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Existence of Solutions of a Convolution Integral Equation

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Abstract

In this study, we prove the of existence of solutions of a convolution Volterra integral equation in the space of the Lebesgue integrable function on the set of positive real numbers and with the standard norm defined on it. An operator P was assigned to the convolution integral operator which was later expressed in terms of the superposition operator and the nonlinear operator. Given a ball B_r belonging to the space L it was established that the operator P maps the ball into itself. The Hausdorff measure of noncompactness was then applied by first proving that given a set $M \in B_r$ the set is bounded, closed, convex and nondecreasing. Finally, the Darbo fixed point theorem was applied on the measure obtained from the set E belonging to M. From this application, it was observed that the conditions for the Darbo fixed point theorem was satisfied. This indicated the presence of at least a fixed point for the integral equation which thereby implying the existence of solutions for the integral equation.

Keywords

Volterra, Integral Equation, Convolution, Fixed Point, Existence, Noncompactness

1. Introduction

The existence of an integral equation solution is a true indicator of whether or not a given integral equation can be solved [1]. Over the years, numerous function space-based methods have been implemented to test the nature of an integral equation solution. Although all these various procedures have the same end goal, the function spaces and the applied fixed-point theorems are the key components that differentiate them. According to [2], fixed-point theorems are essential in determining whether there exists a solution for a given integral equation. The most used fixed point is the Schauder fixed point [3]. Other fixed points confirm that a self-mapping on a set, which is continuous, convex, nonempty, and a compact subset of a Banach space, possesses at least a fixed point [4]. However, finding this subset of a set belonging to a certain function space, which is bounded, convex, closed, and at the same time maps itself by the operator due to an underlying integral equation, is very difficult. For example, some of the approaches that apply either classical Banach or the Schauder fixed point normally result in inaccurate results since strong hypotheses are required for the use of these fixed-point theorems. To address these inaccuracies, the techniques of measures of noncompactness and the Darbo fixed-point theorem have been successfully employed in establishing the existence of solutions, rather than relying solely on methods like the classical Banach or Schauder fixed-point theorems. Measures of noncompactness and Darbo fixed point theorem are highly valuable in functional analysis spanning areas such as metric fixed-point theory and operator theory, differential equations, functional equations, integral and integro-differential equations, optimization, and more.

With the introduction of the concept of the measure of noncompactness, there have been successful applications through the Darbo Fixed point theorem in establishing the existence of the solution of an integral equation. [5] presented an approach that depends on the measure of noncompactness and the Darbo fixed-point theorem.

[6] provided an integral-type generalization of Darbo's theorem and applied it to establish the existence of solutions for functional integral equations. [7] presented another generalization of Darbo's theorem along with an application. Recently, integral equations of fractional orders have been investigated in [8] and [9] using measures of noncompactness. For various types of integral equations, refer to [10] [11] [12] [13]. [14] utilized shifting distance functions to establish several new generalizations.

Although this approach has been applied to prove the existence of monotonic solutions of integral equations of various types in the space of Lebesgue integrable functions, it's application in establishing the solvability of the convolution Volterra integral equation in the space of Lebesgue integrable functions has not been extensively studied [15] [16] [17] [18] [19].

Therefore, in this study, considering a convolution Volterra integral equation in the form of Equation (1):

$$x(t) = f(t) + \int_0^t \alpha(t-s)\, g\big(s, x(\varphi(s))\big)\,\mathrm{d}s,\ t > 0, \tag{1}$$

we establish the proof of existence of solutions of the convolution integral Equation (1) using the measure of noncompactness and the Darbo fixed-point theorem in the space of Lebesgue integrable functions. The mathematical preliminaries and theoretical concepts are in section two, while the main results are presented in section three.

2. Main Concept

2.1. Basic Definitions and Preliminaries

Some of the various mathematical concepts and theorems required for the study are recalled in this section. For the purpose of this study, $\mathbb{R}$ represent the set of real numbers, $\forall$ denotes for all and $\mathbb{N}$ the set of natural numbers.

