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Identities on q-Harmonic Numbers

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Abstract

With the help of the classical Abel's lemma on summation by parts and algorithm of q-hypergeometric summations, we deal with the summation, which can be written as multiplication of a q-hypergeometric term and q-harmonic numbers. This enables us to construct and prove identities on q-harmonic numbers. Several examples are also given.

Keywords

Harmonic Numbers, q-Zeilberger Algorithm, Abel's Lemma

1. Introduction

Harmonic numbers are a class of famous sequences in combinatorics, number theory and computer science. Many properties could be found in the literature. Recall that for positive integers m and n, the classical generalized harmonic numbers are given by

$$H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m}. \tag{1.1}$$

For convenience, we have $H_0^{(m)} = 0$. As usually, $H_n^{(1)} = H_n$.

In recent years, many identities involving the generalized harmonic numbers have been established [1]-[8]. For example, in [2], the author reconsidered the following type of sums

$$S(m) = \sum_{k=1}^{n} \binom{n}{k} (-1)^k k^m H_k. \tag{1.2}$$

With the help of binomial transformations and difference operators, he reproved some well-known identities such as

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^k H_k = -\frac{1}{n}, \quad n > 0, \tag{1.3}$$

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k} kH_{K} = \frac{1}{n-1},\ n>1, \tag{1.4}$$

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k} k^{2}H_{k} = \frac{-n}{(n-1)(n-2)},\ n>2. \tag{1.5}$$

q-analogues of combinatorial identities always appear in other branches such as physics [9]. In this paper, we focus on the q-analogues of identities involving harmonic numbers. The well-known two kinds of q-harmonic numbers are defined as follows.

Definition 1. For positive integers n, two kinds of the q-harmonic numbers are defined as

$$H_q(n)=\sum_{k=1}^{n}\frac{1}{[k]},\ \tilde{H}_q(n)=\sum_{k=1}^{n}\frac{q^k}{[k]}, \tag{1.6}$$

where $[k]=\frac{1-q^k}{1-q}$ is the q-integer. We also set $H_q(0)=\tilde{H}_q(0)=0$.

Note that when $q\to 1$, $[k]\to k$, thus they are q-analogues of classical harmonic numbers.

Definition 2. For non-negative integers n,k and a complex number q with $k\le n$, $|q|<1$, the q-binomial coefficients are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}=\frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \tag{1.7}$$

where $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ is the well-known q-factorial.

Note that we have $(a;q)_0=1$ and $\begin{bmatrix} n \\ k \end{bmatrix}=0$ when $k>n$ or $k<0$.

Mansour *et al.* [10] discovered the following elegant identity by using partial fraction decomposition

$$\sum_{k=0}^{n}(-1)^{n-k} q^{\binom{k-n}{2}}\begin{bmatrix} n+k \\ k \end{bmatrix}\begin{bmatrix} n \\ k \end{bmatrix}H_q(k)=\sum_{k=1}^{n}q^{k^2}\frac{[2k]}{[k]^2},\ n\ge 1, \tag{1.8}$$

which could be known as a q-analogue of the following identity

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{k}\binom{n}{k}H_k=2H_n,\ n\ge 1. \tag{1.9}$$

We should point out that there exist some typos in this result. The correct version is given in Section 3.

Wei and Gu [11] considered the q-analogues of the following type of sums

$$W(m)=\sum_{k=0}^{n}\binom{n}{k}^{m}\left\{1+m(n-2k)H_k\right\}. \tag{1.10}$$

Note that when $-2\le m\le 6$ and $k\ne 0$, the "closed form" of $W(m)$ have been known. By using Watson's q-Whipple transformation, they discovered the q-analogues of these identities. For example,

$$\begin{aligned}&\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}^{-1}q^{-k}\left\{[2k-n]\left(2k(1-q)+\tilde{H}_q(k)\right)+(1+2k)q^{2k-n}-2k\right\}\\&=\frac{[1+n]}{q^{n+1}}\tilde{H}_q(n+1),\end{aligned} \tag{1.11}$$

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}q^{k(k-n-1)}\left\{-[2k-n]\tilde{H}_q(k)+(1+k)q^{2k-n}-k\right\}=1, \tag{1.12}$$

which reduce to

$$W(-1)=\sum_{k=0}^{n}\binom{n}{k}^{-1}\left\{1-(n-2k)H_k\right\}=(1+n)H_{n+1}, \tag{1.13}$$

$$W(1)=\sum_{k=0}^{n}\binom{n}{k}\left\{1+(n-2k)H_k\right\}=1. \tag{1.14}$$

Chen, Hou and Jin [12] proposed the Abel-Zeilberger algorithms to prove combinatorial identities on non-hypergeometric terms. In this paper, we observe that summations on q-harmonic numbers fall in the scope of a simplified q-version of this method. This enables us to prove and establish many identities on q-harmonic numbers. For more detail of the q-analogues identities, see [13].

