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Adequate Closed Form Wave Solutions to the Generalized KdV Equation in Mathematical Physics

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Abstract

In this paper, we consider the generalized Korteweg-de-Vries (KdV) equations which are remarkable models of the water waves mechanics, the shallow water waves, the quantum mechanics, the ion acoustic waves in plasma, the electro-hydro-dynamical model for local electric field, signal processing waves through optical fibers, etc. We determine the useful and further general exact traveling wave solutions of the above mentioned NLDEs by applying the $\exp(-\tau(\xi))$-expansion method by aid of traveling wave transformations. Furthermore, we explain the physical significance of the obtained solutions of its definite values of the involved parameters with graphic representations in order to know the physical phenomena. Finally, we show that the $\exp(-\tau(\xi))$-expansion method is convenient, powerful, straightforward and provide more general solutions and can be helping to examine vast amount of travelling wave solutions to the other different kinds of NLDEs.

Keywords

The Generalized KdV Equation, The $\exp(-\tau(\xi))$-Expansion Method, Travelling Wave, Solitary Wave

1. Introduction

Differential equations are very important branch of modern mathematics. Differential equations are basically two kinds, such as, ordinary differential equation and partial differential equation for classical mechanics. But in generalization, fractional differential equations are an important part of differential equations. Nonlinear differential equations (NLDEs) and fractional nonlinear differential equations (FNLDEs) are important through the applications in integer and fractional calculus and also gained much importance to the researchers in different branches of sciences and engineering, for instance, in mathematical physics, engineering and also it arises in signal processing, control theory, fractal dynamics, optical fibers, chemical kinematics, physics, applied physics, medicine, aerodynamics, hydrology, pharmacy, material science, the modeling of earthquake, electricity, biological science, fractal dynamics, population model of the motion of a projectile, rocket, planet or satellite, the charge or current in an electric circuit, the reactions of chemicals, the rate of growth of a population, spring mass systems, bending of beams condition of heat in a rod or in a slap, etc. The mathematical formulations of all of the above problems give rise to differential equations and fractional differential equations. Basically, most of the differential equations involving physical phenomena are nonlinear. It is simple to solve the differential equations which are linear, but the solutions of nonlinear equations are laborious, and, in some cases, it is impossible to solve them analytically. In such critical situation $\exp\left(-\tau\left(\xi\right)\right)$-expansion processes the investigators attempt to solve the nonlinear differential equations. Nonlinear wave phenomena appear in various scientific and engineering fields, such as, fluid mechanics, plasma physics, high energy physics, condensed matter physics, quantum mechanics, elastic media, biology, solid state physics, chemical kinematics optical fibers, biophysics, geochemistry, electricity, propagation of shallow water waves, chemical physics and so on. To understand better the nonlinear phenomena as well as further application in practical life, it is important to seek their more exact travelling wave solutions. The fundamental equations in physical sciences are nonlinear and in general NLPDEs are often very complicated to solve explicitly exact solutions of NLPDEs that play an important role in the study of nonlinear physical phenomena. Therefore, in the past three decades, many significant methods have been enhanced and developed to get exact solutions of NLPDEs, such as, integer and fractional types NLDEs [1] [2] [3]. Most of these methods are the homogeneous balance method, likely, the Kudryashov method [4], the generalized Kudryashov method [5], the Modified Kudryashov Method [6], the first integral method [7], the improved modified extended tanh-function method [3] [8], the (G′/G,1/G)-expansion technique [9] [10], advanced $\exp\left(-\phi\left(\xi\right)\right)$-expansion method [1] [11] [12], the modified extended tanh-function method [13] [14] [15], the Jacobi elliptic function method [16], the (G'/G^2)-expansion technique [17] [18], the sine-cosine methods [19] [20], the tanh-coth method [21], the simplified Hirota's method [22], the Hirota bilinear method [23] [24] [25], Soret and Dufour effects [26], the modified simple equation method [27], the exp function method [28], the sine-Gordon expansion method [29], the rational sine-Gordon expansion method [2] [30], Wang's Bäcklund transformation-based method [31], the variational iteration method [32] [33] [34], the new auxiliary equation method [35], Variational method [36], Deep Learning approach [37], the method of characteristics [38], Dixon resultant method [39], the three-dimensional molecular structure model [40], etc.

