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Iterated Commutators for Multilinear Singular Integral Operators on Morrey Space with Non-Doubling Measures

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Abstract

Let μ be a non-negative Radon measure on $\mathbb{R}^d$ which only satisfies the following growth condition that there exists a positive constant C such that $\mu(B(x,r)) \le Cr^n$ for all $x \in \mathbb{R}^d$, $r > 0$ and some fixed $n \in (0,d]$. This paper is interested in the properties of the iterated commutators of multilinear singular integral operators on Morrey spaces $M_q^p(\mu)$. Precisely speaking, we show that the iterated commutators generated by multilinear singular integrals operators $(T_m)_{\Pi\vec{b}}$ are bounded from $M_{q_1}^{p_1}(\mu)\times\cdots\times M_{q_m}^{p_m}(\mu)$ to $M_q^p(\mu)$ where $\vec{b} \in RBMO^m$ (Regular Bounded Mean Oscillation space) and $1 < q_j \le p_j < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$ and $1/q = 1/q_1 + \cdots + 1/q_m$.

Keywords

Non-Doubling Measures, Morrey Space, Multilinear Singular Integral Operators, RBMO, Commutator

1. Introduction

Let μ be a positive Radon measures on $\mathbb{R}^d$ satisfying only the growth condition, that is, there exists a constant $C > 0$ and $n \in (0,d]$ such that

$$\mu(Q) \le Cl(Q)^n \tag{1}$$

for any cube $Q \subset \mathbb{R}^d$ with sides parallel to the coordinate axes. $Q(x,l(Q))$ will be the cube centered at x with side length $l(Q)$. For $r > 0$, rQ will denote the cube with the same center as Q and with $l(rQ) = rl(Q)$. The set of all cubes $Q \subset \mathbb{R}^d$, satisfying $\mu(Q) > 0$ is denoted by $\mathfrak{L}(\mu)$. In this note, we do not assume that μ is doubling.

Nazarov, Treil and Volberg developed the theory of the singular integrals for the measures with growth condition to investigate the analytic capacity on the complex plane [1] [2]. Tolsa showed that the analytic capacity is subadditive and that it is bi-Lipschitz invariant [3] [4] and defined for the growth measures *RBMO* (regular bounded mean oscillation) space, the Hardy space $H_1(\mu)$ and the Littlewood-Paley decomposition [5] [6]. He also gave his $H_1(\mu)$ space in terms of the grand maximal operator [7]. Recently many people paid attention to the measure with growth condition because of recovering the Calderón-Zygmund theory and solving the long-standing open Painlevé problem.

The boundedness of fractional integral operators on the classical Morrey spaces was studied by Adams [8], Chiarenza *et al.* [9]. In [9], by establishing a pointwise estimate of fractional integrals in terms of the Hardy-Littlewood maximal function, they showed the boundedness of fractional integral operators on the Morrey spaces. In 2005, Sawano and Tanaka [10] gave a natural definition of Morrey spaces for Radon measures which might be non-doubling but satisfied the growth condition, and they investigated the boundedness in these spaces of some classical operators in harmonic analysis. Later on, Sawano [11] defined the generalized Morrey spaces on $\mathbb{R}^n$ for non-doubling measure and showed the properties of maximal operators, fractional integral operators and singular operators in this space.

A classical result of commutator is due to Coifman, Rochberg and Weiss [12], if $b \in BMO$ and T is a Calderón-Zygmund operator, then the commutator $[b,T]$ is bounded on L^p spaces for $1 < p < \infty$. The same result for the multilinear commutator was obtained by Pérez and Trujillo-Gonzalez [13]. Tolsa [5] developed the theory of Calderón-Zygmund operators and their commutators with *RBMO* functions in the setting of non-doubling measures. Hu, Meng and Yang [14] considered the multilinear commutator on Lebesgue spaces with non-doubling measures. Chen and Sawyer [15] modified the definition of *RBMO* to investigate the commutators of the potential operators and *RBMO* functions.

In the last decade, multilinear singular integrals of Calderón-Zygmund type have attracted great attentions. Some interesting results refer to [16] [17] [18] [19] [20] in the text of Lebesgue measures. It points out that Perez and Pradolini [21] introduced a said iterated commutators generated by the multilinear singular integral operators with Calderón-Zygmund type and vector function $\vec{b} \in RBMO^m$ and obtained the boundedness from $L^{p_1} \times \cdots \times L^{p_m}$ to L^p with $1/p = 1/p_1 + \cdots + 1/p_m$ for $1 < p_1, \cdots, p_m < \infty$ (in fact, they considered the weighted case). Xu [22] extended the result to the case of the non-doubling measures. Very recently, Tao and He [23] obtained the boundedness of the multilinear Calderón-Zygmund operators on the generalized Morrey spaces over the quasi-metric space of non-homogeneoustype. The aim of this paper is to study the iterated commutators of multilinear singular integral operators on Morrey spaces with non-doubling measures.

Before stating our result, we recall some definitions and notation. Given $\beta_d \geq 2^{d+1}$ large enough but depending only on the dimension *d*, we say that a cube $Q \subset \mathbb{R}^d$ is doubling if $\mu(2Q) \leq \beta_d \mu(Q)$. For any fixed cube $Q \subset \mathbb{R}^d$, let *N* be the smallest nonnegative integer such that $2^N Q$ is doubling. We denote this cube by $\tilde{Q}$.

For two cubes $Q \subset R$ in $\mathbb{R}^d$, we suppose

$$K_{Q,R} = 1 + \sum_{k=1}^{N_{Q,R}} \frac{\mu(2^k Q)}{l(2^k Q)^n}, \tag{2}$$

where $N_{Q,R}$ is the first positive integer *k* such that $l(2^k Q) \geq l(R)$. This was introduced by Tolsa in [5].

Let $m_Q f$ be the mean value of *f* on *Q*, namely, $m_Q f = \frac{1}{\mu(Q)} \int_Q f(x) \mathrm{d}\mu$. The regularity bounded mean oscillations function spaces were introduced by Tolsa [5].

Definition 1.1. *Let* $\eta > 1$ *be a fixed constant. We say that* $f \in L^1_{loc}(\mu)$ *is in RBMO if there exists a constant* $\mathcal{A}$ *such that*

$$\frac{1}{\mu(\eta Q)} \int_Q \left| f(y) - m_{\tilde{Q}} f \right| \mathrm{d}\mu(y) \leq \mathcal{A}, \tag{3}$$

for any cube *Q*, and

$$\left| m_Q f - m_R f \right| \leq \mathcal{A} K_{Q,R}, \tag{4}$$

for any two doubling cubes $Q \subset R$. The mininal constant $\mathcal{A}$ is the $RBMO(\mu)$ norm of *f*, and it will be denoted by $\|f\|_*$.

The definition of the Morrey space with non-doubling measure is given in the following [10].

Definition 1.2. *Let* $k > 1$ *and* $1 \leq q \leq p < \infty$, *the Morrey space* $M_q^p(k,\mu)$ *is defined as*

$$M_q^p(k,\mu) := \left\{ f \in L^q_{loc}(\mu) : \|f\|_{M_q^p(k,\mu)} < \infty \right\},$$

where

$$\|f\|_{M_q^p(k,\mu)} := \sup_{Q \in \mathfrak{L}(\mu)} \mu(kQ)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f|^q \mathrm{d}\mu \right)^{\frac{1}{q}}. \tag{5}$$

It is easy to observe that $L^p(\mu) = M_p^p(k,\mu)$, and Hölder's inequality tells us $\|f\|_{M_{q_2}^p(k,\mu)} \leq \|f\|_{M_{q_1}^p(k,\mu)}$ for all $1 \leq q_2 \leq q_1 \leq p$, then we have $L^p(\mu) = M_p^p(k,\mu) \subset M_{q_1}^p(k,\mu) \subset M_{q_2}^p(k,\mu)$. The space $M_q^p(k,\mu)$ is a Banach space with its norm $\|f\|_{M_q^p(k,\mu)}$ and the parameter $k > 1$ appearing in the definition does not affect it. The Morrey space norm reflects local regularity of *f* more precisely than the Lebesgue space norm. See [10] [11] [24] for details. We will denote $M_q^p(2,\mu)$ by $M_q^p(\mu)$.

