave functions modified by the quantum defects δ_l with the condition for the principal number: $n_* = n - \delta_l \geq 1$. It is well known that $\delta_l \to 0$ when the kinetic momentum $l \geq 4$, and for such momenta the spectra turns out to be hydrogenic. Modern software such as Mathematica, allows us to efficiently generate the polynomes defining wave functions with fractional quantum numbers. This leads to a good theoretical representation of these wave functions. To get numerically the quantum defects, a simple code is given to obtain these quantities when the levels assigned to a transition are known. Then, the quantum defects are inserted into the arguments of the correct modified wave functions for the outer electron of an atom or ion undergoing the short range polarization potential.

Keywords

Polarization Potential, Experimental Quantum Defects, Quantum Wave Functions, Mathematica Software

1. Introduction

If one admits that the linear response to an external electric field $\boldsymbol{E}$ is: $\boldsymbol{p} = \alpha_D \boldsymbol{E}$

the energy associated with this external field is then:

$$\mathrm{d}W = -\boldsymbol{p}\cdot\boldsymbol{E}\mathrm{d}\boldsymbol{E} \quad \text{that is:} \quad W = -\frac{\alpha_D \times \boldsymbol{E}^2}{2} = -\frac{\alpha_D e^2}{2r^4}.$$

This gives the correct expression for the polarization potential, a short range attractive potential deeper than the Coulomb potential, for short radial distance *r*.

Using experimental spectroscopy and neglecting fine structure splittings, it is possible to measure the wavelengths of the transitions of these heavier atoms. These transitions depart from those of the non-relativistic of the hydrogen atom:

$$E_n - E_m = h\nu_{nm} = \frac{R_{Hyd}}{2}\left(\frac{1}{n^2} - \frac{1}{m^2}\right).$$

The regular spectrum of hydrogen is modified to deal with the experimental alkaline atoms spectra, such as Na, Li, or Mg, with an analytical approach introducing the quantum defects: $n_* = n - \delta_{\delta_l}$.

The modification for an $l+1 \to l$ electron jump is given by:

$$E_n - E_m = \frac{R_{Hyd}}{2}\left(\frac{1}{\left(n-\delta_l\right)^2} - \frac{1}{\left(m-\delta_{(l+1)}\right)^2}\right).$$

It is possible to model the atoms with the so-called "optical electron" scheme, meaning that far from the cloud of the remaining electrons, appears a dipolar force. The compound cloud with the Z protons of the nucleus and the remaining $Z-1$ electrons interacts with the outer "optical" electron, acts as a dipolar interaction, thus repelling the compound cloud.

Using quantum defect theory, the quantum defects can be calculated two fold if the transition in an alkaline atom is observed experimentally two equations, suffice to specify the values of δ_s and δ_p.

Thus for an $p \to s$ transition,

$$h\nu_{nm}^* = \frac{R_{Hyd}}{2} \times \left(\frac{1}{\left(n-\delta_s\right)^2} - \frac{1}{\left(m-\delta_p\right)^2}\right).$$

The theory of these defects is explained when one introduces the polarization potential, which acts in the same sign as the Coulomb potential. Its expression is: $V_p\left(r\right) = -\frac{\alpha_D}{2r^4}$ atomic units and $\alpha_D \propto a_0^3$ is the static dipole polarizability. The polarization effect on the outer electron, assuming nearly hydrogenic states, can be expressed as:

$$\Delta E\left(nl\right) = -\frac{\alpha_D}{2} \times \left\langle \frac{1}{r^4} \right\rangle.$$

Using known results [1] of the expectation value of: $\left\langle nl \middle| r^{-4} \middle| nl \right\rangle$, the correction to the Coulomb potential is:

$$\begin{aligned}\Delta E(nl) &= -\frac{R_{Hyd}}{2}\times\frac{1}{n^2} - R_{Hyd}\cdot\frac{\alpha_D}{2}\times\frac{Z_{core}^4\left(3n^2 - l(l+1)\right)}{\left(l-\frac{1}{2}\right)l\left(l+\frac{1}{2}\right)(l+1)\left(l+\frac{3}{2}\right)n^5} \\ &= -\frac{R_{Hyd}}{2}\times\frac{1}{(n-\delta_l)^2}\end{aligned} \quad (1)$$

Solving this equation using a limited development, thus assuming that $\frac{\delta_l}{n} < 1$, the theorical quantum defect δ_l is fixed. Once the theorical δ_l is set, a simple code furnishes the numerical quantum defects associated with a spectroscopically identified transition wavelength of an alkaline atom. The comparison of these two values enables us an estimate the static polarizability α_D for the considered atom species.

