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Thermomechanical Dynamics (TMD) and Bifurcation-Integration Solutions in Nonlinear Differential Equations with Time-Dependent Coefficients

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Abstract

The new independent solutions of the nonlinear differential equation with time-dependent coefficients (NDE-TC) are discussed, for the first time, by employing experimental device called a drinking bird whose simple back-and-forth motion develops into water drinking motion. The solution to a drinking bird equation of motion manifests itself the transition from thermodynamic equilibrium to nonequilibrium irreversible states. The independent solution signifying a nonequilibrium thermal state seems to be constructed as if two independent bifurcation solutions are synthesized, and so, the solution is tentatively termed as the *bifurcation-integration solution*. The bifurcation-integration solution expresses the transition from mechanical and thermodynamic equilibrium to a nonequilibrium irreversible state, which is explicitly shown by the nonlinear differential equation with time-dependent coefficients (NDE-TC). The analysis established a new theoretical approach to nonequilibrium irreversible states, *thermomechanical dynamics* (TMD). The TMD method enables one to obtain thermodynamically consistent and time-dependent progresses of thermodynamic quantities, by employing the bifurcation-integration solutions of NDE-TC. We hope that the basic properties of bifurcation-integration solutions will be studied and investigated further in mathematics, physics, chemistry and nonlinear sciences in general.

Keywords

The Nonlinear Differential Equation with Time-Dependent Coefficients, The Bifurcation-Integration Solution, Nonequilibrium Irreversible States, Thermomechanical Dynamics (TMD)

1. The Bifurcation Solutions and Bifurcation-Integration Solutions in Nonlinear Differential Equations

The bifurcation phenomena and solutions in nonlinear differential equations were first reported and discussed by Henri Poincaré in 1885 [1], and the bifurcation theory and classification of solutions have been studied as the property of the *nonlinear differential equation with constant coefficients* (NDE-CC). The nonlinear dynamical systems have been studied in the vast fields of physical, chemical, molecular and biological systems [2] [3] [4] [5]. Though several independent solutions to nonlinear differential equations are often found, it is not possible to combine the known solutions to construct new solutions. In other words, the lack of the superposition principle in nonlinear differential equations prevents the construction of new solutions, and a branching of solutions suddenly appears even with a small change made to coefficient values, which is known as bifurcation phenomena. Although mathematical and topological analyses of bifurcation solutions have been extensively studied in mathematics, the meanings of bifurcation solutions are not clear enough in terms of physics.

More fundamentally, new independent solutions, the *bifurcation-integration solution*, discovered in the nonlinear differential equation with time-dependent coefficients (NDE-TC) are explained, for the first time, in the current paper. It is remarkable that there exists a simple experimental device to study relations between bifurcation solutions of NDE-CC and corresponding solutions expressed by NDE-TC. The device is known as a *drinking bird* whose motion is expressed by a nonlinear differential equation of motion (see **Figure 1**). The equation of motion of a drinking bird is found by a *nonlinear differential equation with time-dependent coefficients* (NDE-TC), and mechanical motion and the drinking bird's motion are explicitly studied in terms of transitions from thermodynamic equilibrium to nonequilibrium irreversible states.

The simple back-and-forth motion of a drinking bird can be expressed by NDE-CC, but it is not possible to produce a drinking bird's water-drinking motion. The correct drinking bird's equation of motion is obtained by changing constant coefficients to time-dependent coefficients required by physical arguments. The time-dependent coefficients are the moment of inertia, $I(t)$, and the effective mass, $m^*(t)$, of a drinking bird; the change of constant, $I_0 \to I(t)$ and $m \to m^*(t)$, can only express the bird's drinking motion, which leads to important conclusions for mathematics and physics [6]. A drinking motion is discovered in the NDE-TC, and since the solution seems to be constructed by integrating two bifurcation solutions, the new solution is termed as the *bifurcation-integration solution*.