Denoting by $\vec{f}=(f_1,f_2,\cdots,f_m)$, we consider the multilinear singular integral operator T_m as follows,

$$T_m(\vec{f})(x) = \int_{(\mathbb{R}^d)^m} K(x,y_1,\cdots,y_m) f_1(y_1)\cdots f_m(y_m)\,\mathrm{d}\mu(y_1)\cdots\mathrm{d}\mu(y_m) \text{ for } x\in\mathbb{R}^d, \tag{6}$$

whenever $f_1,\cdots,f_m$ are C^∞-functions with compact support and $x\notin\bigcap_{j=1}^m \operatorname{supp} f_j$. Moreover,

$$\left|K(y_0,y_1,\cdots,y_m)\right| \le A\left(\sum_{k,l=0}^m |y_k-y_l|\right)^{-mn} \tag{7}$$

and, for some $\epsilon>0$,

$$\left|K(y_0,y_1,\cdots,y_j,\cdots,y_m)-K(y_0,y_1,\cdots,y_j',\cdots,y_m)\right| \le \frac{A|y_j-y_j'|^\epsilon}{\left(\sum_{k,l=0}^m |y_k-y_l|\right)^{mn+\epsilon}} \tag{8}$$

provided that $0\le j\le m$ and $|y_j-y_j'|\le \frac{1}{2}\max_{0\le k\le m}|y_j-y_k|$.

Let $b_i\in RBMO(\mu)$ for $i=1,2,\cdots,m$ and let $\vec{b}=(b_1,b_2,\cdots,b_m)$, then the iterated commutator $(T_m)_{\prod\vec{b}}$ is formally defined as

$$\begin{aligned}(T_m)_{\prod\vec{b}}(\vec{f})(x) = \int_{(\mathbb{R}^d)^m} \prod_{j=1}^m \left(b_j(x)-b_j(y_j)\right) K(x,y_1,\cdots,y_m) \\ \times f_1(y_1)\cdots f_m(y_m)\,\mathrm{d}\mu(y_1)\cdots\mathrm{d}\mu(y_m).\end{aligned} \tag{9}$$

Suppose $\|\mu\|=\infty$ the main result in this paper can be stated as follow.

Theorem 1.1. *Let* $(T_m)_{\prod\vec{b}}$ *as in* (9) *and satisfying conditions* (7) *and* (8). *Let* $1<q_j\le p_j<\infty$ *with* $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$ *and* $\frac{1}{q}=\frac{1}{q_1}+\cdots+\frac{1}{q_m}$. *Suppose* $b_i\in RBMO(\mu)$ *for* $i=1,2,\cdots,m$. *If* T_m *maps* $L^1(\mu)\times\cdots\times L^1(\mu)$ *to* $L^{1/m,\infty}(\mu)$, *then the commutators* $(T_m)_{\prod\vec{b}}$ *are bounded from* $M_{q_1}^{p_1}(\mu)\times\cdots\times M_{q_m}^{p_m}(\mu)$ *to* $M_q^p(\mu)$, *that is,*

$$\left\|(T_m)_{\prod\vec{b}}(\vec{f})(x)\right\|_{M_q^p(\mu)} \le C\prod_{i=1}^m \|b_i\|_* \|f_i\|_{M_{q_i}^{p_i}(\mu)}. \tag{10}$$

More generally, denote by C_i^m the family of all subsets $\sigma=\{\sigma_1,\sigma_2,\cdots,\sigma_i\}$ of i different elements of $\{1,2,\cdots,m\}$, and let $\sigma'=\{1,2,\cdots,m\}\setminus\sigma$ and $\vec{b}_\sigma=\{b_{\sigma_1},b_{\sigma_2},\cdots,b_{\sigma_i}\}$. For any $\sigma\in C_i^m$, we define

$$\begin{aligned}(T_m)_{\prod\vec{b}_\sigma}(\vec{f})(x) = \int_{(\mathbb{R}^d)^m} \prod_{\sigma_j\in\sigma} \left(b_{\sigma_j}(x)-b_{\sigma_j}(y_{\sigma_j})\right) K(x,y_1,\cdots,y_m) \\ \times f_1(y_1)\cdots f_m(y_m)\,\mathrm{d}\mu(y_1)\cdots\mathrm{d}\mu(y_m).\end{aligned} \tag{11}$$

In case $\sigma=\{1,2,\cdots,m\}$, one sees that $(T_m)_{\prod\vec{b}_\sigma}$ is just the commutator $(T_m)_{\prod\vec{b}}$. So we have a more generalization version of the theorem as following.

Theorem 1.2. *Let* $(T_m)_{\prod\vec{b}_\sigma}$ *as in* (11) *and satisfying conditions* (7) *and* (8). *Let* $1<q_j\le p_j<\infty$ *with* $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$ *and* $\frac{1}{q}=\frac{1}{q_1}+\cdots+\frac{1}{q_m}$. *Suppose*

$b_i \in RBMO(\mu)$ *for* $i=1,2,\cdots,m$. *If* T_m *maps* $L^1(\mu)\times\cdots\times L^1(\mu)$ *to* $L^{1/m,\infty}(\mu)$, *then for all* $\sigma\subset\{1,2,\cdots,m\}$, *the commutators* $(T_m)_{\prod\vec{b}_\sigma}$ *are bounded from* $M^{p_1}_{q_1}(\mu)\times\cdots\times M^{p_m}_{q_m}(\mu)$ *to* $M^p_q(\mu)$, *that is,*

$$\left\|(T_m)_{\prod\vec{b}_\sigma}(\vec{f})(x)\right\|_{M^p_q(\mu)} \le C\prod_{j\in\sigma}\|b_j\|_*\prod_{i=1}^m\|f_i\|_{M^{p_i}_{q_i}(\mu)}. \tag{12}$$

2. Proof of Main Results

Before proving our theorem, we recall the following maximal operator,

$$M_k f(x)=\sup_{x\in Q\subset\mathfrak{L}(\mu)}\frac{1}{\mu(kQ)}\int_Q|f(y)|\mathrm{d}\mu(y). \tag{13}$$

we will use the sharp maximal estimates. Let f be a function in $L^1_{loc}(\mu)$, the sharp maximal function of f is defined by

$$M^\# f(x)=\sup_{Q\ni x}\frac{1}{\mu\left(\frac{3}{2}Q\right)}\int_Q\left|f(y)-m_{\tilde{Q}}f\right|\mathrm{d}\mu(y)+\sup_{\substack{R\supset Q\ni x\\ Q,R\text{ doubling}}}\frac{\left|m_Q f-m_R f\right|}{K_{Q,R}} \tag{14}$$

The non-centered doubling maximal operator is defined by

$$Nf(x)=\sup_{Q\ni x,\,Q\text{ doubling}}\frac{1}{\mu(Q)}\int_Q|f(y)|\mathrm{d}\mu(y). \tag{15}$$

By the Lebesgue differential theorem, it is easy to see that $|f(x)|\le Nf(x)$ for any $f\in L^1_{loc}(\mu)$ and $\mu-a.e.x\in\mathbb{R}^d$. Define the non-centered maximal operator,

$$M_{\tau,\eta}f(x)=\sup_{Q\ni x}\left\{\frac{1}{\mu(\eta Q)}\int_Q|f(y)|^\tau\mathrm{d}\mu(y)\right\}^{\frac{1}{\tau}}. \tag{16}$$

for $\eta>1$ and $\tau>1$, where the supremum is taking over all the cubes Q containing the point *x*.

To prove Theorem 1.2 is reduced to the following lemmas.

