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Imaginary Whittaker Modules of the Twisted Affine Nappi-Witten Lie Algebra $\widetilde{nw}[\theta]$

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Abstract

The Nappi-Witten Lie algebra was first introduced by C. Nappi and E. Witten in the study of Wess-Zumino-Novikov-Witten (WZNW) models. They showed that the WZNW model (NW model) based on a central extension of the two-dimensional Euclidean group describes the homogeneous four-dimensional space-time corresponding to a gravitational plane wave. The associated Lie algebra is neither abelian nor semisimple. Recently K. Christodoulopoulou studied the irreducible Whittaker modules for finite- and infinite-dimensional Heisenberg algebras and for the Lie algebra obtained by adjoining a degree derivation to an infinite-dimensional Heisenberg algebra, and used these modules to construct a new class of modules for non-twisted affine algebras, which are called imaginary Whittaker modules. In this paper, imaginary Whittaker modules of the twisted affine Nappi-Witten Lie algebra are constructed based on Whittaker modules of Heisenberg algebras. It is proved that the imaginary Whittaker module with the center acting as a non-zero scalar is irreducible.

Keywords

Twisted Affine Nappi-Witten Lie Algebras, Heisenberg Algebras, Imaginary Whittaker Modules

1. Introduction

The conform field theory (CFT) plays an important role in mathematics and physics. Current algebra [1] [2] has proved to be a valuable tool in understanding CFT and String Theory. All the CFTs we know so far can be constructed one way or another from current algebras. The simplest is the WZW models [3] [4], which realize current algebra as their full symmetry. For obvious reasons, the first type of algebras to be analysed was compact ones, used for compactification purposes in String Theory. Later on, non-compact algebras (of the type SL(N, R), SU(M, N) and SO(M, N)) and their cosets have been considered [5] [6] [7] in order to describe curved Minkowski signature spacetimes. Only recently did current algebras of the non-semisimple type receive some attention [8]. The Nappi-Witten model is a WZW model based on a non-semisimple group. It was discovered by C. Nappi and E. Witten [8] that the WZW model based on the Heisenberg group coincides with the σ-model of the maximally symmetric gravitational wave in four dimensions. The corresponding Lie algebra is called the Nappi-Witten Lie algebra nw, which is neither abelian nor semisimple. More results on NW model were presented in [9] [10] [11] [12].

The Lie algebra nw is a four-dimensional vector space over $\mathbb{C}$ with generators $\{P^+, P^-, J, T\}$ and the following Lie bracket:

$$[P^+, P^-] = T,\ [J, P^+] = P^+,\ [J, P^-] = -P^-,\ [T, \cdot] = 0.$$

There is a non-degenerate invariant symmetric bilinear form $(\ ,\ )$ on nw defined by

$$(P^+, P^-) = 1,\quad (T, J) = 1,\ \text{otherwise}, (\ ,\ ) = 0.$$

Just as the non-twisted affine Kac-Moody Lie algebras given in [13], the non-twisted affine Nappi-Witten Lie algebra is defined as

$$\widetilde{nw} = nw \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$$

with the bracket defined as follows:

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m(x, y)\delta_{m+n,0}K,$$

$$[\widetilde{nw}, K] = 0,\quad [D, x \otimes t^m] = mx \otimes t^m$$

for $x, y \in nw$ and $m, n \in \mathbb{Z}$.

There exist Lie algebra automorphisms θ of nw and $\tilde{\theta}$ of $\widetilde{nw}$:

$$\theta(P^+) = -P^+, \theta(P^-) = -P^-, \theta(T) = T, \theta(J) = J,$$

$$\tilde{\theta}(x \otimes t^n + \lambda K + \mu D) = (-1)^n \theta(x) \otimes t^n + \lambda K + \mu D,$$

for $n \in \mathbb{Z}, x \in nw$, and $\lambda, \mu \in \mathbb{C}$. The twisted affine Nappi-Witten Lie algebra is defined as follows:

$$\widetilde{nw}[\theta] = \{v \in \widetilde{nw} \mid \tilde{\theta}(v) = v\}$$
$$= \left(\sum_{n\in\mathbb{Z}} (\mathbb{C}T + \mathbb{C}J) \otimes \mathbb{C}t^{2n}\right) \oplus \left(\sum_{n\in\mathbb{Z}} (\mathbb{C}P^+ + \mathbb{C}P^-) \otimes \mathbb{C}t^{2n+1}\right) \oplus \mathbb{C}K \oplus \mathbb{C}D.$$