2. Our Method

The classical Abel's lemma on summation by parts is as follows.

Lemma 1. For any two sequences $\{a_n\}$, $\{b_n\}$, we have

$$\sum_{k=m}^{n-1}(a_{k+1}-a_k)b_k=\sum_{k=m}^{n-1}a_{k+1}(b_k-b_{k+1})+a_nb_n-a_mb_m. \tag{2.1}$$

Using the difference operator, Abel's lemma can be rewritten as:

$$\sum_{k=m}^{n-1}b_k\Delta a_k=-\sum_{k=m}^{n-1}a_{k+1}\Delta b_k+a_nb_n-a_mb_m. \tag{2.2}$$

Given a q-hypergeometric term $F(n,k)$, namely, $\frac{F(n+1,k)}{F(n,k)},\frac{F(n,k+1)}{F(n,k)}$ are both rational functions in q^n and q^k, the q-Zeilberger algorithm tries to find polynomials $a_0(q^n),\cdots,a_d(q^n)$ and rational functions $R(q^n,q^k)$ such that

$$\begin{aligned}&a_0(q^n)F(n,k)+a_1(q^n)F(n+1,k)+\cdots+a_d(q^n)F(n+d,k)\\&=\Delta R(q^n,q^k)F(n,k).\end{aligned} \tag{2.3}$$

Then summing this skew-recurrence relation over k, one can find a recurrence relation of the sum

$$S(n)=\sum_k F(n,k). \tag{2.4}$$

For more detail of hypergeometric algorithms, see [14].

Let us consider the sum $S(n)=\sum_k F(n,k)b_k$, where $F(n,k)$ is q-hypergeometric term, b_k is a sequence satisfying Δb_k is q-hypergeometric. The method can be described as follows.

Step 1. Applying the q-Zeilberger algorithm to $F(n,k)$, we find the skew-recurrence relation

$$a_0(q^n)F(n,k)+a_1(q^n)F(n+1,k)+\cdots+a_d(q^n)F(n+d,k)=\Delta G(n,k), \tag{2.5}$$

where $G(n,k)=R(q^n,q^k)F(n,k)$.

Step 2. Multiplying both sides of this relation by b_k and summing over k, we have

$$a_0(q^n)S(n)+a_1(q^n)S(n+1)+\cdots+a_d(q^n)S(n+d)=\sum_k(\Delta G(n,k))b_k. \tag{2.6}$$

Step 3. Applying Abel's lemma to the right-hand side, we will transform it to a hypergeometric sum since Δb_k is q-hypergeometric.

$$\begin{aligned}&a_0(q^n)S(n)+a_1(q^n)S(n+1)+\cdots+a_d(q^n)S(n+d)\\&=-\sum_k G(n,k+1)\Delta b_k+W(n).\end{aligned} \tag{2.7}$$

Note that $W(n)=0$ in many cases.

Step 4. Denote by $T(n)=\sum_k G(n,k+1)\Delta b_k$. If a closed form for $T(n)$ could be found by using the q-Zeilberger algorithm, we thus obtain a recurrence relation for the original sum $S(n)$.

Remark. The Maple package for the q-Zeilberger algorithms can also be found in [14]. Here we use the package APCI, which is written by Hou Q. H. [15]. After loading the package, we input the command qZeil (F, n, k, q, "cert") and then get the skew-recurrence relation (2.5).

Let us give an example to illustrate this method.