Lemma 2.1. *Let* $\tau>1, s_i>1, b_i\in RBMO(\mu)$ *and* $f_i\in L^{q_i}(\mu), i=1,2,\cdots,m$ *and* $\sigma\subset\{1,2,\cdots,m\}$. *If* T_m *maps* $L^1(\mu)\times\cdots\times L^1(\mu)$ *to* $L^{1/m,\infty}(\mu)$ *and satisfying conditions* (7) *and* (8). *Then we have*

$$\begin{aligned}&M^\#\left((T_m)_{\prod\vec{b}_\sigma}(\vec{f})\right)(x)\\&\le C\Bigg[\prod_{j\in\sigma}\|b_j\|_* M_{\tau,\frac{3}{2}}\left(T_m(\vec{f})\right)(x)\\&\quad+\sum_{\substack{\sigma_1\cup\sigma_2=\sigma\\ \sigma_1\ne\varnothing,\,\sigma_2\ne\varnothing}}\prod_{j\notin\sigma_1}\|b_j\|_* M_{\tau,\frac{3}{2}}(T_m)_{\prod\vec{b}_{\sigma_2}}(\vec{f})(x)\\&\quad+\prod_{j\in\sigma}\|b_j\|_*\prod_{j=1}^m M_{s_j,\frac{9}{8}}f_j(x)\Bigg].\end{aligned} \tag{17}$$

We postpone the proof of Lemma 2.1 after of Theorem 1.2.

Lemma 2.2. [10] *Let* $q > \tau > 1, \eta > 1$, *and* $1 < q \le p < \infty$, *then the operator* $M_{\tau,\eta}$ *is bounded on* $M_q^p(\mu)$ *and*

$$\left\| M_{\tau,\eta}(f) \right\|_{M_q^p(\mu)} \le C \|f\|_{M_q^p(\mu)}.$$

with the constant C independent of f.

Lemma 2.3. [24] *Suppose that* $1 < q \le p < \infty$, *and there exists an increasing sequence of concentric doubling cubes,* $I_0 \subset I_1 \subset \cdots \subset I_k \subset \cdots$, *such that*

$$\lim_{k\to\infty} m_{I_k}(f) = 0 \text{ and } \bigcup_{k=0}^{\infty} I_k = \mathbb{R}^d.$$

Then there exist a constant $C > 0$ independent on f such that

$$\|Nf\|_{M_q^p(\mu)} \le C \left\| M^\# \right\|_{M_q^p(\mu)}.$$

Lemma 2.4. [25] *Let* $1 < p_i < \infty$ *and* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. *If* T_m *maps* $L^1(\mu) \times \cdots \times L^1(\mu)$ *to* $L^{1/m,\infty}(\mu)$ *and satisfying conditions* (7) *and* (8). *Then there exists a constant C independent of* f_i *such that*

$$\left\| T_m(\vec{f}) \right\|_{L^p(\mu)} \le C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\mu)}.$$

Remark 2.1. *If* T_m *maps* $L^1(\mu) \times \cdots \times L^1(\mu)$ *to* $L^{1/m,\infty}(\mu)$ *and satisfying conditions* (7) *and* (8), *with* $1 < q_i \le p_i < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ *and* $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$. *From Corollary* 1.8 *in* [23], *we can easily get that* T_m *is bounded from* $M_{q_1}^{p_1}(\mu) \times \cdots \times M_{q_m}^{p_m}(\mu)$ *to* $M_q^p(\mu)$.

Proof of Theorem **1.2.** Using Lemma 2.1, Lemma 2.2, Lemma 2.3 and Lemma 2.4, we get that

$$\begin{aligned}
&\left\| (T_m)_{b_i}(\vec{f}) \right\|_{M_q^p(\mu)} \\
&\le C \left\| N\left((T_m)_{b_i}(\vec{f}) \right) \right\|_{M_q^p(\mu)} \le C \left\| M^\#\left((T_m)_{b_i}(\vec{f}) \right) \right\|_{M_q^p(\mu)} \\
&\le C \|b_i\|_* \left\| M_{\tau,\frac{3}{2}}\left(T_m(\vec{f}) \right)(x) + \prod_{i=1}^{m} M_{s_i,\frac{9}{8}} f_i \right\|_{M_q^p(\mu)} \\
&\le C \|b_i\|_* \left\| M_{\tau,\frac{3}{2}}\left(T_m(\vec{f}) \right) \right\|_{M_q^p(\mu)} + C \|b_i\|_* \left\| \prod_{i=1}^{m} M_{s_i,\frac{9}{8}} f_i \right\|_{M_q^p(\mu)} \\
&\le C \|b_i\|_* \left\| T_m(\vec{f}) \right\|_{M_q^p(\mu)} + C \|b_i\|_* \prod_{i=1}^{m} \|f_i\|_{M_{q_j}^{p_j}(\mu)} \\
&\le C \|b_i\|_* \prod_{j=1}^{m} \|f_i\|_{M_{q_j}^{p_j}(\mu)}.
\end{aligned}$$

Applying the inequality (17) in Lemma 2.1, for $\sigma \subset \{1, 2, \cdots, m\}$, we have

$$
\begin{aligned}
&\left\|\left(T_m\right)_{\prod \vec{b}_\sigma}\left(\vec{f}\right)\right\|_{M_q^p(\mu)} \\
&\le C\left\|N\left(\left(T_m\right)_{\prod \vec{b}_\sigma}\left(\vec{f}\right)\right)\right\|_{M_q^p(\mu)} \le C\left\|M^{\#}\left(\left(T_m\right)_{\prod \vec{b}_\sigma}\left(\vec{f}\right)\right)\right\|_{M_q^p(\mu)} \\
&\le C\left\|\prod_{j\in\sigma}\left\|b_j\right\|_* M_{\tau,\frac{3}{2}}\left(T_m\left(\vec{f}\right)\right)+\sum_{\substack{\sigma_1\cup\sigma_2=\sigma \\ \sigma_1\neq\varnothing\ \sigma_2\neq\varnothing}}\prod_{j\in\sigma_1}\left\|b_j\right\|_* M_{\tau,\frac{3}{2}}\left(\left(T_m\right)_{\prod \vec{b}_{\sigma_2}}\left(\vec{f}\right)\right)\right. \\
&\quad \left.+\prod_{j\in\sigma}\left\|b_j\right\|_*\prod_{j=1}^{m} M_{s_j,\frac{9}{8}} f_j\right\|_{M_q^p(\mu)} \\
&\le C\prod_{j\in\sigma}\left\|b_j\right\|_*\prod_{j=1}^{m}\left\|f_j\right\|_{M_{q_j}^{p_j}(\mu)}+C\sum_{\substack{\sigma_1\cup\sigma_2=\sigma \\ \sigma_1\neq\varnothing\ \sigma_2\neq\varnothing}}\prod_{j\in\sigma_1}\left\|b_j\right\|_*\left\|M_{\tau,\frac{3}{2}}\left(\left(T_m\right)_{\prod \vec{b}_{\sigma_2}}\left(\vec{f}\right)\right)\right\|_{M_q^p(\mu)} \\
&\le C\prod_{j\in\sigma}\left\|b_j\right\|_*\prod_{j=1}^{m}\left\|f_j\right\|_{M_{q_j}^{p_j}(\mu)}+C\sum_{\substack{\sigma_1\cup\sigma_2=\sigma \\ \sigma_1\neq\varnothing\ \sigma_2\neq\varnothing}}\prod_{j\in\sigma_1}\left\|b_j\right\|_*\left\|\left(T_m\right)_{\prod \vec{b}_{\sigma_2}}\left(\vec{f}\right)\right\|_{M_q^p(\mu)}
\end{aligned}
$$

where σ_1 and σ_2 are two nonempty subsets of σ and $\sigma_1\cap\sigma_2=\varnothing$. Hence, we can make use of induction on $\sigma\subset\{1,2,\cdots,m\}$ to get that

$$\left\|\left(T_m\right)_{\prod \vec{b}_\sigma}\left(\vec{f}\right)\right\|_{M_q^p(\mu)} \le C\prod_{j\in\sigma}\left\|b_j\right\|_*\prod_{i=1}^{m}\left\|f_i\right\|_{M_{q_i}^{p_i}(\mu)}. \tag{18}$$

This completes the proof of Theorem 1.2.□

For simplicity of the notation, we only show the special case $\sigma=\{1,\cdots,m\}$ of Lemma 2.1. The similar process with minor modification will be to able prove Lemma 2.1 for the general case.