Definition 1 *A set $G \subset \mathbb{R}^n$ is convex if and only if for every two points in G, the line segment that connects them is entirely contained within G. That is $\lambda x_1 + (1-\lambda)x_2 \in G \quad \forall \lambda \in [0,1] \quad x_1, x_2 \in G$* [20].

Definition 2 *Let X be a vector space over the field $\mathbb{R}$ then X is said to be a Banach space if and only if X is equipped with a norm and is also complete.*

The space X is complete if for every Cauchy sequence $\{x_n\}$ in X, there exist a subsequence $\{x_n\}$ which converges to $x \in X$.

Definition 3 *Let $\mathbb{R}$ denote the set of real numbers and $[0,\infty)$ be an interval on $\mathbb{R}$. For a given nonempty, non-bounded and Lebesgue measurable subset $[0,\infty)$ of $\mathbb{R}$, denoted by L^1, as the space of Lebesgue integrable functions on $[0,\infty)$ the standard norm is given by $\|x\| = \int_0^\infty |x(t)| \mathrm{d}t$.*

Definition 4 *Suppose $0 < p < \infty$ and (X, M, μ) represent a measure space. If $f : X \to \mathbb{R}$ is said to be a measurable function, then we define $\|f\|_{L^p(X)} := \left(\int_X |f|^p \mathrm{d}x\right)^{\frac{1}{p}}$ and $\|f\|_{L^\infty(x)} := \operatorname{ess\,sup}_{x \in X} |f(x)|$.*

The Lebesgue space can therefore be restated in the following definition.

Definition 5 *Let $L^p(x)$ be a space then this is defined as a set of $L^p(x) = \left\{ f : x \to R \,|\, \|f\|_{L^p(x)} < \infty \right\}$*

2.2. Volterra Integral Equations

Volterra Integral equation is a type of integral equation which has one of its limits to be a variable. The standard form of a Volterra integral equation is given by:

$$\vartheta(t)u(t) = f(x) + \lambda \int_0^x k(x,t)u(t)\mathrm{d}t \tag{2}$$

The Volterra integral equation can be of either the first or second kind, depending on the position of the unknown variable inside or outside the integral sign. When the unknown function $u(x)$ appears inside and outside the integral sign and $\vartheta(t) = 1$ in Equation (2), the resulting integral equation is called a Volterra integral equation of the second kind and is represented by:

$$u(x) = f(x) + \lambda \int_0^x k(x,t)u(t)\mathrm{d}t. \tag{3}$$

The convolution integral equation results from the nature of the kernel of the integral equation.

2.3. Convolution and Regularization

Theorem 1 *Let $f \in L^1(\mathbb{R}^N)$ and $g \in L^p(\mathbb{R}^N)$ with $1 < p < \infty$. Then for* *almost everywhere* $x \in \mathbb{R}^N$ *the function* $y \to f(x-y)g(y)$ *is integrable on* $\mathbb{R}^N$ *and so* $(f*g)(x) = \int_{\mathbb{R}^N} f(x-y)g(y)\mathrm{d}y$ *in addition* $f*g \in L^p(\mathbb{R}^N)$ *and* $\|f*g\|_p \le \|f\|_1 \|g\|_p$ [21].

Theorem 2 *Suppose that* $g: I \to R$ *is differentiable on an open interval I and* g' *is integrable on I. Let* $J = g(I)$. *If* $f: J \to R$ *is continuous then for every* $a, b \in I$ $\int_a^b f(g(x))g'(x)\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\mathrm{d}u$.

2.4. Carathéodory Conditions

The Carathéodory Conditions stipulate that in the domain of the (t,x) space, the following conditions are fulfilled:

1) The function $f(t,x)$ be defined and continuous in x for almost all t;

2) The function $f(t,x)$ be measurable in *t* for each *x*;

3) $\|f(t,x)\| \le m(t)$ where the function $m(t)$ is integrable in the Lebesgue sense on each finite interval.

2.5. Superposition Operator

Suppose that a function $f(t,x) = f:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ satisfies the Carathéodory conditions, then for the function $(Tx)t = f(t, x(t))$ where $t \in [0,\infty)$ is assigned for every $x = x(t)$ which is measurable on $[0,\infty)$. The operator T is called the superposition operator generated by *f*. Functions of several variables are converted to a single variable by the superposition operator T [22]. The superposition operator converts the functions of several variables to a single variable function which can be observed for the L norm.

