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On Non Liénard Type with One Limit Cycle

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Abstract

In the present paper, we have investigated the non Liénard system. We have shown that limit cycles may bifurcate at the origin. Bendixons theorem has been used in our study to prove non-existence of limit cycles. We have also proved that the system has unique limit cycle through change of the parameters.

Keywords

Limit Cycle, Non-Liénard Equation, Hopf-Bifucation

1. Introduction

In the present investigation, we revisit the problem of bifurcation of limit cycles. The problem of limit cycle was studied intensively. For Liénard, we can read [1]-[8], and for non Liénard we can read [9]-[19].

We give criterion for the non Liénard system to have or not to have limit cycles with some parameters. We also demonstrate that the system exhibits a Hopf-bifurcation. Now we consider the following Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{1}$$

The above equation may be written in two dimensional autonomous dynamical system

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y. \tag{2}$$

Therefore, the above equations can be written in the Liénard plane as

$$\begin{aligned} \dot{x} &= y - F(x), \\ \dot{y} &= -g(x), \end{aligned} \tag{3}$$

where $F(x) = \int_0^x f(t)\,\mathrm{d}t$.

Theorem 1.1 [11] Suppose that for system (1.1), there exist $r_1 < a_1 < 0 < a_2 < r_2$ such that $F(a_1) = F(0) = F(a_2) = 0$, $g(x)F(x) \le 0$ for $x \in (a_1, a_2)$, $f(x) \ge 0$ for $x \notin (a_1, a_2)$, $xg(x) \ge 0$ for $x \ne 0$, and $G(a_1) = G(a_2)$, then (1.1) has at most one limit cycle in D, which is simple and stable, if exists.

Theorem 1.2 [11] If, in system (1.1), $g(x)F(x) \ge 0$ (or ≤ 0), and equality holds only for at most a finite number of points, then (1.1) has no closed orbits in closed region $D = \{(x, y) : a \le x \le b, c \le y \le d\}$.

In Section 2, the main system equations results have been presented, the section has been divided in two cases.

The case I considered the conditions that the system has a limit cycle when $O(0,0)$ is an anti saddle.

Finally, the case of saddle point with limit cycle is presented in theorems and lemmas in Section 4 along with the concluding remakes.

2. The Basic System Equations and Results

The main part of this paper is devoted to explain the existence and uniqueness of limit cycles of the following differential equations system

$$\begin{aligned} \dot{x} &= -x + ay, \\ \dot{y} &= bx - ay - x^2 y, \end{aligned} \tag{4}$$

the singular points of the system are $x = 0$ and $x = \pm\sqrt{a - ab}$.

The Jacobian matrix

$$A = \begin{bmatrix} -1 & a \\ b & -a \end{bmatrix}$$

has the determinant $|A| = a - ab$ for $a - ab \ge 0$ the origin O is anti saddle and for $a - ab < 0$ the origin O is saddle for more details (see [5]).

The system (2.2) needs to change to the Liénard system (1.1).

Let $z = -x + ay$ so $\dot{z} = -\dot{x} + a\dot{y}$ after simplify and substitute $ay = x + z$ so that we have

$$\begin{aligned} \dot{x} &= z, \\ \dot{z} &= -(1 + a + x^2)z - ((a - ab)x + x^3). \end{aligned} \tag{5}$$

After change z to y we can get system (1.1) as follows

$$\begin{aligned} \dot{x} &= y - \left((1 + a)x + \frac{1}{3}x^3\right), \\ \dot{y} &= -((a - ab)x + x^3). \end{aligned} \tag{6}$$

The system (2.4) is considered in two cases.

Case I: The Origin is an anti Saddle

The case under consideration is $a - ab \ge 0$, in this case and as above the system (2.4) has unique equilibrium point $O(0,0)$ which is an anti saddle.

Lemma 2.1

For $a = 0$ system (2.4) has no limit cycle.

