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Simple Formulas of π in Terms of Φ

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Abstract

The paper presents a novel exploration of π through a re-calculation of formulas using Archimedes' algorithm, resulting in the identification of a general family equation and three new formulas involving the golden ratio Φ, in the form of infinite nested square roots. Some related geometrical properties are shown, enhancing the link between the circle and the golden ratio. Applying the same criteria, a fourth formula is given, that brings to the known Dixon's squaring the circle approximation, thus an easier approach to this problem is suggested, by a rectangle with both sides proportional to the golden ratio Φ.

Keywords

π, Φ, Golden Ratio, Squaring the Circle

1. Introduction

In last century it was challenging and interesting to find formulas of π in terms of the golden ratio Φ (so involving together two of most famous irrational constants), without transcendent functions, as the well known $\pi = \frac{10}{3}\arcsin\left(\frac{\phi}{2}\right)$.

Among these works, few examples are here reported:

$$\pi = \frac{5\sqrt{2+\phi}}{2\phi}\sum_{n=0}^{\infty}\left(\frac{1}{2\phi}\right)^{5n}\left(\frac{1}{5n+1}+\frac{1}{2\phi^2(5n+2)}-\frac{1}{2^2\phi^3(5n+3)}-\frac{1}{2^3\phi^3(5n+4)}\right) \quad (1)$$

Equation (1) has been presented by Chan in [1], inspired by the work of Bailey, Borwein and Plouffe (so called BBP-formulas) in [2].

$$\frac{\pi^2}{50} = \sum_{k=0}^{\infty}\left(\frac{\phi^2}{(5k+1)^2}-\frac{\phi}{(5k+2)^2}-\frac{\phi^2}{(5k+3)^2}+\frac{\phi^5}{(5k+4)^2}+\frac{2\phi^2}{(5k+5)^2}\right)\phi^{-5k} \quad (2)$$

Equation (2) has been discovered by B. Cloitre and reported by Chan in [3], also inspired by BBP-formulas.

The aim of this work, presented in next pages, is focused on identifying other simple formulas of π in terms of Φ.

2. Nested Square Roots Formulas of π

In order to show other simple formulas of π in terms of Φ, first it needs to easily share the calculation behind the family of the known formulas of π in the form of nested square roots. The approach starts from the idea of Archimedes, resumed several centuries later by F. Viete (as reported by Beckmann in [4]), then more recently completely restructured by Servi in [5].

Let us start considering a regular polygon inscribed in a circle with unitarian radius, with N sides ($N \in \mathbb{N}$, $N \geq 3$) of length L_1 ($L_1 = 2\sin\frac{\alpha}{2}$, $\alpha = \frac{2\pi}{N}$), with perimeter $P_1 = NL_1$, as in **Figure 1** (where, as example, is represented a pentagon).

With the polygon obtained doubling the sides, considering $a_1 = \sqrt{1-\left(\frac{L_1}{2}\right)^2}$, $d_1 = 1 - a_1$, the perimeter become $P_2 = N2L_2$ with

$$\begin{aligned} L_2 &= \sqrt{\frac{L_1^2}{4} + \left(1-\sqrt{1-\frac{L_1^2}{4}}\right)^2} = \sqrt{\frac{L_1^2}{4} + 1 + 1 - \frac{L_1^2}{4} - 2\sqrt{1-\frac{L_1^2}{4}}} \\ &= \sqrt{2-2\sqrt{1-\frac{L_1^2}{4}}} = \sqrt{2-\sqrt{4-L_1^2}} \end{aligned} \tag{3}$$

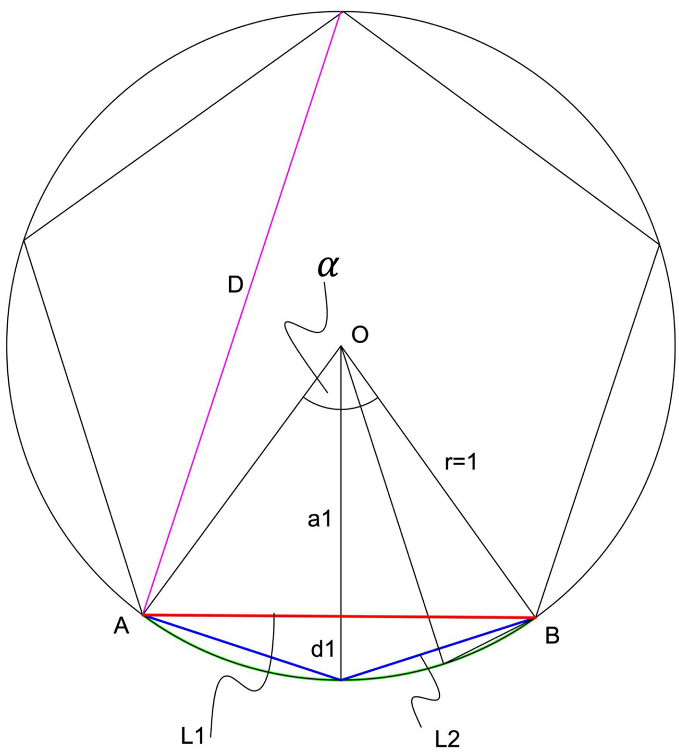

Figure 1. Bisection of chord and arc-example on a pentagon.

