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Analysis of the Boundary Stability of a Diffusion-Reaction System on a Nanolayer

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Abstract

In this paper, the focus is on the boundary stability of a nanolayer in diffusion-reaction systems, taking into account a nonlinear boundary control condition. The authors focus on demonstrating the boundary stability of a nanolayer using the Lyapunov function approach, while making certain regularity assumptions and imposing appropriate control conditions. In addition, the stability analysis is extended to more complex systems by studying the limit problem with interface conditions using the epi-convergence approach. The results obtained in this article are then tested numerically to validate the theoretical conclusions.

Keywords

Limit Behavior, Limit Problems, Feedback Control, Epi-Convergence Method, Lyapunov Method, Nanolayer

1. Introduction

A crucial component of physical and biological systems that involve diffusion and response phenomena is boundary stability. Lyapunov functions are a crucial tool for researchers in applied mathematics and engineering in this situation for demonstrating stability. On the other hand, the direct Lyapunov technique was developed by Lyapunov in the 19th century and is not limited to a local character. It makes use of an energy function to ascertain the stability qualities of a nonlinear system [1]. This method uses a Lyapunov function, which is positive and decreasing along the trajectories of the system, to establish the stability of the system. [2] examines the connections between a system's asymptotic behaviour, the spectral characteristics of its dynamics, and the presence of a Lyapunov functional. The methods employed are based on these connections, the Lyapunov function, or the Riccati equation, as in [2]-[8]. While the asymptotic and exponential stabilizability are explored in [2] and [4], respectively, via the Riccati equation, the exponential stabilizability is investigated in [3] [9] via a suitable decomposition of the state space. The stability of dynamical systems has been the subject of numerous studies, with a focus on the use of Lyapunov functions. Through examination of the system trajectories' convergence to a stable equilibrium state, this work has demonstrated how Lyapunov functions can be utilized to demonstrate the stability of dynamical systems.

The majority of previous research on boundary stability, however, has been conducted in the context of macroscopic or mesoscopic boundaries, where the boundary is a few millimeters or larger. Nanolayer boundary stability has not received much attention. To do so, let us considers the problem of quasi-linear evolution in a body occupying a domain $\Omega \subset \mathbb{R}^3$ with a Lipschitz boundary $\partial\Omega$, a surface Σ_ε which is a part of $\partial\Omega$ and located on the boundary $\partial\Omega$ (see Figure 1), the last-mentioned body is subjected to an external temperature f, and cooled at the boundary $\partial\Omega$, and given a function f bounded on $L^2\left(0,\infty;H^{-1}(\Omega)\right)$. The domain is defined as follows: $\Sigma_\varepsilon = \left\{x \in \partial\Omega \mid |x_3| \le \varepsilon^2\right\}$ is a surface located at a distance of ε^2 from the upper and lower boundaries of $\partial\Omega$, with ε being a parameter intended to tend towards 0. $\Gamma_\varepsilon = \left\{x \in \partial\Omega \mid |x_3| > \varepsilon^2\right\}$ is the remaining part of the boundary.

The system of equations is as follows:

$$\begin{cases} \dot{z} - \Delta z = f & \text{in } \Omega^\infty \\ \dfrac{\partial z}{\partial n} = \dfrac{1}{\varepsilon^\alpha}|u|^{p-2}u & \text{on } \Sigma_\varepsilon^\infty \\ z = 0 & \text{on } \Gamma_\varepsilon^\infty \\ z(t=0,x) = z_0 & \text{on } \Omega, \end{cases}$$

with, $\Omega^\infty = [0,\infty[\times\Omega$, $\Sigma_\varepsilon^\infty = [0,\infty[\times\Sigma_\varepsilon$, $\Gamma_\varepsilon^\infty = [0,\infty[\times\Gamma_\varepsilon$.

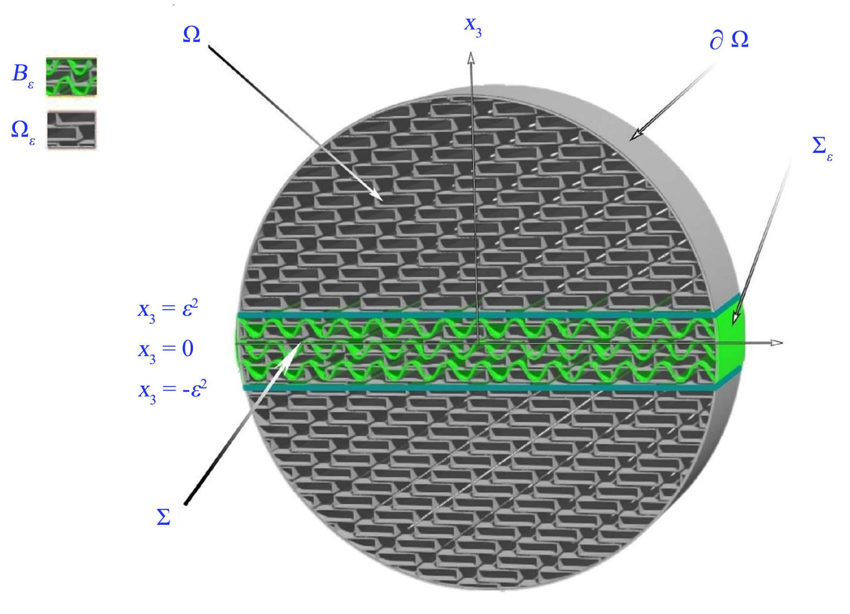

Figure 1. Domain Ω.

Where u is a control that stabilizes the z state of the given dynamic system on the Σ_ε boundary, given that u belongs to the set of admissible controls $U_{ad} = \left\{ u \in L^p\left(0,\infty; L^p\left(\Sigma_\varepsilon\right)\right) : \left\| u(t) \right\|_{L^p(\Sigma_\varepsilon)} \le C, \forall t \ge 0 \right\}$. Here, $p \ge 2$, $\alpha \ge 0$ and C is a positive constant. Now, this study focuses on the nonlinear boundary control condition in diffusion-reaction systems with nanolayer boundary stability. We demonstrate how Lyapunov functions can be applied to these systems to demonstrate nanolayer boundary stability. We also expand our examination of diffusion-reaction systems as more complicated systems for our investigation of nanolayer boundary stability. The understanding of nanolayer boundary stability in diffusion-reaction systems with a nonlinear boundary control condition is significantly advanced by this article. In order to acquire the limit problem and arrive at the topic of this article, which is grouped as follows, the aim would be to search for another equivalent approximation model to work with the finite element method in an exact fashion.

This study examines the impact of nonlinear boundary control conditions on the stability of the nanolayer in diffusion-reaction systems. The paper is structured as follows: Section 2 addresses the preliminary elements necessary to understand the rest of the article. These preliminaries are essential for establishing the context and laying the groundwork for the problem studied. Section 3 demonstrates the stability of the diffusion-reaction system for the approximate problem related to the initial problem using the Lyapunov method, such as energy estimates or variational techniques, and we present the a priori estimates. With the help of the initial findings, definitions, and some properties of the minimization problem, we proceed to the limit. In order to solve the limit problem with interface conditions and acquire a better understanding of the system's behavior near the nanolayer boundary, the method of epi-convergence is taken into account. The results obtained enrich the understanding of the stability of the nanolayer in diffusion-reaction systems, with potential practical applications, and provide an update on recent results on the Lyapunov function approach for nonlinear boundary control. Finally, Section 4 presents a numerical test that illustrates the theoretical results and shows the applicability and accuracy of the proposed strategy.

2. Preliminaries

2.1. Notations

- Let us define the operator m^ε which transforms functions defined z on Σ_ε into functions defined on Σ, like in [10]

$$m^\varepsilon z\left(t, x_1, x_2\right) = \frac{1}{2\varepsilon^2} \int_{-\varepsilon^2}^{\varepsilon^2} z\left(t, x_1, x_2, x_3\right) \mathrm{d}x_3.$$

- $\mathrm{d}\sigma$: represents the surface measure on Σ_ε.
- $(t, x) = (t, x', x_3)$, where $x' = (x_1, x_2)$, $\nabla' = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right)$, $\eta(\alpha) = \lim_{\varepsilon \to 0} \varepsilon^{2-\alpha}$,

with $\alpha \geq 0$.

In the following, C will denote any constant with respect to ε.