Example 1. For positive integer n, we have

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}-n(k+1)}\begin{bmatrix}n\\k\end{bmatrix}\tilde{H}_q(k)=-\frac{1}{[n]}, \tag{2.8}$$

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}H_q(k)=-\frac{1}{[n]}. \tag{2.9}$$

The first result appeared in [16] and the second appeared in [17]. Clearly, both of them could be seen as q-analogue of

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^k H_k=-\frac{1}{n},\ n>0. \tag{2.10}$$

Proof. For brevity, we only prove the first identity. Denote the left-hand side sum by $S(n)$ and let

$$F(n,k)=(-1)^k q^{\binom{k+1}{2}-n(k+1)}\begin{bmatrix}n\\k\end{bmatrix}. \tag{2.11}$$

By the q-Zeilberger algorithm, we find that

$$F(n,k)=G(n,k+1)-G(n,k), \tag{2.12}$$

where $G(n,k)=-\dfrac{q^n(1-q^k)}{(1-q^n)q^k}F(n,k)$. Then with the help of Abel's lemma, we obtain the following formula by multiplying both sides of the above equation by $\tilde{H}_q(k)$ and summing over k from 0 to $+\infty$

$$S(n)=\sum_{k=0}^{n}\Delta G(n,k)\tilde{H}_q(k)=-\sum_{k=0}^{+\infty}\frac{G(n,k+1)}{1-q^{k+1}}(1-q)q^{k+1}. \tag{2.13}$$

Denote the new sum by $T(n)$ and $F_1(n,k)=\frac{G(n,k+1)}{1-q^{k+1}}(1-q)q^{k+1}$, by the q-Zeilberger algorithm, we further find $F_1(n,k)=G_1(n,k+1)-G_1(n,k)$, where $G_1(n,k)=-\frac{q^n(1-q^{k+1})}{(1-q^n)q^{k+1}}F_1(n,k)$.

By summing over k from 0 to ∞, we obtain

$$T(n)=-G_1(n,0)=\frac{1-q}{1-q^n}=\frac{1}{[n]}. \tag{2.14}$$

Thus, we finally have $S(n)=-T(n)=-\frac{1}{[n]}$.

3. Applications

By using the above method, we can construct or prove many identities on q-harmonic numbers. Here, we only give two examples. The first is the correct version of identity

$$\sum_{k=0}^{n}(-1)^{n-k}q^{\binom{k-n}{2}}\begin{bmatrix}n+k\\k\end{bmatrix}\begin{bmatrix}n\\k\end{bmatrix}H_q(k)=\sum_{k=1}^{n}q^{k^2}\frac{[2k]}{[k]^2},\quad n\ge 1. \tag{3.1}$$

Theorem 1. For positive integer n, we have

$$\sum_{k=0}^{n}(-1)^{n-k}q^{\binom{k-n}{2}}\begin{bmatrix}n+k\\k\end{bmatrix}\begin{bmatrix}n\\k\end{bmatrix}H_q(k)=\sum_{k=1}^{n}\frac{[2k]}{[k]^2}. \tag{3.2}$$

Proof. Denote the left-hand side sum by $S(n)$ and let $F(n,k)=(-1)^{n-k}q^{\binom{k-n}{2}}\begin{bmatrix}n+k\\k\end{bmatrix}\begin{bmatrix}n\\k\end{bmatrix}$. By the q-Zeilberger algorithm, we find that

$$F(n,k)-F(n+1,k)=G(n,k+1)-G(n,k), \tag{3.3}$$

where $G(n,k)=\frac{q^{n+1}(1-q^k)^2(1+q^{n+1})}{(1-q^{n+1})(q^{n+1}-q^k)}F(n,k)$. Then with the help of Abel's lemma, we obtain the following formula by multiplying both sides of Equation (3.3) by $H_q(k)$ and summing over k from 0 to $+\infty$,

$$S(n)-S(n+1)=\sum_{k=0}^{n}\Delta G(n,k)H_q(k)=-\sum_{k=0}^{+\infty}\frac{G(n,k+1)}{1-q^{k+1}}(1-q). \tag{3.4}$$

Denote the new sum by $T(n)$ and $F_1(n,k)=\frac{G(n,k+1)}{1-q^{k+1}}(1-q)$, by the q-Zeilberger algorithm, we further find

$$\begin{aligned}&\left(1+q^{n+2}\right)\left(q^{n+1}-1\right)F_1(n,k)-\left(1+q^{n+1}\right)\left(q^{n+2}-1\right)F_1(n+1,k)\\&=G_1(n,k+1)-G_1(n,k),\end{aligned} \tag{3.5}$$

where $G_1(n,k)=-\dfrac{\left(1+q^{n+2}\right)\left(q^k-1\right)\left(q^{k+1}-1\right)q^{n+1}\left(q^{2n+3}-1\right)}{\left(q^{n+1}-q^k\right)\left(q^{n+2}-1\right)}F_1(n,k)$. By summing over k from 0 to ∞, we obtain

$$\left(1+q^{n+2}\right)\left(q^{n+1}-1\right)T(n)-\left(1+q^{n+1}\right)\left(q^{n+2}-1\right)T(n+1)=0, \tag{3.6}$$