Theorem 3 (*Superposition*) *The space* L^1 *map continuously onto itself by the superposition operator T if and only if*

$$|f(t,x)| \le a(t) + b|x| \tag{4}$$

$\forall t \in [0,\infty)$ *and* $x \in \mathbb{R}$, *where* $b \ge 0$ *and* $a(t)$ *is a function from* L^1 [23].

Next, we recall a theorem on the compactness of a measure subset X of L^1.

Theorem 4 *X is a compact measure if and only if X is a bounded subset of* L^1 *comprising of function which are almost everywhere nondecreasing or nonincreasing on the interval* [24].

Also, we recall some facts on the convolution operator as indicated in [25]. Let $p \in L^1(R)$ and $g \in L^1(R)$ then the integral

$$(Hx)t = \int_0^\infty p(t-s)g(s)\mathrm{d}s \tag{5}$$

exists for almost every $t \in [0,\infty)$. $(Hx)t$ belongs to the space L^1 where it is the linear operator which maps the space of L^1 to L^1. The linear operator H is also bounded and continuous since the norm

$$\|Hx\| \le \|H\|_{L^1(R)} \|g\| \tag{6}$$

For every $x \in L^1(R)$. Thus, $\|H\|$ is a convolution operator which is majored by $\|H\|_{L^1(R)}$.

Theorem 5 *Suppose* $p(t,s)=p:[0,\infty)^2\to[0,\infty)$ *is measurable on* $[0,\infty)^2$ *such that the integral operator*

$$(Hx)t=\int_0^\infty p(t,s)g(s)\mathrm{d}s,\ t\ge 0 \tag{7}$$

maps L^1 *into* L^1, *then* H *transforms the set of nonincreasing functions from* L^1 *into* L^1 *if and only if for* $\forall A>0$ *and*, $t_1,t_2\in[0,\infty)$ *then the assertion* $t_1<t_2$ $\Rightarrow\int_0^A p(t_1-s)\mathrm{d}s\ge\int_0^A p(t_2-s)\mathrm{d}s$ *is valid.*

2.6. Measure of Non-Compactness

One of the most widely used techniques for proving that certain operator equation has a solution is, to reformulate the problem as a fixed-point problem and see if the latter can be solved via a fixed-point argument. Measure of non-compactness is a function defined as the family of all non-empty and bounded subset of a metric space such that it is equal to zero on the whole family of relatively compact sets [26].

2.7. Hausdorff Measure of Non-Compactness

The Hausdorff measure of noncompactness of a nonempty and bounded subset Q of X denoted by $\chi(Q)$ according to [27] is defined as the infimum of all numbers $r>0$ such that Q has r-net in X.

$$\chi(Q)=\inf\{r>0:Q\subset S+B_r,S\text{ is finite}\} \tag{8}$$

Also, [10] defined the Hausdorff measure in space L as for $\varepsilon>0$, let

$$c(x)=\lim_{\varepsilon\to 0}\left\{\sup_{x\in X}\left\{\sup\left[\int_D|x(t)|\mathrm{d}t,D\subset[0,\infty),\,meas\,D\le\varepsilon\right]\right\}\right\} \tag{9}$$

and

$$d(x)=\lim_{T\to\infty}\left\{\sup\left[\int_T^\infty|x(t)|\mathrm{d}t:x\in Q\right]\right\} \tag{10}$$

where $meas\,D$ denotes the Lebesgue measure of a subset D. Also given that $\gamma(Q)=c(Q)+d(Q)$, then these two measures $\chi(Q)$ and $\gamma(Q)$ are connected by the following theorem.

Theorem 6 *Let* Q *be a nonempty, bounded and compact measure subset of* L^1, *then*

$$\chi(Q)\le\gamma(Q)\le 2\chi(Q) \tag{11}$$

Since these measures of noncompactness are used alongside certain fixed-point theorem, the next theorem considers the fixed point which will be used in this paper.

2.8. Darbo Fixed Point Theorem

The Darbo fixed point theorem is an extension of the classical Banach contraction mapping and the Schauder fixed point theorem.