The obtained solutions are the remarkable mathematical model of the turbulent motion, the electro-hydro-dynamical model for the local electric field, the ion acoustic waves in plasma, the fluid flow of motion in shallow water waves under gravity, the propagation waves, the waves particle duality that is noteworthy, the signal processing waves through optical fibers, the variation over time of a physical structure on the fractional fluid mechanics system, ion acoustic waves, the varies in temperature from one place to another, the conservation of mass and acceleration due to gravity, the viscoelasticity waves, the traffic flow model, etc.. We have discussed the physical consequence of the attained solutions by setting definite values of the involved parameters by depicting figures. We also have established that the $\exp(-\tau(\xi))$-expansion method is potential, efficient, straightforward, further general, and rising method to search huge amounts of traveling wave solutions to the NLDEs and FNLDEs.

1.1. Research Objectives

- The main goal is to get closed-form solutions for the generalized KdV problem, which is a significant nonlinear partial differential equation in mathematical physics.
- To create or employ analytical techniques to acquire precise results.
- The objective is to categorize the many forms of wave solutions, such as solitons, periodic waves, and breathers, and analyze their characteristics and significance within the framework of the generalized KdV equation.
- The resulting solutions may be used to mimic real-world phenomena, such as shallow water waves, plasma waves, or other physical systems defined by the generalized KdV equation.
- This aims to explore the wider consequences of these answers in the realm of mathematical physics, encompassing possible practical uses and avenues for future investigation.

1.2. Research Gaps or Limitations

It is conceivable that certain types of nonlinear problems may not be solvable using this method. While it may not have the capability to solve solutions for complex and diverse nonlinear systems, it operates efficiently for specific types of equations. Employ other methodologies to verify the obtained solutions, such as asymptotic analysis, numerical simulations, or, if accessible, comparison with empirical data. This enhances the reliability and accuracy of the generated solutions. Utilize the logical Sine-Gordon expansion strategy as part of a broader set of tools. Integrate further perturbation, analytical, or numerical methods to mitigate its limitations and enhance its strengths.

2. Description of the $\exp(-\tau(\xi))$-Expansion Method

The $\exp(-\tau(\xi))$-expansion method is a powerful analytical technique used to find exact solutions to nonlinear differential equations (NLDEs). This method is particularly useful for obtaining traveling wave solutions of various forms, such as solitary waves, periodic waves, and other types of wave structures. The $\exp(-\tau(\xi))$-expansion method provides a systematic approach to derive exact solutions for NLDEs. By transforming the NLDE into an ODE, assuming an appropriate ansatz, and solving the resulting algebraic equations, this method can uncover a variety of exact wave solutions, enhancing our understanding of nonlinear phenomena in mathematical physics. Here's a detailed overview of the theoretical framework and mathematical principles underlying this method:

Here we briefly discuss the major characteristics of the $\exp(-\tau(\xi))$-expansion method. Let us suppose the general nonlinear partial differential equation of the form:

$$H\left(u,u_x,u_y,u_z,u_t,u_{xx},u_{xy},\cdots\right)=0, \tag{2.1}$$

where $u=u(x,y,z,t)$ is an unknown function, H is a polynomial in $u(x,y,z,t)$ and its derivatives in which highest order derivatives and nonlinear terms are occupied and the subscripts indicate partial derivatives.

Also, we consider the general nonlinear fractional partial differential equation of the form:

$$H\left(u,D_t^{\alpha}u,D_x^{\beta}u,D_y^{\gamma}u,D_z^{\varepsilon}u,D_t^{2\alpha}u,D_x^{2\beta}u,\cdots\right)=0, \tag{2.2}$$

where $u=u(x,y,z,t)$ is an unidentified function, H is a polynomial in $u(x,y,z,t)$ and its fractional derivatives, which include the highest order derivative and nonlinear terms of the highest order where in $\alpha,\beta,\gamma,\varepsilon$ are non-integer and the subscripts denote the partial derivatives.