Re-iterating the process, doubling the polygon at each step, from Equation (3), considering that for $n \to \infty$ the perimeter of poligons $\to 2\pi$, the following succession is obtained

$$\begin{cases} L_1 = 2\sin\dfrac{\pi}{N} & N \in \mathbb{N},\, N \geq 3 \\ L_n = \sqrt{2 - \sqrt{4 - L_{n-1}^2}} & n \in \mathbb{N},\, n \geq 2 \\ \pi = N \lim\limits_{n\to\infty} 2^{n-2} L_n \end{cases} \tag{4}$$

Expanding Equations (3) and (4):

$$L_3 = \sqrt{2 - \sqrt{4 - L_2^2}} = \sqrt{2 - \sqrt{4 - \left(2 - \sqrt{4 - L_1^2}\right)}} = \sqrt{2 - \sqrt{2 + \sqrt{4 - L_1^2}}}$$

$$L_4 = \sqrt{2 - \sqrt{4 - L_3^2}} = \sqrt{2 - \sqrt{4 - \left(2 - \sqrt{2 + \sqrt{4 - L_1^2}}\right)}} = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{4 - L_1^2}}}}$$

$$L_n = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots \sqrt{2 + \sqrt{4 - L_1^2}}}}}} \quad n \text{ square roots} \tag{5}$$

Using the results of Equation (5), a first family equation of π in the form of nested square roots follows, valid for any regular polygons with N sides and side length L_1.

$$\begin{cases} L_1 = 2\sin\dfrac{\pi}{N} & N \in \mathbb{N},\, N \geq 3 \\ \pi = N \lim\limits_{n\to\infty} 2^{n-2} \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots \sqrt{2 + \sqrt{4 - L_1^2}}}}}} & n \text{ square roots} \end{cases} \tag{6}$$

Now we would like to extend this approach, freeing from the regular polygons.

Let us focus on the arc $\overset{\frown}{AB} = \alpha < \pi \text{ rad}$, implied with the chord $\overline{AB} = 2\sin\dfrac{\alpha}{2}$ (**Figure 1**); with this in mind, we can call $\mu \in \mathbb{R}$ the ratio between the circumference length and the arc $\overset{\frown}{AB}$:

$$\mu = \frac{2\pi}{\alpha} \rightarrowtail \pi = \frac{1}{2}\mu\alpha \quad \text{with } 0 < \alpha < \pi \rightarrowtail 2 < \mu < +\infty \tag{7}$$

Applying a similar strategy, it is proven that, dividing iteratively halfway the arc $\overset{\frown}{AB}$ (as in **Figure 1**), the sum of the chords implied converge to the length of the arc $\overset{\frown}{AB} = \alpha$, thus

$$\begin{cases} \mu = \dfrac{2\pi}{\alpha} & 0 < \alpha < \pi \\ \pi = \mu \lim\limits_{n\to\infty} 2^{n-1} \sin\left(\dfrac{\alpha}{2^n}\right) \end{cases} \tag{8}$$

Applying several times the goniometric bisection formulas $\sin\dfrac{\gamma}{2} = \sqrt{\dfrac{1-\cos\gamma}{2}}$, $\cos\dfrac{\gamma}{2} = \sqrt{\dfrac{1+\cos\gamma}{2}}$, and the formula $\sin^2\gamma + \cos^2\gamma = 1$ to the expression $\sin\dfrac{\alpha}{2^n}$

we obtain:

$$\sin\left(\frac{\alpha}{2^n}\right)=\sin\left(\frac{\frac{\alpha}{2^{n-1}}}{2}\right)=\sqrt{\frac{1-\cos\left(\frac{\alpha}{2^{n-1}}\right)}{2}}$$

$$=\frac{1}{2}\sqrt{2-2\cos\left(\frac{\alpha}{2^{n-1}}\right)}=\frac{1}{2}\sqrt{2-2\cos\left(\frac{\frac{\alpha}{2^{n-2}}}{2}\right)}$$

$$=\frac{1}{2}\sqrt{2-2\sqrt{\frac{1+\cos\left(\frac{\alpha}{2^{n-2}}\right)}{2}}}=\frac{1}{2}\sqrt{2-\sqrt{2+2\cos\left(\frac{\alpha}{2^{n-2}}\right)}}$$

$$=\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\cdots\sqrt{2+2\cos\left(\frac{\alpha}{2}\right)}}}}$$

$$=\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\cdots\sqrt{2+2\sqrt{1-\sin^2\left(\frac{\alpha}{2}\right)}}}}}$$

$$=\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\cdots\sqrt{2+\sqrt{4-\left[2\sin\left(\frac{\alpha}{2}\right)\right]^2}}}}}\quad n \text{ square roots} \tag{9}$$

Equation (9) coincides with Equation (9) found by Abrarov in [6], when $\alpha=\pi$. Using Equation (9) in Equation (8), noting that $L_1=\overline{AB}=2\sin\left(\frac{\alpha}{2}\right)$, finally the following general family Equation (10) is obtained:

$$\begin{cases} \mu=\dfrac{2\pi}{\alpha} & 0<\alpha<\pi,\ \mu\in\mathbb{R},\ \mu>2 \\ L_1=2\sin\left(\dfrac{\alpha}{2}\right) & \\ \pi=\mu\lim\limits_{n\to\infty}2^{n-2}\sqrt{2-\sqrt{2+\sqrt{2+\cdots\sqrt{2+\sqrt{4-L_1^2}}}}} & n \text{ square roots} \end{cases} \tag{10}$$

Equation (10) generalizes and confirms Equation (6). Equation (10) coincides with Equation (6) when $\mu=N\in\mathbb{N}$, or in other words when α is an integer divisor of the circle, and in this case $L_1=2\sin\left(\frac{\alpha}{2}\right)=2\sin\left(\frac{\pi}{N}\right)$. Also the succession in Equation (4) can be extended substituting N with $\mu=\frac{2\pi}{\alpha}$ and L_1 with $L_1=2\sin\left(\frac{\alpha}{2}\right)$ for values not integer of $\frac{2\pi}{\alpha}$, with $\alpha\in(0,\pi)$.

Equation (10), as Equation (6), arises some interest when applied for known values not transcendent of $\sin\left(\frac{\alpha}{2}\right)$. Some of these instances follow.