The representation theory for the non-twisted affine Nappi-Witten Lie algebra has been well studied in [14]. The irreducible restricted modules for the non-twisted affine Nappi-Witten Lie algebra with some natural conditions have been classified and the extension of the vertex operator algebra $V_{\hat{H}_4}(l, 0)$ by the even lattice L has been considered in [15]. Verma modules and vertex operator representations for the twisted affine Nappi-Witten Lie algebra have also been investigated in [16]. Recently K. Christodoulopoulou defined Whittaker modules for Heisenberg algebras and used these modules to construct a new class of modules for non-twisted affine algebras (imaginary Whittaker modules) [17]. [18] studied virtual Whittaker modules of the non-twisted affine Nappi-Witten Lie algebra. Inspired by the works mentioned above, the aim of the present paper is to give a characterization of the imaginary Whittaker modules for the twisted affine Nappi-Witten Lie algebra $\widetilde{nw}[\theta]$.

Here is a brief outline of Section 2. First, we obtain a Heisenberg subalgebra $\tilde{H}$ by the decomposition of the Lie algebra $\widetilde{nw}[\theta]$. Second, we construct the imaginary Whittaker module $W_{\psi,\varphi}$ of $\widetilde{nw}[\theta]$ by the Whittaker module of $\tilde{H}$. Finally, we give the properties of the module $W_{\psi,\varphi}$ (see Propositions 2.2 and 2.3) and prove that $W_{\psi,\varphi}$ with K acting as a non-zero scalar is irreducible (see Theorem 2.4).

Throughout the paper, denote by $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Z}_+$ the sets of the complex numbers, the non-zero complex numbers, the non-negative integers, the integers and the positive integers, respectively. All linear spaces and algebras in this paper are over $\mathbb{C}$ unless indicated otherwise.

2. The Imaginary Whittaker Modules

In the following, for $x \in nw$ and $n \in \mathbb{Z}$, we will denote $x \otimes t^n$ by $x(n)$. It is clear that $\widetilde{nw}[\theta]$ has the following decomposition

$$\widetilde{nw}[\theta] = \widetilde{nw}[\theta]^+ \oplus \left(\tilde{H} \oplus \dot{\eta}\right) \oplus \widetilde{nw}[\theta]^-,$$

where

$$\widetilde{nw}[\theta]^+ = \mathrm{Span}_{\mathbb{C}}\left\{P^+(n) \mid n \in 2\mathbb{Z}+1\right\}, \quad \widetilde{nw}[\theta]^- = \mathrm{Span}_{\mathbb{C}}\left\{P^-(n) \mid n \in 2\mathbb{Z}+1\right\}$$

$$\tilde{H} = \mathrm{Span}_{\mathbb{C}}\left\{T(m), J(m), K, D \mid m \in 2\mathbb{Z}\setminus\{0\}\right\}, \quad \dot{\eta} = \mathrm{Span}_{\mathbb{C}}\left\{T(0), J(0)\right\}.$$

We first review the Whittaker modules of the Heisenberg algebra $\tilde{H}$ in [17].

Let $\tilde{H} = \bigoplus_{i \in 2\mathbb{Z}} \tilde{H}_i$, where

$$\tilde{H}_i = \mathrm{Span}_{\mathbb{C}}\left\{\frac{1}{i}T(i), \frac{1}{i}J(i)\right\}, \quad \tilde{H}_{-i} = \mathrm{Span}_{\mathbb{C}}\left\{J(-i), T(-i)\right\}, i \in 2\mathbb{Z}_+,$$

$$\tilde{H}_0 = \mathbb{C}K \oplus \mathbb{C}D, \quad \tilde{H}^{\pm} = \bigoplus_{i \in 2\mathbb{Z}_+} \tilde{H}_{\pm i}.$$

Thus $\tilde{H}$ is an infinite-dimensional Heisenberg subalgebra of $\widetilde{nw}[\theta]$.