In order to obtain exact wave solutions of Equation (2.1) or Equation (2.2) by applying the $\exp(-\tau(\xi))$-expansion method, we have to execute the following noteworthy steps:

Step-1. We combine the real variables x, y and t by a compound variable ξ

$$u(x,y,y,t)=u(\xi),\ \ \xi=x+y\pm wt, \tag{2.3}$$

where w is the velocity of the traveling wave and we consider the following traveling wave variable,

$$u(x,y,z,y,z,t)=u(\xi),\ \ \xi=\frac{kt^{\alpha}}{\Gamma(1+\alpha)}+\frac{mx^{\alpha}}{\Gamma(1+\alpha)}+\frac{ny^{\gamma}}{\Gamma(1+\gamma)}+\frac{lz^{\varepsilon}}{\Gamma(1+\varepsilon)}, \tag{2.4}$$

for fractional differential equations.

Now by making use of the traveling wave transformation Equation (3.3) or Equation (3.4) the partial differential Equation (2.1) or Equation (2.2) turns into ordinary differential equation (ODE) as below:

$$G\left(u,u',u'',u''',\cdots\right)=0, \tag{2.5}$$

where G is a polynomial of u and its derivatives, and the superscripts refer to the ordinary derivatives with respect to ξ.

Step-2. We advise that the solution of Equation (3.5) can be exposed in the form:

$$u(\xi)=\sum_{i=0}^{N} A_i\left(\exp\left(-\tau(\xi)\right)\right)^i, \tag{2.6}$$

where $A_i\left(0\le i\le N\right)$ are constants to be determined, such that $A_N\ne 0$ and $\tau=\tau(\xi)$ and satisfied the following ordinary differential equation:

$$\tau'(\xi)=\exp\left(-\tau(\xi)\right)+\mu\exp\left(\tau(\xi)\right)+\lambda \tag{2.7}$$

Equation (2.7) documented the following solutions:

Set-1: When $\mu\ne 0$, $\lambda^2-4\mu>0$,

$$\tau(\xi)=\ln\left(\frac{-\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi+c)\right)-\lambda}{2\mu}\right) \tag{2.8}$$

Set-2: When $\mu\ne 0$, $\lambda^2-4\mu<0$,

$$\tau(\xi)=\ln\left(\frac{\sqrt{4\mu-\lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi+c)\right)-\lambda}{2\mu}\right) \tag{2.9}$$

Set-3: When $\mu=0$, $\lambda\ne 0$ and $\lambda^2-4\mu>0$,

$$\tau(\xi)=-\ln\left(\frac{\lambda}{\exp\left(\lambda(\xi+c)\right)-1}\right) \tag{2.10}$$

Set-4: When $\mu\ne 0$, $\lambda\ne 0$, and $\lambda^2-4\mu=0$,

$$\tau(\xi)=\ln\left(-\frac{2\left(\lambda(\xi+c)+2\right)}{\lambda^2(\xi+c)}\right) \tag{2.11}$$

Set-5: When $\mu=0$, $\lambda\ne 0$, and $\lambda^2-4\mu=0$,

$$\tau(\xi)=\ln(\xi+c) \tag{2.12}$$

Step-3. The positive integer N can be calculated by considering the homogeneous balance between the highest order derivatives and the nonlinear terms of the highest order appearing in Equation (2.5).

Step-4. We utilize Equation (2.6) into Equation (2.5) and then we consider the function $\exp\left(-\tau(\xi)\right)$. Therefore, of this substitution, we attain a polynomial in $\exp\left(-\tau(\xi)\right)$ and equalize to zero express a system of algebraic equations whichever can be solved to find A_N, …, w, λ, μ. The values of A_N, …, w, λ, μ in company with general solution of Equation (3.7) inclusive the determination of the solution of Equation (2.1) or Equation (2.2).