2.2. Functional Setting

$$\mathbb{H}_{\varepsilon,0} = \left\{ z \in H^1(\Omega) \mid z = 0 \text{ on } \Gamma_\varepsilon \right\}, \text{ sach that } H_0^1(\Omega) \subset \mathbb{H}_{\varepsilon,0} \subset H^1(\Omega).$$

The demonstration concerns the convergence of Σ_ε to Σ in the Hausdorff sense. To do this, we define a family of subvarieties $M_\varepsilon = \left\{ x \in \partial\Omega \mid |x_3| \leq \varepsilon^2 \right\}$ and we calculate the Hausdorff distance between Σ_ε and Σ. We find that for all x in Σ_ε, the distance to Σ is at most ε^2, and for all y in Σ, the distance to Σ_ε is 0. Thus, the Hausdorff distance is equal to ε^2. Finally, we show that when ε tends to 0, the Hausdorff distance also tends to 0, demonstrating the convergence of Σ_ε to Σ in the Hausdorff sense.

$$\mathbb{H}_\Sigma = \left\{ z \in H^1(\Omega) \mid z = 0 \text{ on } \partial\Omega \setminus \Sigma \right\}$$

$$\mathbb{G} = \left\{ z \in L^2(0,\infty;\mathbb{H}_\Sigma) : z_{|\Sigma} \in L^2\left(0,\infty;L^2(\Sigma)\right) \right\}$$

$$\mathbb{D} = \left\{ z \in \mathcal{D}\left( ]0,\infty[\times \Omega \right) \right\}$$

We know that $\overline{\mathbb{D}} = \mathbb{G}$.

2.3. Functional Framework

We'll put out the epi-convergence notion of operator's sequence convergence;

Definition 2.1 ([11], Definition 1.9.). Let (X,τ) be a reflexive Banach space, $F_\varepsilon : X \to \mathbb{R} \cup +\infty$ a family of convex functionals, and $F : X \to \mathbb{R} \cup +\infty$ a convex functional. Suppose that

1) $\liminf_{\varepsilon \to 0} F_\varepsilon(x) \geq F(x)$ for all $x \in X$.

2) For any sequence $(x_\varepsilon) \subset X$ such that $x_\varepsilon \rightharpoonup x$ weakly in X, we have $\limsup_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon) \leq F(x)$. Then, we have $F_\varepsilon \xrightarrow{\tau-\text{epi}} F$.

We present the function spaces used in the study and go over some of their fundamental characteristics.

Remark 2.1. [12] Since the Sobolev space $H^{1/2}(\Omega)$ is compactly embedded in the Lebesgue space $L^p(\Omega)$ for all $\frac{3}{2} < p < \infty$, then the space $L^p\left(0,\infty;H^{1/2}(\Omega)\right)$ is also compactly embedded in the space $L^p\left(0,\infty;L^p(\Omega)\right)$ for all $\frac{3}{2} < p < \infty$.

Remark 2.2. [12] Let Ω be a bounded open subset of $\mathbb{R}^n$ with a boundary of class C^1, and let Σ_ε be a part of the boundary of Ω ($\Sigma_\varepsilon \subset \partial\Omega$). If $z_\varepsilon(t) \in H^1(\Omega)$ for all $t \geq 0$, then:

1) The Sobolev trace theorem guarantees that the trace of $z_\varepsilon(t)$ on Σ_ε is well-defined and belongs to the Sobolev space $H^{1/2}(\Sigma_\varepsilon)$.

2) Furthermore, the normal derivative $\frac{\partial z_\varepsilon}{\partial n}$ of z_ε on Σ_ε belongs to

$L^2\left(0,\infty;H^{-1/2}\left(\Sigma_\varepsilon\right)\right)$. This property follows from the fact that the trace of z_ε on Σ_ε is in $H^{1/2}\left(\Sigma_\varepsilon\right)$, in accordance with the Neumann condition imposed on Σ_ε, which implies that $\frac{\partial z_\varepsilon}{\partial n}$ belongs to $H^{-1/2}\left(\Sigma_\varepsilon\right)$ as a distribution.

Remark 2.3. [13] Continuous injection of the Sobolev space $H^{-1/2}\left(\Sigma_\varepsilon\right)$ into the space $L^2\left(\Sigma_\varepsilon\right)$. It asserts that the norm in $H^{-1/2}\left(\Sigma_\varepsilon\right)$ is equivalent to the norm in $L^2\left(\Sigma_\varepsilon\right)$, up to a constant factor of $\sqrt{2}$.

Indeed, we have the following relation for all $z_\varepsilon(t)\in H^{-1/2}\left(\Sigma_\varepsilon\right)$:

$$\left\|z_\varepsilon(t)\right\|_{H^{-1/2}(\Sigma_\varepsilon)} = \max_{\substack{\phi\in H^{1/2}(\Sigma_\varepsilon)\\ \|\phi\|_{H^{1/2}(\Sigma_\varepsilon)}\le 1}} \left\langle z_\varepsilon(t),\phi\right\rangle_{H^{-1/2}(\Sigma_\varepsilon),H^{1/2}(\Sigma_\varepsilon)}.$$

Using the definition of the L^2 norm and the trace operator, we can show that:

$$\left\|z_\varepsilon(t)\right\|_{L^2(\Sigma_\varepsilon)} = \left(\int_{\Sigma_\varepsilon}\left|z_\varepsilon(t)\right|^2\mathrm{d}s\right)^{1/2} = \frac{1}{\sqrt{2}}\left(\int_{\Sigma_\varepsilon}\left|z_\varepsilon(t)\right|^2\mathrm{d}\sigma\right)^{1/2} = \frac{1}{\sqrt{2}}\left\|z_\varepsilon(t)\right\|_{H^{-1/2}(\Sigma_\varepsilon)}.$$

3. Main Results

3.1. Stability Study

We consider the following approximate problem:

$$\begin{cases} \dot{z}_\varepsilon - \Delta z_\varepsilon = f_\varepsilon & \text{in } \Omega^\infty \\ \dfrac{\partial z_\varepsilon}{\partial n} = \dfrac{1}{\varepsilon^\alpha}\left|u_\varepsilon\right|^{p-2}u_\varepsilon & \text{on } \Sigma_\varepsilon^\infty \\ z_\varepsilon = 0 & \text{on } \Gamma_\varepsilon^\infty \\ z_\varepsilon(t=0,x) = z_{0,\varepsilon} & \text{on } \Omega \end{cases}$$

Using the Lyapunov method, stabilize the border $\Sigma_\varepsilon^\infty$ with a control u_ε.

First, we choose the Lyapunov function $V\left(z_\varepsilon\right)$ as follows:

$$V\left(z_\varepsilon\right) = \frac{1}{2}\int_{\Omega^\infty}\left|z_\varepsilon\right|^2\mathrm{d}x\mathrm{d}t$$

1) Since $\left|z_\varepsilon\right|^2\ge 0$ for all $z_\varepsilon\in L^2\left(0,\infty;H^1(\Omega)\right)$, we have:

$$\frac{1}{2}\int_{\Omega^\infty}\left|z_\varepsilon\right|^2\mathrm{d}x\mathrm{d}t\ge 0$$

2) If $z_\varepsilon=0$, then $V\left(z_\varepsilon\right)=0$. Conversely, if $V\left(z_\varepsilon\right)=0$, then $\int_{\Omega^\infty}\left|z_\varepsilon\right|^2\mathrm{d}x\mathrm{d}t=0$, which implies $z_\varepsilon=0$ almost everywhere in Ω^∞ (since $z_\varepsilon\in L^2\left(0,\infty;H^1(\Omega)\right)$). Thus, $V\left(z_\varepsilon\right)=0$ if and only if $z_\varepsilon=0$.