2. Experimental Quantum Defects for MgI and Ions MgII MgIII, NaI, NaII, NaII

For alkaline atoms, such as Na, Ca, Li, Mg, K, and their ions, it is possible to measure the wavelengths associated with identified transitions. These can be found in tables Moore (1949) [2]. It is easy to use symbolic software like Mathematica, and solve two equations, if one wants to have two quantum defects: δ_s and δ_p, that requires two identified transitions of any alkaline, for instance for the Na (sodium) atom and Mg (magnesium). These data are listed in Table 1 and Table 2.

Using Mathematica syntax gives the way to proceed with the experimental determination of quantum defects δ_s and δ_s and is quite simple. Neglecting

Table 1. Values of NaI quantum defects $l = 0, 1, 2$.

NaI δ_l	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$l = 0$	1.373	1.357	1.352	1.349
$l = 1$	0.883	0.867	0.862	0.859
$l = 2$	0.010	0.011	0.013	0.011
$l = 3$	-	0.000	−0.001	−0.008

Table 2. Values of MgI quantum defects $l = 0, 1$.

MgI δ_l	$n = 4$	$n = 5$	$n = 6$	$n = 7$
$l = 0$	1.507	1.526	1.521	1.519
$l = 1$	1.2068	1.258	1.253	1.249

---

[1] At this stage, when the quantum defect is defined one can equalize the quantity: $\Delta E(nl) = 0.5\times\left(\frac{1}{n^2} - \frac{1}{(n-\delta_l)^2}\right)$. This exactly represents the difference between the hydrogen energy levels from the deeper energy levels associated with the effect of the polarization potential. This leads to an estimate of the static polarizability α_D by solving the following set of equations. Dealing with charged ions one has to change the factor $0.5 \to 0.5\times Z^2$.

fine structure, one can find in tables of data [2], two transitions, observed in experimental spectroscopy such as for MgI that is neutral Magnesium:

MgI $3s \to 3p$ and $4p \to 3s$ that is in atomic units:

$\Delta E_{3s3p} = 0.159715$ and $\Delta E_{3s4p} = 0.224959$ assuming that the quantum defects depend on the value of the momenta $l = 0$ and $l = 1$ that is an $p \to s$ transition.

One can perform the calculation of the two quantum defects by using the Solve function common in many symbolic softwares (Mathematica, Mapple, Matlab, Python). It is necessary to use two independent equations for two distinct quantum defects.

$$myapp = \text{Solve}\left[0.5\times\left(\frac{1}{\left(3-\delta_s\right)^2}-\frac{1}{\left(3-\delta_p\right)^2}\right) == 0.159715,\right. \tag{2}$$

$$\left. 0.5\times\left(\frac{1}{\left(3-\delta_s\right)^2}-\frac{1}{\left(4-\delta_p\right)^2}\right) == 0.224959, \left\{\delta_s, \delta_p\right\}\right] \tag{3}$$

The two variables are: δ_s, δ_p. A lot of solutions appear, it is easy two isolate the good set of variables.

The numerical results are the following:

$$\delta_p = 0.95021, \delta_s = 4.33938, \delta_p = 0.95021 \tag{4}$$

$$\delta_s = 1.66062, \delta_p = 3.49592, \delta_s = 3.47752 \tag{5}$$

$$\delta_s = 3.49592, \delta_s = 2.52248, \delta_p = 4.77693 - 2.09315\times i \tag{6}$$

The complete set of solutions contains imaginary terms for the defects: thus ignored, $\delta_s = 4.33938 \geq n = 3,4$ is out of the range. It remains the correct $\delta_s = 1.66062$ and $\delta_p = 0.95021$.

This solve function can be used for any couple of identified transitions, for example charged ions: MgII, or MgIII MgIV, that is Mg^{+}, $Z = 1$, Mg^{++}, $Z = 2$, Mg^{+++}, $Z = 3$.

$$\Delta E(nl) = -\frac{\delta_l}{n^3} - \frac{3\delta_l^2}{2n^4} + O\left(\delta_l^3\right) \tag{7}$$

Using the first order of the development, requiring a small quantum defect $\delta_{l=2}$ (a D state), and setting $-R_{Hyd} = 0.5$ atomic units and $Z_{core} = 1$.

$$\Delta E(nl) = -\frac{\delta_l}{n^3} \tag{8}$$

$$\Delta E(nl) = -0.5\frac{\alpha_D}{2}\times\frac{Z_{core}^4\left(3n^2 - l(l+1)\right)}{\left(l-\frac{1}{2}\right)l\left(l+\frac{1}{2}\right)(l+1)\left(l+\frac{3}{2}\right)n^5} \tag{9}$$