3. Application of the Generalized KdV Equation

In the literature, examined the exact solutions of the Korteweg-de-Vries equation is of the form $u_t + u_{xxx} - 6uu_x = 0$. So far we know, the general form the Korteweg-de-Vries equation which is not solved by applying the $\exp(-\tau(\xi))$-expansion method. Therefore, in this sub-section, we have examined the exact solutions of the generalized Korteweg-de-Vries equation which arranged below:

Let us consider the generalized Korteweg-de-Vries equation in the form,

$$u_t + uu_x + \delta u_{xxx} = 0, \tag{3.1}$$

where δ is an arbitrary constant and different values of δ the equation represents generalization form of the equation.

Now, we have applied the traveling wave transformation Equation (2.3) to reduce Equation (3.1) into the following ordinary differential equation:

$$-wu' + uu' + \delta u''' = 0 \tag{3.2}$$

Integrating Equation (4.2.2) with respect to ξ and choosing the integrating constant zero, we have attained

$$-wu + \frac{1}{2}u^2 + \delta u'' = 0 \tag{3.3}$$

Now, balancing between the highest order nonlinear term and linear terms occurring in Equation (3.3), yields $N = 2$. Therefore, the solution of Equation (3.3) takes the following form

$$u(\xi) = A_0 + A_1 \exp(-\tau(\xi)) + A_2 \left(\exp(-\tau(\xi))\right)^2, \tag{3.4}$$

where $A_0 A_0, A_1, A_2$ are arbitrary constants such that $A_2 \neq 0$.

We substitute Equation (3.4) into Equation (3.3) and taking consideration Equation (3.4), it generates a polynomial and then setting the coefficients of $\exp(-\tau(\xi))$ to zero, yields

$$-wA_0 + \frac{A_0^2}{2} + \delta\lambda\mu A_1 + 2\delta\mu^2 A_2 = 0,$$

$$-wA_1 + A_0 A_1 + \delta\lambda^2 A_1 + 2\delta\mu A_1 + 6\delta\lambda\mu A_2 = 0,$$

$$\frac{A_1^2}{2} - wA_2 + A_0 A_2 + 3\delta\lambda A_1 + 4\delta\lambda^2 A_2 + 8\delta\mu A_2 = 0,$$

$$2\delta A_1 + 10\delta\lambda A_2 + A_1 A_2 = 0,$$

$$\frac{A_2^2}{2} + 6\delta A_2 = 0.$$

Solving the above system of equation, we get two sets of solutions:

Set-1

$$A_0 = -12\delta\mu,\ A_1 = -12\delta\lambda,\ A_2 = -12\delta,\ w = \delta\left(\lambda^2 - 4\mu\right).$$

Set-2

$$A_0 = -2\delta\left(\lambda^2 + 2\mu\right),\ A_1 = -12\delta\lambda,\ A_2 = -12\delta,\ w = -\delta\left(\lambda^2 - 4\mu\right).$$

where λ, μ, δ are arbitrary constants.

For simplicity we have discussed the solution Set-1 of the mentioned equation.

Now replacing the value of Set-1 into Equation (3.4) gives the following.

$$u(\xi) = -12\delta\mu - 12\delta\lambda \exp(-\tau(\xi)) - 12\delta(\exp(-\tau(\xi)))^2 \tag{3.5}$$

Now substituting Equations (3.6)-(3.10) into Equation (4.2.12) respectively we get the following five traveling wave solution of the nonlinear generalized Korteweg-de-Vries equation.

While $\mu \neq 0$, $\lambda^2 - 4\mu > 0$,

$$u_1(\xi) = -12\delta\mu - 12\delta\lambda \left(\frac{-2\mu}{\left(\sqrt{\lambda^2 - 4\mu}\right) \tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + c) \right) + \lambda} \right)$$

$$-12\delta \left(\frac{-2\mu}{\left(\sqrt{\lambda^2 - 4\mu}\right) \tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + c) \right) + \lambda} \right)^2$$