Applying Equation (6) to an equilateral triangle ($N=3$, $L_1=2\sin\frac{\pi}{3}=\sqrt{3}$, calculating the last three square roots in Equation (10) $\sqrt{2+\sqrt{2+\sqrt{4-L_1^2}}} = \sqrt{2+\sqrt{2+1}} = \sqrt{2+\sqrt{3}}$), or an hexagon ($N=6$, $L_1 = 2\sin\frac{\pi}{6} = 1$, calculating the last two square roots in Equation (10) $\sqrt{2+\sqrt{4-L_1^2}} = \sqrt{2+\sqrt{3}}$), or a dodecagon ($N=12$, $L_1 = 2\sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{2}$, calculating the last square roots in Equation (10) $\sqrt{4-L_1^2} = \sqrt{4-\left(\frac{\sqrt{6}-\sqrt{2}}{2}\right)^2} = \sqrt{4-\frac{6+2-\sqrt{12}}{4}} = \sqrt{4-\left(2-\sqrt{3}\right)} = \sqrt{2+\sqrt{3}}$) the following Equation (11), coinciding with the Formula (3) presented by Servi in [5], is obtained:

$$\pi = 3\lim_{n\to\infty} 2^{n-1}\sqrt{2-\sqrt{2+\sqrt{2+\cdots\sqrt{2+\sqrt{3}}}}} \quad n \text{ square roots} \tag{11}$$

Applying Equation (6) to a square, with $N = 4$, $L_1 = 2\sin\frac{\pi}{4} = \sqrt{2}$, calculating the last two square roots in Equation (10) $\sqrt{2+\sqrt{4-L_1^2}} = \sqrt{2+\sqrt{4-2}} = \sqrt{2+\sqrt{2}}$, the following Equation (12), coinciding with the Formula (1) presented by Servi in [5], is obtained:

$$\pi = 2\lim_{n\to\infty} 2^{n-1}\sqrt{2-\sqrt{2+\sqrt{2+\cdots\sqrt{2+\sqrt{2}}}}} \quad n \text{ square roots} \tag{12}$$

Applying Equation (10) to the arc $\widehat{AB} = \alpha = \frac{3}{4}\pi$ (135°), $\mu = \frac{2\pi}{\alpha} = \frac{8}{3}$, calculating $L_1 = 2\sin\left(\frac{\alpha}{2}\right) = \sqrt{2+\sqrt{2}}$ and the last square root in Equation (10) $\sqrt{4-L_1^2} = \sqrt{4-\left(2+\sqrt{2}\right)} = \sqrt{2-\sqrt{2}}$, the following Equation (13), coinciding with the Formula (2) presented by Servi in [5], is obtained:

$$\pi = \frac{2}{3}\lim_{n\to\infty} 2^{n-1}\sqrt{2-\sqrt{2+\sqrt{2+\cdots\sqrt{2+\sqrt{2-\sqrt{2}}}}}} \quad n \text{ square roots} \tag{13}$$

Applying Equation (10) to the arc $\widehat{AB} = \alpha = \frac{5}{6}\pi$ (150°), $\mu = \frac{2\pi}{\alpha} = \frac{12}{5}$, calculating $L_1 = 2\sin\left(\frac{\alpha}{2}\right) = 2\frac{\sqrt{6}+\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{2}$ and the last square root in Equation (10) $\sqrt{4-L_1^2} = \sqrt{4-\frac{\left(\sqrt{6}+\sqrt{2}\right)^2}{4}} = \sqrt{4-\frac{6+2+2\sqrt{12}}{4}} = \frac{1}{2}\sqrt{8-4\sqrt{3}} = \sqrt{2-\sqrt{3}}$, the following Equation (14), coinciding with the Formula (4) presented by Servi in [5], is obtained:

$$\pi = \frac{3}{5}\lim_{n\to\infty} 2^{n-1}\sqrt{2-\sqrt{2+\sqrt{2+\cdots\sqrt{2+\sqrt{2-\sqrt{3}}}}}} \quad n \text{ square roots} \tag{14}$$

3. New Simple Formulas of π in Terms of Φ

Now applying specifically Equations (10) and (6), the following three simple formulas of π in terms of Φ are identified:

$$\pi = 5\lim_{n\to\infty} 2^{n-1}\sqrt{2-\sqrt{2+\sqrt{2+\cdots\sqrt{2+\sqrt{2+\phi}}}}} \quad n \text{ square roots} \tag{15}$$

$$\pi = \frac{5}{3}\lim_{n\to\infty} 2^{n-1}\sqrt{2-\sqrt{2+\sqrt{2+\cdots\sqrt{2+\sqrt{3-\phi}}}}} \quad n \text{ square roots} \tag{16}$$

$$\pi = \frac{5}{4}\lim_{n\to\infty} 2^{n-1}\sqrt{2-\sqrt{2+\sqrt{2+\cdots\sqrt{2+\sqrt{2-\phi}}}}} \quad n \text{ square roots} \tag{17}$$

Equation (15) is obtained applying the Equation (6) to a pentagon inscribed in the circumference (as in **Figure 1**) with unitarian radius ($N=5$, $\alpha=\frac{2}{5}\pi$, with simple passages $L_1 = 2\sin\left(\frac{\pi}{5}\right) = 2\frac{\sqrt{10-2\sqrt{5}}}{4} = \sqrt{\frac{10-2\sqrt{5}}{4}} = \sqrt{\frac{5-\sqrt{5}}{2}} = \sqrt{3-\frac{1+\sqrt{5}}{2}}$ $=\sqrt{3-\phi}$). From Equation (6), and remembering that $\phi^n = \phi^{n-1}+\phi^{n-2}$, the last two square roots become $\sqrt{2+\sqrt{4-L_1^2}} = \sqrt{2+\sqrt{4-(3-\phi)}} = \sqrt{2+\sqrt{1+\phi}} = \sqrt{2+\phi}$. Since between the diagonal D and the side L of a pentagon results $D=\phi L$ (Ghyka in [7]), it is noted the fact quite singular that in this case the last square root in Equation (15) represents exactly the length of the diagonal $D_1 = \phi L_1 =$ $\phi\sqrt{3-\phi} = \sqrt{3\phi^2-\phi^3} = \sqrt{3\phi^2-(\phi^2+\phi)} = \sqrt{2\phi^2-\phi} = \sqrt{2(\phi+1)-\phi} = \sqrt{2+\phi}$.