Lemma 2.5. *Let* $\tau>1, s_i>1, b_i\in RBMO(\mu)$ *and* $f_i\in L^{q_i}(\mu), i=1,2,\cdots,m$. *If* T_m *maps* $L^1(\mu)\times\cdots\times L^1(\mu)$ *to* $L^{1/m,\infty}(\mu)$ *and satisfying conditions* (7) *and* (8), *then there exists a constant* $C>0$ *independent of* b_i *and* f_i *such that*

$$
\begin{aligned}
&M^{\#}\left(\left(T_m\right)_{\prod \vec{b}}\left(\vec{f}\right)\right)(x) \\
&\le C\left[\prod_{i=1}^{m}\left\|b_i\right\|_* M_{\tau,\frac{3}{2}}\left(T_m\left(\vec{f}\right)\right)(x)\right. \\
&\quad +\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}\prod_{j\in\sigma}\left\|b_j\right\|_* M_{\eta,\frac{3}{2}}\left(\left(T_m\right)_{\prod \vec{b}_{\sigma'}}\left(\vec{f}\right)\right)(x) \\
&\quad \left.+\prod_{i=1}^{m}\left\|b_i\right\|_* M_{s_i,\frac{9}{8}} f_i(x)\right]
\end{aligned} \tag{19}
$$

In order to prove Lemma 2.5, we have the following decomposition for the commutators $\left(T_m\right)_{\prod \vec{b}}$. For any $\vec{\lambda}=\left(\lambda_1,\lambda_2,\cdots,\lambda_m\right)\in\mathbb{R}^m$, writing $\vec{b}_\sigma(x)-\vec{b}_\sigma(y)=\left(\vec{b}_\sigma(x)-\vec{\lambda}_\sigma\right)-\left(\vec{b}_\sigma(y)-\vec{\lambda}_\sigma\right)$, thus it is clear that $\left(T_m\right)_{\prod \vec{b}_\sigma}\left(\vec{f}\right)=\left(T_m\right)_{\prod\left(\vec{b}_\sigma-\vec{\lambda}_\sigma\right)}\left(\vec{f}\right)$. Moreover,

$$
\begin{aligned}
&\left(T_m\right)_{\prod \vec{b}}\left(\vec{f}\right)(x) \\
&=\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}(-1)^{m-i}\prod_{\sigma_j\in\sigma}\left(b_{\sigma_j}(x)-\lambda_{\sigma_j}\right)\times\int_{\left(\mathbb{R}^d\right)^m}\prod_{\sigma_k\in\sigma'}\left(b_{\sigma_k}(y)-\lambda_{\sigma_k}\right) \\
&\quad\times K\left(x,y_1,\cdots,y_m\right)f_1\left(y_1\right)\cdots f_m\left(y_m\right)\mathrm{d}\mu\left(y_1\right)\cdots\mathrm{d}\mu\left(y_m\right).
\end{aligned} \tag{20}
$$

By expanding $b_{\sigma_k}(y)-\lambda_{\sigma_k}=\left(b_{\sigma_k}(y)-b_{\sigma_k}(x)\right)-\left(b_{\sigma_k}(x)-\lambda_{\sigma_k}\right)$ and

$$\prod_{\sigma_k\in\sigma'}\left(b_{\sigma_k}(y)-\lambda_{\sigma_k}\right)=\sum_{\substack{\sigma^{(1)}\cup\sigma^{(2)}=\sigma'\\ \sigma^{(1)}\cap\sigma^{(2)}=\varnothing}}\prod_{\sigma_u\in\sigma^{(1)}}\left(b_{\sigma_u}(y)-b_{\sigma_u}(x)\right)\prod_{\sigma_v\in\sigma^{(2)}}\left(b_{\sigma_v}(x)-\lambda_{\sigma_v}\right),$$

hence we can obtain from the equality (20) that

$$\begin{aligned}&(T_m)_{\prod\vec{b}}\left(\vec{f}\right)(x)\\&=\prod_{j=1}^{m}\left(b_j(x)-\lambda_j\right)T_m\left(\vec{f}\right)(x)+(-1)^m T_m\left(\left(\vec{b}-\vec{\lambda}\right)\vec{f}\right)(x)\\&\quad+\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}C_{m,i}\prod_{\sigma_j\in\sigma}\left(b_{\sigma_j}(x)-\lambda_{\sigma_j}\right)\times(T_m)_{\prod\vec{b}_{\sigma'}}\left(\vec{f}\right)(x)\end{aligned}\tag{21}$$

where $C_{m,i}$ are constants depending only on m and i.

Proof of Lemma 2.5. For simplicity, we denote by $\mathcal{R}\left(\vec{b},\vec{f}\right)(x)$ the quantities on the right hand side of the inequality (19). Recall the definition of the sharp maximal operator $M^{\#}$, and use the standard technique, see [15] for example, we only need to prove that

$$\frac{1}{\mu\left(\frac{3}{2}Q\right)}\int_Q\left|(T_m)_{\prod\vec{b}}\left(\vec{f}\right)(z)-h_Q\right|\mathrm{d}\mu(z)\le C\mathcal{R}\left(\vec{b},\vec{f}\right)(x)\tag{22}$$

and

$$\left|h_Q-h_R\right|\le C\left(K_{Q,R}\right)^{2m}\mathcal{R}\left(\vec{b},\vec{f}\right)(x)\tag{23}$$

with the absolute constant C independent of $\vec{b},\vec{f},Q$ and R, where R is any doubling cube with $Q\subset R$. In fact, we take

$$h_Q=(-1)^m m_Q\left(T_m\left(\left(m_{\tilde{Q}}(b_1)-b_1\right)f_1\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q},\cdots,\left(m_{\tilde{Q}}(b_m)-b_m\right)f_m\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}\right)\right)\tag{24}$$

and clearly

$$h_R=(-1)^m m_R\left(T_m\left(\left(m_{\tilde{R}}(b_1)-b_1\right)f_1\chi_{\mathbb{R}^d\setminus\frac{4}{3}R},\cdots,\left(m_{\tilde{R}}(b_m)-b_m\right)f_m\chi_{\mathbb{R}^d\setminus\frac{4}{3}R}\right)\right)\tag{25}$$

Recall the equality (21), for any $z\in Q$, we have that

$$\begin{aligned}&\left|(T_m)_{\prod\vec{b}}\left(\vec{f}\right)(z)-h_Q\right|\\&\le\left|\prod_{j=1}^{m}\left(b_j(x)-m_{\tilde{Q}}(b_j)\right)T_m\left(\vec{f}\right)(z)\right|\\&\quad+\left|\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}C_{i,m}\prod_{\sigma_j\in\sigma}\left(b_{\sigma_j}(z)-m_{\tilde{Q}}\left(b_{\sigma_j}\right)\right)(T_m)_{\prod\vec{b}_{\sigma'}}\left(\vec{f}\right)(z)\right|\\&\quad+\left|(-1)^m T_m\left(\left(\vec{b}-m_{\tilde{Q}}\left(\vec{b}\right)\right)\vec{f}\right)(z)-h_Q\right|\\&=:I(z)+II(z)+III(z).\end{aligned}\tag{26}$$

In order to show the inequality (22), we will calculate the integrals for the three functions above, respectively. Firstly, for $\tau > 1$, by the Hölder inequality one sees that

$$\begin{aligned}&\frac{1}{\mu\left(\frac{3}{2}Q\right)}\int_Q |I(z)|\,\mathrm{d}\mu(z)\\&\le \prod_{i=1}^{m}\left(\frac{1}{\mu\left(\frac{3}{2}Q\right)}\int_Q \left|b_i(z)-m_{\tilde{Q}}(b_i)\right|^{\tau_i}\mathrm{d}\mu(z)\right)^{\frac{1}{\tau_i}}\left(\frac{1}{\mu\left(\frac{3}{2}Q\right)}\int_Q \left|T_m(\vec{f})\right|^{\tau}\mathrm{d}\mu(z)\right)^{\frac{1}{\tau}}\\&\le C\|b_1\|_*\|b_2\|_*\cdots\|b_m\|_* M_{\tau,\frac{3}{2}}\left(T_m(\vec{f})\right)(x)\end{aligned}\tag{27}$$

where we have choose $\tau_i > 1$ such that $\frac{1}{\tau_1}+\frac{1}{\tau_2}+\cdots+\frac{1}{\tau_m}+\frac{1}{\tau}=1$.