Some simple Mathematica instructions, reproduce quickly the relation written in Equation (10) [1].

$$\delta_{l=2} = \alpha_D \times \frac{n^2-2}{105n^2} \tag{10}$$

The results obtained with a small code are in accordance with Equation (10) in [1]:

$$l = 2 \tag{11}$$

$$nx \geq 3 \tag{12}$$

$$\Delta Enl\left(nx, \delta_l\right) = -\frac{3\delta^2}{2nx^4} - \frac{\delta}{nx^3} + O\left(\delta^3\right) \tag{13}$$

$$\Delta Enl\left(nx, \delta_l\right) = \frac{\delta_l}{nx^3} \tag{14}$$

$$\text{D states } l = 2 \tag{15}$$

$$\text{Jungen formula} \tag{16}$$

$$\delta_D = \frac{\alpha_D \times nx^2 - 2}{105nx^2} \tag{17}$$

$$\text{Mathematica} \tag{18}$$

$$U_{pol}\left(\alpha_D, nx, l\right) = -\frac{\alpha_D R_{hyd} Z_{core}^4 \times \left(3nx^2 - l\left(l+1\right)\right)}{\left(l - \frac{1}{2}\right) l \left(l + \frac{1}{2}\right)\left(l+1\right)\left(l + \frac{3}{2}\right) nx^5} \tag{19}$$

$$\text{Solve}\left[\Delta Enl\left(nx, \delta_l\right) = U_{pol}\left(\alpha_D, nx, l\right), \delta_l\right] \tag{20}$$

$$\text{solution is: } \delta_l = \left(0.00952381 - \frac{0.0190476}{nx^2} \times \alpha_D\right) \tag{21}$$

For $n = 4$ and $l = 2$

$$\text{my solution } \delta_{l=2} = 0.00833333 \times \alpha_D \tag{22}$$

$$\text{Jungen value } \delta_{l=2} = 0.00833333 \times \alpha_D \tag{23}$$

Assuming a small quantum defect: $\delta_{l=2} = 0.011$ and $n = 4$ for Na+ atom, this equation enables us to give an estimate of the static polarizability $\alpha_D = 1.4a_0^3$.

3. Some Remarks on the Laguerre Polynomial Mathematica Function LaguerreL

Table 3 below shows how the quantum defects vary for an atom like Magnesium and its ions. The greater Z_{core} is, the greater the quantum defects are. These data are taken from the Topbase on line database.

To deal with quantum wave functions, one uses the following form $L_n^a\left(x\right)$ called the generalized Laguerre polynomial, which satisfies the differential equation:

$$xy'' + \left(a + 1 - x\right)y' + ny = 0 \tag{24}$$

The equation below shows the hydrogenic wave functions; it is given because it serves to show how the quantum defect wave functions are modified from $\Psi_H\left(n,l,r\right)$.

$$\Psi_H\left(n,l,r\right) = \sqrt{\frac{\left(n-l-1\right)!}{\left(n+l\right)!}} e^{-}\left(\frac{r}{n}\right) \times \left(\frac{2r}{n}\right)^l \times \frac{2}{n^2} LaguerreL\left[n-l-1, 2l+1, \frac{2r}{n}\right] \tag{25}$$

The LaguerreL polynomial is written as given in the Mathematica software

Table 3. Values of Mg ions quantum defects $l = 0, 1$ as function of the Z_{core}.

Mg ions Z_{core}	δ_s	δ_p
12	1.544	0.982
11	1.069	0.700
10	0.829	0.417
9	0696	0.696
7	0.517	0.426
6	0.307	0.233
5	0.225	0.154
4	0.138	0.071
3	0.071	0.015
2	0.071	0.015

[a] $(Z_{core} \times e)$ is the charge that moves the optical electron. The building of the wave function for low quantum numbers: $n_* = n - \delta \leq 1$ breaks down. A relativistic theory is needed and is an open research problem.

where the $\Psi_H(n,l,r)$ serves to represent the hydrogen wave functions. If one deals with an hydrogenic ion with a nuclear charge Z the radial r has to be changed to $\frac{r}{Z}$. When solved with correct physical parameters using atomic units: $e = \hbar = m_e = 1$, this equation for the motion of a bound electron gives the discrete spectrum for the bound states:

$$\mathcal{E} = \frac{1}{n^2}.$$

The total energy H is $H = T + V$ and the full solution for the radial part is obtained using spherical coordinates (r, θ, ϕ). This is called the Schrodinger wave equation and the bound states are obtained by solving:

$$\text{Solve}\left[H\left(\Psi_H(n,l,r)\right) - \mathcal{E}\Psi_H(n,l,r)\right] = [0, \mathcal{E}]$$

is easily solved, using two Mathematica functions. The Solve function, when correctly written, gives the correct solution for the attractive Coulomb potential $-\frac{1}{r^2}$ with the centrifugal term $-\frac{l(l+1)}{r^2}$.