3) Next, we compute the time derivative of $V\left(z_\varepsilon\right)$ along the solutions of the system:

$$\frac{\mathrm{d}}{\mathrm{d}t}V\left(z_\varepsilon\right) = \int_{\Omega^\infty} z_\varepsilon\cdot\frac{\mathrm{d}}{\mathrm{d}t}z_\varepsilon\mathrm{d}x\mathrm{d}t = \int_{\Omega^\infty} z_\varepsilon\cdot\left(\Delta z_\varepsilon + f_\varepsilon\left(z_\varepsilon\right)\right)\mathrm{d}x\mathrm{d}t.$$

Using integration by parts and the boundary conditions, we can simplify the above expression as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V\left(z_\varepsilon\right) &= -\int_{\Omega^\infty}\left|\nabla z_\varepsilon\right|^2\mathrm{d}x\mathrm{d}t + \int_{\Sigma_\varepsilon^\infty}\frac{1}{\varepsilon^\alpha}\left|u_\varepsilon\right|^{p-2}u_\varepsilon z_\varepsilon\mathrm{d}\sigma\mathrm{d}t + \int_{\Omega^\infty} z_\varepsilon f_\varepsilon\left(z_\varepsilon\right)\mathrm{d}x\mathrm{d}t \\ &\le -\int_{\Omega^\infty}\left|\nabla z_\varepsilon\right|^2\mathrm{d}x\mathrm{d}t + \int_{\Sigma_\varepsilon^\infty}\frac{1}{\varepsilon^\alpha}\left|u_\varepsilon\right|^{p-2}u_\varepsilon z_\varepsilon\mathrm{d}\sigma\mathrm{d}t + M_3\int_0^\infty\left\|z_\varepsilon\right\|_{H^1(\Omega)}^2\mathrm{d}t.\end{aligned}$$

To achieve this, we used the fact that $\int_0^\infty \left\| f_\varepsilon(z_\varepsilon) \right\|_{H^{-1}(\Omega)}^2$ is bounded by a constant $M_3 \le 1$ that is independent of z_ε.

The control's choice must ensure that the Lyapunov function's derivative is negative and that the integral $\int_{\Sigma_\varepsilon} \frac{1}{\varepsilon^\alpha} |u_\varepsilon|^{p-2} u_\varepsilon z_\varepsilon \mathrm{d}\sigma$ is negative and finite. Therefore, for $-1 < \beta \le \frac{2-p}{p-1}$, set $u_\varepsilon = -\varepsilon^a |z_\varepsilon|^\beta z_\varepsilon$, with $a \ge 1$ a positive constant.

$$
\begin{aligned}
\int_{\Sigma_\varepsilon^\infty} \frac{1}{\varepsilon^\alpha} |u_\varepsilon|^{p-2} u_\varepsilon z_\varepsilon \mathrm{d}\sigma \mathrm{d}t &= \int_{\Sigma_\varepsilon^\infty} \frac{1}{\varepsilon^\alpha} \left| -\varepsilon^a |z_\varepsilon|^\beta z_\varepsilon \right|^{p-2} \left(-\varepsilon^a |z_\varepsilon|^\beta z_\varepsilon \right) z_\varepsilon \mathrm{d}\sigma \mathrm{d}t \\
&= \int_{\Sigma_\varepsilon^\infty} \frac{1}{\varepsilon^\alpha} \left(\varepsilon^a \right)^{p-2} |z_\varepsilon|^{(\beta+1)(p-2)} \left(-\varepsilon^a |z_\varepsilon|^{\beta+2} \right) \mathrm{d}\sigma \mathrm{d}t \\
&= -\varepsilon^{a(p-1)-\alpha} \int_{\Sigma_\varepsilon^\infty} |z_\varepsilon|^{\beta(p-1)+p} \mathrm{d}\sigma \mathrm{d}t.
\end{aligned}
$$

By substituting this term in the expression for the time derivative of $V(z_\varepsilon)$, we obtain:

$$
\frac{\mathrm{d}}{\mathrm{d}t} V(z_\varepsilon) \le (-1 + M_3) \int_{\Omega^\infty} |\nabla z_\varepsilon|^2 \mathrm{d}x \mathrm{d}t - \varepsilon^{a(p-1)-\alpha} \int_{\Sigma_\varepsilon^\infty} |z_\varepsilon|^{\beta(p-1)+p} \mathrm{d}\sigma \mathrm{d}t.
$$

So the time derivative of $V(z_\varepsilon)$ is negative, hence $V(z_\varepsilon)$ satisfies the assumptions, which implies that $V(z_\varepsilon)$ is a Lyapunov function and the system is stable.

3.2. Limit Behavior of Solution

The set $V = H^1(\Omega)$ is a Banach and reflexive space, with $H^1(\Omega)$ has the norm $\|\cdot\|_{H^1(\Omega)}$, according to the separability of *V*, hence it admits a countable basis $\{w_1, w_2, w_3, \cdots, w_n, \cdots\}$, with $w_i \in V$, $\forall m$ $\{w_1, w_2, w_3, \cdots, w_n\}$ is a free family, $H = Vect\{w_1, w_2, w_3, \cdots, w_n, \cdots\}$ is dense in *V*.

Let us consider in the spaces $V_m = Vect\{w_1, w_2, w_3, \cdots, w_m\}$ the following approximate problem;

We put $z_\varepsilon(t) = \sum_{i=1}^m h_{i\varepsilon}(t) w_i \in V_m$.

$$
\begin{cases}
\dot{z}_\varepsilon - \Delta z_\varepsilon = f_\varepsilon & \text{in } \Omega^\infty \\
\frac{\partial z_\varepsilon}{\partial n} = \frac{1}{\varepsilon^\alpha} |u_\varepsilon|^{p-2} u_\varepsilon & \text{on } \Sigma_\varepsilon^\infty \\
z_\varepsilon = 0 & \text{on } \Gamma_\varepsilon^\infty \\
z_\varepsilon(t = 0, x) = z_{0,\varepsilon} & \text{on } \Omega
\end{cases}
$$

Existence of the Solution

To solve this problem, we aim to find a solution by minimizing the energy functional $J(z_\varepsilon)$ given by:

$$
J(z_\varepsilon) = \frac{1}{2} \int_{\Omega^\infty} |\nabla z_\varepsilon|^2 \mathrm{d}x \mathrm{d}t - \int_{\Omega^\infty} f_\varepsilon z_\varepsilon \mathrm{d}x \mathrm{d}t - \frac{1}{\varepsilon^\alpha \left(\beta(p-1) + p \right)} \int_{\Sigma_\varepsilon^\infty} |u_\varepsilon|^{p-2} u_\varepsilon z_\varepsilon \mathrm{d}\sigma \mathrm{d}t. \tag{1}
$$

To show the existence of critical points for the function *J*, we need to verify the Palais-Smale condition, which states that a bounded sequence with a gradient converging to zero has a convergent sub-sequence in the energy space.

- Convergence of $z_{\varepsilon,k}$:

The sequence z_{ε} is sequentially bounded in the reflexive space $L^2(0,\infty;H^1(\Omega))$. Consequently, there exists a sub-sequence $z_{\varepsilon,k}$ such that $z_{\varepsilon,k}$ converges to z in $L^2(0,\infty;H^1(\Omega))$.

- Boundedness of $J(z_{\varepsilon,k})$:

Since $z_{\varepsilon,k}$ is bounded in $L^2(0,\infty;H^1(\Omega))$, we have $\|z_{\varepsilon,k}\|_{L^2(0,\infty;H^1(\Omega))} \le C$ for some constant $C>0$ independent of *k*.

Using the boundedness of *f* in $L^2(0,\infty;H^{-1}(\Omega))$, we can estimate the second term of $J(z_{\varepsilon,k})$ as

$$\left|\int_{\Omega^\infty} f_{\varepsilon,k} z_{\varepsilon,k}\mathrm{d}x\mathrm{d}t\right| \le \|f_{\varepsilon,k}\|_{L^2(0,\infty;H^{-1}(\Omega))}\|z_{\varepsilon,k}\|_{L^2(0,\infty;H^1(\Omega))} \le C.$$

For the third term, for $\beta \le \frac{2-p}{p-1}$;

$$\frac{1}{p\varepsilon^\alpha}\left|\int_{\Sigma_\varepsilon^\infty}|u_{\varepsilon,k}|^{p-2}u_{\varepsilon,k}z_{\varepsilon,k}\mathrm{d}\sigma\mathrm{d}t\right| = \frac{\varepsilon^{a(p-1)-\alpha}}{p}\left|\int_{\Sigma_\varepsilon^\infty}|z_\varepsilon|^{\beta(p-1)+p}\mathrm{d}\sigma\mathrm{d}t\right| \le \varepsilon^{a(p-1)-\alpha}C.$$

Therefore, we can bound $J(z_{\varepsilon,k})$ as

$$J(z_{\varepsilon,k}) \le \frac{1}{2}\int_0^\infty \|z_{\varepsilon,k}\|_{H^1(\Omega)}^2 + C + \varepsilon^{a(p-1)-\alpha}C \le +\infty.$$

This shows that $J(z_{\varepsilon,k})$ is bounded.