Theorem 7 (Darbo Fixed Point) *Suppose Q is a nonempty, bounded, closed and convex subset of X and let* $P:Q\to Q$ *be a continuous transformation which is a contraction with respect to the measure of noncompactness μ, i.e. there exist* $k\in[0,1)$ *such that* $\mu(PE)=k\mu(E)$ *for any nonempty subset E of Q. Then P has at least one fixed point in the set Q* [28].

2.9. Lebesgue Integration

A measurable real-valued function φ defined on a set E is said to be simple provided it takes only a finite number of real values. Suppose φ assumes distinct values $a_1,\cdots,a_n$ on *E*, then the measurability of φ, its level set $\varphi^{-1}(a_i)$ are measurable and the canonical representation of φ on E is given by $\varphi=\sum_{i=1}^{n}a_i\chi E_i$ on E [29].

The following definitions from [29] are also recalled.

Definition 6 A bounded function f on a domain E of finite measure is said to be Lebesgue integrable over E provided its upper and lower integrals is called the Lebesgue integral and is denoted by $\int_E f$.

Definition 7 (Measurable function) let (X,μ) be a measure space. A function $f:X\to[-\infty,\infty]$ is said to be measurable if the set $f((a,\infty])=\{x\in X \mid f(x)>a\}$ is measurable for each $a\in\mathbb{R}$.

Suppose that X is a measure space, then the Lebesgue integral $\int_X f\mathrm{d}\mu$ can be defined for any non-negative measurable function $f:X\to[0,\infty]$. Although, this will depend more on the function, the integral can be infinite but will always be well-defined as $[0,\infty)$.

3. Main Results

According to [30], in order to establish the existence of a solution of an integral equation in the form of Equation (1), if the integral equation has a convolution kernel, then the right-hand side of Equation (1) can be defined under more general hypotheses. Therefore, the following assumptions are made for establishing the proof of existence of solution of the convolution Volterra integral equation in Equation (1):

1) Let $f(t)\in L$ be such that f is continuous and bounded on $\mathbb{R}_+$.

2) $\alpha:\mathbb{R}\to\mathbb{R}_+\in L(\mathbb{R})$

3) $g:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ satisfies the Carathéodory condition.

4) $\varphi:\mathbb{R}_+\to\mathbb{R}$ is increasing and absolutely continuous such that there exist u such that $\varphi'(t)\ge u$.

5) $b\|C\|<u$

Theorem 8 *There exists at least one solution for* Equation (*1*) *that is* $x\in L$ *which is almost everywhere nondecreasing on* $\mathbb{R}_+$ *if and only if the assumptions* (*1*) - (*5*) *are satisfied.*

Proof Let the right-hand side of Equation (1) be represented by operator *P*, therefore,

$$(Px)t = f(t) + \int_0^t \alpha(t-s) g(s, x(\varphi(s))) \tag{12}$$

which implies that:

$$x(t) = (Px)t \tag{13}$$

Let the nonlinear Volterra integral operator as a result of Equation (1) be presented by Equation (14):

$$(Tx)t = \int_0^t \alpha(t-s) g(s, x(s)) \mathrm{d}s \tag{14}$$

According to [21] the nonlinear Volterra integral operator can be expressed in terms of the convolution operator as a result of Equation (1) which is given by:

$$(Cx)t = \int_0^t \alpha(t-s) g(s) \mathrm{d}s \tag{15}$$

and F, the superposition operator due to Equation (1) is also given by

$$(Fx)t = g(t, x(t)) \tag{16}$$

Therefore, Equation (1) can be written in the form:

$$Px = f + Tx = f + CFx \tag{17}$$

Next, in order to show that the operator Px will transform any ball of radius r, (B_r) into itself, it is established that for $x \in L$, the function Px belongs to L when assumptions (1) - (5) are satisfied and will also imply that if there exist a ball B_r, Px transforms the ball into itself. Therefore,

$$\int_0^\infty |(Px)t| \mathrm{d}t = \int_0^\infty \left| f(t) + \int_0^t \alpha(t-s) g(s, x(\varphi(s))) \right| \mathrm{d}s\mathrm{d}t \tag{18}$$

$$\le \int_0^\infty |f(t)| \mathrm{d}t + \int_0^\infty \left| \int_0^t \alpha(t-s) g(s, x(\varphi(s))) \mathrm{d}s \right| \mathrm{d}t \tag{19}$$

$$\|Px\| \le \|f\| + \|CFx\| \tag{20}$$

In order to apply the superposition theorem, Equation (20) is expanded to separate the norms of the convolution and the superposition operators.