Equation (16) is obtained applying the Equation (10) to the arc $\overset{\frown}{AB}$ (in **Figure 1**) for $\alpha=\frac{3}{5}\pi$ (that is the arc $\overset{\frown}{BF}$ of 108° in **Figure 2**), $\mu=\frac{2\pi}{\alpha}=\frac{10}{3}$. Calculating $L_1 = 2\sin\left(\frac{\alpha}{2}\right) = 2\frac{1+\sqrt{5}}{4} = \frac{1+\sqrt{5}}{2} = \phi$ (in fact in this case the chord $\overline{BF}$ with two radius forms the triangle BFO gnomon of the golden triangle), thus the last square root in Equation (10) become $\sqrt{4-L_1^2} = \sqrt{4-\phi^2} =$ $\sqrt{4-(\phi+1)} = \sqrt{3-\phi}$. It is noted the fact quite singular that in this case the last square root in the formula Equation (16) represents exactly the length of the side of a pentagon inscribed in the circle.

Equation (17) is obtained applying the Equation (10) to the arc $\overset{\frown}{AB} = \alpha = \frac{4}{5}\pi$ (that is the arc $\overset{\frown}{AD}$ of 144° in **Figure 2**), $\mu=\frac{2\pi}{\alpha}=\frac{10}{4}$. Calculating $L_1 = 2\sin\left(\frac{\alpha}{2}\right) = \frac{1}{2}\sqrt{10+2\sqrt{5}} = \sqrt{\frac{4+1+\sqrt{5}}{2}} = \sqrt{2+\phi}$ (coinciding with the diagonal $\overline{AD}$ in **Figure 2**), thus the last square root in Equation (10) become $\sqrt{4-L_1^2} = \sqrt{2-\phi}$. It is noted the fact quite singular that in this case the last square root in Equation (17) represents exactly the length of the side of a decagon inscribed in the circle, as $2\sin\frac{36^\circ}{2} = \frac{1}{2}\left(\sqrt{5}-1\right) = \sqrt{\left(\frac{\sqrt{5}-1}{2}\right)^2} = \sqrt{\frac{6-2\sqrt{5}}{4}} =$ $\sqrt{\frac{4-1-\sqrt{5}}{2}} = \sqrt{2-\left(\frac{1+\sqrt{5}}{2}\right)} = \sqrt{2-\phi}$ ($\overline{FD}$ in **Figure 2**, side of a golden triangle *FDO*).

A numerical simulations for Equations (15), (16), (17) follow, performed with Excel ver 16.84 for Mac, utilizing for ϕ a value with 10 digits after comma (1.6180339887), each row representing the calculation of π for increasing value of n, till 13th iteration.

For Equation (15) it is obtained:

n	*π estimated*	*Error*
2	3.1286893010	0.0129033526
3	3.1383638293	0.0032288243
4	3.1407852609	0.0008073927
5	3.1413907939	0.0002018597
6	3.1415421881	0.0000504655
7	3.1415800373	0.0000126163
8	3.1415894997	0.0000031539
9	3.1415918653	0.0000007883
10	3.1415924566	0.0000001969
11	3.1415926044	0.0000000492
12	3.1415926415	0.0000000121
13	3.1415926489	0.0000000047

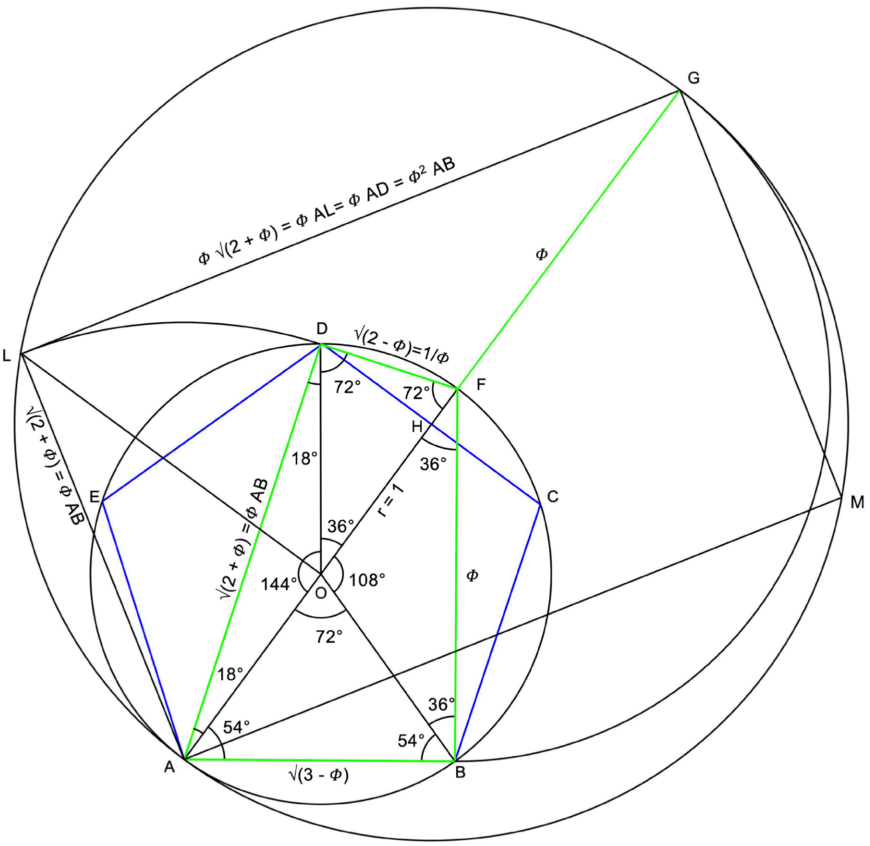

Figure 2. Geometrical properties between circle, π and Φ.

