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Existence and Stability Results for Impulsive Fractional *q*-Difference Equation

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Abstract

In this paper, we study the boundary value problem for an impulsive fractional *q*-difference equation. Based on Banach's contraction mapping principle, the existence and Hyers-Ulam stability of solutions for the equation which we considered are obtained. At last, an illustrative example is given for the main result.

Keywords

Impulsive Fractional *q*-Difference Equation, Hyers-Ulam Stability, Existence, *q*-Calculus

1. Introduction

The *q*-calculus or quantum calculus is an old subject that was initially developed by Jackson [1]; basic definitions and properties of *q*-calculus can be found in [2]. The fractional *q*-calculus had its origin in the works by Al-Salam [3] and Agarwal [4]. But the definitions mentioned above about *q*-calculus can't be applied to impulse points $t_k, k \in \mathbb{Z}$, such that $t_k \in (qt, t)$. In [5], the authors defined the concepts of fractional *q*-calculus by defining a *q*-shifting operator ${}_a\Phi_q(m) = qm + (1-q)a,\ m, a \in \mathbb{R}$. Using the *q*-shifting operator, the fractional impulsive *q*-difference equation was defined. In paper [5] [6] [7], the authors discussed the existence of solutions for the fractional impulsive *q*-difference equation with Riemann-Liouville and Caputo fractional derivatives respectively. Some other results about *q*-difference equations can be found in papers [8]-[16] and the references cited therein. Dumitru Baleanu *et al.* discussed the stability of non-autonomous systems with the *q*-Caputo fractional derivatives in reference [17]. However, the existence and stability of solutions for the fractional impulsive q-difference have not been yet studied.

Motivated greatly by the above mentioned excellent works, in this paper we investigate the following fractional impulsive q-difference equation with q-integral boundary conditions:

$$\begin{cases} {}_{t_k}^{c}D_{q_k}^{\alpha_k}x(t)=f(t,x(t)), t\in J_k\subseteq J=[0,T], t\neq t_k, \\ \Delta x(t_k)=x(t_k^+)-x(t_k)=\varphi_k(x(t_k)), k=1,2,\cdots,m, \\ \eta_1 x(0)+\eta_2 x(T)=\mu\sum_{k=0}^{m}{}_{t_k}I_{q_k}^{\beta_k}x(t_{k+1}). \end{cases} \quad (1)$$

where ${}_{t_k}^{c}D_{q_k}^{\alpha_k}$ is the fractional q_k-derivative of the Caputo type of order α_k on J_k, $0<\alpha_k<1$, $0<q_k<1$, $J_0=[0,t_1]$, $J_0=[0,t_1]$, $k=1,2,\cdots,m$, $\varphi_k\in C(\mathbb{R},\mathbb{R})$, $f\in C(J\times\mathbb{R},\mathbb{R})$. ${}_{t_k}I_{q_k}^{\beta_k}$ denotes the Riemann-Liouville q_k-fractional integral of order $\beta_k>0$ on $J_k, k=0,1,2,\cdots,m$ and η_1,η_2,μ are three constants.

2. Preliminaries on q-Calculus and Lemmas

Here we recall some definitions and fundamental results on fractional q-integral and fractional q-derivative, for the full theory for which one is referred to [5] [6] [7].

For $q\in(0,1)$, we define a q-shifting operator as ${}_a\Phi_q(m)=qm+(1-q)a$. The new power of q-shifting operator is defined as ${}_a(n-m)_q^{(0)}=1$, ${}_a(n-m)_q^{(k)}=\prod_{i=0}^{k-1}(n-{}_a\Phi_q^i(m))$, $k\in\mathbb{N}\cup\{0\}$, $n\in\mathbb{R}$. If $\nu\in\mathbb{R}$, then

$${}_a(n-m)_q^{(\nu)}=n^\nu\prod_{i=0}^{\infty}\frac{1-\frac{a}{n}\Phi_q^i\left(\frac{m}{n}\right)}{1-\frac{a}{n}\Phi_q^{i+\nu}\left(\frac{m}{n}\right)}.$$

The q-derivative of a function f on interval $[a,b]$ is defined by

$$({}_aD_qf)(t)=\frac{f(t)-f({}_a\Phi_q(t))}{(1-q)(t-a)}, t\neq a, ({}_aD_qf)(a)=\lim_{t\to a}({}_aD_qf)(t).$$

The q-integral of a function f defined on the interval $[a,b]$ is given by

$$({}_aI_qf)(t)=\int_a^t f(s)\,{}_a\mathrm{d}s=(1-q)(t-a)\sum_{i=0}^{\infty}q^i f({}_a\Phi_{q^i}(t)), t\in[a,b].$$

Some results about operator ${}_aD_q$ and ${}_aI_q$ can be found in references [5]. Let us define fractional q-derivative and q-integral on interval $[a,b]$ and outline some of their properties [5] [6] [7].