- Convergence of $\nabla J(z_{\varepsilon,k})$:

Expanding the expression for *J* using:

$$J(z_{\varepsilon,k}) = \frac{1}{2}\int_{\Omega^\infty}|\nabla z_{\varepsilon,k}|^2\mathrm{d}x\mathrm{d}t - \int_{\Omega^\infty} f_{\varepsilon,k}z_{\varepsilon,k}\mathrm{d}x\mathrm{d}t - \frac{\varepsilon^{a(p-1)-\alpha}}{\beta(p-1)+p}\int_{\Sigma_\varepsilon^\infty}|z_{\varepsilon,k}|^{\beta(p-1)+p}\mathrm{d}\sigma\mathrm{d}t.$$

To demonstrate the Palais-Smale condition, we need to consider the variation of the functional *J* with respect to $z_{\varepsilon,n}$. Let *v* be a trial function in $L^2(0,\infty;H^1(\Omega))$ such that $v(t)=0$ for *t* outside a bounded interval. Then, for $h \ne 0$, we have:

$$J'(z_{\varepsilon,k};v) = \lim_{h\to 0}\frac{J(z_{\varepsilon,k}+hv)-J(z_{\varepsilon,k})}{h} = \frac{\mathrm{d}}{\mathrm{d}h}J(z_{\varepsilon,k}+hv)\Big|_{h=0}.$$

Now let's calculate the derivative terms one by one:

- Derivative of the first term:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}h}\left(\int_{\Omega^\infty}|\nabla(z_{\varepsilon,k}+hv)|^2\mathrm{d}x\mathrm{d}t\right)\\ &= \lim_{h\to 0}\frac{1}{h}\left(2h\int_{\Omega^\infty}\nabla z_{\varepsilon,k}\cdot\nabla v\mathrm{d}x\mathrm{d}t + h^2\int_{\Omega^\infty}|\nabla v|^2\mathrm{d}x\mathrm{d}t\right)\\ &= 2\int_{\Omega^\infty}\nabla z_{\varepsilon,k}\cdot\nabla v\mathrm{d}x\mathrm{d}t.\end{aligned}$$

- Derivative of the second term:

$$\frac{\mathrm{d}}{\mathrm{d}h}\left(-\int_{\Omega^\infty} f_{\varepsilon,k}(z_{\varepsilon,k}+hv)\mathrm{d}x\mathrm{d}t\right) = -\int_{\Omega^\infty} f_{\varepsilon,k}v\mathrm{d}x\mathrm{d}t.$$

- Derivative of the third term:

Let's correctly compute the derivative of the third term with respect to h and evaluate it at $h=0$:

$$\frac{\mathrm{d}}{\mathrm{d}h}\left(\int_{\Sigma_\varepsilon^\infty}\left|z_{\varepsilon,k}+hv\right|^{\beta(p-1)+p}\mathrm{d}\sigma\mathrm{d}t\right)\Bigg|_{h=0}$$

To evaluate this derivative, we can use the chain rule, we have:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}h}\left(\left|z_{\varepsilon,k}+hv\right|^{\beta(p-1)+p}\right)\Bigg|_{h=0} &= \frac{\mathrm{d}}{\mathrm{d}h}\left|z_{\varepsilon,k}+hv\right|^{\beta(p-1)+p}\Bigg|_{h=0}\\
&=\left(\beta(p-1)+p\right)sign\left(z_{\varepsilon,k}+hv\right)\left|z_{\varepsilon,k}+hv\right|^{\beta(p-1)+p-1}\frac{\mathrm{d}}{\mathrm{d}h}\left(z_{\varepsilon,k}+hv\right)\Bigg|_{h=0}\\
&=\left(\beta(p-1)+p\right)sign\left(z_{\varepsilon,k}\right)\left|z_{\varepsilon,k}\right|^{\beta(p-1)+p-1}v.
\end{aligned}$$

Therefore, the correct expression for the derivative of the third term is $\left(\beta(p-1)+p\right)sign\left(z_{\varepsilon,k}\right)\left|z_{\varepsilon,k}\right|^{\beta(p-1)+p-1}v$.

Using the definition of the derivative of J with respect to h, we have:

$$\begin{aligned}
J'\left(z_{\varepsilon,k};v\right) &= \frac{\mathrm{d}}{\mathrm{d}h}J\left(z_{\varepsilon,k}+hv\right)\Bigg|_{h=0}\\
&=\int_{\Omega^\infty}\nabla z_{\varepsilon,k}\cdot\nabla v\mathrm{d}x\mathrm{d}t-\int_{\Omega^\infty}f_{\varepsilon,k}v\mathrm{d}x\mathrm{d}t\\
&\quad-\frac{\varepsilon^{a(p-1)-\alpha}}{\beta(p-1)+p}\int_{\Sigma_\varepsilon^\infty}\left(\beta(p-1)+p\right)sign\left(z_{\varepsilon,k}\right)\left|z_{\varepsilon,k}\right|^{\beta(p-1)+p-1}v\mathrm{d}\sigma\mathrm{d}t.
\end{aligned}$$

We can rewrite the above expression as:

$$\begin{aligned}
\nabla J\left(z_{\varepsilon,k}\right)\cdot v &= \int_{\Omega^\infty}\nabla z_{\varepsilon,k}\cdot\nabla v\mathrm{d}x\mathrm{d}t-\int_{\Omega^\infty}f_{\varepsilon,k}v\mathrm{d}x\mathrm{d}t\\
&\quad-\varepsilon^{a(p-1)-\alpha}\int_{\Sigma_\varepsilon^\infty}sign\left(z_{\varepsilon,k}\right)\left|z_{\varepsilon,k}\right|^{\beta(p-1)+p-1}v\mathrm{d}\sigma\mathrm{d}t.
\end{aligned}$$

Since $\nabla J\left(z_{\varepsilon,k}\right)$ converges to zero, it implies that:

$$\int_{\Omega^\infty}\nabla z_{\varepsilon,k}\cdot\nabla v\mathrm{d}x\mathrm{d}t-\int_{\Omega^\infty}f_{\varepsilon,k}v\mathrm{d}x\mathrm{d}t-\varepsilon^{a(p-1)-\alpha}\int_{\Sigma_\varepsilon^\infty}sign\left(z_{\varepsilon,k}\right)\left|z_{\varepsilon,k}\right|^{\beta(p-1)+p-1}v\mathrm{d}\sigma\mathrm{d}t\to 0,$$

as $k\to+\infty$. This holds for all trial functions v in $L^2\left(0,\infty;H^1(\Omega)\right)$. By the definition of weak convergence, when $k\to+\infty$. According to the classical result, the diagonalization lemma, there is a function $k(\varepsilon):\mathbb{R}^+\to\mathbb{N}$ increasing to $+\infty$ when $\varepsilon\to 0$, we can conclude that:

$$\Delta z_{\varepsilon,k}+f_{\varepsilon,k}-\frac{1}{\varepsilon^\alpha}\left|u_{\varepsilon,k}\right|^{p-2}u_{\varepsilon,k}\to 0$$

weakly in $L^2\left(0,\infty;H^{-1}(\Omega)\right)$ as $k\to\infty$.

- The lower semi-continuity of $J\left(z_{\varepsilon,k}\right)$:

We need to show that for any sequence $z_{\varepsilon,k}$ converging weakly to z in the energy space, we have:

$$\liminf_{k\to\infty}J\left(z_{\varepsilon,k}\right)\geq J(z)$$

We start by considering the first term of the functional: $\frac{1}{2}\int_{\Omega^\infty}\left|\nabla z_\varepsilon\right|^2\mathrm{d}x\mathrm{d}t$.

Since $z_{\varepsilon,k}$ converges weakly to z, we have $\nabla z_{\varepsilon,k}\rightharpoonup\nabla z$ weakly in

$L^2\left(]0,\infty[\times\Omega\right)$. Using Fatou's lemma;

$$\frac{1}{2}\int_{\Omega^\infty}\left|\nabla z\right|^2\mathrm{d}x\mathrm{d}t\le\liminf_{k\to\infty}\frac{1}{2}\int_{\Omega^\infty}\left|\nabla z_{\varepsilon,k}\right|^2\mathrm{d}x\mathrm{d}t.$$

Next, let us analyze the second term of the functional: $-\int_{\Omega^\infty}f_{\varepsilon,k}z_{\varepsilon,k}\mathrm{d}x\mathrm{d}t$.