Assume that $\lambda \in (\mathbb{C}K)^*$ and $\psi : U\left(\tilde{H}^+\right) \to \mathbb{C}$ is an algebra homomorphism such that $\psi|_{\tilde{H}^+} \neq 0$. Set $\tilde{\mathfrak{b}} = \tilde{H}^+ \oplus \mathbb{C}K$. Let $\mathbb{C}_{\psi,\lambda(K)} = \mathbb{C}\tilde{\tilde{\omega}}$ be a one-dimensional vector space viewed as a $\tilde{\mathfrak{b}}$-module by

$$K\tilde{\tilde{\omega}} = \lambda(K)\tilde{\tilde{\omega}}, \quad x\tilde{\tilde{\omega}} = \psi(x)\tilde{\tilde{\omega}}, \text{ for all } x \in U\left(\tilde{H}^+\right).$$

Set

$$\tilde{M}_{\psi,\lambda(K)} = U\left(\tilde{H}\right) \bigotimes_{U(\tilde{\mathfrak{b}})} \mathbb{C}_{\psi,\lambda(K)}, \tilde{\omega} = 1 \otimes \tilde{\tilde{\omega}}.$$

Define an action of $U\left(\tilde{H}\right)$ on $\tilde{M}_{\psi,\lambda(K)}$ by left multiplication. Then $\tilde{M}_{\psi,\lambda(K)} = U\left(\tilde{H}\right)\tilde{\omega}$ and $\tilde{M}_{\psi,\lambda(K)}$ is a Whittaker module of type ψ for $\tilde{H}$.

Lemma 2.1 ([17]) *Let* $\lambda(K)\neq 0$. *If* $\psi|_{\tilde{H}_i}\neq 0$ *for infinitely many* $i\in 2\mathbb{Z}_+$, *then* $\tilde{M}_{\psi,\lambda(K)}$ *is irreducible as a* $U(\tilde{H})$*-module.*

In the following we define the imaginary Whittaker module of $\widetilde{nw}[\theta]$ according to [17]. We will assume that $\varphi\in(\dot{\eta}\oplus\mathbb{C}K)^*$ is such that $\varphi|_{\mathbb{C}K}=\lambda$ and $\varphi(K)\neq 0$. Let $\psi:U(\tilde{H}^+)\to\mathbb{C}$ is an algebra homomorphism such that $\psi|_{\tilde{H}_i}\neq 0$ for infinitely many $i\in 2\mathbb{Z}_+$.

Set $\mathfrak{p}=\widetilde{nw}[\theta]^+\oplus(\tilde{H}\oplus\dot{\eta})$. $\mathfrak{p}$ is a parabolic subalgebra of $\widetilde{nw}[\theta]$. It is obvious that $[\tilde{H},\dot{\eta}]=0$ and $\widetilde{nw}[\theta]^+$ is an ideal of $\mathfrak{p}$. Let $\tilde{\omega}\in\tilde{M}_{\psi,\varphi(K)}$ be a Whittaker vector of type ψ. Define a $U(\mathfrak{p})$-module structure on $\tilde{M}_{\psi,\varphi(K)}$ by letting

$$hv=\varphi(h)v,\ \widetilde{nw}[\theta]^+v=0 \text{ for all } h\in\dot{\eta}\oplus\mathbb{C}K, \text{ any } v\in\tilde{M}_{\psi,\varphi(K)}.$$

Set

$$W_{\psi,\varphi}=U(\widetilde{nw}[\theta])\underset{U(\mathfrak{p})}{\otimes}\tilde{M}_{\psi,\varphi(K)},\quad \omega=1\otimes\tilde{\omega}.$$

Define an action of $U(\widetilde{nw}[\theta])$ on $W_{\psi,\varphi}$ by left multiplication. Then $W_{\psi,\varphi}$ is called an imaginary Whittaker module of type (ψ,φ) for $\widetilde{nw}[\theta]$.