Similarly, for $\tau > 1$, by the Hölder inequality, we also deduce that

$$\begin{aligned}&\frac{1}{\mu\left(\frac{3}{2}Q\right)}\int_Q |II(z)|\,\mathrm{d}\mu(z)\\&\le C\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}\prod_{\sigma_j\in\sigma}\left\|b_{\sigma_j}\right\|_* M_{\tau,\frac{3}{2}}\left(\left(T_m\right)_{b_{\sigma'}}(\vec{f})\right)(x).\end{aligned}\tag{28}$$

To estimate the integral related to the function $III(z)$, we split f_i as $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f\chi_{\frac{4}{3}Q}$ and $f_i^\infty = f_i - f_i^0$, this yields

$$\begin{aligned}\prod_{j=1}^{m} f_j(y_j) &= \sum_{\alpha_1,\alpha_2,\cdots,\alpha_m\in\{0,\infty\}} f_1^{\alpha_1}(y_1)\cdots f_m^{\alpha_m}(y_m)\\&=\prod_{j=1}^{m} f_j^0(y_j)+\sum_{*} f_1^{\alpha_1}(y_1)\cdots f_m^{\alpha_m}(y_m)+\prod_{j=1}^{m} f_j^\infty(y_j),\end{aligned}\tag{29}$$

where each term in $\sum_{*}$ satisfies that $\alpha_{j_1}=\alpha_{j_2}=\cdots=\alpha_{j_\lambda}=0$, for some $1\le\lambda<m$ and some $\{j_1,j_2,\cdots,j_\lambda\}\subset\{1,2,\cdots,m\}$. So we can decompose the function $III(z)$ further into three parts as follows

$$\begin{aligned}III(z) &\le \left|T_m\left(\left(b_1-m_{\tilde{Q}}(b_1)\right)f_1^0,\cdots,\left(b_m-m_{\tilde{Q}}(b_m)\right)f_m^0\right)(z)\right|\\&\quad+\sum_{*}\left|T_m\left(\left(b_1-m_{\tilde{Q}}(b_1)\right)f_1^{\alpha_1},\cdots,\left(b_m-m_{\tilde{Q}}(b_m)\right)f_1^{\alpha_m}\right)(z)\right|\\&\quad+\left|(-1)^m T_m\left(\left(b_1-m_{\tilde{Q}}(b_1)\right)f_1^\infty,\cdots,\left(b_m-m_{\tilde{Q}}(b_m)\right)f_1^\infty\right)(z)-h_Q\right|\\&=: III_1(z)+III_2(z)+III_3(z).\end{aligned}\tag{30}$$

For $s_i > 1$,we can take $1<\mu_i<s_i$ such that $\frac{1}{v}=\frac{1}{u_1}+\frac{1}{u_2}+\cdots+\frac{1}{u_m}$ and $v>1$. Let $\frac{1}{u_i}=\frac{1}{s_i}+\frac{1}{t_i}$ for each $i=1,2,\cdots,m$, then $1<t_i<\infty$. Using Lemma 2.4, we know that the T_m is bounded from $L^{u_1}(\mu)\times L^{u_2}(\mu)\times\cdots\times L^{u_m}(\mu)$ to $L^{v}(\mu)$. Hence, by this boundedness and Hölder inequality, we have

$$
\begin{aligned}
&\frac{1}{\mu\left(\frac{3}{2}Q\right)}\int_Q \left|III_1(z)\right| \mathrm{d}\mu(z) \\
&\le \frac{\mu(Q)^{1-\frac{1}{v}}}{\mu\left(\frac{3}{2}Q\right)}\left\|T_m\left(\left(b_1-m_{\tilde{Q}}(b_1)\right)f_1^0,\cdots,\left(b_m-m_{\tilde{Q}}(b_m)\right)f_m^0\right)(z)\right\|_{L^v(\mu)} \\
&\le C\mu\left(\frac{3}{2}Q\right)^{-\frac{1}{v}}\prod_{i=1}^m\left\|\left(b_i-m_{\tilde{Q}}(b_i)\right)f_i^0\right\|_{L^{u_i}(\mu)} \\
&\le C\prod_{i=1}^m\|b_i\|_* M_{s_i,\frac{9}{8}}f_i(x).
\end{aligned}
\tag{31}
$$

In order to estimate the integral of terms $III_2(z)$ and $III_3(z)$ over Q, we will give their point-wise estimates. In fact, for $z\in Q$, since $1\le\lambda\le m-1$ we observe that

$$
\begin{aligned}
III_2(z) &= \sum_*\left|T_m\left(\left(b_1-m_{\tilde{Q}}(b_1)\right)f_1^{\alpha_1},\cdots,\left(b_m-m_{\tilde{Q}}(b_m)\right)f_1^{\alpha_m}\right)(z)\right| \\
&\le C\sum_*\prod_{j\in\{j_1,\cdots,j_\lambda\}}\int_{\frac{4}{3}Q}\left|b_j(y_j)-m_{\tilde{Q}}(b_j)\right|\left|f_j(y_j)\right|\mathrm{d}\mu(y_j) \\
&\quad\times\int_{\left(\mathbb{R}^d\setminus\frac{4}{3}Q\right)^{m-\lambda}}\frac{\sum_{j\notin\{j_1,\cdots,j_\lambda\}}\left|b_j(y_j)-m_{\tilde{Q}}(b_j)\right|\left|f_j(y_j)\right|\mathrm{d}\mu(y_j)}{\left(\sum_{j\notin\{j_1,\cdots,j_\lambda\}}\left|z-y_j\right|\right)^{mn}} \\
&\le C\sum_*\prod_{j\in\{j_1,j_2,\cdots,j_\lambda\}}\left\|b_j\right\|_* M_{s_j,\frac{9}{8}}f_j(x) \\
&\quad\times\sum_{k=1}^\infty 2^{-k\lambda n}\prod_{j\notin\{j_1,j_2,\cdots,j_\lambda\}}\left\|b_j\right\|_*(k+1)M_{s_j,\frac{9}{8}}f_j(x) \\
&\le C\prod_{j=1}^\infty\left\|b_j\right\|_* M_{s_j,\frac{9}{8}}f_j(x)
\end{aligned}
\tag{32}
$$

where we have used the fact (see[5]) that, there is an absolute constant C such that, for any $b\in RBMO$, integer $k\ge 0$ and cubes Q,

$$
\begin{aligned}
\left|m_{\widetilde{2^k\frac{4}{3}Q}}(b)-m_{\tilde{Q}}(b)\right| &\le C\|b\|_* K_{\tilde{Q},\widetilde{2^k\frac{4}{3}Q}} \\
&\le C\|b\|_* K_{Q,2^k\frac{4}{3}Q} \\
&\le Ck\|b\|_*.
\end{aligned}
\tag{33}
$$

On the other hand, for $III_3(z)$, we note for any $z,y\in Q$ that

$$
\begin{aligned}
&\left|T_m\left(\left(b_1-m_{\tilde{Q}}(b_1)\right)f_1^\infty,\cdots,\left(b_m-m_{\tilde{Q}}(b_m)\right)f_m^\infty\right)(z)\right. \\
&\left.-T_m\left(\left(b_1-m_{\tilde{Q}}(b_1)\right)f_1^\infty,\cdots,\left(b_m-m_{\tilde{Q}}(b_m)\right)f_m^\infty\right)(y)\right| \\
&\le\int_{\left(\mathbb{R}^d\setminus\frac{4}{3}Q\right)^m}\left|K(z,y_1,\cdots,y_m)-K(y,y_1,\cdots,y_m)\right| \\
&\quad\times\left|\prod_{i=1}^m\left(b_i(y_i)-m_{\tilde{Q}}(b_i)\right)f_i^\infty\right|\mathrm{d}\mu(y_1)\mathrm{d}\mu(y_2)\cdots\mathrm{d}\mu(y_m)
\end{aligned}
$$

$$\le \int_{\left(\mathbb{R}^d\setminus\frac{4}{3}Q\right)^m} \frac{|z-y|^{\epsilon}\prod_{i=1}^m\left|\left(b_i(y_i)-m_{\tilde{Q}}(b_i)\right)f_i^{\infty}\right|}{\left(\sum_{k,l=0}^m|y_k-y_l|\right)^{mn+\epsilon}}\mathrm{d}\mu(y_1)\cdots\mathrm{d}\mu(y_m)$$

$$\le C\prod_{i=1}^m\sum_{k=1}^{\infty}\int_{2^k\frac{4}{3}Q\setminus 2^{k-1}\frac{4}{3}Q}\frac{l(Q)^{\frac{\epsilon}{m}}}{l\left(2^k\frac{3}{2}Q\right)^{n+\frac{\epsilon}{m}}}\left|\left(b_i(y_i)-m_{\tilde{Q}}(b_i)\right)f_i\right|\mathrm{d}\mu(y_i) \tag{34}$$

$$\le C\prod_{i=1}^m\sum_{k=1}^{\infty}2^{-\frac{k}{m}}(k+1)\|b_i\|_* M_{s_i,\frac{9}{8}}f_i(x)$$

$$\le C\prod_{i=1}^m\|b_i\|_* M_{s_i,\frac{9}{8}}f_i(x),$$

where we have use the inequation (33) again.