Since $f_{\varepsilon,k}$ is a bounded function, we can use the weak convergence of $z_{\varepsilon,k}$ to z to obtain:

$$-\int_{\Omega^\infty}fz\mathrm{d}x\mathrm{d}t\le\liminf_{k\to\infty}-\int_{\Omega^\infty}f_{\varepsilon,k}z_{\varepsilon,k}\mathrm{d}x\mathrm{d}t.$$

Finally, let's consider the third term of the functional:

$$-\frac{1}{\varepsilon^\alpha\left(\beta\left(p-1\right)+p\right)}\int_{\Sigma_\varepsilon^\infty}\left|u_{\varepsilon,k}\right|^{p-2}u_{\varepsilon,k}z_{\varepsilon,k}\mathrm{d}\sigma\mathrm{d}t.$$

With regard to the third term, we can demonstrate on the basis of the proof of the epi-convergence theorem 3.1 that we can establish the inequality in question.

Combining these inequalities, we obtain:

$$J\left(z\right)\le\liminf_{k\to\infty}J\left(z_{\varepsilon,k}\right).$$

- Convergence of $J\left(z_{\varepsilon,k}\right)$ to its infimum:

We have established that z_ε^* is a minimizing sequence of $J\left(z_\varepsilon\right)$. We want to show that $J\left(z_{\varepsilon,k}\right)$ converges to $\inf_{z\in L^2\left(0,\infty;H^1(\Omega)\right)}J\left(z\right)$ as $k\to\infty$.

To prove this, we can utilize the lower semicontinuity property of J that we established earlier. Since z_ε^* is a minimizing sequence, we have:

$$J\left(z_\varepsilon^*\right)\le J\left(z_{\varepsilon,k}\right)$$

for all k. Taking the liminf on both sides, we obtain:

$$\liminf_{k\to\infty}J\left(z_\varepsilon^*\right)\le\liminf_{k\to\infty}J\left(z_{\varepsilon,k}\right)$$

Since $J\left(z_\varepsilon^*\right)$ is the infimum of J over $L^2\left(0,\infty;H^1\left(\Omega\right)\right)$, we have:

$$\liminf_{k\to\infty}J\left(z_\varepsilon^*\right)\le\inf_{z\in L^2\left(0,\infty;H^1(\Omega)\right)}J\left(z\right)$$

Combining these inequalities, we obtain:

$$\inf_{z\in L^2\left(0,\infty;H^1(\Omega)\right)}J\left(z\right)\le\liminf_{k\to\infty}J\left(z_{\varepsilon,k}\right)$$

Since we have already shown that J is lower semicontinuous, we can conclude that:

$$\lim_{k\to\infty}J\left(z_{\varepsilon,k}\right)=\inf_{z\in L^2\left(0,\infty;H^1(\Omega)\right)}J\left(z\right).$$

Therefore, we have demonstrated the convergence of $J\left(z_{\varepsilon,k}\right)$ towards its infimum, completing the proof.

These results guarantee the existence of a solution to the initial problem. The convergence of the minimizing sequence and the functional to their limit suggests that this solution is stable and indeed represents the energy minimum. In addition, we have verified the Palais-Smale conditions that are essential to guarantee the existence of minimizing solutions.

Lemma 3.1. *The family* $(z_\varepsilon)_{\varepsilon>0}$ *satisfies*:

$$\|z_\varepsilon\|_{L^2(0,\infty;H^1(\Omega))} \le C \tag{2}$$

$$\|u_\varepsilon\|_{L^p(0,\infty;L^p(\Sigma_\varepsilon))} \le C$$

Proof. Let us consider the approximate problem; we multiply the equations defined on Ω^∞ by $h_{i\varepsilon}(t)$ and sum from $i=1$ to m for a fixed k. This leads to the variational formulation of our problem;

$$\int_\Omega \partial_t z_\varepsilon v\mathrm{d}x + \int_\Omega \nabla z_\varepsilon \cdot \nabla v\mathrm{d}x = \int_\Omega f_\varepsilon v\mathrm{d}x + \frac{1}{\varepsilon^\alpha}\int_{\Sigma_\varepsilon} |u_\varepsilon|^{p-2} u_\varepsilon v\mathrm{d}\sigma.$$

For all $v \in L^2(0,\infty;H^1(\Omega))$. By choosing $v = z_\varepsilon$ in this formulation, we obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_\varepsilon\|^2_{L^2(\Omega)} + \|\nabla z_\varepsilon\|^2_{L^2(\Omega)} = \int_\Omega f_\varepsilon z_\varepsilon \mathrm{d}x + \frac{1}{\varepsilon^\alpha}\int_{\Sigma_\varepsilon} |u_\varepsilon|^{p-2} u_\varepsilon z_\varepsilon \mathrm{d}\sigma$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_\varepsilon\|^2_{L^2(\Omega)} + \|\nabla z_\varepsilon\|^2_{L^2(\Omega)} = \int_\Omega f_\varepsilon z_\varepsilon \mathrm{d}x - \frac{1}{\varepsilon^\alpha}\int_{\Sigma_\varepsilon} \left|-\varepsilon^a z_\varepsilon |z_\varepsilon|^\beta\right|^{p-2} \varepsilon^a z_\varepsilon |z_\varepsilon|^\beta z_\varepsilon \mathrm{d}\sigma$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_\varepsilon\|^2_{L^2(\Omega)} + \|\nabla z_\varepsilon\|^2_{L^2(\Omega)} = \int_\Omega f_\varepsilon z_\varepsilon \mathrm{d}x - \varepsilon^{a(p-1)-\alpha}\int_{\Sigma_\varepsilon^\infty} |z_\varepsilon|^{\beta(p-1)+p} \mathrm{d}\sigma.$$

Using the Holder Inequality, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_\varepsilon\|^2_{L^2(\Omega)} + \|\nabla z_\varepsilon\|^2_{L^2(\Omega)} \le \frac{1}{2}\|f_\varepsilon\|^2_{H^{-1}(\Omega)} + \frac{1}{2}\|z_\varepsilon\|^2_{H^1(\Omega)}$$

By integrating this inequality with respect to time, we obtain the following a priori estimate for z_ε:

$$\frac{1}{2}\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\|z_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^\infty \|\nabla z_\varepsilon\|^2_{L^2(\Omega)} \le \frac{1}{2}\int_0^\infty \|f_\varepsilon\|^2_{H^{-1}(\Omega)}$$

$$\frac{1}{2}\int_0^\infty \|\nabla z_\varepsilon\|^2_{L^2(\Omega)} \le C.$$

Which proves that;

$$\|z_\varepsilon\|_{L^2(0,\infty;H^1(\Omega))} \le C.$$

This a priori estimate shows that the norm of z_ε is in the Bochner space $L^2(0,\infty;H^1(\Omega))$.

On the other hand, we seek to obtain an a priori estimate of the control norm u_ε. An admissible control for this problem is a function u_ε that satisfies the control constraint on Σ_ε, using Remark 2.3.

$$\begin{aligned}\|u_\varepsilon\|^{2p-2}_{L^p(\Sigma_\varepsilon)} &\le C\|u_\varepsilon\|^{2p-2}_{L^{2p-2}(\Sigma_\varepsilon)} = C\int_{\Sigma_\varepsilon} |u_\varepsilon|^{2p-2}\mathrm{d}\sigma = C\int_{\Sigma_\varepsilon}\left(|u_\varepsilon|^{p-2}u_\varepsilon\right)^2\mathrm{d}\sigma \\ &= C\left\||u_\varepsilon|^{p-2}u_\varepsilon\right\|^2_{L^2(\Sigma_\varepsilon)} = \frac{C}{2}\left\||u_\varepsilon|^{p-2}u_\varepsilon\right\|^2_{H^{-1/2}(\Sigma_\varepsilon)} \\ &= \frac{C\varepsilon^{2\alpha}}{2}\left\|\frac{1}{\varepsilon^\alpha}|u_\varepsilon|^{p-2}u_\varepsilon\right\|^2_{H^{-1/2}(\Sigma_\varepsilon)} \le \frac{C\varepsilon^{2\alpha}}{2}\left\|\frac{\partial z_\varepsilon}{\partial n}\right\|^2_{H^{-1/2}(\Sigma_\varepsilon)} < \infty.\end{aligned}$$

Thus, we have:

$$\int_0^\infty \|u_\varepsilon\|^p_{L^p(\Sigma_\varepsilon)} \le C\int_0^\infty \|u_\varepsilon\|^{2p-2}_{L^{2p-2}(\Sigma_\varepsilon)}\mathrm{d}t \le C\varepsilon^{2\alpha}\int_0^\infty\left\|\frac{\partial z_\varepsilon}{\partial n}\right\|^2_{H^{-1/2}(\Sigma_\varepsilon)}\mathrm{d}t < \infty.$$

This a priori estimate shows that the norm of u_ε in Bochner space $L^p\left(0,\infty;L^p\left(\Sigma_\varepsilon\right)\right)$.