In the complete Schrodinger equation, the angles θ and ϕ separate, from the variable r, it is well known that the eigen function for the angles is the spherical harmonics: $Y_{lm}(\theta, \phi)$.

4. Building of the Wave Function with a Principal Quantum Number $n_* = n - \delta_l$

The Case of the Alkaline Optical Electron Radial Wave Functions

For what concerns the construction of a wave function using the non integer principal quantum number, defined as: $n_* = n - \delta_l$, we need to change the arguments, of the well built LaguerreL polynomial. In any version of the Mathematica software, the l momentum has to be changed to $l_* = l - \delta_l$ together with for the projection of the momentum $-l_* \le m_* \le l_*$. With this software, it possible is to obtain a correct result, for non integer arguments: for instance the $LaguerreL\left[\frac{1}{2}, 1.4\right] = 0.141185$ gives good numbers.

I will simply change the arguments of the Laguerre polynomial:

$$\Psi_H(n,l,r) = \sqrt{\frac{(n-l-1)!}{(n+l)!}} e^{-}\left(\frac{r}{n}\right) \times \left(\frac{2r}{n}\right)^l \times \frac{2}{n^2} LaguerreL\left[n-l-1, 2l+1, \frac{2r}{n}\right] \quad (26)$$

$$n \to n_* = n - \delta_l \quad (27)$$

$$l \to l_* = l - \delta_l \quad (28)$$

$$\Psi_{Alk}(n_*,l_*,r) = \sqrt{\frac{(n-l-1)!}{(n_*+l_*)!}} e^{-}\left(\frac{r}{n_*}\right) \times \left(\frac{2r}{n_*}\right)^{l_*} \times \frac{2}{n_*^2} LaguerreL\left[n-l-1, 2l_*+1, \frac{2r}{n_*}\right] \quad (29)$$

One may notice that some arguments do not change with the transformation $n \to n_* = n - \delta_l \to l_* = l - \delta_l \to l_* = l - \delta_l$ that is $(n-l-1)!$.

The factorial function with Mathematica, works for any number: $Factorial[2.325] = N[2.325!] = 2.75407$ and the Laguerre polynomial: that is the generalized $LaguerreL[n,a,x]$ function *n*,*l*, fractional arguments.

If one deals with an ionic core of charge *Z*, one has to change the *r* radial variable into $\frac{r}{Z}$ in the $\Psi_{Alk}(n_*,l_*,r)$.

For what concerns the calculation of the quantum defect wave function it is possible to extend continuously the transformation to $n \to n_* = n - \delta_l$.

5. Conclusions

The author shows how the quantum theory, with the use of wave functions taking into account the modification of the quantum numbers, as explained by Kostelecky and Nieto [3] provides a simple framework for computing atomic and ionic wave functions using current symbolic software.

This is a consequence of the extreme refinement of symbolic softwares now available: Mathematica, Mapple, MatLab, Python. The old quantum theory exposed by Max Born [4] explains that the effect of the polarization potential results in a precession of the elliptical trajectory of the electron around its perihelion. This work shows how to simply produce, with the help of a modified Schrodinger equation, the probability distribution for the optical electron near the nucleus of alkaline atoms and their ions. Our approach to this problem is self-consistent that is: a simple code is given, this enables to calculate the quantum defects associated with a spectroscopically identified transition of an alkaline atom such as MgI Equation (2), then another code is used to construct the quantum wave functions using these set of quantum defects Equation (29), as shown in the figures.

Our approach is coherent: it conciliates the calculation of the quantum defects coming from spectroscopical data, for which exist databases, with the building of the wave functions. These change from those of hydrogen with the the n, l set of quantum numbers. The change means that the set quantum numbers used are $n_* = n - \delta_l, l_* = l - \delta_l$. It is the simple and easy way to determine the static polarizability α_D of many atoms, these quantities are still an open research problem, beyond the scope of this paper.

Figure 1 shows the hydrogen wave function $\Psi_{Hyd}\left(n=4, l=0, r\right)$, (a S state), and two alkaline wave functions $\Psi_{Alk}\left(n_* = 3.75, l_* = 0.5, r\right)$ and $\Psi_{Alk}\left(n_* = 3.1, l_* = 0.5, r\right)$.