The drinking bird's equation with time-dependent coefficients gradually develops to a threshold time when a drinking bird's simple mechanical back-and-forth motion changes to a water drinking mode. In other words, it signifies the transition from a state described by NDE-CC to a state NDE-TC, which corresponds to the transition from mechanical equilibrium states to nonequilibrium irreversible states (NISs) in physical terms. The self-consistent and thermomechanical

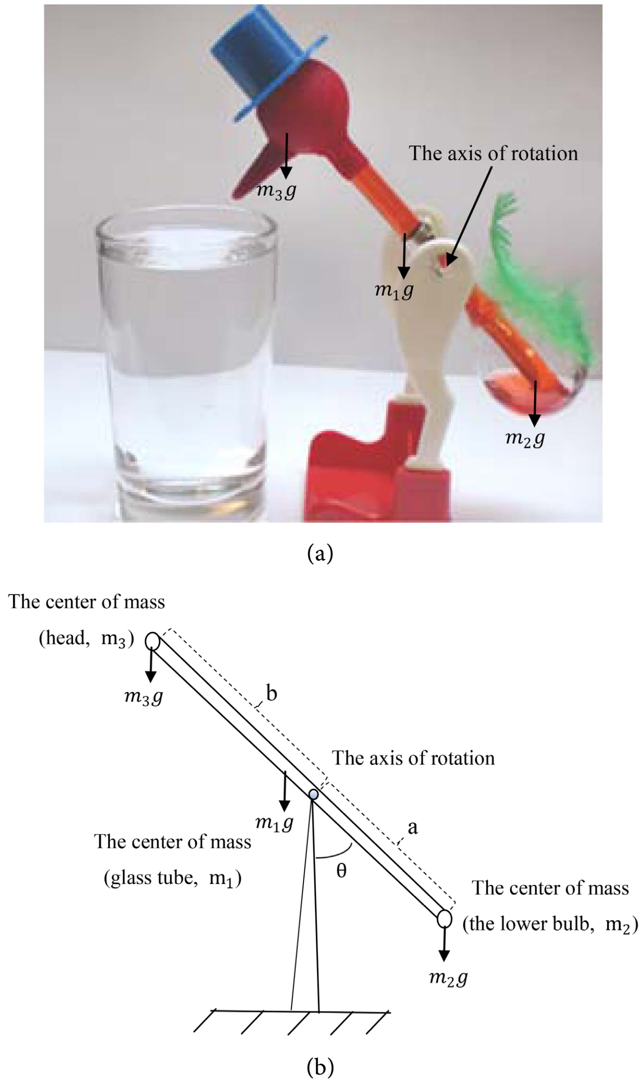


Figure 1. A mechanical modeling and deformation of a drinking bird. (a) A drinking bird (DB) divided into the head m_3, glass tube m_1, and bottom m_2 which includes volatile water. (b) A topological deformation for the drinking bird of (a) by employing the concept of centers of mass.

equation of motion for a drinking bird is explained for the first time by applying the method of TMD [7].

The corect drinking bird's equation of motion is constructed by considering the coupling of heat flows and thermal dissipation processes, and so, the derived equation of motion is termed as the *dissipative equation of motion*. The time-dependent progresses of thermodynamic quantities, such as internal energy $\mathcal{E}(t)$, thermodynamic work $W_{th}(t)$, entropy $S(t)$ and temperature $\tilde{T}(t)$ in nonequilibrium irreversible states (NISs) are self-consistently calculated by the method of TMD. The water drinking motion exhibits the phase transition from equilibrium states to NISs, and it is essential to recognize that the equation of motion in NISs is only expressible by the nonlinear equations with time-dependent coefficients (NDE-TC).