3.3. Proof of Theorem 3.1

To prove our theorem, we will need to establish the two lemmas 3.2 and 3.3 and the proposition 3.1.

Lemma 3.2. The operator m^ε is linear and bounded of $L^2\left(0,\infty;L^2\left(\Sigma_\varepsilon\right)\right)$ respectively $L^2\left(0,\infty;H^1\left(\Sigma_\varepsilon\right)\right)$ in $L^2\left(0,\infty;L^2\left(\Sigma\right)\right)$ (respectively $L^2\left(0,\infty;H^1\left(\Sigma\right)\right)$, moreover, for all $z\in L^2\left(0,\infty;H^1\left(\Sigma_\varepsilon\right)\right)$, we have

$$\left\|m^\varepsilon z - z_{|\Sigma}\right\|^2_{L^2(]0,\infty[\times\Sigma)} \le C\varepsilon^2\int_0^\infty\int_{\Sigma_\varepsilon}\left|\nabla z\right|^2. \tag{3}$$

Proof. Let $z\in\bar{\mathcal{D}}\left(\left]0,\infty\right[\times\Sigma_\varepsilon\right)$, so that

$$\left\|m^\varepsilon z - z_{|\Sigma}\right\|^2_{L^2(\Sigma)} = \int_\Sigma\left|\left(\frac{1}{2\varepsilon^2}\int_{-\varepsilon^2}^{\varepsilon^2} z\left(t,x_1,x_2,x_3\right)dx_3\right) - z\left(t,x_1,x_2,0\right)\right|^2 \mathrm{d}x_1\mathrm{d}x_2.$$

Using the Hölder inequality,

$$\begin{aligned}\left\|m^\varepsilon z - z_{|\Sigma}\right\|^2_{L^2(\Sigma)} &\le \frac{1}{2\varepsilon^2}\int_\Sigma\left(\int_{-\varepsilon^2}^{\varepsilon^2}\left|z\left(t,x_1,x_2,x_3\right)-z\left(t,x_1,x_2,0\right)\right|^2\mathrm{d}x_3\right)\mathrm{d}x_1\mathrm{d}x_2\\ &\le \frac{1}{2\varepsilon^2}\int_\Sigma\left(\int_{-\varepsilon^2}^{\varepsilon^2}\left|\int_0^{x_3}\frac{\partial z}{\partial x_3}\left(t,x_1,x_2,w\right)\mathrm{d}w\right|^2\mathrm{d}x_3\right)\mathrm{d}x_1\mathrm{d}x_2\\ &\le \frac{1}{2\varepsilon^2}\int_\Sigma\left(\int_{-\varepsilon^2}^{\varepsilon^2}\left|x_3\right|\left(\int_{-\varepsilon^2}^{\varepsilon^2}\left|\frac{\partial z}{\partial x_3}\left(t,x_1,x_2,w\right)\right|^2\mathrm{d}w\right)\mathrm{d}x_3\right)\mathrm{d}x_1\mathrm{d}x_2\\ &\le C\varepsilon^2\int_\Sigma\left(\int_{-\varepsilon^2}^{\varepsilon^2}\left|\frac{\partial z}{\partial x_3}\right|^2\mathrm{d}x_3\right)\mathrm{d}x_1\mathrm{d}x_2 \le C\varepsilon^2\int_{\Sigma_\varepsilon}\left|\nabla z\right|^2\mathrm{d}x.\end{aligned}$$

By density arguments, we have for all $z\in L^2\left(0,\infty;H^1\left(\Sigma_\varepsilon\right)\right)$

$$\left\|m^\varepsilon z - z_{|\Sigma}\right\|^2_{L^2(]0,\infty[\times\Sigma)} \le C\varepsilon^2\int_0^\infty\int_{\Sigma_\varepsilon}\left|\nabla z\right|^2\mathrm{d}x\mathrm{d}t.$$

Hence the result.

Lemma 3.3. Let $\left(z_\varepsilon\right)_{\varepsilon>0}\subset L^2\left(0,\infty;H^1\left(\Omega\right)\right)$ which satisfies (2). Then

$$\left\|m^\varepsilon z_\varepsilon\right\|^2_{L^2(]0,\infty[\times\Sigma)} \le C. \tag{4}$$

In addition, $m^\varepsilon z_\varepsilon$ have a bounded sub-sequence in $L^2\left(\left]0,\infty\right[\times\Sigma\right)$.

Proof. From lemma 3.2, we get

$$\left\|m^\varepsilon z - z_{|\Sigma}\right\|^2_{L^2(]0,\infty[\times\Sigma)} \le C\varepsilon^2\int_0^\infty\int_{\Sigma_\varepsilon}\left|\nabla z\right|^2 \le C\varepsilon^2.$$

z_ε is bounded in $L^2\left(0,\infty;H^1\left(\Omega\right)\right)$, it follows that there exists $z^*\in L^2\left(0,\infty;H^1\left(\Omega\right)\right)$ and a sub-sequence z_ε, always noted z_ε, such as $z_\varepsilon \rightharpoonup z^*$ in $L^2\left(0,\infty;H^1\left(\Omega\right)\right)$, then $z_{\varepsilon|\Sigma}$ is a bounded sequence in $L^2\left(\left]0,\infty\right[\times\Sigma\right)$.

Since,

$$\left\|m^\varepsilon z_\varepsilon\right\|_{L^2(]0,\infty[\times\Sigma)} \le \left\|m^\varepsilon z_\varepsilon - z_{\varepsilon|\Sigma}\right\|_{L^2(]0,\infty[\times\Sigma)} + \left\|z_{\varepsilon|\Sigma}\right\|_{L^2(]0,\infty[\times\Sigma)},$$

then there exists C such that $\left\|m^\varepsilon z_\varepsilon\right\|^2_{L^2(]0,\infty[\times\Sigma)} \le C$.

Proposition 3.1. $(z_\varepsilon)_\varepsilon$, has a weakly convergent sub-sequence to an element z^* in $L^2(0,\infty;H^1(\Omega))$ satisfactory, $z^*|_\Sigma \in L^2(0,\infty;L^2(\Sigma))$.

Proof. The sequence z_ε is bounded in $L^2(0,\infty;H^1(\Omega))$, it follows that there is an element $z^* \in L^2(0,\infty;H^1(\Omega))$ and a sub-sequence of z_ε, always designated by z_ε such as $z_\varepsilon \rightharpoonup z^*$ in $L^2(0,\infty;H^1(\Omega))$. We have

$$\left\| m^\varepsilon z_\varepsilon - z_{\varepsilon|\Sigma} \right\|^2_{L^2(]0,\infty[\times\Sigma)} \le C\varepsilon^2 \text{ and } z_{\varepsilon|\Sigma} \rightharpoonup z^*_{|\Sigma} \text{ in } L^2(]0,\infty[\times\Sigma).$$

According to the evaluation (4), as $m^\varepsilon z_\varepsilon \rightharpoonup z^*_{|\Sigma}$ in $L^2(0,\infty;L^2(\Sigma))$. Hence $z^*_{|\Sigma} \in L^2(0,\infty;L^2(\Sigma))$.

Hence the results.

The prior findings have allowed us to emphasize our core finding (theorem 3.1).

In this article, we focus on establishing the following main result, which demonstrates the limit behavior presented in the theorem below:

We consider the energy operator

$$F_\varepsilon(z_\varepsilon) = \frac{1}{2}\int_{\Omega^\infty} |\nabla z_\varepsilon|^2 \,\mathrm{d}x\mathrm{d}t - \frac{\varepsilon^{a(p-1)-\alpha}}{\beta(p-1)+p}\int_{\Sigma_\varepsilon^\infty} |z_\varepsilon|^{\beta(p-1)+p} \,\mathrm{d}\sigma\mathrm{d}t$$

One denotes by τ_f the weak topology on $L^2(0,\infty;\mathbb{H}_\Sigma)$.