Taking the mean over $y\in Q$, we can obtain that

$$\frac{1}{\mu\left(\frac{3}{2}Q\right)}\int_Q\left(|III_2(z)|+|III_3(z)|\right)\mathrm{d}\mu(z)\le C\prod_{i=1}^m\|b_i\|_* M_{s_i,\frac{9}{8}}f_i(x). \tag{35}$$

Combing the inequalities (26) (27), (28), (30), (31) and (33), we see from the estimates of I, II, III_1, III_2 and III_3 that the desired inequality (22) holds.

Next we turn to estimate the inequality (23). For any cubes $Q\subset R$ with $x\in Q$, where R is doubling. We denote $N_{Q,R}+2$ by N, then $2^NQ\supset 2Q$ and $2^NQ\supset 2R$. We recall the equality (29) and let $f_i^0=f_i\chi_{2^NQ\setminus\frac{4}{3}Q}$ and $f_i^R=f_i\chi_{2^NQ\setminus\frac{4}{3}R}$, and let $f_i^{\infty}=f_i\chi_{\mathbb{R}^d\setminus 2^NQ}$. Then we can write

$$\begin{aligned}
|h_Q-h_R| &= \Big|m_Q\Big[T_m\Big(\left(m_{\tilde{Q}}(b_1)-b_1\right)f_1\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q},\cdots,\left(m_{\tilde{Q}}(b_m)-b_m\right)f_m\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}\Big)\Big]\\
&\quad -m_R\Big[T_m\Big(\left(m_{\tilde{R}}(b_1)-b_1\right)f_1\chi_{\mathbb{R}^d\setminus\frac{4}{3}R},\cdots,\left(m_{\tilde{R}}(b_m)-b_m\right)f_m\chi_{\mathbb{R}^d\setminus\frac{4}{3}R}\Big)\Big]\Big|\\
&\le \Big|m_Q\Big[T_m\Big(\left(m_{\tilde{Q}}(b_1)-b_1\right)f_1^{\infty},\cdots,\left(m_{\tilde{Q}}(b_m)-b_m\right)f_m^{\infty}\Big)\Big]\\
&\quad -m_R\Big[T_m\Big(\left(m_{\tilde{Q}}(b_1)-b_1\right)f_1^{\infty},\cdots,\left(m_{\tilde{Q}}(b_m)-b_m\right)f_m^{\infty}\Big)\Big]\Big|\\
&\quad +\Big|m_R\Big[T_m\Big(\left(m_{\tilde{Q}}(b_1)-b_1\right)f_1^{\infty},\cdots,\left(m_{\tilde{Q}}(b_m)-b_m\right)f_m^{\infty}\Big)\Big]\\
&\quad -m_R\Big[T_m\Big(\left(m_R(b_1)-b_1\right)f_1^{\infty},\cdots,\left(m_R(b_m)-b_m\right)f_m^{\infty}\Big)\Big]\Big|\\
&\quad +\sum_{\substack{\alpha_1,\alpha_2,\cdots,\alpha_m\in\{0,\infty\}\\ \text{at least one }\alpha_i\ne\infty}}\Big|m_Q\Big[T_m\Big(\left(m_{\tilde{Q}}(b_1)-b_1\right)f_1^{\alpha_1},\cdots,\left(m_{\tilde{Q}}(b_m)-b_m\right)f_m^{\alpha_m}\Big)\Big]\Big|\\
&\quad +\sum_{\substack{\alpha_1,\alpha_2,\cdots,\alpha_m\in\{R,\infty\}\\ \text{at least one }\alpha_i\ne\infty}}\Big|m_R\Big[T_m\Big(\left(m_R(b_1)-b_1\right)f_1^{\alpha_1},\cdots,\left(m_R(b_m)-b_m\right)f_m^{\alpha_m}\Big)\Big]\Big|\\
&=: A_1+A_2+A_3+A_4.
\end{aligned} \tag{36}$$

For the term A_1, noting

$$\left|m_R(b_i)-m_{\tilde{Q}}(b_i)\right|\le C\|b_i\|_* K_{Q,R} \tag{37}$$

and the similar argument as that for the estimate of III_3, we can obtain that

$$A_1\le C\prod_{i=1}^m K_{Q,R}\|b_i\|_* M_{s_i,\frac{9}{8}}f_i(x)\le C\left(K_{Q,R}\right)^m\prod_{i=1}^m\|b_i\|_* M_{s_i,\frac{9}{8}}f_i(x). \tag{38}$$

To estimate A_2, we recall the notations and note that, for any sequences ξ_j and ζ_j,

$$\prod_{j=1}^m\left(\xi_j+\zeta_j\right)=\sum_{i=0}^m\sum_{\sigma\in C_i^m}\prod_{j\in\sigma}\xi_j\prod_{j'\in\sigma'}\zeta_{j'}. \tag{39}$$

Using this equality and expanding $m_{\tilde{Q}}(b_j)-b_j(y)=\left[m_{\tilde{Q}}(b_j)-b_j(z)\right]+\left[b_j(z)-b_j(y)\right]$, we observe that

$$\begin{aligned}&T_m\left(\left(m_{\tilde{Q}}(b_1)-b_1\right)f_1^\infty,\cdots,\left(m_{\tilde{Q}}(b_m)-b_m\right)f_m^\infty\right)(z)\\&=\sum_{i=0}^m\sum_{\sigma\in C_i^m}\prod_{j\in\sigma}\left(m_{\tilde{Q}}(b_j)-b_j(z)\right)\left(T_m\right)_{\prod\vec{b}_{\sigma'}}\left(\vec{f}\chi_{\mathbb{R}^d\setminus 2^NQ}\right)(z)\end{aligned} \tag{40}$$

Similarly,

$$\begin{aligned}&T_m\left(\left(m_R(b_1)-b_1\right)f_1^\infty,\cdots,\left(m_R(b_m)-b_m\right)f_m^\infty\right)(z)\\&=\sum_{i=0}^m\sum_{\sigma\in C_i^m}\prod_{j\in\sigma}\left(m_R(b_j)-b_j(z)\right)\left(T_m\right)_{\prod\vec{b}_{\sigma'}}\left(\vec{f}\chi_{\mathbb{R}^d\setminus 2^NQ}\right)(z).\end{aligned} \tag{41}$$

Thus

$$\begin{aligned}&\Big|T_m\left(\left(m_{\tilde{Q}}(b_1)-b_1\right)f_1^\infty,\cdots,\left(m_{\tilde{Q}}(b_m)-b_m\right)f_m^\infty\right)(z)\\&-T_m\left(\left(m_R(b_1)-b_1\right)f_1^\infty,\cdots,\left(m_R(b_m)-b_m\right)f_m^\infty\right)(z)\Big|\\&\le\left|\sum_{i=1}^m\sum_{\sigma\in C_i^m}\prod_{j\in\sigma}\left(m_{\tilde{Q}}(b_j)-b_j(z)\right)\left(T_m\right)_{\prod\vec{b}_{\sigma'}}\left(\vec{f}\chi_{\mathbb{R}^d\setminus 2^NQ}\right)(z)\right|\\&\quad+\left|\sum_{i=1}^m\sum_{\sigma\in C_i^m}\prod_{j\in\sigma}\left(m_R(b_j)-b_j(z)\right)\left(T_m\right)_{\prod\vec{b}_{\sigma'}}\left(\vec{f}\chi_{\mathbb{R}^d\setminus 2^NQ}\right)(z)\right|\\&=:B_1(z)+B_2(z).\end{aligned} \tag{42}$$