The water drinking motion is fundamental in terms of nonlinear mathematics and physics of phase transitions. The drinking bird's motion in mechanical and thermodynamic equilibrium states is described by NDE-CC (the simple back-and-forth motion), and the time-dependent progress of nonequilibrium irreversible states corresponds to NDE-TC (the water drinking motion) [7]. Therefore, the drinking bird's motion helps us understand the fundamental property of phase transitions from mechanical and thermal equilibrium to nonequilibrium irreversible states [8] [9] [10], and vice versa. By employing the simple device of a drinking bird, we explain the phase transitions, the dissipative equation of motion for a drinking bird and the property of NDE-TC. The solutions of NDE-CC are discussed in Section 2, and the correct equation of motion (NDE-TC) and the new independent, *bifurcation-integration* solutions are discussed in Section 3, and conclusions are in Section 4.

2. The Bifurcation Solutions of the Drinking Bird Equation of Motion with Constant Coefficients

The traditional, simple explanations for drinking bird's oscillating motion employ pressure, temperature and volume in gas phase (Boyle-Charles' law, or the ideal gas law), which is exact only in thermodynamic equilibrium. A drinking bird is a heat engine driven by thermal heat flows; its motion evolves from simple back-and-forth oscillations to water drinking motion through transitions to nonequilibrium irreversible states (NISs). Hence, a drinking bird's equation of motion is required to be consistent with thermodynamic laws and NISs. It is a difficult problem to construct the basic equation of motion, and we start from the analyses of bifurcation solutions of a drinking bird's NDE-CC, while it expresses no drinking motion.

It is essential to know that the constant coefficients in a drinking bird's nonlinear equation of motion are composed of mechanical quantities in physics: masses, length of glass tube, the moment of inertia, friction of materials and working fluid. Some of these constants are physically controllable and adjustable quantities. In order to construct equation of motion of a drinking bird, one can start from Lagrangian or Hamiltonian method in mechanics with the friction term, $c\dot{\theta}(t)$, known as the Rayleigh dissipation function. One obtains:

$$\ddot{\theta}(t)+c\dot{\theta}(t)+\frac{glm^{*}}{I_0}\sin\theta(t)=0, \tag{2.1}$$

and constant coefficients are, g,c,l,m^{*},I_0, for gravity, friction, length, effective mass and moment of inertia of the drinking bird system.

A decreasing back-and-forth solution can be obtained, which converges in $\theta(t)=0$ at $t\to\infty$. The solution is written as $\theta_0(t)$, which is often justified by linearizing (2.1) by assuming $|\theta(t)|\sim 0$. One can find the independent solutions of (2.1), denoted by $\theta_{n\pi}(t)$ ($n=\pm1,\pm2,\cdots$), corresponding to oscillating solutions converging in $n\pi$ at $t\to\infty$, as shown in **Figure 2** in case of ($n=0,1,2$). The other independent solutions could be obtained mathematically, while they