We assume that $\alpha\in(\dot{\eta}\oplus\mathbb{C}K)^*$ is such that $\alpha(T(0))=0$, $\alpha(J(0))=1$, $\alpha(K)=0$. Let $\chi\in\mathbb{N}\alpha$. Set

$$U(\widetilde{nw}[\theta]^-)^{-\chi}=\{v\in U(\widetilde{nw}[\theta]^-)\mid[h,v]=-\chi(h)v \text{ for all } h\in\dot{\eta}\oplus\mathbb{C}K\}.$$

It is easy to see that $U(\widetilde{nw}[\theta]^-)^0\cong\mathbb{C}$.

$U(\widetilde{nw}[\theta]^-)^{-r\alpha}\cong\mathrm{Span}_{\mathbb{C}}\left\{\prod_{i=1}^{r}P^-(n_i)\mid n_i\in 2\mathbb{Z}+1\right\}$,

$U(\widetilde{nw}[\theta]^-)=\underset{\chi\in\mathbb{N}\alpha}{\oplus}U(\widetilde{nw}[\theta]^-)^{-\chi}$. Set

$$W_{\psi,\varphi}^{\rho}=\{\omega\in W_{\psi,\varphi}\mid h\omega=\rho(h)\omega \text{ for all } h\in\dot{\eta}\oplus\mathbb{C}K\},$$

for any $\rho\in(\dot{\eta}\oplus\mathbb{C}K)^*$.

Proposition 2.2

1) $W_{\psi,\varphi}$ *is a free* $U(\widetilde{nw}[\theta]^-)$*-module, and*

$$W_{\psi,\varphi}\cong U(\widetilde{nw}[\theta]^-)\otimes_{\mathbb{C}}\tilde{M}_{\psi,\varphi(K)}.$$

2) $\tilde{M}_{\psi,\varphi(K)}\cong U(\mathfrak{p})\omega$ *as* $\mathfrak{p}$*-modules and we can view* $\tilde{M}_{\psi,\varphi(K)}$ *as the* $\mathfrak{p}$*-submodule* $U(\mathfrak{p})\omega$ of $W_{\psi,\varphi}$ *under this isomorphism.*

3) $W_{\psi,\varphi}=\underset{\chi\in\mathbb{N}\alpha}{\oplus}W_{\psi,\varphi}^{\varphi-\chi}$ *and as modules for* $\dot{\eta}\oplus\mathbb{C}K$,

$$W_{\psi,\varphi}^{\varphi-\chi}\cong U(\widetilde{nw}[\theta]^-)^{-\chi}\otimes_{\mathbb{C}}\tilde{M}_{\psi,\varphi(K)}.$$

In particular, $W_{\psi,\varphi}^{\varphi}\cong\tilde{M}_{\psi,\varphi(K)}$.

Proof. 1) Since $\widetilde{nw}[\theta]=\widetilde{nw}[\theta]^-\oplus\mathfrak{p}$, the PBW Theorem implies that $U(\widetilde{nw}[\theta])=U(\widetilde{nw}[\theta]^-)\otimes_{\mathbb{C}}U(\mathfrak{p})$ and thus $W_{\psi,\varphi}\cong U(\widetilde{nw}[\theta]^-)\otimes_{\mathbb{C}}\tilde{M}_{\psi,\varphi(K)}$ as vector space over $\mathbb{C}$. So the map $g: U\left(\widetilde{nw}[\theta]^-\right)\otimes_{\mathbb{C}} \tilde{M}_{\psi,\varphi(K)} \to W_{\psi,\varphi}$ defined by $(v,\tau)\to v\tau$ is an isomorphism of left $U\left(\widetilde{nw}[\theta]^-\right)$-modules.