Applying $\mu \neq 0$, $\lambda^2 - 4\mu < 0$,

$$u_2(\xi) = -12\delta\mu - 12\delta\lambda \left(\frac{2\mu}{\left(\sqrt{4\mu - \lambda^2}\right) \tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + c) \right) - \lambda} \right)$$

$$-12\delta \left(\frac{2\mu}{\left(\sqrt{4\mu - \lambda^2}\right) \tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + c) \right) - \lambda} \right)^2$$

When $\mu = 0$, $\lambda \neq 0$, $\lambda^2 - 4\mu > 0$,

$$u_3(\xi) = -12\delta\mu - 12\delta\lambda \left(\frac{1}{\exp(\lambda(\xi + c)) - 1} \right) - 12\delta \left(\frac{1}{\exp(\lambda(\xi + c)) - 1} \right)^2$$

Using $\mu \neq 0$, $\lambda \neq 0$, $\lambda^2 - 4\mu > 0$,

$$u_4(\xi) = -12\delta\mu - 12\delta\lambda \left(-\frac{\lambda(\xi + c)}{2(\lambda(\xi + c)) + 2} \right) - 12\delta \left(-\frac{\lambda(\xi + c)}{2(\lambda(\xi + c)) + 2} \right)^2$$

When $\mu = 0$, $\lambda = 0$, $\lambda^2 - 4\mu = 0$,

$$u_5(\xi) = -12\delta\mu - 12\delta\lambda \left(\frac{1}{\xi + c} \right) - 12\delta \left(\frac{1}{\xi + c} \right)^2$$

where c is an arbitrary constant.

4. Result and Discussion

4.1. Graphical Presentation

In this section, we have delineated the shape of figures of the obtained solutions to the generalized Korteweg-de-Vries (KdV) equation, which are given below:

Figure 1 shows the shape of the figure of the obtained solution $u_1(\xi)$ for $\lambda=3$, $\mu=1$, $\delta=2$, $c=1$ of the parameters is shown below.

Figure 2 shows the shape of the figure of the obtained solution $u_2(\xi)$ for $\lambda=3$, $\mu=1$, $\delta=2$, $c=1$ of the parameters is shown below.

Figure 3 shows the shape of the figure of the obtained solution $u_3(\xi)$ for $\lambda=3$, $\mu=1$, $\delta=2$, $c=1$ of the parameters is shown below.

Figure 4 shows the shape of the figure of the obtained solution $u_4(\xi)$ for $\lambda=3$, $\mu=1$, $\delta=2$, $c=1$ of the parameters is shown below.