Proof

Let $a=0$, then $F(x)=x+\frac{1}{3}x^3$ and $g(x)=x^3$. Thus $g(x)F(x)=x^4\left(1+\frac{1}{3}x^2\right)\geq 0$ and by using theorem (1.2) there is no limit cycle so we just look for $a\neq 0$.

In the case of $a<0$ O becomes saddle also for $a^2-4>0$ or ($a>2$) O is node in two cases no limit cycles surround *O*. Thus, in the sequel, we only need to consider $0<a<2$.

Consider the polynomial Liénard system of degree n

$$\begin{aligned}\dot{x}&=y-\left(a_1x+a_2x^2+\cdots+a_mx^n\right),\\ \dot{y}&=-\left(b_1x+b_2x^2+\cdots+b_kx^k\right).\end{aligned}\tag{7}$$

Lemma 2.2 [11]

For system (2.5) with $b_1=1$, the first three focal values at $O(0,0)$ are

$$\eta_2=-a_1,\quad \eta_4=\frac{1}{8}\left(2a_2b_2-3a_3\right),$$

$$\eta_6=c_0\left(6a_2a_4+20a_4b_2-15a_3b_3-15a_5\right).$$

where c_0 is positive constant.

By scaling $x\to x\sqrt{a-ab}$ and $t\to t\sqrt{a-ab}$ [where new $x=\frac{x}{\sqrt{a-ab}}$, $\dot{x}=\dot{x}$ and $\dot{y}=\sqrt{a-ab}\,\dot{y}$], then system (2.4) becomes

$$\begin{aligned}\dot{x}&=y-\left(\frac{a-1}{\sqrt{a-ab}}x+\frac{1}{3(a-ab)^3\sqrt{a-ab}}x^3\right),\\ \dot{y}&=-x\left(1+\frac{1}{(a-ab)^2}x^2\right).\end{aligned}\tag{8}$$

Therefore the three focal values of $O(0,0)$ and by using Lemma 2.2 namely are

$$\eta_2=-\frac{1+a}{\sqrt{a-ab}},\qquad \eta_4=-\frac{1}{(a-ab)\sqrt{a-ab}},$$

$$\eta_6=-\frac{5c_0}{(a-ab)^2\sqrt{a-ab}}.$$

If $a\neq -1$ then *O*, is strong focus which is unstable for $a<-1$ and stable if $a>-1$, and for $a=1$, then O is weak focus of order one which is stable.

By using Hopf-bifurcation (by changing of stability), for $a>-1$ no limit cycle because no change of stability if $a=-1$, then O is weak focus of order one which is stable. Thus as a decreasing from -1 O becomes unstable and one stable limit cycle appears from Hopf-bifurcation.

Theorem 2.3

For $1<a<2$ the system (2.4) has a unique stable limit cycle.

Proof:

Now we apply theorem (1.1) consider $g(x)=(a-ab)x+x^3$ since

$a-ab>0$ So $g(x)$ has only one root which is $x=0$. For $F(x)=\frac{1}{3}x^3+(a+1)x$ the roots are $a_1=-\sqrt{-3(a+1)}<0<a_2=\sqrt{-3(a+1)}$. The roots of $f(x)$ are $-\sqrt{-(a+1)}<0<\sqrt{-(a+1)}$ and $f(x)$ has minimum at $(0,a+1)$.

Since we have $-\sqrt{-3(a+1)}<-\sqrt{-(a+1)}<0<\sqrt{-(a+1)}<\sqrt{-3(a+1)}$, then we deduce that $f(x)\geq 0$ for $x\notin(a_1,a_2)$
$g(x)F(x)=x^2\left(a-ab+x^2\right)\left(a+1+\frac{1}{3}x^3\right)$ since $a-ab>0$ then the term $a-ab+x^2>0$ and the value of $\left(a+1+\frac{1}{3}x^3\right)<0$ in the interval $\left(-\sqrt{-3(a+1)},\sqrt{-3(a+1)}\right)$ so $g(x)F(x)\leq 0$ for $x\in(a_1,a_2)$. Finally since $a_1=-a_2$ so we have $G(a_1)=G(a_2)$.