Definition 1 [5] The fractional q-derivative of Riemann-Liouville type of order $\nu\geq 0$ on interval $[a,b]$ is defined by $({}_aD_q^0f)(t)=f(t)$ and

$$({}_aD_q^\nu f)(t)=({}_aD_q^l\,{}_aI_q^{l-\nu}f)(t), \nu>0,$$

where l is the smallest integer greater than or equal to ν.

Definition 2 [5] Let $\alpha\geq 0$ and f be a function defined on $[a,b]$. The fractional q-integral of Riemann-Liouville type is given by $\left({}_aI_q^0 f\right)(t) = f(t)$ and

$$\left({}_aI_q^\alpha f\right)(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t {}_a\left(t - {}_a\Phi_q(s)\right)_q^{\alpha-1} f(s)\,{}_a\mathrm{d}_q s, \alpha > 0, t \in [a,b].$$

Lemma 1 [5] Let $\alpha, \beta \in \mathbb{R}^+$ and f be a continuous function on $[a,b], a \ge 0$. The Riemann-Liouville fractional q-integral has the following semi-group property

$${}_aI_q^\beta\, {}_aI_q^\alpha f(t) = {}_aI_q^\alpha\, {}_aI_q^\beta f(t) = {}_aI_q^{\alpha+\beta} f(t).$$

Lemma 2 [5] Let f be a q-integrable function on $[a,b]$. Then the following equality holds

$${}_aD_q^\alpha\, {}_aI_q^\alpha f(t) = f(t), \text{ for } \alpha > 0, t \in [a,b].$$

Lemma 3 [5] Let $\alpha > 0$ and p be a positive integer. Then for $t \in [a,b]$ the following equality holds

$${}_aI_q^\alpha\, {}_aD_q^p f(t) = {}_aD_q^p\, {}_aI_q^\alpha f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)}\, {}_aD_q^k f(a).$$

Definition 3 [7] The fractional q-derivative of Caputo type of order $\alpha \ge 0$ on interval $[a,b]$ is defined by ${}_a^cD_q^0 f(t) = f(t)$ and

$$\left({}_a^cD_q^\alpha f\right)(t) = \left({}_aI_q^{n-\alpha}\, {}_aD_q^n f\right)(t), \alpha > 0,$$

where n is the smallest integer greater than or equal to α.

Lemma 4 [7] Let $\alpha > 0$ and n be the smallest integer great than or equal to α. Then for $t \in [a,b]$ the following equality holds

$${}_aI_q^\alpha\, {}_a^cD_q^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^k}{\Gamma_q(k+1)}\, {}_aD_q^k f(a).$$

3. Main Results

In this section, we will give the main results of this paper.

Let $PC(J,\mathbb{R}) = \{x : J \to \mathbb{R}, x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist, and $x(t_k^-) = x(t_k), k = 1,2,\cdots,m\}$. $PC(J,\mathbb{R})$ is a Banach space with the norm

$$\|x\| = \sup\{|x(t)| : t \in J\}.$$

First, for the sake of convenience, we introduce the following notations:

$$\Lambda = \eta_1 + \eta_2 - \mu \sum_{i=0}^m \Omega_{\beta_i} \ne 0,\ \Omega_{\sigma_i} = \frac{{}_{t_i}\left(t_{i+1} - t_i\right)_{q_i}^{(\sigma_i)}}{\Gamma_{q_i}(\sigma_i + 1)},$$

where $\sigma_i \in \{\alpha_i, \beta_i, \alpha_i + \beta_i\}, q_i \in (0,1), i = 0,1,2,\cdots,m$.

To obtain our main results, we need the following lemma.

Lemma 5 Let $\mu \sum_{i=0}^m \Omega_{\beta_i} \ne \eta_1 + \eta_2$ and $h(t) \in C(J,\mathbb{R})$. Then for any $t \in J_k$, the solution of the following problem

$$\begin{cases} {}_{t_k}^{c}D_{q_k}^{\alpha_k} x(t) = h(t), t \in J_k \subseteq J = [0,T], t \neq t_k, \\ \Delta x(t_k) = x(t_k^+) - x(t_k) = \varphi_k(x(t_k)), k = 1,2,\cdots,m, \\ \eta_1 x(0) + \eta_2 x(T) = \mu \sum_{k=0}^{m} {}_{t_k}I_{q_k}^{\beta_k} x(t_{k+1}) \end{cases} \tag{2}$$