Figure 2 concerns P state, $\Psi_{Hyd}\left(n=4, l=1, r\right)$ and three wave functions $\Psi_{Alk}\left(n_* = 4-\delta, l_* = 1-\delta, r\right)$ (with different δ).

The probability for the outer electron to be near the nucleus is enhanced, and illustrated as shown in **Figure 1** and **Figure 2**, it is a consequence of the attractive polarization potential $V_p\left(r\right) = -\frac{\alpha_D}{2r^4}$ effect dominating the Coulomb potential $V_c\left(r\right) = -\frac{e^2}{r}$.

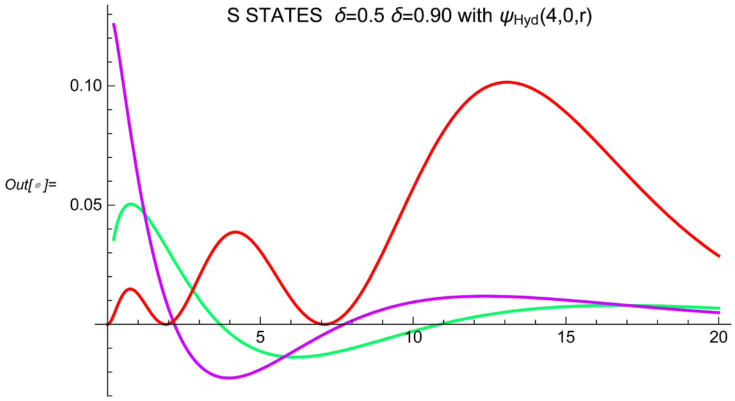


Figure 1. Red curve is the n = 4 l = 0 $\Psi_{Hyd}\left(n,l,r\right)$, green curve corresponds to $\Psi_{Alk}\left(n_*,l_*,r\right)$ $n_* = 3.75$, $l_* = 0.5$ with $I\left(l\right) = 1$, the scarlet curve corresponds to $\Psi_{Alk}\left(n_*,l_*,r\right)$ $n_* = 3.1$, $l_* = 0.5$ with $I\left(l\right) = 1$, this curve shows clearly that the probability for the electron to be near the nucleus is greater than the hydrogen electron.

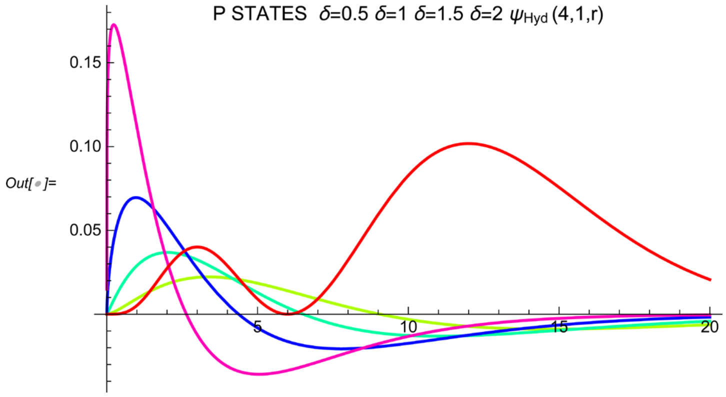
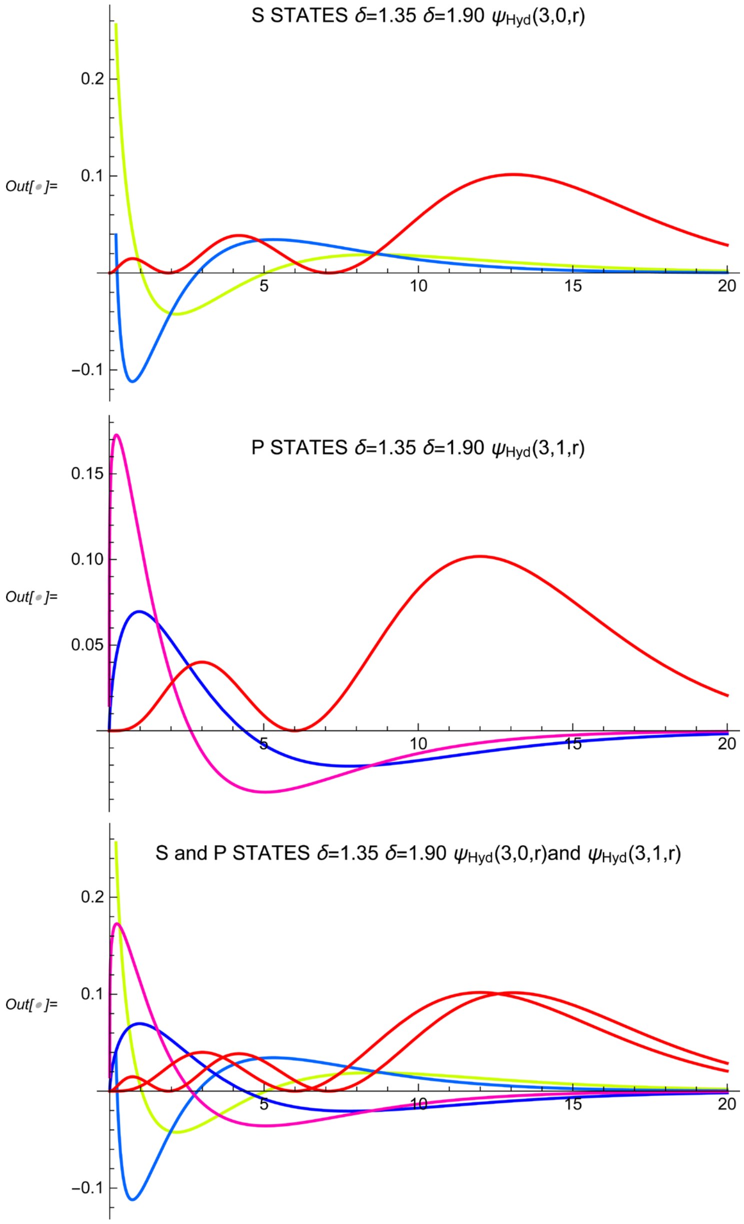