Taking the initial value $T(1)=G(1,0)+G(1,1)=\dfrac{1+q^2}{1+q}$ into account, we obtain

$$S(n)-S(n+1)=-T(n)=-\frac{1+q^{n+1}}{1-q^{n+1}}(1-q). \tag{3.7}$$

Furthermore, $S(1)=1+q$, we thus have

$$S(n)=S(1)+\sum_{k=2}^{n}\frac{1-q^{2k}}{\left(1-q^k\right)^2}(1-q)=\sum_{k=1}^{n}\frac{[2k]}{[k]^2}. \tag{3.8}$$

The second example is the following q-analogues of (1.4) and (1.5).

Theorem 2. For positive integer n, we have

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}-n(k+1)}[k]\begin{bmatrix}n\\k\end{bmatrix}\tilde{H}_q(k)=\frac{1}{q[n-1]},\ (n>1), \tag{3.9}$$

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}-n(k+1)}[k]^2\begin{bmatrix}n\\k\end{bmatrix}\tilde{H}_q(k)=-\frac{[n]}{q^2[n-1][n-2]},\ (n>2). \tag{3.10}$$

Proof. We only prove the first identity. Denote the summation by $S(n)$ and

$$F(n,k)=(-1)^k q^{\binom{k+1}{2}-n(k+1)}[k]\begin{bmatrix}n\\k\end{bmatrix}. \tag{3.11}$$

Then, by the q-Zeilberger algorithm, we know

$$F(n,k)=G(n,k+1)-G(n,k), \tag{3.12}$$

where

$$G(n,k)=-\frac{q^n\left(q-q^k\right)}{\left(q-q^n\right)q^k}F(n,k). \tag{3.13}$$

Noting that the summation range of the series is equivalent to summing from 0 to ∞, we multiply both sides of Equation (3.13) by $\tilde{H}_q(k)$ and sum over k from 0 to ∞. Using Abel's lemma, we get (with the reminder term being 0).

$$\begin{aligned}S(n)&=\sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}-n(k+1)}[k]\begin{bmatrix}n\\k\end{bmatrix}\tilde{H}_q(k)\\&=\sum_{k=0}^{n}\Delta G(n,k)\tilde{H}_q(k)\\&=-\sum_{k=0}^{+\infty}\frac{G(n,k+1)}{1-q^{k+1}}(1-q)q^{k+1}\\&=-T(n).\end{aligned} \tag{3.14}$$

Let

$$F_1(n,k)=\frac{G(n,k+1)}{1-q^{k+1}}(1-q)q^{k+1}. \tag{3.15}$$

Furthermore, by the q-Zeilberger algorithm, we know

$$F_1(n,k)=G_1(n,k+1)-G_1(n,k), \tag{3.16}$$

where

$$G_1(n,k)=\frac{\left(q^{k+n+1}+q^2-q^{k+1}-2q^{n+1}+q^n\right)q^n\left(q^{k+1}-1\right)}{\left(-1+q^k\right)\left(q-q^n\right)q^k\left(q^n-1\right)q^2}F_1(n,k). \tag{3.17}$$

Taking the sum over k from 0 to ∞ for Equation (3.17), we obtain:

$$T(n)=-G_1(n,0)=-\frac{1}{q[n-1]}. \tag{3.18}$$

Thus, we have:

$$S(n)=-T(n)=\frac{1}{q[n-1]},\ (n>1). \tag{3.19}$$

4. Conclusion

Sometimes, finding q-analogues for given combinatorial identities may be challenging. In this paper, we used the ideal of the Abel-Zeilberger algorithm to construct and prove several q-analogues of identities involving harmonic numbers. In particular, we obtained the q-analogues identities for the classical kind of $\sum_{k=1}^{n}\binom{n}{k}(-1)^k k^m H_k$ when $m=0,1,2$. For $m\geq 3$, we can also construct the corresponding identities. In fact, we have verified almost all identities on q-harmonic numbers appeared in the references. We also point out that some examples of the method have been included in Zheng huanhuan's master thesis.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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