To estimate the integrals above, we recall that $f_i^\infty=f_i\chi_{\mathbb{R}^d\setminus 2^NQ}$ and let $f_j^Q=f_j\chi_{2^NQ}$, then we can write that $f_j^\infty=f_j-f_j^Q$ and $f_j=f_j^\infty+f_j^Q$, and thus we have

$$\begin{aligned}&\prod_{j=1}^m f_j^\infty(y_j)=\prod_{j=1}^m f_j(y_j)+\sum_{i=1}^m(-1)^i\sum_{\rho\in C_i^m}\prod_{j\in\rho}f_j^Q(y_j)\prod_{j'\in\rho'}f_{j'}(y_{j'})\\&=\prod_{j=1}^m f_j(y_j)+\sum_{\lambda=1}^m\sum_{\{j_1,j_2,\cdots,j_\lambda\}\subset\{1,2,\cdots,m\}}\\&\times C_{j_1,j_2,\cdots,j_m}f_{j_1}^Q(y_{j_1}),\cdots,f_{j_\lambda}^Q(y_{j_\lambda})f_{j_{\lambda+1}}^\infty(y_{j_{\lambda+1}}),\cdots,f_{j_m}^\infty(y_{j_m})\end{aligned} \tag{43}$$

where $C_{j_1,j_2,\cdots,j_m}$ are constant independent of $\vec{f}$ and Q. From the equality (43), we can deduce that

$$\begin{aligned}&(T_m)_{\prod \vec{b}_{\sigma'}}\left(\vec{f}\chi_{\mathbb{R}^d \setminus 2^N Q}\right)(z)\\&=(T_m)_{\prod \vec{b}_{\sigma'}}\vec{f}(z)+\sum_{\lambda=1}^{m}\sum_{\{j_1,j_2,\cdots,j_m\}\subset\{1,2,\cdots,m\}}C_{j_1,j_2,\cdots,j_m}T_m(g_1F_1,g_2F_2,\cdots,g_mF_m)(z)\end{aligned} \tag{44}$$

where

$$g_j(y)=\begin{cases}b_j(z)-b_j(y), & \text{if } j\in\sigma'\\ 1, & \text{if } j\in\sigma\end{cases}$$

and

$$F_j(y)=\begin{cases}f_j^Q(y), & \text{if } j\in\{j_1,\cdots,j_\lambda\}\\ f_j^\infty(y), & \text{if } j\notin\{j_1,\cdots,j_\lambda\}\end{cases}$$

Along the same lines as that of the pointwise estimates of $III_2(z)$,we can obtain that, for $x,z\in R\subset 2^{N-1}Q$ and if $1\le\lambda\le m$,

$$\begin{aligned}&\left|T_m(g_1F_1,g_2F_2,\cdots,g_mF_m)(z)\right|\\&\le C\prod_{j\in\{j_1,j_2,\cdots,j_\lambda\}}\frac{1}{l(2^NQ)^n}\int_{2^NQ}\left|g_j(y_j)f_j(y_j)\right|\mathrm{d}\mu(y_j)\\&\times\sum_{k=1}^{\infty}2^{-k\lambda n}\prod_{\substack{j\notin\{j_1,j_2,\cdots,j_\lambda\}\\1\le j\le m}}\frac{1}{l(2^k2^NQ)^n}\int_{2^k2^NQ}\left|g_j(y_j)f_j(y_j)\right|\mathrm{d}\mu(y_j).\end{aligned} \tag{45}$$

Let $(g_j)^*_{k,s'_j}=1$ if $j\in\sigma$; and

$$(g_j)^*_{k,s'_j}=\begin{cases}\left(\dfrac{1}{l(2^NQ)^n}\displaystyle\int_{2^NQ}\left|b_j(z)-b_j(y)\right|^{s'_j}\mathrm{d}\mu(y)\right)^{\frac{1}{s'_j}}, & \text{if } j\in\sigma'\cap\{j_1,j_2,\cdots,j_\lambda\}\\ \left(\dfrac{1}{l(2^{k+N}Q)^n}\displaystyle\int_{2^{k+N}Q}\left|b_j(z)-b_j(y)\right|^{s'_j}\mathrm{d}\mu(y)\right)^{\frac{1}{s'_j}}, & \text{if } j\in\sigma'\setminus\{j_1,j_2,\cdots,j_\lambda\}\end{cases}$$

thus for $j\in\sigma'$,

$$(g_j)^*_{k,s'_j}\le C\|b_j\|_*+C\left|m_{2^NQ}(b_j)-b_j(z)\right|+C\left|m_{2^{k+N}Q}(b_j)-b_j(z)\right|.$$

Hence we get from (45) that, for $\tau>1$,

$$\begin{aligned}&\left(\frac{1}{\mu(R)}\int_R\left|T_m(g_1F_1,g_2F_2,\cdots,g_mF_m)(z)\right|^\tau\mathrm{d}\mu(z)\right)^{\frac{1}{\tau}}\\&\le C\left(\prod_{j=1}^{m}M_{s_j,\frac{9}{8}}f_j(x)\right)\sum_{j=1}^{\infty}2^{-k\lambda n}\times\left(\frac{1}{\mu(R)}\int_R\left|\prod_{j\in\sigma'}(g_j)^*_{k,s'_j}\right|^\tau\mathrm{d}\mu(z)\right)^{\frac{1}{\tau}}\\&\le C\left(\prod_{j=1}^{m}M_{s_j,\frac{9}{8}}f_j(x)\right)\sum_{j=1}^{\infty}2^{-k\lambda n}\prod_{j\in\sigma'}\|b_j\|_*\left(1+K_{R,2^NQ}+K_{R,2^{k+N}Q}\right)\\&\le C\left(\prod_{j=1}^{m}M_{s_j,\frac{9}{8}}f_j(x)\right)\prod_{j\in\sigma'}\|b_j\|_*,\end{aligned} \tag{46}$$

where we have used the fact that the cubes R and 2^NQ are comparable, which implies $K_{R,2^N Q} \le C$ and $K_{R,2^{k+N} Q} \le C(1+k)$. Using the inequality (46) above and the identity (44), we obtain that, for $\tau > 1$,

$$\begin{aligned}
A_2 &\le \frac{1}{\mu(R)}\int_R \left(|B_1(z)| + |B_2(z)|\right)\mathrm{d}\mu(z) \\
&\le C\sum_{i=1}^{m}\sum_{\sigma\in C_i^m}\left(\prod_{j\in\sigma} K_{Q,R}\|b_j\|_*\right) \\
&\quad \times\left(\frac{1}{\mu(R)}\int_R \left|(T_m)_{\prod \vec{b}_{\sigma'}}\left(\vec{f}\chi_{\mathbb{R}^d\setminus 2^N Q}\right)\right|^{\tau}\mathrm{d}\mu(z)\right)^{\frac{1}{\tau}} \\
&\le C\sum_{i=1}^{m}\sum_{\sigma\in C_i^m}\left(\prod_{j\in\sigma} K_{Q,R}\|b_j\|_*\right) M_{\tau,\frac{3}{2}}\left((T_m)_{\prod \vec{b}_{\sigma'}}(\vec{f})\right)(x) \\
&\quad + C\prod_{j=1}^{m} K_{Q,R}\|b_j\|_* M_{s_j,\frac{9}{8}} f_j(x).
\end{aligned}$$