$$\|Px\| \le \|f\| + \|C\| \|Fx\| \tag{21}$$

Applying the superposition theorem on the superposition operator in Equation (21) results in:

$$\|Px\| \le \|f\| + \|C\| \int_0^\infty \left[a(t) + b \left| x(\varphi(t)) \right| \right] \mathrm{d}t \tag{22}$$

$$\|Px\| \le \|f\| + \|C\| \|a\| + b\|C\| \int_0^\infty \left| x(\varphi(t)) \right| \mathrm{d}t \tag{23}$$

from assumption (4), Equation (23) is rewritten as:

$$\|Px\| \le \|f\| + \|C\| \|a\| + \frac{b\|C\|}{u} \int_0^\infty \left| x(\varphi(t)) \varphi'(t) \right| \mathrm{d}t \tag{24}$$

The theorem for Lebesgue integration by substitution (Theorem 2) is applied in order to convert the function of several variables under the integral sign in Equation (24) to a function of single variable as indicated in Equation (25)

$$\|Px\| \leq \|f\| + \|C\|\|a\| + \frac{b\|C\|}{u}\int_0^\infty |x(t)|\mathrm{d}t \tag{25}$$

Expressing $\int_0^\infty |x(t)|\mathrm{d}t$ in terms of the norm on the space of Lebesgue integrable functions, results in Equation (26)

$$\|Px\| \leq \|f\| + \|C\|\|a\| + \frac{\|C\|b}{u}\|x\| \tag{26}$$

Since the operator Px maps the space of Lebesgue integrable functions into itself by the superposition operator, then for any $x \in L$, if $\|x\| \leq r$, then the operator P assumes the radius of x that is $\|Px\| \leq r$. This is due to the fact that P is an operator and cannot have a norm so it assumes the norm defined on the space of Lebesgue integrable functions.

Therefore, the exact value of the radius, r of the ball B_r is deduced from Equation (26) by assuming that $\|x\| \leq r$ and also $\|Px\| \leq r$.

Hence from Equation (26):

$$r \leq \|f\| + \|C\|\|a\| + \frac{b}{u}\|C\|r \tag{27}$$

$$r - \frac{b}{u}\|C\|r \leq \|f\| + \|C\|\|a\| \tag{28}$$

$$r\left(1 - \frac{b}{u}\|C\|\right) \leq \|f\| + \|C\|\|a\| \tag{29}$$

Hence

$$r = \frac{\|f\| + \|C\|\|a\|}{1 - \frac{b}{u}\|C\|} \tag{30}$$

Therefore, let the radius r of a ball be defined by:

$$r = \frac{\|f\| + \|C\|\|a\|}{1 - \frac{\|C\|b}{u}}, \text{ where } \frac{\|C\|b}{u} \neq 1 \tag{31}$$

Then, given that $x \in B_r$ then it can be concluded that $Px \in B_r$. This means that P maps the ball B_r into itself since substituting the value of the radius r in Equation (31) in place of x in Equation (26) results in the inequality,

$$\|Px\| \leq \|f\| + \|C\|\|a\| + \frac{b\|C\|}{u}\left(\frac{\|f\| + \|C\|\|a\|}{1 - \frac{\|C\|b}{u}}\right) \tag{32}$$

$$\|Px\| \leq \|f\| + \|C\|\|a\| + \frac{\frac{b\|C\|\|f\|}{u} + \frac{b\|C\|^2\|a\|}{u}}{1 - \frac{\|C\|b}{u}} \tag{33}$$

$$\|Px\| \le \frac{\left(\|f\|+\|C\|\|a\|\right)\left(1-\frac{\|C\|b}{u}\right)+\frac{b\|C\|\|f\|}{u}+\frac{b\|C\|^2\|a\|}{u}}{1-\frac{\|C\|b}{u}} \tag{34}$$

$$\|Px\| \le \frac{\|f\|-\|f\|\frac{\|C\|b}{u}+\|C\|\|a\|-\frac{b}{u}\|C\|^2\|a\|+\frac{b\|C\|\|f\|}{u}+\frac{b\|C\|^2\|a\|}{u}}{1-\frac{\|C\|b}{u}} \tag{35}$$