2) The map $u\to 1\otimes u$ defines a $\mathfrak{p}$-isomorphism of $\tilde{M}_{\psi,\varphi(K)}$ onto the $\mathfrak{p}$-submodule $U(\mathfrak{p})\omega$ of $W_{\psi,\varphi}$.

3) $\dot{\eta}\oplus\mathbb{C}K$ acts semisimply on $U\left(\widetilde{nw}[\theta]^-\right)$ via the adjoint action and $U\left(\widetilde{nw}[\theta]^-\right)=\oplus_{\chi\in\mathbb{N}\alpha} U\left(\widetilde{nw}[\theta]^-\right)^{-\chi}$. It is clear that the isomorphism g of (1) maps $U\left(\widetilde{nw}[\theta]^-\right)^{-\chi}\otimes_{\mathbb{C}}\tilde{M}_{\psi,\varphi(K)}$ isomorphically to $W_{\psi,\varphi}^{\varphi-\chi}$ for every $\chi\in\mathbb{N}\alpha$. In particular, if $\chi=0$, then $W_{\psi,\varphi}^{\varphi}=\tilde{M}_{\psi,\varphi(K)}$ because $U\left(\widetilde{nw}[\theta]^-\right)^0\cong\mathbb{C}$. Thus (3) holds.

The following proposition is evident for weight modules.

Proposition 2.3 *Any* $U\left(\widetilde{nw}[\theta]\right)$*-submodule* V *of* $W_{\psi,\varphi}$ *has a weight space decomposition*

$$V=\oplus_{\chi\in\mathbb{N}\alpha} V\cap W_{\psi,\varphi}^{\varphi-\chi}$$

relative to $\dot{\eta}\oplus\mathbb{C}K$.

Proof. Set $\varphi-\chi=\rho$. Then by Proposition 2.2 (3), we have

$$W_{\psi,\varphi}=\oplus_{\rho\in(\dot{\eta}\oplus\mathbb{C}K)^*} W_{\psi,\varphi}^{\rho},\quad W_{\psi,\varphi}^{\rho}=\left\{\omega\in W_{\psi,\varphi}\mid h\omega=\rho(h)\omega \text{ for all } h\in\dot{\eta}\oplus\mathbb{C}K\right\}.$$

Any $v\in W_{\psi,\varphi}$ can be written in the form $v=\sum_{j=1}^{n} v_j$, where $v_j\in W_{\psi,\varphi}^{\rho_j}$, and there exists $h\in\dot{\eta}\oplus\mathbb{C}K$ such that $\rho_j(h)(j=1,2,\cdots,n)$ are distinct. We have for $v\in V$,

$$h^l(v)=\sum_{j=1}^{n}\rho_j(h)^l v_j\in V\left(l=0,1,2,\cdots,n-1\right).$$

This is a system of linear equations with a nondegenerate matrix. Hence all v_j lie in *V*.

We are now in a position to give the main result of this paper as follows.

Theorem 2.4 *Let* $\varphi\in(\dot{\eta}\oplus\mathbb{C}K)^*$ *and* $\psi: U\left(\tilde{H}^+\right)\to\mathbb{C}$ *be an algebra homomorphism such that* $\varphi(K)\neq 0$ *and* $\psi|_{\tilde{H}_i}\neq 0$ *for infinitely many* $i\in 2\mathbb{Z}_+$. *Then* $W_{\psi,\varphi}$ *is irreducible as a* $U\left(\widetilde{nw}[\theta]\right)$*-module.*

Proof. Let $0\neq V$ be a $U\left(\widetilde{nw}[\theta]\right)$-submodule of $W_{\psi,\varphi}$. We next show that $V=W_{\psi,\varphi}$. By Proposition 2.2 (2), we can identify $\tilde{M}_{\psi,\varphi(K)}$ with $U(\mathfrak{p})\omega$. Since $\tilde{M}_{\psi,\varphi(K)}=U\left(\tilde{H}\right)\omega$ is irreducible as a $U\left(\tilde{H}\right)$-module and $W_{\psi,\varphi}=U\left(\widetilde{nw}[\theta]\right)\omega$, it suffices to show that $V\cap\tilde{M}_{\psi,\varphi(K)}\neq 0$.