Case II: The Origin is a saddle

In this case, we discuss system (2.4) when $a-ab<0$ and as above the system has three equilibrium points $O(0,0)$ and $\pm\alpha$ where $\alpha=\sqrt{ab-a}$ trance the $(\alpha,0)$ to the Origin by the relation $x\to(x-\alpha)$

$$\begin{aligned}\dot{x}&=y-\left((1+ab)x-\left(\sqrt{ab-a}\right)x^2+\frac{1}{3}x^3\right),\\ \dot{y}&=-x\left(2(ab-a)-3\sqrt{ab-a}x+x^2\right).\end{aligned}\tag{9}$$

Let $t=-\tau$, $y\to-y$, (2.4) is converted into

$$\begin{aligned}\dot{x}&=y-\left(-(1+ab)x+\left(\sqrt{ab-a}\right)x^2-\frac{1}{3}x^3\right),\\ \dot{y}&=-x\left(2(ab-a)-3\sqrt{ab-a}x+x^2\right).\end{aligned}\tag{10}$$

$$\eta_2=\frac{ab+1}{\sqrt{2(ab-a)}},\quad \eta_4=-\frac{1}{4(ab-a)\sqrt{a-ab}},$$

$$\eta_6=-\frac{5}{8(ab-a)\sqrt{2(ab-a)}}.$$

By using Hopf-bifurcation, for $ab+1<0$ no limit cycle because no change of stability if $ab+1=0$, then O becomes weak focus of order one which is stable. Thus for fixe b $a^*=-\frac{1}{b}$ is bifurcate value so as a^* increasing, O becomes unstable and one stable limit cycle appear from Hopf-bifurcation.

Lemma 3.1

$ab+a+2>0$ equivalent to $ab+1>0$.

Proof

Assume that $ab+1<0$ since $a+1<0$, then we have $ab+a+2=ab+1+a+1<0$ contradiction. Thus for $ab+a+2>0$ also we get $ab+1>0$.

Lemma 3.2 [10]

If there exists a constant $m\geq 0$ such that $F'(x)G(x)-mF(x)g(x)\geq 0$ for

$x(\neq 0)$, System (2.8) has at most one limit cycle.

We have by putting $c = ab+1$, $\alpha = \sqrt{ab-a}$ and after simplify we have

$$\phi(x,m) = \left(\frac{m}{3}-\frac{1}{4}\right)x^4 + \left(\frac{3}{2}\alpha - 2m\alpha\right)x^3 + \left(\left(c+\frac{11}{3}\alpha^2\right)m - \frac{c}{4} - 3\alpha^2\right)x^2$$
$$\left(c\alpha + 2\alpha^3 - \left(3c\alpha + 2\alpha^3\right)m\right)x + \frac{1}{2}c\alpha^2$$

Let $m = \frac{3}{4}$ so we have

$$\phi\left(x,\frac{3}{4}\right) = \left(\frac{1}{2}c - \frac{1}{4}\alpha^2\right)x^2 + \left(\frac{1}{2}\alpha^3 - \frac{1}{4}c\alpha\right)x + \frac{1}{2}c\alpha^2$$

$$\Delta = \frac{1}{4}(ab-a)^3 + \frac{1}{4}(ab+1)(ab-a)^2 - \frac{15}{16}(ab+1)^2(ab-a).$$

Since $(ab-a) > 0$, we can delete from upper equation and for suitable a as small enough we have

$$\Delta = \frac{1}{4}(ab-a)^2 + \frac{1}{4}(ab+1)(ab-a) - \frac{15}{16}(ab+1)^2 < 0.$$

3. Conclusion

A non-Liénard system is studied and analyzed by adapting Hopf-bifurcation theory. It has been proved that the system has unique limit cycle under some change of parameters under two cases. Bendixons theorem is used to prove non-existence of limit cycles.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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