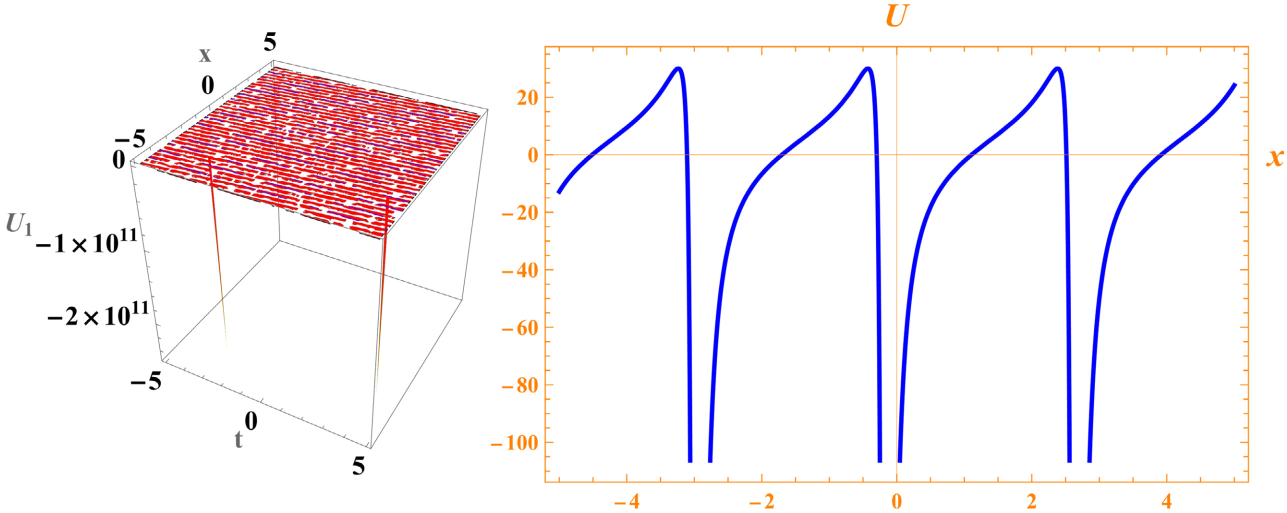

Figure 1. 3D and 2D plot of solution $u_1(\xi)$ which is singular periodic within the interval $-5 \le x,t \le 5$ for 3D and $t=0$ for 2D.

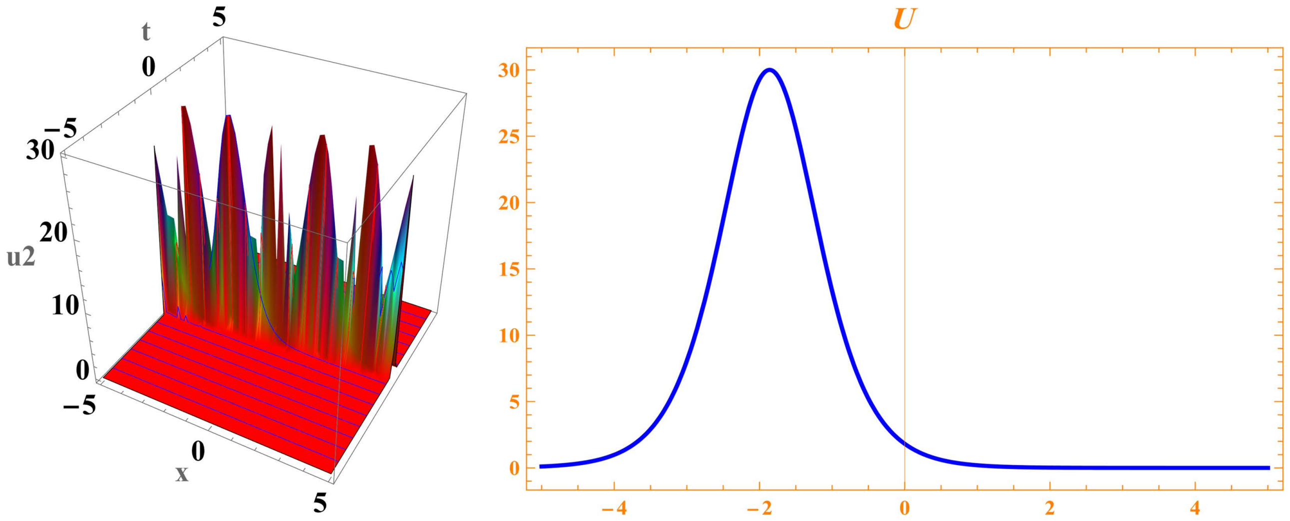

Figure 2. The 3D and 2D plots of solution $u_2(\xi)$ which is kink type within the interval $-5 \le x,t \le 5$ for 3D and $t=0$ for 2D.

Figure 5 shows the shape of the figure of the obtained solution $u_5(\xi)$ for $\lambda = 3$, $\mu = 1$, $\delta = 2$, $c = 1$ of the parameters is shown below.