Figure 2. Red curve is the $n = 4$ $l = 1$ hydrogen wave function $\Psi_{Hyd}\left(n,l,r\right)$, green, blue and scarlet curves correspond to the wave function $\Psi_{Alk}\left(n_*,l_*,r\right)$ $n_* = 4-\delta$, $l_* = 1-\delta+I\left(l\right)$, r) with $I\left(l\right) = 1$, the higher is δ, the higher is the contribution of the alkaline wave function near the nucleus, here obtained near the origin $r = 0$.

Finally it is possible to build the difference function, here called: $dif\left(n,l,\delta_l,r\right)=\left|\Psi_{Alk}\left(n-\delta_l,l-\delta_l,r\right)^2-\Psi_H\left(n,l,r\right)^2\right|$ and to give a correct illustration of the effect inherent to the polarization potential $V_p\left(r\right)$. These are the main purposes of this article.

6. Comments on the Figures

Figure 3 contains S and P states representations of the hydrogen wave functions


S STATES δ=1.35 δ=1.90 $\psi_{\text{Hyd}}(3,0,r)$

P STATES δ=1.35 δ=1.90 $\psi_{\text{Hyd}}(3,1,r)$

S and P STATES δ=1.35 δ=1.90 $\psi_{\text{Hyd}}(3,0,r)$and $\psi_{\text{Hyd}}(3,1,r)$

Figure 3. This shows how the introduction of the quantum defects in the wave function $\Psi_{Alk}\left(n_*,l_*,r\right)$ differs from the hydrogen wave functions $\Psi_H\left(n,l,r\right)$ red curves. The first plot and the second represent respectively a S state and a P state for the hydrogen electron (red curve) and the other color curves describe the model alkaline electron. The third plot superposes all these.

and those of alkaline wave functions modified by the introduction of the quantum defects $\delta_l > 0$.

Figure 4 represents the square of the wave functions $\left|\Psi_{Hyd}\left(n=3, l=1, r\right)\right|^2$ and $\left|\Psi_{Alk}\left(n=1.96, l=0.96, r\right)\right|^2$, these squared wave functions are the true probabilities for the electron of the hydrogen atom, and the optical electron to be near the nucleus, a proton or an ionic core with $Z_{core}=1$.

Figure 5 shows that the optical electron with its squared wave function $\left|\Psi_{Alk}\left(3-\delta, l-\delta, r\right)\right|^2$, with $\delta=1.04$ and $n_*=3-\delta=1.96$ gets nearer to the nucleus than the electron of the hydrogen atom $\left|\Psi_{Hyd}\left(3,1,r\right)\right|^2$.