is given by

$$\begin{aligned} x(t) = \frac{1}{\Lambda} \Bigg\{ & \sum_{i=0}^{m} \left(\mu \, {}_{t_i}I_{q_i}^{\alpha_i+\beta_i} h(t_{i+1}) - \eta_2 \, {}_{t_i}I_{q_i}^{\alpha_i} h(t_{i+1}) \right) \\ & + \sum_{i=1}^{m} \left[\mu \left(\sum_{j=1}^{i} \varphi_j(x(t_j)) + \sum_{j=0}^{i-1} {}_{t_j}I_{q_j}^{\alpha_j} h(t_{j+1}) \right) \Omega_{\beta_i} - \eta_2 \varphi_i(x(t_i)) \right] \Bigg\} \\ & + \sum_{i=1}^{k} \varphi_i(x(t_i)) + \sum_{i=0}^{k-1} {}_{t_i}I_{q_i}^{\alpha_i} h(t_{i+1}) + {}_{t_k}I_{q_k}^{\alpha_k} h(t). \end{aligned} \tag{3}$$

Proof. Applying the operator ${}_{t_0}I_{q_0}^{\alpha_0}$ on both sides of the first equation of (2) for $t \in J_0$ and using Lemma 4, we have

$$x(t) = x(t_0) + {}_{t_0}I_{q_0}^{\alpha_0} h(t).$$

Then we get for $t = t_1$ that

$$x(t_1) = x(t_0) + {}_{t_0}I_{q_0}^{\alpha_0} h(t_1). \tag{4}$$

For $t \in J_1$, again taking the ${}_{t_1}I_{q_1}^{\alpha_1}$ to (4) and using the above process, we get

$$x(t) = x(t_1^+) + {}_{t_1}I_{q_1}^{\alpha_1} h(t).$$

Applying the impulsive condition $x(t_1^+) = x(t_1) + \varphi_1(x(t_1))$, we get

$$x(t) = x(t_0) + \varphi_1(x(t_1)) + {}_{t_0}I_{q_0}^{\alpha_0} h(t_1) + {}_{t_1}I_{q_1}^{\alpha_1} h(t).$$

By the same way, for $t \in J_2$, we have

$$x(t) = x(t_0) + \varphi_1(x(t_1)) + \varphi_2(x(t_2)) + {}_{t_0}I_{q_0}^{\alpha_0} h(t_1) + {}_{t_1}I_{q_1}^{\alpha_1} h(t_2) + {}_{t_2}I_{q_2}^{\alpha_2} h(t).$$

Repeating the above process for $t \in J_k \subseteq J, k = 0,1,2,\cdots,m$, we get

$$x(t) = x(t_0) + \sum_{i=1}^{k} \varphi_i(x(t_i)) + \sum_{i=0}^{k-1} {}_{t_i}I_{q_i}^{\alpha_i} h(t_{i+1}) + {}_{t_k}I_{q_k}^{\alpha_k} h(t). \tag{5}$$

From (5), we find that

$$x(T) = x(t_0) + \sum_{i=1}^{k} \varphi_i(x(t_i)) + \sum_{i=0}^{k-1} {}_{t_i}I_{q_i}^{\alpha_i} h(t_{i+1}) + {}_{t_k}I_{q_k}^{\alpha_k} h(T).$$

From the boundary condition of (2), we get

$$\begin{aligned} x(t_0) = \frac{1}{\Lambda} \Bigg\{ & \sum_{i=0}^{m} \left(\mu \, {}_{t_i}I_{q_i}^{\alpha_i+\beta_i} h(t_{i+1}) - \eta_2 \, {}_{t_i}I_{q_i}^{\alpha_i} h(t_{i+1}) \right) \\ & + \sum_{i=1}^{m} \left[\mu \left(\sum_{j=1}^{i} \varphi_j(x(t_j)) + \sum_{j=0}^{i-1} {}_{t_j}I_{q_j}^{\alpha_j} h(t_{j+1}) \right) \Omega_{\beta_i} - \eta_2 \varphi_i(x(t_i)) \right] \Bigg\}. \end{aligned} \tag{6}$$

Substituting (6) to (5), we obtain the solution (3). This completes the proof.

We define an operator $\mathcal{G}: PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ as follows:

$$\begin{aligned}\mathcal{G}x(t) = \frac{1}{\Lambda}\Bigg\{ & \sum_{i=0}^{m}\left(\mu\, {}_{t_i}I_{q_i}^{\alpha_i+\beta_i} f(s,x)(t_{i+1}) - \eta_2\, {}_{t_i}I_{q_i}^{\alpha_i} f(s,x)(t_{i+1})\right) \\ & + \sum_{i=1}^{m}\left[\mu\left(\sum_{j=1}^{i}\varphi_j\left(x(t_j)\right) + \sum_{j=0}^{i-1} {}_{t_j}I_{q_j}^{\alpha_j} f(s,x)(t_{j+1})\right)\Omega_{\beta_i} - \eta_2\varphi_i\left(x(t_i)\right)\right]\Bigg\} \quad (7) \\ & + \sum_{i=1}^{k}\varphi_i\left(x(t_i)\right) + \sum_{i=0}^{k-1} {}_{t_i}I_{q_i}^{\alpha_i} f(s,x)(t_{i+1}) + {}_{t_k}I_{q_k}^{\alpha_k} f(s,x)(t).\end{aligned}$$

Then, the existence of solutions of system (1) is equivalent to the problem of fixed point of operator $\mathcal{G}$ in (7).