Theorem 3.1. According to the values of α, there exists a functional F^α defined on $L^2(0,\infty;\mathbb{H}_\Sigma)$ with a value in $\mathbb{R}\cup\{+\infty\}$ such that $\tau_f - \lim_e F_\varepsilon = F^\alpha$ in $L^2(0,\infty;\mathbb{H}_\Sigma)$, where the functional F^α is given by;

1) If $\alpha < 2 + a(p-1)$:

$$F^\alpha(z) = \frac{1}{2}\int_{\Omega^\infty} |\nabla z|^2 \,\mathrm{d}x\mathrm{d}t.$$

2) If $\alpha \ge 2 + a(p-1)$:

$$F^\alpha(z) = \frac{1}{2}\int_{\Omega^\infty} |\nabla z|^2 \,\mathrm{d}x\mathrm{d}t - \frac{\eta(\alpha - a(p-1))}{\beta(p-1)+p}\oint_{\Sigma^\infty} |z_{|\Sigma}|^{\beta(p-1)+p} \,\mathrm{d}\sigma\mathrm{d}t.$$

Proof. First, we write the energy functional $F_\varepsilon(z_\varepsilon)$ associated with the problem as follows; Let $z_\varepsilon \in L^2(0,\infty;\mathbb{H}_\Sigma)$, we have:

$$F_\varepsilon(z_\varepsilon) = \frac{1}{2}\int_{\Omega^\infty} |\nabla z_\varepsilon|^2 \,\mathrm{d}x\mathrm{d}t - \frac{1}{\varepsilon^\alpha(\beta(p-1)+p)}\int_{\Sigma_\varepsilon^\infty} |u_\varepsilon|^{p-2} u_\varepsilon z_\varepsilon \mathrm{d}\sigma\mathrm{d}t.$$

And,

$$G(z_\varepsilon) = -\int_{\Omega^\infty} f_\varepsilon z_\varepsilon \mathrm{d}x\mathrm{d}t.$$

Given $u_\varepsilon(z_\varepsilon) = -\varepsilon^a z_\varepsilon |z_\varepsilon|^\beta$, we want to apply the method of epi-convergence.

1) We will determine the upper epi-limit:

From a density result, let $z \in \mathbb{G} \subset L^2(0,\infty;\mathbb{H}_\Sigma)$, there is a sequence (z_k) in $\mathbb{D}$ such that

$$z_k \to z \text{ in } \mathbb{G}, \text{ as } k \to +\infty.$$

So that $z_k \to z$ in $L^2(0,\infty;\mathbb{H}_\Sigma)$.

Let θ be a smooth function verifying $\theta(x_3) = 1$ if $|x_3| \le 1$, $\theta(x_3) = 0$ if

$|x_3| \geq 2$ and $|\theta'(x_3)| \leq 2$, $\forall x \in \mathbb{R}$.

We define

$$\theta_\varepsilon(x) = \theta\left(\frac{x_3}{\varepsilon^2}\right).$$

And $z_{\varepsilon,k} = \theta_\varepsilon(x) z_{k|\Sigma} + (1-\theta_\varepsilon(x)) z_k$.

It is easy to show that $z_{\varepsilon,k} \in L^2(0,\infty;\mathbb{H}_\Sigma)$ and $z_{\varepsilon,k} \to z_k$ in $\mathbb{G}$, when $\varepsilon \to 0$. Since

$$F_\varepsilon(z_{\varepsilon,k}) = \frac{1}{2}\int_{\Omega^\infty} |\nabla z_{\varepsilon,k}|^2 \, \mathrm{d}x\mathrm{d}t - \frac{\varepsilon^{a(p-1)-\alpha}}{\beta(p-1)+p}\int_{\Sigma_\varepsilon^\infty} |z_{\varepsilon,k}|^{\beta(p-1)+p} \, \mathrm{d}\sigma\mathrm{d}t.$$

So that

$$\begin{aligned} F_\varepsilon(z_{\varepsilon,n}) &= \frac{1}{2}\int_{\Omega^\infty} |\nabla z_{\varepsilon,k}|^2 \, \mathrm{d}x\mathrm{d}t - \frac{\varepsilon^{a(p-1)-\alpha}}{\beta(p-1)+p}\int_{\Sigma_\varepsilon^\infty} |z_{\varepsilon,k}|^{\beta(p-1)+p} \, \mathrm{d}\sigma\mathrm{d}t \\ &= \frac{1}{2}\int_{\Omega^\infty} |\nabla z_k|^2 \, \mathrm{d}x\mathrm{d}t - \frac{\varepsilon^{2+a(p-1)-\alpha}}{\beta(p-1)+p}\oint_{\Sigma^\infty} |z_{k|\Sigma}|^{\beta(p-1)+p} \, \mathrm{d}\sigma\mathrm{d}t. \end{aligned}$$

Since $\varepsilon^{2+a(p-1)-\alpha} \to \eta(\alpha - a(p-1))$,

$$\begin{aligned} &\limsup_{\varepsilon\to 0} F_\varepsilon(z_{\varepsilon,k}) \\ &\leq \limsup_{\varepsilon\to 0}\left(\frac{1}{2}\int_{\Omega^\infty} |\nabla z_{\varepsilon,k}|^2 \, \mathrm{d}x\mathrm{d}t - \frac{\varepsilon^{2+a(p-1)-\alpha}}{\beta(p-1)+p}\oint_{\Sigma^\infty} |z_{k|\Sigma}|^{\beta(p-1)+p} \, \mathrm{d}\sigma\mathrm{d}t\right) \\ &\leq \frac{1}{2}\int_{\Omega^\infty} |\nabla z_k|^2 \, \mathrm{d}x\mathrm{d}t - \begin{cases} 0 & \text{if } \alpha < 2 + a(p-1) \\ \dfrac{\eta(\alpha - a(p-1))}{\beta(p-1)+p}\oint_{\Sigma^\infty} |z_{k|\Sigma}|^{\beta(p-1)+p} \, \mathrm{d}\sigma\mathrm{d}t & \text{if } \alpha \geq 2 + a(p-1) \end{cases} \end{aligned}$$

Since $z_k \to z$ in $L^2(0,\infty,\mathbb{H}_\Sigma)$, when $k \to +\infty$. According to the classical result, the diagonalization lemma ([11], Lemma 1.15), there is a function $k(\varepsilon): \mathbb{R}^+ \to \mathbb{N}$ increasing to $+\infty$ when $\varepsilon \to 0$, such as $z_{\varepsilon,k(\varepsilon)} \to z$ in $L^2(0,\infty,\mathbb{H}_\Sigma)$, when $\varepsilon \to 0$. While k approaches $+\infty$;

$$\begin{aligned} &\limsup_{\varepsilon\to 0} F_\varepsilon(z_{\varepsilon,k(\varepsilon)}) \\ &\leq \lim_{k\to+\infty}\sup \limsup_{\varepsilon\to 0} F_\varepsilon(z_{\varepsilon,n}) \\ &\leq \frac{1}{2}\int_{\Omega^\infty} |\nabla z|^2 \, \mathrm{d}x\mathrm{d}t - \begin{cases} 0 & \text{if } \alpha \neq 2 + a(p-1) \\ \dfrac{\eta(\alpha - a(p-1))}{\beta(p-1)+p}\oint_{\Sigma^\infty} |z_{|\Sigma}|^{\beta(p-1)+p} \, \mathrm{d}\sigma\mathrm{d}t & \text{if } \alpha = 2 + a(p-1) \end{cases} \end{aligned}$$

2) We will determine the lower epi-limit:

Let $z \in \mathbb{G}$ and (z_ε) be a sequence in $L^2(0,\infty;\mathbb{H}_\Sigma)$ such that $z_\varepsilon \rightharpoonup z$ in $L^2(0,\infty;\mathbb{H}_\Sigma)$, so that

$$\nabla z_\varepsilon \rightharpoonup \nabla z \quad \text{in } L^2\left(0,\infty,L^2(\Omega)\right)^3. \tag{5}$$

Using Fatou's lemma and the fact that z_ε converges weakly to z in $L^2(0,\infty;\mathbb{H}_\Sigma)$, we obtain

$$\liminf_{\varepsilon\to 0} \frac{1}{2}\int_{\Omega^\infty} |\nabla z_\varepsilon|^2 \, \mathrm{d}x\mathrm{d}t \geq \frac{1}{2}\int_{\Omega^\infty} |\nabla z|^2 \, \mathrm{d}x\mathrm{d}t.$$

For $z_\varepsilon \in L^2(0,\infty;\mathbb{H}_\Sigma)$, we have

$$F_\varepsilon(z_\varepsilon) = \frac{1}{2}\int_{\Omega^\infty} |\nabla z_\varepsilon|^2 \,\mathrm{d}x\mathrm{d}t - \frac{\varepsilon^{a(p-1)-\alpha}}{\beta(p-1)+p}\int_{\Sigma_\varepsilon^\infty} |z_\varepsilon|^{\beta(p-1)+p}\,\mathrm{d}\sigma\mathrm{d}t$$
$$= \frac{1}{2}\int_{\Omega^\infty} |\nabla z_\varepsilon|^2 \,\mathrm{d}x\mathrm{d}t - \frac{\varepsilon^{2+a(p-1)-\alpha}}{\beta(p-1)+p}\oint_{\Sigma^\infty} |m^\varepsilon z_\varepsilon|^{\beta(p-1)+p}\,\mathrm{d}\sigma\mathrm{d}t$$