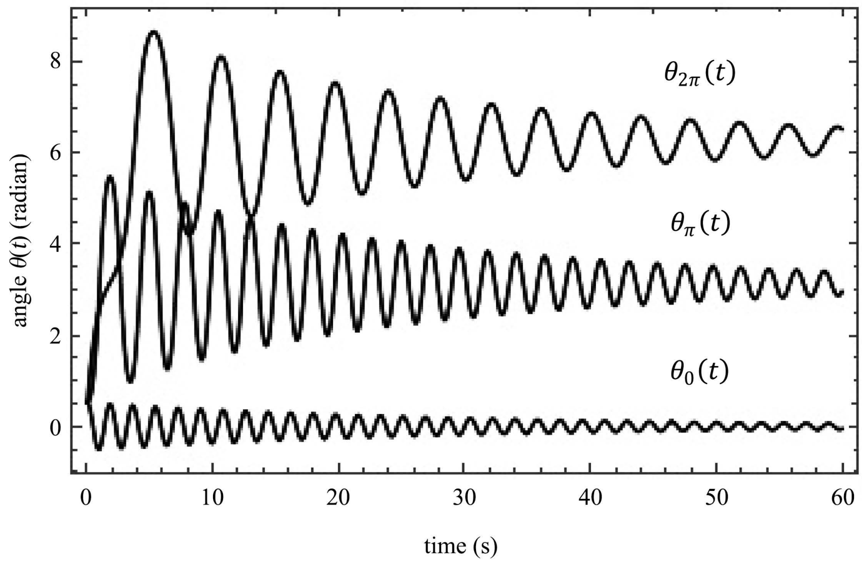
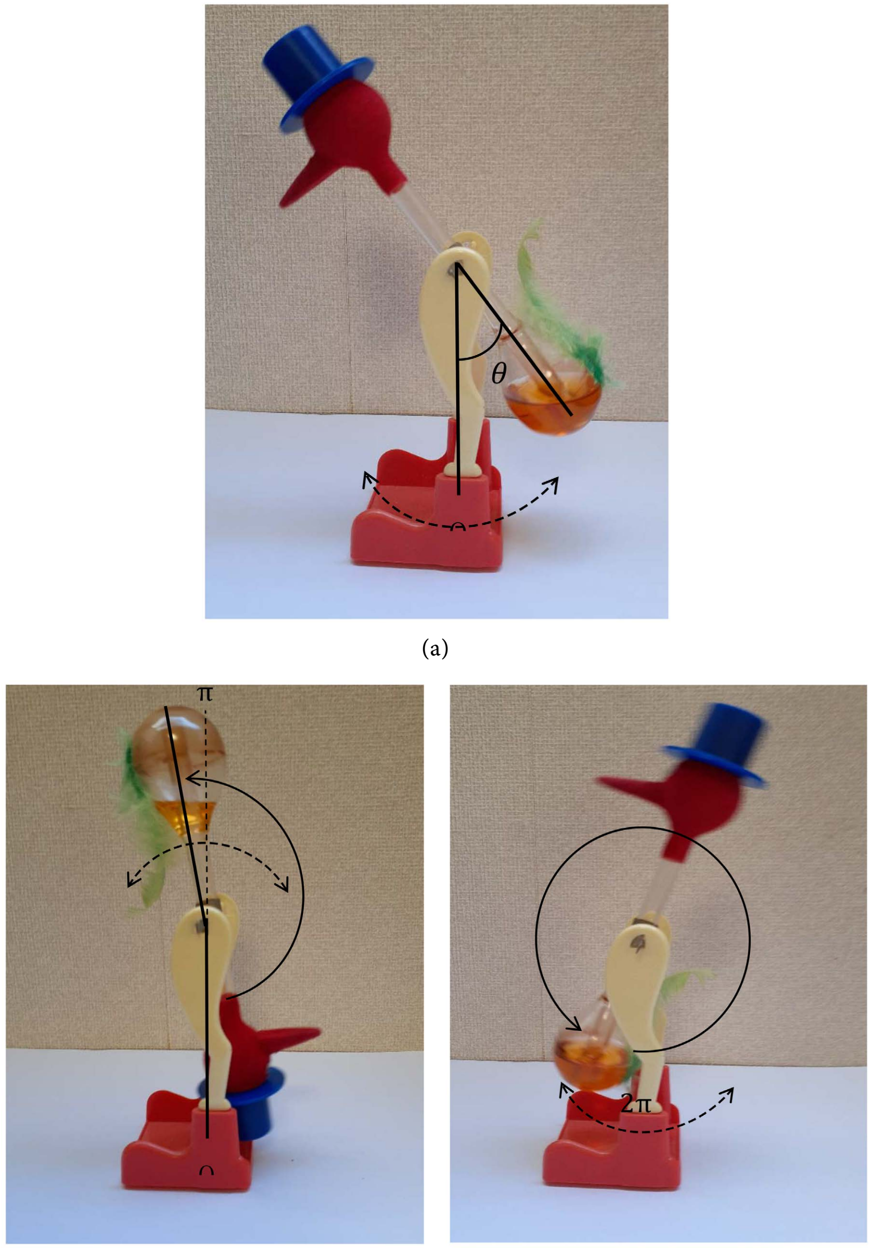