Theorem 1 Let $f: J\times\mathbb{R}\to\mathbb{R}$ and $\varphi_k:\mathbb{R}\to\mathbb{R}, k=1,2,\cdots,m$ be continuous functions. Assume that $\mu\sum_{i=0}^{m}\Omega_{\beta_i} \neq \eta_1+\eta_2$ and the following conditions are satisfied:

(H_1) There exists a positive constant L such that $|\varphi_k(x)-\varphi_k(y)| \le L|x-y|$ for each $x,y\in\mathbb{R}$ and $k=1,2,\cdots,m$.

(H_2) There exists a function $M(t)\in C(J,\mathbb{R}^+)$ such that

$$|f(t,x)-f(t,y)| \le M(t)|x-y|, \forall t\in J, x,y\in\mathbb{R}.$$

(H_3) $\Delta<1$.

Then problem (1) has a unique solution on *J*, where $M=\sup_{t\in J}|M(t)|$ and

$$\begin{aligned}\Delta = & \frac{1}{\Lambda}\sum_{i=1}^{m}\left(\mu M\Omega_{\alpha_i+\beta_i} + (\eta_2+M)\Omega_{\alpha_i} + \mu M\sum_{j=0}^{i-1}\Omega_{\alpha_j}\Omega_{\beta_i} + \mu Li\Omega_{\beta_i}\right) \\ & + \frac{1}{\Lambda}\left(\mu\Omega_{\alpha_0+\beta_0} + \eta_2\Omega_{\alpha_0}\right) + mL\left(\frac{1}{\Lambda}\eta_2+1\right).\end{aligned}$$

Proof. The conclusion will follow once we have shown that the operator $\mathcal{G}$ defined (7) is a construction with respect to a suitable norm on $PC(J,\mathbb{R})$.

For any functions $x,y\in PC(J,\mathbb{R})$, we have

$$\begin{aligned}& |(\mathcal{G}x)(t)-(\mathcal{G}y)(t)| \\ \le & \frac{1}{\Lambda}\Bigg\{\sum_{i=0}^{m}\left(\mu\, {}_{t_i}I_{q_i}^{\alpha_i+\beta_i}|f(s,x)-f(s,y)|(t_{i+1}) + \eta_2\, {}_{t_i}I_{q_i}^{\alpha_i}|f(s,x)-f(s,y)|(t_{i+1})\right) \\ & + \sum_{i=1}^{m}\Bigg[\mu\left(\sum_{j=1}^{i}\left|\varphi_j\left(x(t_j)\right)-\varphi_j\left(y(t_j)\right)\right| + \sum_{j=0}^{i-1} {}_{t_j}I_{q_j}^{\alpha_j}|f(s,x)-f(s,y)|(t_{j+1})\right)\Omega_{\beta_i} \\ & + \eta_2\left|\varphi_i\left(x(t_i)\right)-\varphi_i\left(y(t_i)\right)\right|\Bigg]\Bigg\} + \sum_{i=1}^{m}\left|\varphi_i\left(x(t_i)\right)-\varphi_i\left(y(t_i)\right)\right| \\ & + \sum_{i=0}^{m-1} {}_{t_i}I_{q_i}^{\alpha_i}|f(s,x)-f(s,y)|(t_{i+1}) + {}_{t_m}I_{q_m}^{\alpha_m}|f(s,x)-f(s,y)|(t).\end{aligned}$$

By conditions (H_1) and (H_2), we get

$$\begin{aligned}& |(\mathcal{G}x)(t)-(\mathcal{G}y)(t)| \\ \le & \frac{1}{\Lambda}\Bigg\{\sum_{i=0}^{m}\left(\mu\, {}_{t_i}I_{q_i}^{\alpha_i+\beta_i}\left(M\|x-y\|\right)(t_{i+1}) + \eta_2\, {}_{t_i}I_{q_i}^{\alpha_i}\left(M\|x-y\|\right)(t_{i+1})\right) \\ & + \sum_{i=1}^{m}\left[\mu\left(\sum_{j=1}^{i}L\|x-y\| + \sum_{j=0}^{i-1} {}_{t_j}I_{q_j}^{\alpha_j}\left(M\|x-y\|\right)\right)\Omega_{\beta_i} + \eta_2 L\|x-y\|\right]\Bigg\}\end{aligned}$$