This is a consequencce of the attractive effect of the polarisation potential. It is not difficult to write a small code, taking into account the modification of the wave functions, using non integer $n_*=n-\delta$ and $l_*=l-\delta$, the limits of validity of the modified wave function:

$$\Psi_{Alk}\left(n_*, l_*, r\right)=\sqrt{\frac{(n-l-1)!}{\left(n_*+l_*\right)!}}e^{-}\left(\frac{r}{n_*}\right)\times\left(\frac{2r}{n_*}\right)^{l_*}\times\frac{2}{n_*^2}LaguerreL\left[n-l-1, 2l_*+1, \frac{2r}{n_*}\right]$$

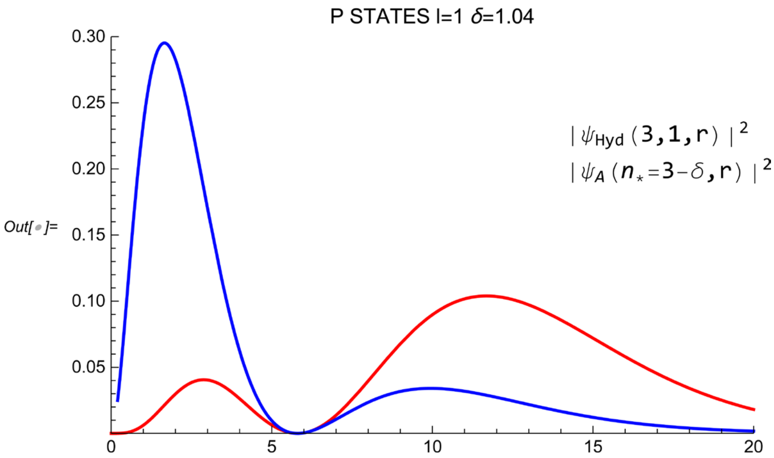
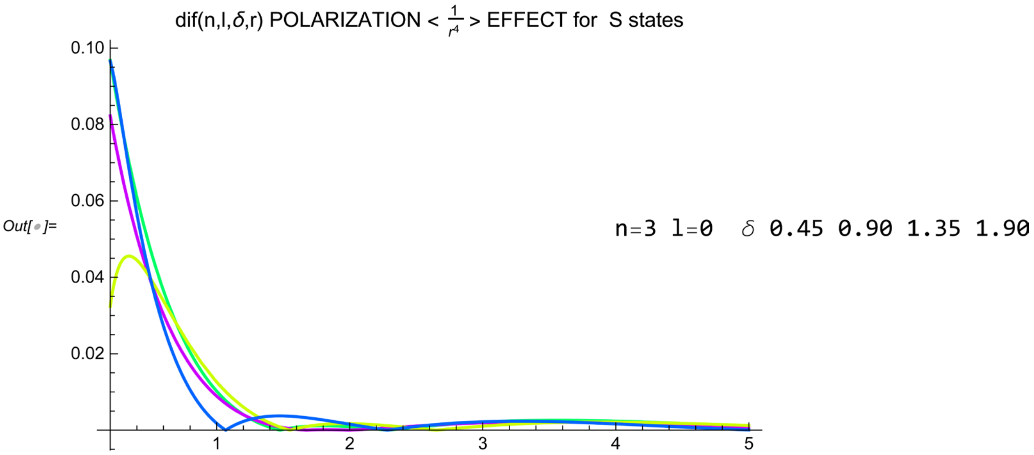


Figure 4. The red curve is $\left|\Psi_{Hyd}\left(3,1,r\right)\right|^2$ with the blue curve $\left|\Psi_{Alk}\left(3-\delta, l-\delta, r\right)\right|^2$ with $\delta=1.04$. These curves are everywhere positive, showing the real probabilty functions for the electron of the hydrogen atom, and the optical electron of a model alkaline atom.