By Proposition 2.3, for some $\chi\in\mathbb{N}\alpha$, we have $V\cap W_{\psi,\varphi}^{\varphi-\chi}\neq 0$. Let $\chi=r\alpha$ and $0\neq v\in V\cap W_{\psi,\varphi}^{\varphi-r\alpha}$. We assume

$$v=\sum_{i\in I}\xi_i P^-\left(n_1^i\right)\cdots P^-\left(n_r^i\right)T\left(-s_1^i\right)\cdots T\left(-s_{p_i}^i\right)J\left(-t_1^i\right)\cdots J\left(-t_{q_i}^i\right)D^{k_i}\omega,$$

where $\xi_i\in\mathbb{C}^*$, $n_1^i\geq\cdots\geq n_r^i$, $s_1^i\leq\cdots\leq s_{p_i}^i$, $t_1^i\leq\cdots\leq t_{q_i}^i$, I is a finite index set,

$n_j^i \in 2\mathbb{Z}+1$, $s_j^i, t_j^i \in 2\mathbb{Z}_+$, $r, p_i, q_i, k_i \in \mathbb{N}$.

Claim There exists $n \in 2\mathbb{N}+1$ such that $P^+(-n)v \neq 0$.

Set

$$\overline{n}_r = \min\left\{n_r^i \mid i \in I\right\},$$

$$n = \max\left\{n_1^i \mid i \in I\right\} + \max\left\{\sum_{j=1}^{p_i} s_j^i \mid i \in I\right\} + 2\left|\max\left\{n_1^i \mid i \in I\right\}\right| \in 2\mathbb{N}+1.$$

It is clear that

$$\overline{n}_r - n \leq -\left(n_1^i - n_r^i\right) - \sum_{j=1}^{p_i} s_j^i - 2\left|\max\left\{n_1^i \mid i \in I\right\}\right| < 0.$$

Moreover,

$$\begin{aligned}\overline{n}_r - n + s_{p_i}^i &\leq n_r^i - n_1^i - \sum_{j=1}^{p_i} s_j^i - 2\left|\max\left\{n_1^i \mid i \in I\right\}\right| + s_{p_i}^i \\ &= -\left(n_1^i - n_r^i\right) - \sum_{j=1}^{p_i-1} s_j^i - 2\left|\max\left\{n_1^i \mid i \in I\right\}\right| < 0,\end{aligned}$$

for all $i \in I$. It is easy to check that, for each $i \in I$ such that $n_r^i = \overline{n}_r$, the coefficient of the basis element

$$P^-\left(n_1^i\right)\cdots P^-\left(n_{r-1}^i\right)T\left(\overline{n}_r - n\right)T\left(-s_1^i\right)\cdots T\left(-s_{p_i}^i\right)J\left(-t_1^i\right)\cdots J\left(-t_{q_i}^i\right)D^{k_i}\omega$$

in $P^+(-n)v$ is $\xi_i \times \#\left\{j \mid 1 \leq j \leq r, n_j^i = \overline{n}_r\right\} \neq 0$. Thus $P^+(-n)v \neq 0$.

Since $0 \neq P^+(-n)v \in V \cap W_{\psi,\varphi}^{\varphi-(r-1)\alpha}$, we can use induction on r and conclude that there exists $u \in U\left(\widetilde{nw}[\theta]\right)$ such that $0 \neq uP^+(-n)v \in V \cap \tilde{M}_{\psi,\varphi(K)}$. The theorem is proved.

3. Conclusion

We construct the imaginary Whittaker module $W_{\psi,\varphi}$ of the twisted affine Nappi-Witten Lie algebra $\widetilde{nw}[\theta]$ by its Heisenberg subalgebra $\tilde{H}$. We study the structure of the module $W_{\psi,\varphi}$ and prove that $W_{\psi,\varphi}$ with the center acting as a non-zero scalar is irreducible. Our future work is to determine the maximal submodule of $W_{\psi,\varphi}$ when it is reducible.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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