$$+\sum_{i=1}^{m} L\|x-y\| + \sum_{i=0}^{m-1} {}_{t_i}I_{q_i}^{\alpha_i}\left(M\|x-y\|\right)(t_{i+1}) + {}_{t_m}I_{q_m}^{\alpha_m}\left(M\|x-y\|\right)(t_{m+1})$$

$$\le \left\{\frac{1}{\Lambda}\sum_{i=1}^{m}\left(\mu M\Omega_{\alpha_i+\beta_i} + \eta_2\Omega_{\alpha_i} + \mu Li\Omega_{\beta_i} + \mu M\sum_{j=0}^{i-1}\Omega_{\alpha_j}\Omega_{\beta_i} + M\Omega_{\alpha_i}\right)\right.$$

$$\left. + \frac{1}{\Lambda}\left(\mu\Omega_{\alpha_0+\beta_0} + \eta_2\Omega_{\alpha_0}\right) + mL\left(\frac{1}{\Lambda}\eta_2 + 1\right)\right\}\|x-y\|,$$

which implies that

$$\|\mathcal{G}x - \mathcal{G}y\| \le \Delta\|x-y\|.$$

Thus the operator $\mathcal{G}$ is a contraction in view of the condition (H_3). By Banach's contraction mapping principle, the problem (1) has a unique solution on *J*. This completes the proof.

In the following, we study the Hyers-Ulam stability of impulsive fractional q-difference Equation (1). Let $\varepsilon > 0, \epsilon > 0$ and $\delta : [0,T] \to \mathbb{R}$ be a continuous function. Consider the inequalities:

$$\begin{cases} \left|{}_{t_k}^{c}D_{q_k}^{\alpha_k}\bar{x}(t) - f(t,\bar{x}(t))\right| \le \delta(t)\varepsilon, t \in J_k \subseteq J = [0,T], t \ne t_k, k = 0,1,\cdots,m, \\ \left|\Delta\bar{x}(t_k) - \phi_k(\bar{x}(t_k))\right| \le \epsilon\varepsilon, k = 1,2,\cdots,m, \\ \eta_1\bar{x}(0) + \eta_2\bar{x}(T) = \mu\sum_{k=0}^{m} {}_{t_k}I_{q_k}^{\beta_k}\bar{x}(t_{k+1}). \end{cases} \tag{8}$$

Now, we give out the definition of Hyers-Ulam stability of system (1).

Definition 4 System (1) is Hyers-Ulam stable with respect to system (8), if there exists $A_f > 0$ such that

$$|\bar{x} - \tilde{x}| \le A_f\varepsilon$$

for all $t \in J$, where $\bar{x}$ is the solution of (8), and $\tilde{x}$ of the solution for system (1).

Theorem 2 Assume $f : J \times \mathbb{R} \to \mathbb{R}$ satisfy assumption (H_2), $\varphi_i : \mathbb{R} \to \mathbb{R}, i = 1,2,\cdots,m$ are continuous functions and satisfy assumption (H_1) and the condition (H_3) holds, $\sup_{t\in J}\delta(t) \le 1$. Then the system (1) is Hyers-Ulam stable with respect to system (8).

Proof. Let ${}_{t_k}^{c}D_{q_k}^{\alpha_k}\bar{x}(t) = f(t,\bar{x}(t)) + g(t), k = 0,1,\cdots,m$ and $\Delta\bar{x}(t_k) = \varphi_k(\bar{x}(t_k)) + g_k, k = 1,2,\cdots,m$. Consider the system

$$\begin{cases} {}_{t_k}^{c}D_{q_k}^{\alpha_k}\bar{x}(t) = f(t,\bar{x}(t)) + g(t), t \in J_k \subseteq J = [0,T], t \ne t_k, \\ \Delta\bar{x}(t_k) = \varphi_k(\bar{x}(t_k)) + g_k, k = 1,2,\cdots,m. \\ \eta_1\bar{x}(0) + \eta_2\bar{x}(T) = \mu\sum_{k=0}^{m} {}_{t_k}I_{q_k}^{\beta_k}\bar{x}(t_{k+1}). \end{cases} \tag{9}$$

Similarly to the system in Theorem 1, system (9) is equivalent to the following integral equation in Lemma 5.