For Equation (16) it is obtained:

n	π *estimated*	*Error*
2	3.0266033316	0.1149893220
3	3.1126048514	0.0289878022
4	3.1343305988	0.0072620548
5	3.1397761947	0.0018164589
6	3.1411384797	0.0004541739
7	3.1414791064	0.0001135472
8	3.1415642665	0.0000283871
9	3.1415855568	0.0000070968
10	3.1415908793	0.0000017742
11	3.1415922100	0.0000004436
12	3.1415925427	0.0000001109
13	3.1415926258	0.0000000278

For Equation (17) it is obtained:

n	π *estimated*	*Error*
2	2.9389262614	0.2026663922
3	3.0901699437	0.0514227099
4	3.1286893008	0.0129033528
5	3.1383638291	0.0032288245
6	3.1407852607	0.0008073929
7	3.1413907936	0.0002018599
8	3.1415421878	0.0000504658
9	3.1415800371	0.0000126165
10	3.1415894994	0.0000031542
11	3.1415918650	0.0000007886
12	3.1415924564	0.0000001972
13	3.1415926044	0.0000000492

As we could expect, the formulas converge slowly (around 1 digit every 2 iterations), with Equation (15) just a bit faster than the others.

It should be clear that these formulas are not intended for π calculation with several digits, both because this nested roots structure is very slow comparing with recent algorithms, and also because there is no sense in calculating π from another irrational number as ϕ.

4. Geometrical Properties

Figure 2 helps to understand the singularities pointed out in last square root of Equations (15), (16) and (17).

For Equation (15), as $L_1 = \overline{AB} = \sqrt{3-\phi}$ (side of the pentagon inscribed in a circle with unitarian radius), applying the last square root $\sqrt{4-L_1^2}$ means finding the other cathetus $\overline{BF}$ of the right triangle *ABF*, being the hypotenuse $\overline{AF} = 2$, then $\overline{BF} = \sqrt{4-(3-\phi)} = \sqrt{1+\phi} = \sqrt{\phi^2} = \phi$ (also as side of a gnomon triangle *OBF* of the golden triangle). Then, for the following square root $\sqrt{2+\sqrt{4-L_1^2}} = \sqrt{2+\phi}$, with the geometric approach it is possible to construct the segment AG with length $2+\phi$, then drawing a circle with center in H e diameter *AG*, $\overline{AL}$ resulting the square root of $\overline{AG}$ (from the equivalence of the right triangles *AGL* and *AOL*, with $\overline{AO} = 1$); finally we get the diagonal $\overline{AD} = \overline{AL} = \sqrt{2+\Phi}$ (that is also equal to $\phi\overline{AB} = \phi\sqrt{3-\phi}$).

For Equation (16), as $L_1 = \overline{BF} = \phi$ (side of the gnomon triangle), applying the last square root $\sqrt{4-L_1^2}$ means finding the other cathetus $\overline{AB}$ of the right triangle *ABF*, of length $\sqrt{3-\phi}$ (side of the pentagon).

For Equation (17), as $L_1 = \overline{AD} = \sqrt{2+\phi}$ (diagonal of the pentagon), applying the last square root $\sqrt{4-L_1^2}$ means finding the other cathetus $\overline{DF}$ of the right triangle *AFD*, then $\overline{DF} = \sqrt{4-(2+\Phi)} = \sqrt{2-\Phi}$ (side of the decagon, for which is also valid $\sqrt{2-\phi} = \sqrt{2-\left(1+\frac{1}{\phi}\right)} = \sqrt{1-\frac{1}{\phi}} = \sqrt{\frac{\phi-1}{\phi}} = \sqrt{\frac{\frac{1}{\phi}}{\phi}} = \sqrt{\frac{1}{\phi^2}} = \frac{1}{\phi}$, as *FDO* is a golden triangle with $\overline{OF} = 1$).

Referring again to Figure 2, from this geometrical approach it is evident that, if we apply Equation (10) with $L_1 = \overline{DF} = \sqrt{2-\Phi} = \frac{1}{\phi}$ (side of a decagon), calculating the last square root $\sqrt{4-L_1^2}$ means finding the other cathetus $\overline{AD}$ of the right triangle *AFD*, of length $\sqrt{2+\phi}$ (diagonal of the pentagon), thus with the same result obtained in Equation (15) starting with $L_1 = \overline{AB} = \sqrt{3-\Phi}$ (side of a pentagon), as we could expect because a decagon comes by doubling the sides of the pentagon on the first iteration of Equation (4).