Figure 2. The solutions of nonlinear differential equation with constant coefficients (NDE-CC), (2.1). The corresponding physical picture of solutions for drinking bird's motion are shown in **Figure 3**.

are better understood by physical argument, searching for certain values of constant coefficients, g, c, l, m^*, I_0 and initial conditions. The trial and error search based on physical argument is possible because the drinking bird's equation of motion (2.1) must have stable solutions: $\theta_\pi(t)$ is the upside down solution, and $\theta_{2\pi}(t)$ is the one-rotation solution, and so forth (**Figure 3**).

However, the values of coefficients and initial conditions for other $\theta_{n\pi}(t)$ solutions are difficult to obtain, because the $\theta_{n\pi}(t)$ solutions appear suddenly even with a tiny change of coefficient values near the threshold of branching values. This is the property of bifurcation phenomena of NDE-CC as explained in the introduction. The conditions to determine parameter values to obtain *bifurcation solutions* are mathematically not known, but the derivation and existence of bifurcation solutions in (2.1) are definitely understood in terms of physics. It concludes that the water-drinking motion is constructed by the transition solution from $\theta_0(t)$ to $\theta_\pi(t)$, which is denoted as, $\theta_{0\to\pi}(t)$ for short, in the next section.

The independent solutions, $\theta_{n\pi}(t)$ ($n = \pm 1, \pm 2, \cdots$), are mathematically all possible and equally realizable solutions if corresponding values of coefficient are supplied, but it is difficult to find these values for large *n* solutions, and they are not meaningful in physical applications except $\theta_0(t)$ and $\theta_{\pm\pi}(t)$, because $\theta_0(t)$ and $\theta_{\pm\pi}(t)$ solutions could suffice a physical application in reality. It should be emphasized that the lack of the superposition principle in nonlinear equations prevents the construction of new solutions. Thus, it is concluded that the continuous transitions from solution $\theta_{n\pi}(t)$ to solution $\theta_{n\pi}(t)$ ($m \neq n$) do not exist. Therefore, the water-drinking motion cannot be constructed or found from bifurcation solutions in (2.1). Since the equation of motion (2.1) is rigorously produced by way of the method of Lagrangian or Hamiltonian mechanics, it suggests that a fundamentally different approach must be employed to explain the water drinking motion, and the method of thermomechanical dynamics (TMD) was proposed [6] [7] [11] for the transition to nonequilibrium states.



(a) (b) (c)

Figure 3. The independent solutions of nonlinear differential Equation (2.1), converging in $0, \pi, 2\pi$ as $t \to \infty$, respectively. Note the corresponding solutions in **Figure 2**. In the real drinking bird toy, the solutions, $\theta_{\pi}(t)$ and $\theta_{2\pi}(t)$, are mechanically forbidden. (a) The solution $\theta_0(t)$ converging in 0 as $t \to \infty$. (b) The solution $\theta_{\pi}(t)$ converging in π. (c) The solution $\theta_{2\pi}(t)$ converging in 2π.

3. The Bifurcation-Integration Solutions of the Drinking Bird's Nonlinear Equation of Motion with Time-Dependent Coefficients

The water drinking motion is constructed by the continuous transition from $\theta_0(t)$ to $\theta_\pi(t)$, but the continuous transition is not possible in Newtonian mechanics as discussed in Section 2. The drinking bird system is a heat engine driven by thermodynamic force and thermal heat flows. Therefore, a drinking bird motion by employing mechanics and partly thermodynamics are, at most, acceptable for approximate explanations, but theoretically speaking, it is essentially inconsistent, because of time-asymmetry and existence of entropy in non-equilibrium irreversible states (NISs), which fundamentally demands a method and a concept different from Newtonian mechanics and equilibrium thermodynamics. We proposed thermomechanical dynamics (TMD), which has consistently solved two-independent systems of heat engines: equations of motion of a drinking bird [7] and a low temperature Stirling engine [11].