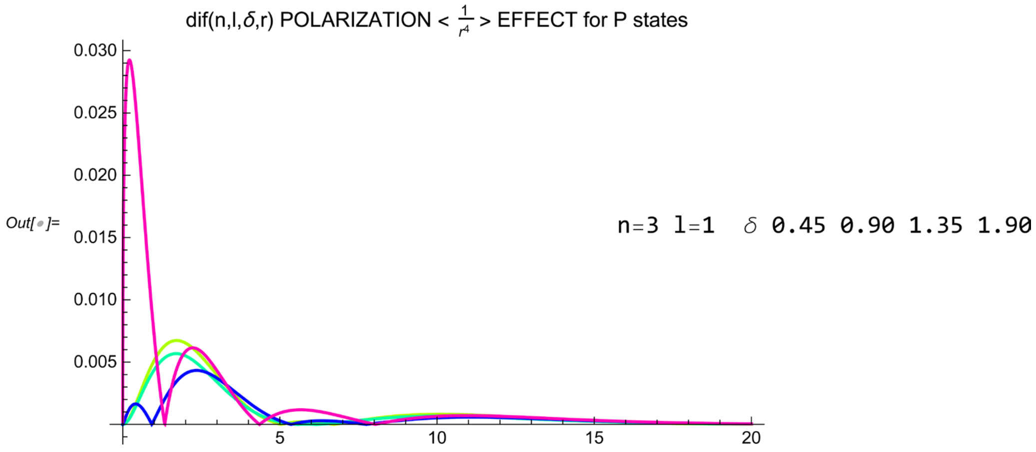


Figure 5. The difference function: $dif(n,l,\delta_l,r) = \left|\Psi_{Alk}(n-\delta_l, l-\delta_l, r)^2 - \Psi_H(n,l,r)^2\right|$ is plotted, that is $n = 3$ S $l = 0$ and P $l = 1$ states of the hydrogen squared wave functions subtracted from the squared wave functions modified by different quantum defects ($\delta_{l=0}, \delta_{l=1}$). The greater the quantum defects are, the higher are the contributions of the polarization potential near the origin $r = 0$, that is near the core nucleus.

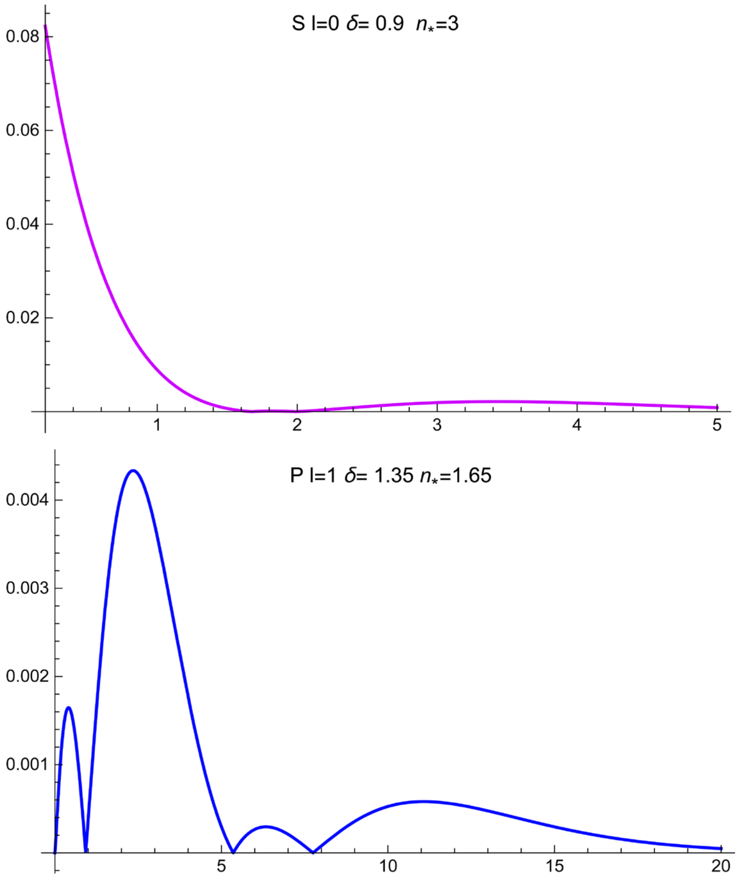


Figure 6. $\left|\Psi_{Hyd}(4,0,r)^2 - \Psi_{Alk}(n_* = 4-0.9, l_* = 0.1, r)^2\right|$ and $\left|\Psi_{Hyd}(3,1,r)^2 - \Psi_{Alk}(n_* = 3-1.35, l_* = 0.0.65, r)^2\right|$. These plots illustrate and isolate the quantum effect of the polarization potential $V_{pol}(r) = \frac{\alpha_D e^2}{2r^4}$ once subtracted the Coulomb potential for hydrogen.

are $n_* = n - \delta \le 1$ and $l_* = l - \delta \le 0$, this is well explained by two authors Kostelecky and Nieto with their factor: $l_* = l - \delta + I(l)$ where the integer factor $I(l) = 1$ or $I(l) = 2$, this transformation enables to obtain a continuous wave function: $\Psi_{Alk}(n_*, l_*, r)$ even when one deals with low quantum numbers n and high quantum defect δ_l. **Figure 6** illustrates the quantum strength of the polarization potential alone, it can be visualized in the old quantum theory as a precession motion added to the elliptical trajectory described by the outer electron.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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