$$\bar{x}(t) = \frac{1}{\Lambda}\left\{\sum_{i=0}^{m}\left(\mu\, {}_{t_i}I_{q_i}^{\alpha_i+\beta_i}\left(f(s,\bar{x}) + g(s)\right)(t_{i+1}) - \eta_2\, {}_{t_i}I_{q_i}^{\alpha_i}\left(f(s,\bar{x}) + g(s)\right)(t_{i+1})\right)\right.$$

$$+\sum_{i=1}^{m}\left[\mu\left(\sum_{j=1}^{i}\left(\varphi_j(\bar{x}(t_j)) + g_j\right) + \sum_{j=0}^{i-1} {}_{t_j}I_{q_j}^{\alpha_j}\left(f(s,\bar{x}) + g(s)\right)(t_{j+1})\right)\Omega_{\beta_i}\right.$$

$$-\eta_2\left(\varphi_i\left(\overline{x}\left(t_i\right)\right)+g_i\right)\Bigg]\Bigg]\Bigg\}+\sum_{i=1}^{k}\left(\varphi_i\left(\overline{x}\left(t_i\right)\right)+g_i\right)$$
$$+\sum_{i=0}^{k-1}{}_{t_i}I_{q_i}^{\alpha_i}\left(f\left(s,\overline{x}\right)+g\left(s\right)\right)\left(t_{i+1}\right)+{}_{t_k}I_{q_k}^{\alpha_k}\left(f\left(t,\overline{x}\right)+g\left(t\right)\right) \tag{10}$$

Now, we define the operator $\tilde{\mathcal{G}}$ as following

$$\tilde{\mathcal{G}}x\left(t\right)=\frac{1}{\Lambda}\Bigg\{\sum_{i=0}^{m}\left(\mu\,{}_{t_i}I_{q_i}^{\alpha_i+\beta_i}f\left(s,x\right)\left(t_{i+1}\right)-\eta_2\,{}_{t_i}I_{q_i}^{\alpha_i}f\left(s,x\right)\left(t_{i+1}\right)\right)$$
$$+\sum_{i=1}^{m}\left[\mu\left(\sum_{j=1}^{i}\varphi_j\left(x\left(t_j\right)\right)+\sum_{j=0}^{i-1}{}_{t_j}I_{q_j}^{\alpha_j}f\left(s,x\right)\left(t_{j+1}\right)\right)\Omega_{\beta_i}-\eta_2\varphi_i\left(x\left(t_i\right)\right)\right]\Bigg\} \tag{11}$$
$$+\sum_{i=1}^{k}\varphi_i\left(x\left(t_i\right)\right)+\sum_{i=0}^{k-1}{}_{t_i}I_{q_i}^{\alpha_i}f\left(s,x\right)\left(t_{i+1}\right)+{}_{t_k}I_{q_k}^{\alpha_k}f\left(s,x\right)\left(t\right)+G\left(t\right)$$
$$=\mathcal{G}x+G\left(t\right).$$

where

$$G\left(t\right)=\frac{1}{\Lambda}\Bigg\{\sum_{i=0}^{m}\left(\mu\,{}_{t_i}I_{q_i}^{\alpha_i+\beta_i}g\left(t_{i+1}\right)-\eta_2\,{}_{t_i}I_{q_i}^{\alpha_i}g\left(t_{i+1}\right)\right)$$
$$+\sum_{i=1}^{m}\left[\mu\left(\sum_{j=1}^{i}g_j+\sum_{j=0}^{i-1}{}_{t_j}I_{q_j}^{\alpha_j}g\left(t_{j+1}\right)\right)\Omega_{\beta_i}-\eta_2 g_i\right]\Bigg\} \tag{12}$$
$$+\sum_{i=1}^{k}g_i+\sum_{i=0}^{k-1}{}_{t_i}I_{q_i}^{\alpha_i}g\left(t_{i+1}\right)+{}_{t_k}I_{q_k}^{\alpha_k}g\left(t\right).$$

Note that

$$\left\|\tilde{\mathcal{G}}x-\tilde{\mathcal{G}}y\right\|=\left\|\mathcal{G}x-\mathcal{G}y\right\|.$$

Then the existence of a solution of (1) implies the existence of a solution to (9), it follows from Theorem 1 that $\tilde{\mathcal{G}}$ is a contraction. Thus there is a unique fixed point $\overline{x}$ of $\tilde{\mathcal{G}}$, and respectively $\tilde{x}$ of $\mathcal{G}$.