Some other geometrical properties could be highlighted: in Figure 2 the rectangle *AMGL* is a golden rectangle, as $\overline{LG} = \sqrt{\overline{AG}^2 - \overline{AL}^2} = \sqrt{(2+\phi)^2 - (2+\phi)}$ $= \sqrt{(2+\phi)(2+\phi-1)} = \sqrt{(2+\phi)\phi^2} = \phi\sqrt{2+\phi} = \phi\overline{AL}$, and H is midpoint of *AG*, as $\overline{AG} = 2+\phi = 2\left(1+\frac{\phi}{2}\right) = 2\left(\overline{AO}+\overline{OH}\right) = 2\overline{AH}$. It noted that, since the area of rectangle *AMGL* is $AreaAMGL = \overline{AL}\times\overline{LG} = \Phi\overline{AB}\times\phi^2\overline{AB} = \phi^3\overline{AB}$, and the area of the pentagon ABCDE is $AreaABCDE = 5\times AreaOCD = 5\left(\frac{\overline{OH}\times\overline{AB}}{2}\right)$ $= \frac{5}{2}\frac{\phi}{2}\overline{AB} = \frac{5}{4}\phi\overline{AB}$, their ratio results $\frac{Area-rectangleAMGL}{Area-pentagonABCDE} = \frac{4}{5}\phi^2$.

Referring to **Figure 2** and **Figure 3**, it should be noted the fact quite singular that Equations (15), (16) and (17) have the same form as in following Equation (18)

$$\pi = k \lim_{n\to\infty} 2^n \sqrt{2-\sqrt{2+\sqrt{2+\cdots\sqrt{2+M}}}} \quad n \text{ square roots} \tag{18}$$

where $k = 5, \frac{5}{3}, \frac{5}{4}$ and M equal to the side or any diagonals (except the diameter) of a decagon (as in **Figure 3**) inscribed in a circle with unitarian radius. This results from Equation (15) applied both for $M = \overline{AE} = \sqrt{2+\phi}$ and $M = \overline{AD} = \phi$; from Equation (16) applied for $M = \overline{AC} = \sqrt{3-\phi}$; from Equation (17) applied for $M = \overline{AB} = \sqrt{2-\phi}$).

It is appropriate to mention here also the other well known relations between the radius and the sides and diagonals of inscribed decagon: $\overline{AO} = \phi\overline{AB}$, $\overline{AD} = \phi\overline{AO}$, $\overline{AE} = \phi\overline{AC}$ (as in **Figure 3** and **Figure 4**).

All these properties (in Equations (15), (16), (17) and (18), and in **Figure 2** and **Figure 3**) remind to the studies reported by Ghyka in [7] par. 3 "The geometric canons of Mediterranean Architecture", where, referring to the works of J. Hambidge, he wrote that "the diagonals in fact do play a major role in the graphics manipulations of the Greek geometers", and, referring to the works of Prof. Mossel, he wrote "Mossel notes that the largest number of drawing were not produced by this natural 'astronomical' subdivision of the circle of orientation into 4, 8, or 16, but by its more subtle division into 10 or 5 parts, that is, by the inscription inside this circle, which then becomes the directing circle of a life-size template, of a regular pentagon or decagon. This applies equally to Egyptian drawing and to those from the great era of classical Greece, and takes us directly to Lund's Gothic drawing with their starry plans and pentagonal polarity", as can be viewed in **Figure 4** and **Figure 5**, extracted from [7].

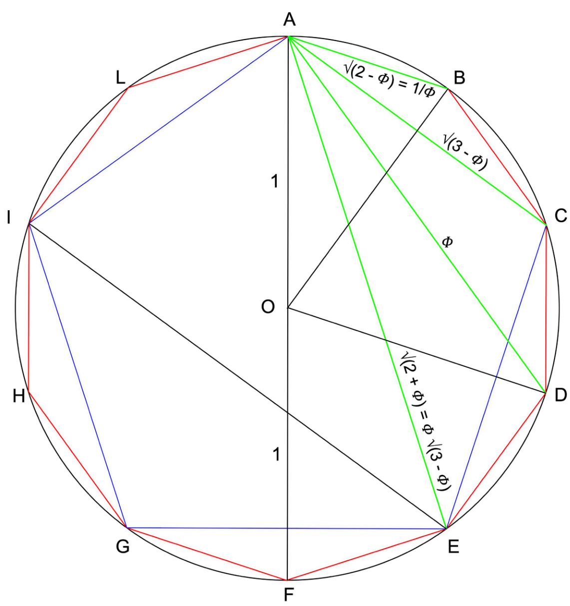

Figure 3. Decagon and its diagonals.

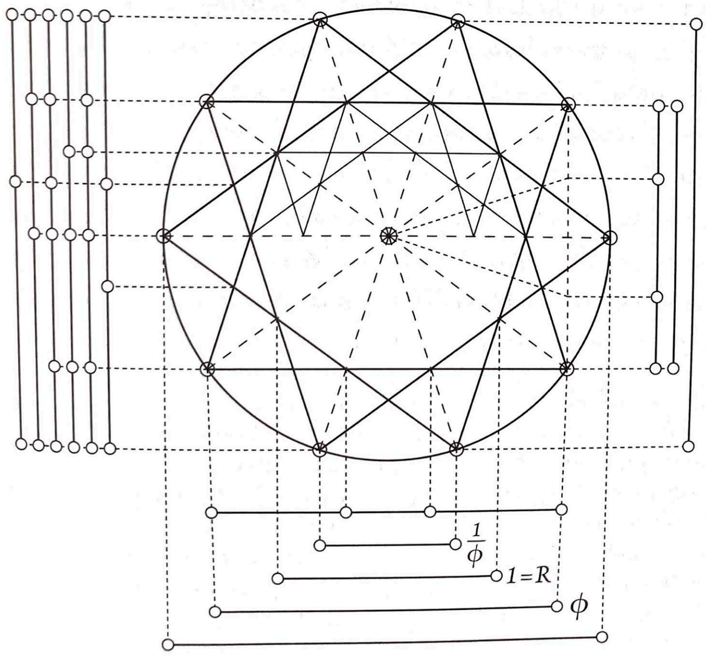

Figure 4. System of proportion identified by prof. Mossel.