Therefore,

$$\|Px\| \le \|f\|+\|C\|\|a\|+\frac{b\|C\|}{u}r = r \tag{36}$$

Next, in order to apply the Hausdorff measure of non-compactness and the Darbo fixed point theorem, Lemma 1 is established since the Darbo fixed point theorem is applied on sets which are closed, bounded, convex and a compact measure.

Lemma 1 *Let* $M \in B_r$ *consisting of all functions which are almost everywhere positive and nondecreasing on* $[0,\infty)$ *then M is closed, bounded, convex subset of* $L(R_+)$ *and a compact measure.*

Proof Suppose for $x \in M$ there exist $r \ge 0$ then x is bounded for all functions of M with respect to time if and only if $\|x\| \le r$.

Then, for M to be closed, there exist a sequence $x_n \in M$ such that $\|x_n - x\| \to 0$ and the sequence x_n converges to a point in $x \in M$ as $n \to \infty$.

Furthermore, to show also that M contains functions which are nondecreasing, let $\varepsilon > 0$ such that $\left|x(t_1)-x_{n_k}(t_1)\right| \le \frac{\varepsilon}{2}$ and $\left|x(t_2)-x_{n_k}(t_2)\right| \le \frac{\varepsilon}{2}$ for $x(t_2)-x(t_1) \le \varepsilon$.

Thus, for every $n_k \in \mathbb{N}$

$$x(t_1)-x(t_2) = -\left(x(t_2)-x(t_1)\right)$$

$$x(t_1)-x(t_2) = -\left(x(t_2)-x_{n_k}(t_2)+x_{n_k}(t_2)-x_{n_k}(t_1)+x_{n_k}(t_1)-x(t_1)\right) \tag{37}$$

$$= |-1|\left|\left(x(t_2)-x_{n_k}(t_2)+x_{n_k}(t_2)-x_{n_k}(t_1)+x_{n_k}(t_1)-x(t_1)\right)\right|$$

$$\le \left|x(t_2)-x_{n_k}(t_2)\right|+\left|x_{n_k}(t_1)-x(t_1)\right| \tag{38}$$

$$x(t_1)-x(t_2) \le \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \tag{39}$$

Since $x_{n_k} \to x$ almost everywhere on $[0,\infty)$ and also $\varepsilon > 0$ then

$$x(t_1)-x(t_2) < \varepsilon \tag{40}$$

Therefore, $x(t_1) \le x(t_2)$. This implies that x is nondecreasing on $[0,\infty)$ and as such M is closed. Next, for M to be convex, Let $x_1, x_2 \in M$ for $r > 0$ then $\|x_i\| \le r$ for all $i = 1,2$

Let $z(t) = \lambda x_1(t)+(1-\lambda)x_2(t)$ for all $t \in [0,\infty)$ and $0 \le \lambda \le 1$ hence

$$\|z\| \le \lambda\|x\|+(1-\lambda)\|x\| \le \lambda r+(1-\lambda)r \tag{41}$$

$$\|z\| \le r \tag{42}$$

Thus, the convexity of M is established.

Again, the subset M is a compact measure as a result of Theorem 4, since it is bounded and contains functions which are nondecreasing almost everywhere on $[0,\infty)$. Therefore, $x \in M$ implies that $x(t)$ is nondecreasing and positive almost everywhere on $[0,\infty)$. Hence Px is also nondecreasing and positive on $[0,\infty)$. Also, since $P: B_r \to B_r$ and P is nondecreasing and positive on $[0,\infty)$, it can be concluded also that $P: M \to M$.