The estimates of A_3 and A_4 is very similar to the one used in the estimate of A_2. In fact, repeating the similar procedures used in (45) and (46) for $\tau > 1$, and noting that $K_{Q,2^{k+N}Q} \le K_{Q,R} + K_{R,2^{k+N}Q} \le C(1+k) + K_{Q,R}$ since $2^{N-3}Q \subset 2R \subset 2^N Q$ by the definition of *N*, we can deduce that

$$\begin{aligned}
&\sum_{\substack{\alpha_1,\alpha_2,\cdots,\alpha_m\in\{0,\infty\} \\ \text{at least one } \alpha_i\ne\infty \\ \text{and one } \alpha_j\ne 0}}\left|m_Q\left[T_m\left(\left(m_{\tilde{Q}}(b_1)-b_1\right)f_1^{\alpha_1},\cdots,\left(m_{\tilde{Q}}(b_m)-b_m\right)f_m^{\alpha_m}\right)\right]\right| \\
&+\sum_{\substack{\alpha_1,\alpha_2,\cdots,\alpha_m\in\{R,\infty\} \\ \text{at least one } \alpha_i\ne\infty}}\left|m_R\left[T_m\left(\left(m_R(b_1)-b_1\right)f_1^{\alpha_1},\cdots,\left(m_R(b_m)-b_m\right)f_m^{\alpha_m}\right)\right]\right| \qquad (47) \\
&\le C\prod_{j=1}^{m} K_{Q,R}\|b_j\|_* M_{s_j,\frac{9}{8}} f_j(x).
\end{aligned}$$

It is left to estimate the term in A_3 of the case $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$. A small modification is needed to estimate this term. For $z \in Q$ and $x \in Q$, one sees

$$\begin{aligned}
&\left|T_m\left(\left(m_{\tilde{Q}}(b_1)-b_1\right)f_1^0,\cdots,\left(m_{\tilde{Q}}(b_j)-b_j\right)f_j^0,\cdots,\left(m_{\tilde{Q}}(b_m)-b_m\right)f_m^0\right)(z)\right| \\
&\le C\sum_{k=1}^{N}\prod_{j=1}^{m}\frac{1}{l\left(2^k\frac{4}{3}Q\right)^n}\int_{2^k\frac{4}{3}Q}\left|\left(m_{\tilde{Q}}(b_j)-b_j(y_j)\right)f_j(y_j)\right|\mathrm{d}\mu(y_j) \\
&\le C\sum_{k=1}^{N}\prod_{j=1}^{m}\frac{\mu\left(2^k\frac{3}{2}Q\right)}{l\left(2^k\frac{4}{3}Q\right)^n}\left(1+K_{Q,2^k\frac{4}{3}Q}\right)\|b_j\|_* M_{s_j,\frac{9}{8}} f_j(x) \\
&\le C\left(K_{Q,R}\right)^{2m}\prod_{j=1}^{m}\|b_j\|_* M_{s_j,\frac{9}{8}} f_j(x).
\end{aligned}$$

This and the inequality (47) follows

$$A_3 + A_4 \le C\left(K_{Q,R}\right)^{2m}\prod_{j=1}^{m}\|b_j\|_* M_{s_j,\frac{9}{8}} f_j(x).$$

Moreover, combing the estimates of A_1, A_2, A_3 and A_4, we obtain the desired inequality (23).

Finally, let us show how to acquire the inequality (19) from the two inequalities (22) and (23). Fix the point x and let Q be any cube that $x \in Q$. notice $K_{Q,\tilde{Q}} \le C$, hence we see from the inequalities (22) and (23) that

$$\begin{aligned}
&\frac{1}{\mu\left(\frac{3}{2}Q\right)}\int_Q \left|(T_m)_{\prod \vec{b}}\left(\vec{f}\right)(z) - m_{\tilde{Q}}\left((T_m)_{\prod \vec{b}}\left(\vec{f}\right)\right)\right| \mathrm{d}\mu(z) \\
&\le \frac{1}{\mu\left(\frac{3}{2}Q\right)}\int_Q \left|(T_m)_{\prod \vec{b}}\left(\vec{f}\right)(z) - h_Q\right| \mathrm{d}\mu(z) + \frac{1}{\mu\left(\frac{3}{2}Q\right)}\int_Q \left|h_Q - h_{\tilde{Q}}\right| \mathrm{d}\mu(z) \\
&+ \frac{1}{\mu\left(\frac{3}{2}Q\right)}\int_Q \left|m_{\tilde{Q}}\left((T_m)_{\prod \vec{b}}\left(\vec{f}\right)\right) - h_{\tilde{Q}}\right| \mathrm{d}\mu(z) \\
&\le C\mathcal{R}\left(\vec{b}, \vec{f}\right)(x).
\end{aligned} \tag{48}$$

On the other hand for all doubling cubes $Q \subset R$ with $x \in Q$ such that $K_{Q,R} \le P_0$, where P_0 is the constant in Lemma 6 in [15], using (23), we have

$$\left|h_Q - h_R\right| \le CK_{Q,R}P_0^{2m-1}\mathcal{R}\left(\vec{b}, \vec{f}\right)(x) \tag{49}$$

and moreover the inequality (49) holds for any doubling cubes Q, R with $Q \subset R$. Therefore,

$$\begin{aligned}
&m_Q\left((T_m)_{\prod \vec{b}}\left(\vec{f}\right)\right) - m_R\left((T_m)_{\prod \vec{b}}\left(\vec{f}\right)\right)| \\
&\le \left|m_Q\left((T_m)_{\prod \vec{b}}\left(\vec{f}\right)\right) - h_Q\right| + \left|h_R - m_R\left((T_m)_{\prod \vec{b}}\left(\vec{f}\right)\right)\right| + \left|h_Q - h_R\right| \\
&\le CK_{Q,R}\mathcal{R}\left(\vec{b}, \vec{f}\right)(x).
\end{aligned} \tag{50}$$

According to the estimates (48) (50) and the definition of the sharp maximal function, we deduce the inequality (19) and so finish the proof of the Lemma 2.5.□

3. Conclusions

The proof of Lemma 2.5 can be slightly modified to prove the conclusion of Lemma 2.1. Therefore we show that the iterated commutators generated by multilinear singular integrals operators $(T_m)_{\prod \vec{b}}$ are bounded from $M_{q_1}^{p_1}(\mu) \times \cdots \times M_{q_m}^{p_m}(\mu)$ to $M_q^p(\mu)$. Suppose $\|\mu\| = \infty$, the detailed conclusion can be described as follows: Let $(T_m)_{\prod \vec{b}}$ as in (9) and satisfying conditions (7) and (8). Let $1 < q_j \le p_j < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$. Suppose $b_i \in RBMO(\mu)$ for $i = 1, 2, \cdots, m$. If T_m maps $L^1(\mu) \times \cdots \times L^1(\mu)$ to $L^{1/m,\infty}(\mu)$, then the commutators $(T_m)_{\prod \vec{b}}$ are bounded from $M_{q_1}^{p_1}(\mu) \times \cdots \times M_{q_m}^{p_m}(\mu)$ to $M_q^p(\mu)$, that is,

$$\left\|(T_m)_{\prod \vec{b}}\left(\vec{f}\right)(x)\right\|_{M_q^p(\mu)} \le C\prod_{i=1}^m \|b_i\|_* \|f_i\|_{M_{q_i}^{p_i}(\mu)}.$$

More generally, let $(T_m)_{\prod \vec{b}_\sigma}$ as in (11) and satisfying conditions (7) and (8).

Let $1 < q_j \le p_j < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$. Suppose $b_i \in RBMO(\mu)$ for $i = 1, 2, \cdots, m$. If T_m maps $L^1(\mu) \times \cdots \times L^1(\mu)$ to $L^{1/m,\infty}(\mu)$, then for all $\sigma \subset \{1, 2, \cdots, m\}$, the commutators $(T_m)_{\prod \vec{b}_\sigma}$ are bounded from $M_{q_1}^{p_1}(\mu) \times \cdots \times M_{q_m}^{p_m}(\mu)$ to $M_q^p(\mu)$, that is,

$$\left\| (T_m)_{\prod \vec{b}_\sigma} (\vec{f})(x) \right\|_{M_q^p(\mu)} \le C \prod_{j \in \sigma} \|b_j\|_* \prod_{i=1}^{m} \|f_i\|_{M_{q_i}^{p_i}(\mu)}.$$

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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