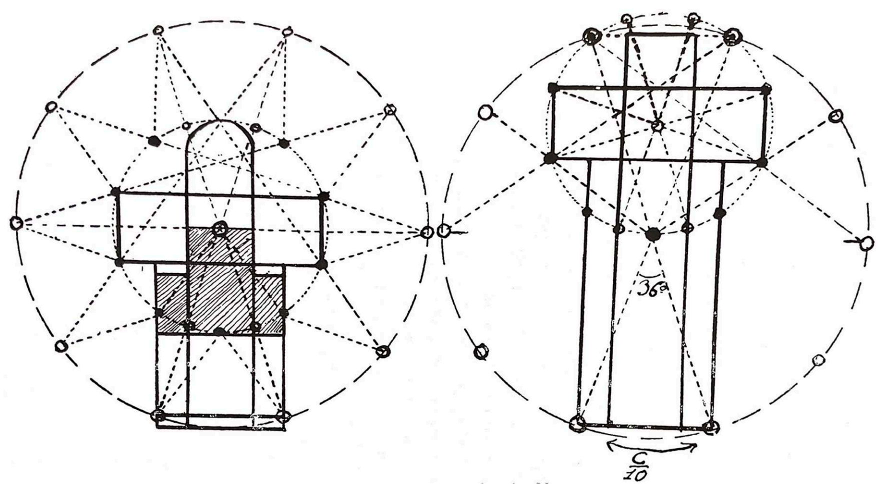

Figure 5. Typical Gothic design showing decadal segmentation (Mossel).

In conclusion, Equations (15), (16), (17) and these geometrical properties show some more links between the circle, π and Φ, thus extending the relations with the decagon (in **Figure 3**), the pentagon (in **Figure 2**) and its pentagram (in **Figure 6**), these well known since Pythagoras ancient times (Ghyka in [7]).

5. New Approximate "Squaring" the Circle Proposal

Let us consider a particular angle, the arc $\widehat{AB} = \alpha = \frac{2\pi}{\phi^2}$ ($\approx$ 137.5° called also "golden angle", that is found many times in nature, for instance in phyllotaxis), with $\mu = \frac{2\pi}{\alpha} = \phi^2$.

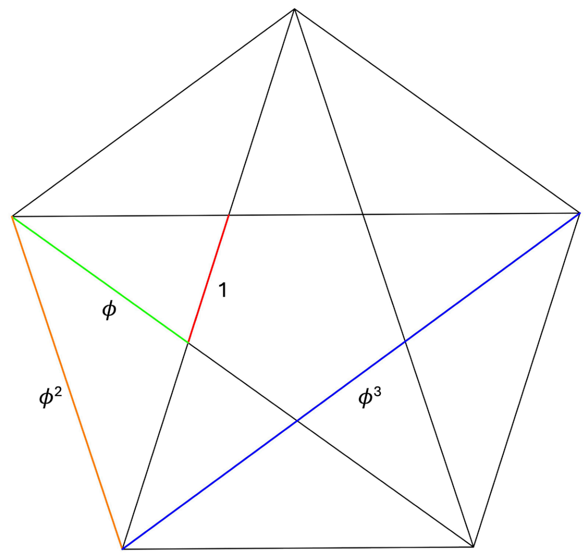

Figure 6. Relations in Pythagorean pentagram.

Calculating $L_1 = 2\sin\left(\frac{\alpha}{2}\right) = 2\sin\left(\frac{\pi}{\phi^2}\right) \simeq 1.86406485$, Equation (10) becomes:

$$\pi = \phi^2 \lim_{n\to\infty} 2^{n-2}\sqrt{2-\sqrt{2+\sqrt{2+\cdots\sqrt{2+\sqrt{4-L_1^2}}}}} \quad n \text{ square roots} \tag{19}$$

The interest on this formula arises noticing that the limit converges to a number 1.199981546…, thus can be approximate to 1.2 with an error lower than 0.00005.

The approximation of Equation (19) provides a mathematical source to the well known approximate formula between π and ϕ in following Equation (20):

$$\pi \simeq \frac{6}{5}\phi^2 = \frac{6}{5}(1+\Phi) = 3.141640\cdots \quad err < 0.00005 \tag{20}$$

This approximation has been pointed out by Dixon in [8], also showing an interesting procedure to draw a square with an area of $\frac{6}{5}(1+\phi) \simeq \pi$ with an error lower than 0.00005 (in **Figure 7**). It could be interesting to mention that the relation in Equation (20) is known at least from the 12th century by the French master masons that built the gothic cathedrals, as proved by Beatrix in [9].

Procedure for the constructions just by rule and compass (**Figure 7**): 1) draw a circle with radius = 1; 2) trace the golden rectangle *OEID*; 3) apply the rule to divide a segment in five equal parts with segment *AG*, identifying the fifth part *HE* of $\overline{AE} = \overline{AO} + \overline{OE} = 1 + \phi$, then add this 1/5 to the right, in order to identify the segment *AL* with length $\frac{6}{5}(1+\phi)$; 4) trace half a circle on the diameter *AL* finding point *M* as the intersection with the vertical line from the centre *O*; 5) construct the square *AMNP* on the segment *AM*. As the triangles *ALM* and *AOM* are similar,

$$\overline{AL}:\overline{AM}=\overline{AM}:\overline{AO}\rightarrow\overline{AM}^2=\overline{AL}\rightarrow\overline{AM}=\sqrt{\overline{AL}}=\sqrt{\frac{6}{5}(1+\phi)}\simeq\sqrt{\pi}\,.$$

We propose here another easier way to approximate the "squaring" the circle based on Equation (20) with not a square but a rectangle, with sides length ϕ and $\frac{6}{5}\phi$, whose area $\frac{6}{5}\phi^2$ is quite close (with error lower than 0.00005) to the area π of the circle, as in **Figure 8**.