Since $t\in\left[0,T\right]$ and $\sup_{t\in J}\delta\left(t\right)\le 1$, we obtain

$$\left\|G\right\|=\max_{t\in J}\left|G\left(t\right)\right|$$
$$=\max_{t\in J}\Bigg|\frac{1}{\Lambda}\Bigg\{\sum_{i=0}^{m}\left(\mu\,{}_{t_i}I_{q_i}^{\alpha_i+\beta_i}g\left(t_{i+1}\right)-\eta_2\,{}_{t_i}I_{q_i}^{\alpha_i}g\left(t_{i+1}\right)\right)$$
$$+\sum_{i=1}^{m}\left[\mu\left(\sum_{j=1}^{i}g_j+\sum_{j=0}^{i-1}{}_{t_j}I_{q_j}^{\alpha_j}g\left(t_{j+1}\right)\right)\Omega_{\beta_i}-\eta_2 g_i\right]\Bigg\}$$
$$+\sum_{i=1}^{k}g_i+\sum_{i=0}^{k-1}{}_{t_i}I_{q_i}^{\alpha_i}g\left(t_{i+1}\right)+{}_{t_k}I_{q_k}^{\alpha_k}g\left(t\right)\Bigg|$$
$$\le\max_{t\in J}\Bigg|\frac{1}{\Lambda}\Bigg\{\sum_{i=0}^{m}\left(\mu\,{}_{t_i}I_{q_i}^{\alpha_i+\beta_i}g\left(t_{i+1}\right)-\eta_2\,{}_{t_i}I_{q_i}^{\alpha_i}g\left(t_{i+1}\right)\right)$$
$$+\sum_{i=1}^{m}\left[\mu\left(\sum_{j=1}^{i}g_j+\sum_{j=0}^{i-1}{}_{t_j}I_{q_j}^{\alpha_j}g\left(t_{j+1}\right)\right)\Omega_{\beta_i}-\eta_2 g_i\right]\Bigg\}$$

$$+\sum_{i=1}^{m} g_i + \sum_{i=0}^{m-1} {}_{t_i}I_{q_i}^{\alpha_i} g\left(t_{i+1}\right) + {}_{t_m}I_{q_m}^{\alpha_m} g\left(t\right)\Bigg|$$

$$\le \left\{ \frac{1}{\Lambda}\sum_{i=1}^{m}\left(\mu\Omega_{\alpha_i+\beta_i} + \eta_2\Omega_{\alpha_i} + \mu\epsilon i\Omega_{\beta_i} + \mu\sum_{j=0}^{i-1}\Omega_{\alpha_j}\Omega_{\beta_i} + \Omega_{\alpha_i} \right) + \frac{1}{\Lambda}\left(\mu\Omega_{\alpha_0+\beta_0} + \eta_2\Omega_{\alpha_0}\right) + m\epsilon\left(\frac{1}{\Lambda}\eta_2 + 1\right) \right\}\varepsilon. \quad (13)$$

Then, we get

$$\left\|\overline{x} - \tilde{x}\right\| = \left\|\tilde{\mathcal{G}}\overline{x} - \mathcal{G}\tilde{x}\right\| = \left\|\mathcal{G}\overline{x} - \mathcal{G}\tilde{x} + G(t)\right\| \le \left\|\mathcal{G}\overline{x} - \mathcal{G}\tilde{x}\right\| + \left\|G\right\|$$

$$\le \Delta\left\|\overline{x} - \tilde{x}\right\| + \left\{ \frac{1}{\Lambda}\sum_{i=0}^{m}\left(\mu\Omega_{\alpha_i+\beta_i} + \eta_2\Omega_{\alpha_i} + \mu\epsilon i\Omega_{\beta_i} + \mu\sum_{j=0}^{i-1}\Omega_{\alpha_j}\Omega_{\beta_i} + \Omega_{\alpha_i} \right) + \frac{1}{\Lambda}\left(\mu\Omega_{\alpha_0+\beta_0} + \eta_2\Omega_{\alpha_0}\right) + m\epsilon\left(\frac{1}{\Lambda}\eta_2 + 1\right) \right\}\varepsilon. \quad (14)$$

By condition (H_3), we have

$$\left\|\overline{x} - \tilde{x}\right\| \le \left(1-\Delta\right)^{-1}\left\{ \frac{1}{\Lambda}\sum_{i=0}^{m}\left(\mu\Omega_{\alpha_i+\beta_i} + \eta_2\Omega_{\alpha_i} + \mu\epsilon i\Omega_{\beta_i} + \mu\sum_{j=0}^{i-1}\Omega_{\alpha_j}\Omega_{\beta_i} + \Omega_{\alpha_i} \right) + m\epsilon\left(\frac{1}{\Lambda}\eta_2 + 1\right) \right\}\varepsilon. \quad (15)$$

Let $A_f = \left(1-\Delta\right)^{-1}\left\{ \frac{1}{\Lambda}\sum_{i=0}^{m}\left(\mu\Omega_{\alpha_i+\beta_i} + \eta_2\Omega_{\alpha_i} + \mu\epsilon i\Omega_{\beta_i} + \mu\sum_{j=0}^{i-1}\Omega_{\alpha_j}\Omega_{\beta_i} + \Omega_{\alpha_i} \right) + m\epsilon\left(\frac{1}{\Lambda}\eta_2 + 1\right) \right\}$, then

$\left\|\overline{x} - \tilde{x}\right\| \le A_f\varepsilon.$

This completes the proof.