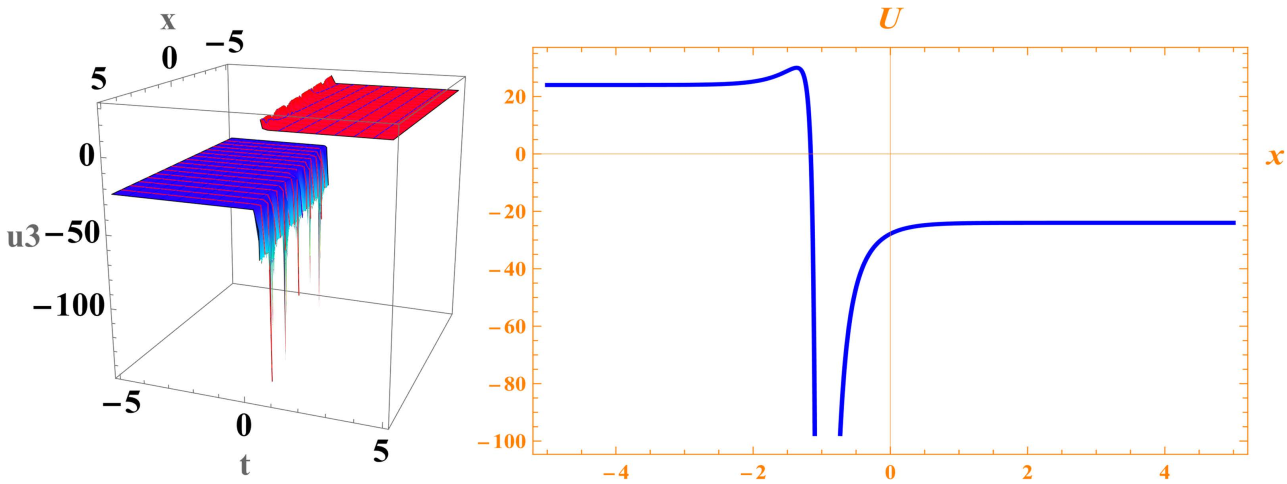

Figure 3. 3D and 2D plot of solution $u_3(\xi)$ which is a singular kink within the interval $-5 \le x,t \le 5$ for 3D and $t = 0$ for 2D.

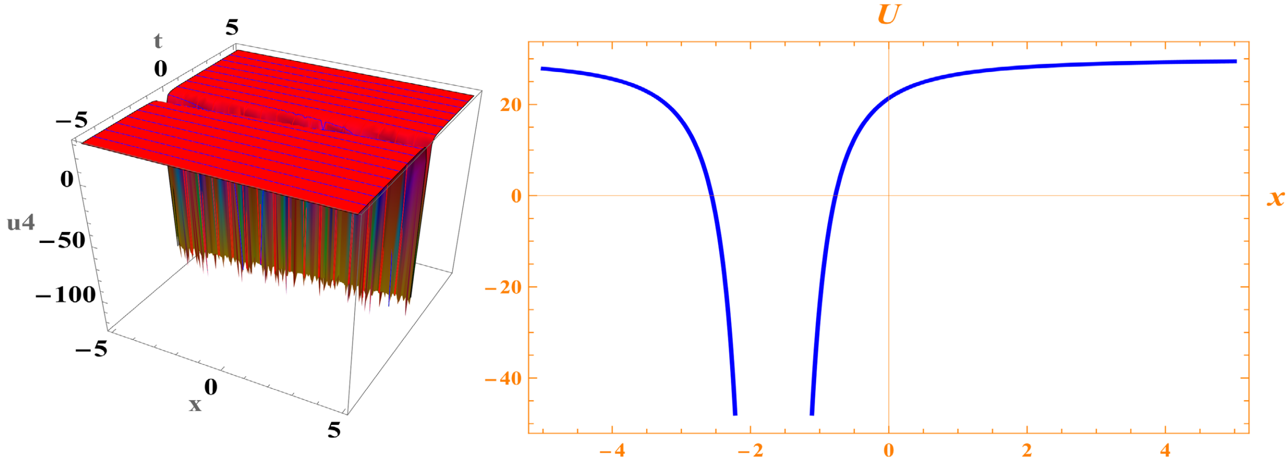

Figure 4. 3D and 2D plot of solution $u_4(\xi)$ which is singular kink within the interval $-5 \le x,t \le 5$ for 3D and $t = 0$ for 2D.