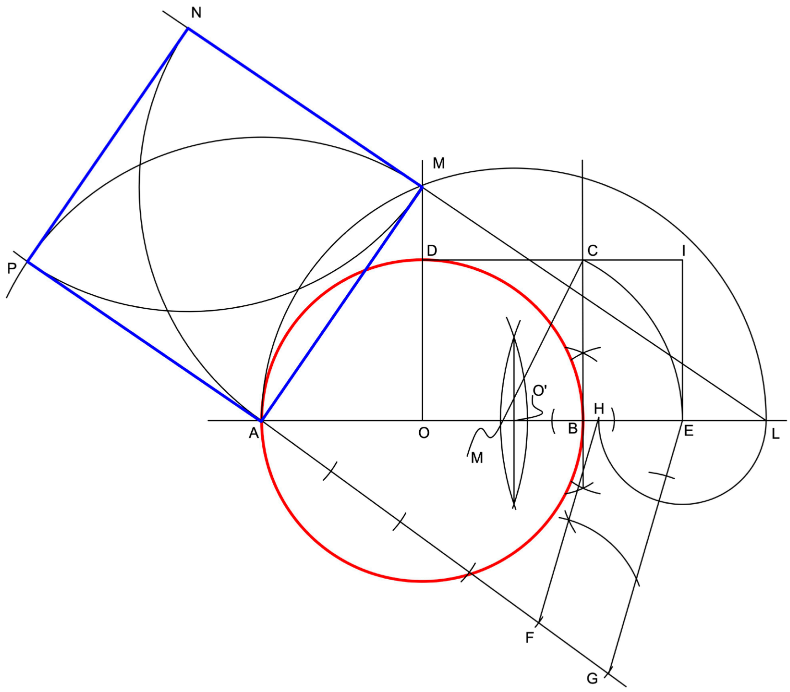

Figure 7. Approximate squaring the circle by R. A. Dixon.

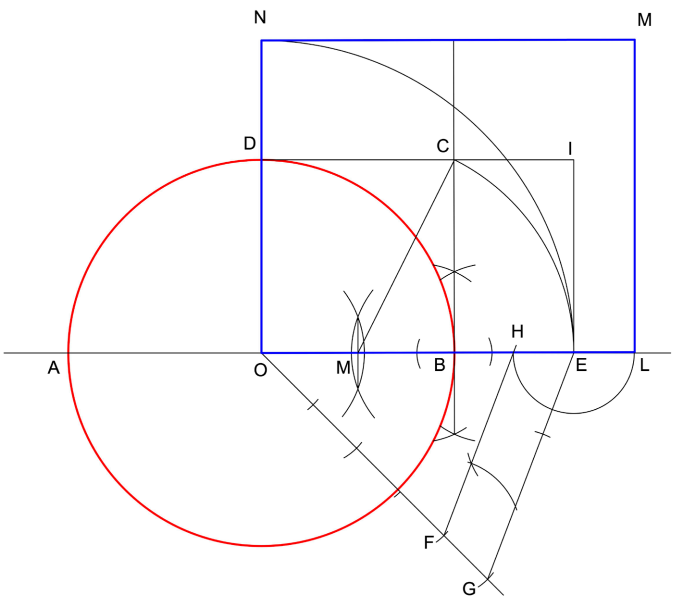

Figure 8. Approximate "squaring" the circle by a "π~rectangle".

Procedure for the constructions just by rule and compass (**Figure 8**): 1) and 2) apply the same previous steps; 3) apply the rule to divide a segment in five equal parts with segment OG, identifying the fifth part HE of $\overline{OE} = \phi$, then add this 1/5 to the right, in order to identify the segment OL with length $\frac{6}{5}\phi$; 4) trace the arc with center on O from E in order to identify the point N as intersection with the vertical line from O, so $\overline{ON} = \phi$; 5) Trace the rectangle $OLMN$, (that we can call the "π~rectangle"), that has sides length $\phi \times \frac{6}{5}\phi$, thus with area $\frac{6}{5}\phi^2 \simeq \pi$.

A numerical simulation for Equations (20) follows, performed with Excel ver 16.84 for Mac, utilizing for ϕ and π values from 2 to 10 digits after comma:

n digit after comma	*ERR* (*circle area*)	*ERR* (*rectangle area*)
2	0.00159265359	−0.007687346
3	−0.00040734641	0.000083854
4	−0.00000734641	0.000083854
5	0.00000265359	−0.000032643
6	−0.00000034641	−0.000048177
7	−0.00000004641	−0.000048177
8	−0.00000000359	−0.000048138
9	−0.00000000041	−0.000048134
10	−0.00000000001	−0.000048133

From simulation results, it can surprise that using just 3 digit after comma, the result with the "π rectangle" ($\frac{6}{5}\phi^2$, with $\phi = 1.618$) has less error than with π (with $\pi = 3.142$), but in the other cases using π the error rapidly decreases, while the error with the "π rectangle" goes asymptotically to a value almost 0.00005, as expected.

6. Conclusion

After sharing the calculation behind the family of the known formulas of π in the form of infinite nested square roots, with the presented general formula in Equation (10) three new simple formulas of π in terms of Φ are found in Equations (15), (16) and (17), arising some interesting geometrical properties, shown in **Figure 2** and **Figure 3**, and Equation (18). These results enhance the links between the circle, the pentagon, the decagon and the golden ratio, that could be deeper investigated in the future. A mathematical basis, in Equation (19), is provided for the well known Dixon's approximation of π in terms of Φ in Equation (20), and a so called "π ~ rectangle", that has sides length ϕr and $\frac{6}{5}\phi r$, is suggested as an approximated "squaring" the circle solution (**Figure 8**).

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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