Remark 1 Note that (1) has a very general form, as special instances results from (1), when, $\eta_1 = \eta_2 = 1, \mu = 0$, (1) reduces to the antiperiodic boundary value problem of the impulsive fractional q-difference equation:

$$\begin{cases} {}_{t_k}^{c}D_{q_k}^{\alpha_k} x(t) = f\left(t, x(t)\right), t \in J_k \subseteq J = [0,T], t \ne t_k, \\ \Delta x(t_k) = x(t_k^+) - x(t_k) = \varphi_k\left(x(t_k)\right), k = 1,2,\cdots,m, \\ x(0) + x(T) = 0. \end{cases}$$

4. Example

Consider the following boundary value problem:

$$\begin{cases} {}_{t_k}^{c}D_{\frac{3k+1}{4k+3}}^{\frac{k+1}{3k+2}} x(t) = \frac{\sin^2 t}{t^2+50}\frac{2\left|x(t)\right|}{1+\left|x(t)\right|} + \frac{3t}{4}, t \in \left[0, \frac{3}{2}\right] \setminus \{t_1, t_2\}, \\ \Delta x(t_k) = \frac{1}{200k}\frac{x^2(t_k) + 2\left|x(t_k)\right|}{1+\left|x(t_k)\right|} + \frac{k}{5}, t_k = \frac{k}{2}, k = 1,2, \\ \frac{8}{3}x(0) + \frac{1}{6}x\left(\frac{3}{2}\right) = \frac{1}{2}\sum_{k=0}^{2} {}_{t_k}I_{\frac{3k+1}{4k+3}}^{\frac{k+1}{k^2+2}} x(t_{k+1}). \end{cases} \quad (16)$$

Corresponding to boundary value problem (1), one see that $\alpha_k = \frac{k+1}{3k+2}$, $\beta_k = \frac{k+1}{k^2+2}$, $q_k = \frac{3k+1}{4k+3}$, $t_k = \frac{k}{2}$, $f(t,x) = \frac{\sin^2 t}{t^2+50}\frac{2|x(t)|}{1+|x(t)|} + \frac{3}{4}$, $\varphi_k(x(t_k)) = \frac{1}{200k}\frac{x^2(t_k)+2|x(t_k)|}{1+|x(t_k)|}$. Through a simple calculation, we get

$$|f(t,x) - f(t,y)| \le \frac{\sin^2 t}{t^2+25}|x-y|, M(t) = \frac{\sin^2 t}{t^2+25} \le \frac{1}{25} = M,$$

$$|\varphi_k(x) - \varphi_k(y)| \le \frac{1}{200k}|x-y| \le \frac{1}{200}|x-y|, L = \frac{1}{200},$$

$$\Lambda \doteq 1.7875 > 0,\ \Delta \doteq 0.4873 < 1.$$

From Theorem 1, the problem (16) has a unique solution x on $\left[0, \frac{3}{2}\right]$. Furthermore, the solution x is Hyers-Ulam stable with respect to the following system

$$\begin{cases} \left| {}_{t_k}^{c}D_{\frac{3k+1}{4k+3}}^{\frac{k+1}{3k+2}} x(t) - \frac{\sin^2 t}{t^2+50}\frac{2|x(t)|}{1+|x(t)|} - \frac{3t}{4} \right| \le \delta(t)\varepsilon, t \in \left[0, \frac{3}{2}\right] \backslash \{t_1, t_2\}, \\ \left| \Delta x(t_k) - \frac{1}{200k}\frac{x^2(t_k)+2|x(t_k)|}{1+|x(t_k)|} - \frac{k}{5} \right| \le \epsilon\varepsilon, t_k = \frac{k}{2}, k = 1, 2, \\ \frac{8}{3}x(0) + \frac{1}{6}x\left(\frac{3}{2}\right) = \frac{1}{2}\sum_{k=0}^{2} {}_{t_k}I_{\frac{3k+1}{4k+3}}^{\frac{k+1}{k^2+2}} x(t_{k+1}), \end{cases} \tag{17}$$

where $\epsilon > 0, \varepsilon > 0$, $\sup_{t \in \left[0, \frac{3}{2}\right]} \delta(t) < 1$.

5. Conclusion

In this paper, we study the existence and Hyers-Ulam stability of solutions for impulsive fractional q-difference equation. We obtain some results as following: 1) Using the q-shifting operator, the results of existence of solutions for impulsive fractional q-difference equation with q-integral boundary conditions are obtained. 2) The Hyers-Ulam stability of the nonlinear impulsive fractional q-difference equations was obtained.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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