Therefore, we have $\alpha \neq 2 + a(p-1)$;

$$\liminf_{\varepsilon\to 0} F_\varepsilon(z_\varepsilon) \geq \frac{1}{2}\int_{\Omega^\infty} |\nabla z|^2 \,\mathrm{d}x\mathrm{d}t.$$

If $\alpha = 2 + a(p-1)$**: If** $\liminf_{\varepsilon\to 0} F^\varepsilon(z_\varepsilon) = +\infty$, there is nothing to prove, because

$$\frac{1}{2}\int_{\Omega^\infty} |\nabla z_\varepsilon|^2 \,\mathrm{d}x\mathrm{d}t - \frac{\varepsilon^{a(p-1)-\alpha}}{\beta(p-1)+p}\int_{\Sigma_\varepsilon^\infty} |z_\varepsilon|^{\beta(p-1)+p}\,\mathrm{d}\sigma\mathrm{d}t \leq +\infty.$$

Otherwise, $\liminf_{\varepsilon\to 0} F_\varepsilon(z_\varepsilon) < +\infty$, there is a sub-sequence of $F_\varepsilon(z_\varepsilon)$ still designated by $F_\varepsilon(z_\varepsilon)$ and a constant $C > 0$, such as $F_\varepsilon(z_\varepsilon) \leq C$. which implies that $\frac{1}{2}\int_{\Omega^\infty} |\nabla z_\varepsilon|^2 \,\mathrm{d}x\mathrm{d}t < C$.

Moreover, thanks to (3) and the continuous inclusion of $L^2(\Sigma)$ in $L^{\beta(p-1)+p}(\Sigma)$ for $\beta \leq \frac{2-p}{p-1}$;

$$\left\|m^\varepsilon z - z_{|\Sigma}\right\|^2_{L^{\beta(p-1)+p}(]0,\infty[\times\Sigma)} \leq C\left\|m^\varepsilon z - z_{|\Sigma}\right\|^2_{L^2(]0,\infty[\times\Sigma)} \leq C\varepsilon^2\int_0^\infty\int_{\Sigma_\varepsilon} |\nabla z|^2 \,\mathrm{d}\sigma\mathrm{d}t.$$

We have weak convergence of $z_{\varepsilon|\Sigma}$ to $z_{|\Sigma}$ in $L^2(]0,\infty[\times\Sigma)$. Since $z_{\varepsilon|\Sigma} \rightharpoonup z_{|\Sigma}$ in $L^{\beta(p-1)+p}(]0,\infty[\times\Sigma)$, we have $m^\varepsilon z_\varepsilon \rightharpoonup z_{|\Sigma}$ in $L^{\beta(p-1)+p}(]0,\infty[\times\Sigma)$, and hence $m^\varepsilon z_\varepsilon \rightharpoonup z_{|\Sigma}$ in $L^{\beta(p-1)+p}\left(0,\infty;L^{\beta(p-1)+p}(\Sigma)\right)$.

$$F_\varepsilon(z_\varepsilon) \geq \frac{1}{2}\int_{\Omega^\infty} |\nabla z_\varepsilon|^2 \,\mathrm{d}x\mathrm{d}t - \frac{\varepsilon^{2+a(p-1)-\alpha}}{\beta(p-1)+p}\oint_{\Sigma^\infty} |m^\varepsilon z_\varepsilon|^{\beta(p-1)+p}\,\mathrm{d}\sigma\mathrm{d}t.$$

Using the subdifferential inequality, we obtain

$$F_\varepsilon(z_\varepsilon) \geq \frac{1}{2}\int_{\Omega^\infty} |\nabla z_\varepsilon|^2 \,\mathrm{d}x\mathrm{d}t - \frac{\varepsilon^{2+a(p-1)-\alpha}}{\beta(p-1)+p}\oint_{\Sigma^\infty} |z_{|\Sigma}|^{\beta(p-1)+p}\,\mathrm{d}\sigma\mathrm{d}t$$
$$- \frac{\varepsilon^{2+a(p-1)-\alpha}}{\beta(p-1)+p}\oint_{\Sigma^\infty} |z_{|\Sigma}|^{\beta(p-1)+p-2} z_{|\Sigma}\left(m^\varepsilon z_\varepsilon - z_{|\Sigma}\right)\mathrm{d}\sigma\mathrm{d}t.$$

By passing to the lower limit, we obtain

$$\liminf_{\varepsilon\to 0} F_\varepsilon(z_\varepsilon) \geq \frac{1}{2}\int_{\Omega^\infty} |\nabla z|^2 \,\mathrm{d}x\mathrm{d}t - \frac{\eta(\alpha - a(p-1))}{\beta(p-1)+p}\oint_{\Sigma^\infty} |z_{|\Sigma}|^{\beta(p-1)+p}\,\mathrm{d}\sigma\mathrm{d}t.$$

Hence the result.

In the sequel, one is interested to limit problem determination partner to the problem (1), when ε approaches zero. Thanks to the epi-convergence results, (see ([11], Proposition p. 40), and according to τ_f-continuity of G in $L^2(0,\infty;\mathbb{H}_\Sigma)$, one has $F_\varepsilon + G$ τ_f-epi-converges toward $F^\alpha + G$ in $L^2(0,\infty;\mathbb{H}_\Sigma)$.

Table 1. Numerical tests for stability on - Σ_ε and Ω.

t	$\|z_\varepsilon\|$	$\|z_{\varepsilon\|\Sigma}\|$	$\|z\|$	$\|z_{\|\Sigma}\|$
$t = 1$	3.9947036053134716e−05	8.461611843695231e−22	3.798728877912462e−05	8.70266010292555e−22
$t = 3$	4.0078456996587055e−05	8.4776107549483e−22	3.808151054788732e−05	8.719448911188138e−22
$t = 6$	4.007915985374034e−05	8.477653057484787e−22	3.808176052445502e−05	8.719492925316215e−22
$t = 10$	4.007915985374034e−05	8.477653057484787e−22	3.808176052445502e−05	8.719492925316215e−22

4. Numerical Tests

For a sufficiently small value of *ε*, the solution z_ε of the approximating problem approaches the solution z of the limit problem. We are interested in the numerical treatment in this section and we will focus on the impact of the control on the surface $\Sigma_\varepsilon^\infty$, with

$$T = 10 \quad \Sigma_\varepsilon = \left\{x \in \partial\Omega \,|\, |x_3| \le \varepsilon^2\right\}$$

$$\Omega = \left\{(x_1, x_2, x_3) \,|\, x_1 \in]-1,1[, x_2 \in]-1,1[, x_3 \in]-1,1[\right\} \quad u_\varepsilon = -\varepsilon^a z_\varepsilon |z_\varepsilon|^\beta$$

Using the Python programming language, with the finite element method and the Newton method, with $p = 7$, $\alpha = 2 + a(p-1)$ and $\varepsilon = 1\mathrm{e}-7$, one will have the results shown in the table.

The solution of the approximation problem converges to that of the limit problem.

Initially, u_ε does not stabilize the state on all of Ω, which is normal because the control is defined only on Σ_ε, so the control will stabilize the state only on Σ_ε.

Table 1 shows that the solution of the approximation problem converges to that of the limit problem and shows that u_ε stabilizes the state $z_{\varepsilon|\Sigma}$, and u stabilizes the state $z_{|\Sigma}$ on the nanolayer, which shows that the model is suitable for control specialists on the nanolayer.

5. Conclusion

In this paper we have focused on the stability of nanolayer boundaries in diffusion-reaction systems, taking into account a nonlinear boundary control condition. We have demonstrated the stability of nanolayer boundaries using the Lyapunov function approach, making certain regularity assumptions and imposing appropriate control conditions. In addition, we have extended the stability analysis to more complex systems by studying the boundary problem with interface conditions using the epi-convergence approach. The results obtained in this paper were then tested numerically to validate the theoretical conclusions. These results pave the way for further research into the stability of boundaries in diffusion-reaction systems with non-linear control conditions.

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Ethical Approval

Since this research exclusively entails theoretical and computational analysis, with no involvement of human or animal subjects, ethical approval was not required.

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This study was not supported by any data.

Conflicts of Interest

There are no conflicts of interest, according to the authors.

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