In order to apply the Hausdorff measure of noncompactness, let $E \subseteq M$, which is nonempty and $\varepsilon > 0$. Then for $x \in E$ and for a set $d \subset [0,\infty)$ if meas $d \le \varepsilon$ then from Equation (17)

$$\int_d |(Px)t| \mathrm{d}t \le \int_d |f(t)| \mathrm{d}t + \int_d |CFx| \mathrm{d}t \tag{43}$$

$$\int_d |(Px)t| \mathrm{d}t \le \int_d |f(t)| \mathrm{d}t + \|CFx\|_{l(d)} \tag{44}$$

Applying the superposition operator on Equation (44), Equation (45) is obtained.

$$\int_d |(Px)t| \mathrm{d}t \le \int_d |f(t)| \mathrm{d}t + \|C\|_d \int_d \left(a(s) + b\left|x(\varphi(s))\right|\right) \mathrm{d}s \tag{45}$$

To convert the function of several variables to a function of single variable under the integral sign in Equation (45), assumption (4) is applied on Equation (45) which generates into Equation (46) as follows:

$$\int_d |(Px)t| \mathrm{d}t \le \int_d |f(t)| \mathrm{d}t + \|C\|_d \int_d a(s) \mathrm{d}s + \frac{b}{u} \|C\|_d \int_d \left|x(\varphi(s))\right| \varphi'(s) \mathrm{d}s \tag{46}$$

$$\int_d |(Px)t| \mathrm{d}t \le \int_d |f(t)| \mathrm{d}t + \|C\|_d \int_d a(s) \mathrm{d}s + \frac{b}{u} \|C\|_d \int_{\varphi(d)} |x(t)| \mathrm{d}t \tag{47}$$

Applying the Hausdorff measure of noncompactness in Equation (9) to Equation (47)

$$\lim_{\varepsilon \to 0} \left\{ \sup \left[\sup_{x \in E} \int_d |f(t)| \mathrm{d}t + \|C\|_d \int_d a(s) \mathrm{d}s : d \subset [0,\infty), meas\, d \le \varepsilon \right] \right\} = 0 \tag{48}$$

Therefore, the measure for the last inequality becomes:

$$c(PE) \le \frac{b}{u} \|C\|_d \, c(E) \tag{49}$$

Furthermore, fixing $S > 0$ so that the lower limit of the integral equation could be any value apart from zero, Equation (47) becomes:

$$\int_S^\infty |(Px)t| \mathrm{d}t \le \int_S^\infty |f(t)| \mathrm{d}t + \|C\|_d \int_S^\infty a(s) \mathrm{d}s + \frac{b}{u} \|C\|_d \int_S^\infty |x(t)| \mathrm{d}t \tag{50}$$

Applying the measure of noncompactness in Equation (10) to Equation (50) result in Equation (51):

$$\lim_{S \to \infty} \left\{ \sup \int_S^\infty |(Px)t| \mathrm{d}t \le \int_S^\infty |f(t)| \mathrm{d}t + \|C\|_d \int_S^\infty a(s) \mathrm{d}s + \frac{b}{u} \|C\|_d \int_S^\infty |x(t)| \mathrm{d}t, x \in E \right\} \tag{51}$$

Therefore,

$$d(PE) \le \frac{b\|C\|_d}{u} d(E) \tag{52}$$

Combining Equations (49) and (52), the measure of noncompactness is given by

$$\beta(PE) \le \frac{b}{u}\|c\|\beta(E) \tag{53}$$

By assumption (4), applying the Darbo fixed point theorem in Theorem 7 implies that, there exist at least one fixed point for the operator P in M. This also implies that, there exist a solution for the integral Equation (1) since the condition for the Darbo fixed point theorem is satisfied.

4. Conclusion

The study proved the existence of solution of the convolution Volterra integral equation in Equation (1). For a set $M \in B_r$ which is a compact measure, bounded and convex, the condition for the Darbo fixed point theorem is satisfied. This indicates the presence of a fixed point for the convolution Volterra integral equation after the Hausdorff measure of noncompactness was applied to obtain the measure of the set $x \in E \subset M$. The presence of the fixed point is an indication that there exists at least one solution to the convolution Volterra integral equation.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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