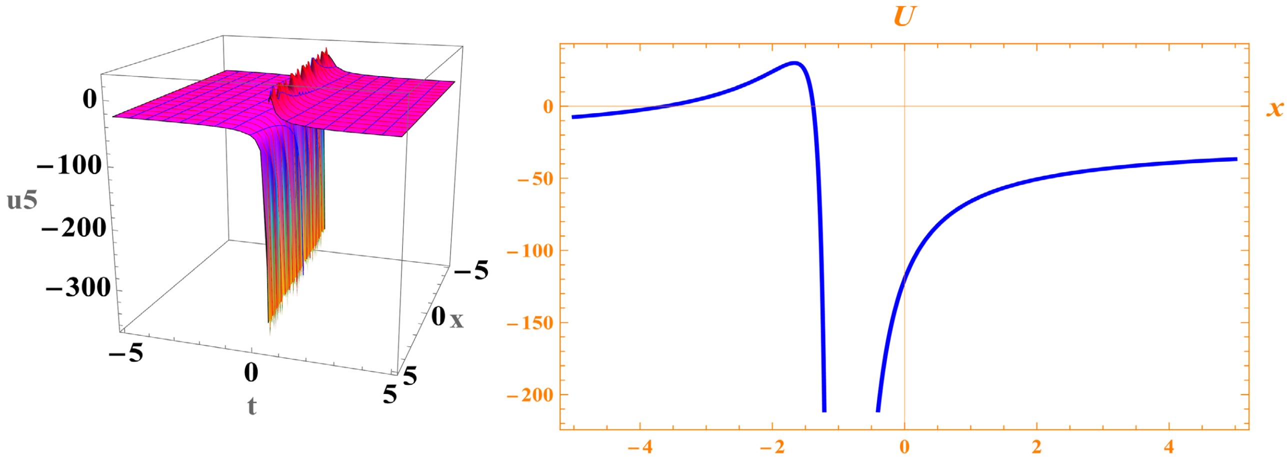

Figure 5. 3D and 2D plots of solution $u_5(\xi)$ which is a singular kink within the interval $-5 \le x,t \le 5$ for 3D and $t = 0$ for 2D.

4.2. Discussion

Lee and Kuo (2015) examined the generalized KdV equations, and they found four solutions (see Appendix) by applying the simplest equation method. Using the $\exp(-\tau(\xi))$-expansion method, we attained five travelling wave solutions, which are not repeated in the simplest equation method. Furthermore, the several choices of the integral constant from Equations (3.8)-(3.12) give different types of exact wave solutions. Therefore, comparing the solutions obtained and those obtained by Lee and Kuo, we might conclude that our attained solutions are practically and further general and give many solutions compared to those obtained by Lee and Kuo.

5. Conclusion

In this research, we have determined the new, valuable and more general exact travelling wave solutions of the generalized Korteweg-de-Vries (KdV) equation by applying the $\exp(-\tau(\xi))$-expansion method by means of the traveling wave transformations. Most of the attained solutions are in the form of trigonometry, hyperbolic and rational functions. These attained solutions might be useful to the physical events related to the fluid motion in shallow water waves, the water waves under gravity in the long-wave regime, the ion acoustic waves in plasma, the quantum mechanics, the electro-hydro-dynamical model for local electric field, the signal processing through optical, etc. We have also discussed the physical significance of the obtained solutions by depicting the graphs. Different types of well-known shape of solutions are examined, likely, the kink shape wave solutions, the singular kink shape wave solutions, the singular periodic shape wave solutions, etc. The attained solutions showed that the $\exp(-\tau(\xi))$-expansion method is straightforward, efficient powerful and more general which can be used to examine exact wave solutions to the different NLDEs and FNLDEs arising in different fields of mathematics and engineering. To obtain these types of solutions, sometimes it arises some difficulties to determine balance number, the using method and other methods does not keep simple and straightforward and it does not give solution straightforwardly to the NLDEs and FNLDEs. Therefore, our future work is how we can employ the $\exp(-\tau(\xi))$-expansion method and the other methods to NLDEs and FNLDEs for balance number and how we can employ the methods to detect these complicated and highly nonlinear DEs for non-integer balance number and also without considering another new transformation (*i.e.*, using directly the balance number).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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