A drinking bird motion is fundamental to study a nonlinear differential equation in mathematics and physics, corresponding to the transition from mechanical and thermal equilibrium to NISs. The key point in the NDE-CC Equation (2.1) is that the coefficients, moment of inertia and effective mass, I_0 and m^*, are constant, whereas the coefficients become time-dependent in the correct equation of motion. The volatile fluid, CH_2Cl_2, rises up through the body tube and changes the constant moment of inertia and effective mass to time-dependent: $I_0 \to I(t)$ and $m^* \to m^*(t)$. The correct nonlinear differential equation with time-dependent coefficients (NDE-TC) is given by:

$$\ddot{\theta} + c\dot{\theta} + \frac{glm^*(t)}{I(t)}\sin\theta = 0. \tag{3.1}$$

The Equations (2.1) and (3.1) and the time-dependency of coefficients, $I(t)$ and $m^*(t)$, should be carefully compared. The time-dependent functions, $I(t)$ and $m^*(t)$ are important to obtain the drinking bird solution, and the transition solution of the water drinking motion, $\theta_{0\to\pi}(t)$, is discussed in detail [6]. The correct solution of water drinking is shown in **Figure 4**.

The solution, $\theta_{0\to\pi}(t)$, does not exist in the NDE-CC (2.1), but the water drinking motion exists in the NDE-TC (3.1). The bird's water dipping ($\theta \simeq \pi/2$) resets volatile liquid in the glass tube, and some mechanical properties are initialized (more or less initialized, because mechanical and thermodynamic state are not the same at all). The initialization in the equation of motion means that time-dependent quantities are set as,

$$\left(x_2(t), m^*(t), I(t)\right) \to \left(a, m^*, I_0\right)_{t_1}, \tag{3.2}$$

at water drinking time t_1 [6], and after the water drinking motion $\theta_{0\to\pi}(t)$, a real drinking bird toy is constructed to return to the initial state and continues back-and-forth motion, as shown in **Figure 5**.

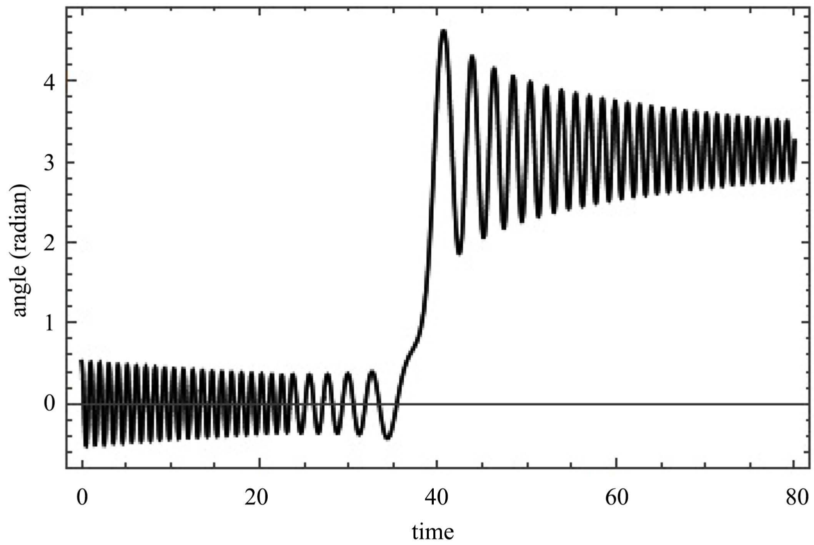


Figure 4. The bifurcation-integration solution, $\theta_{0\to\pi}(t)$, of the nonlinear differential equation with time-dependent coefficients (NDE-TC), for the correct drinking bird's motion (3.1).

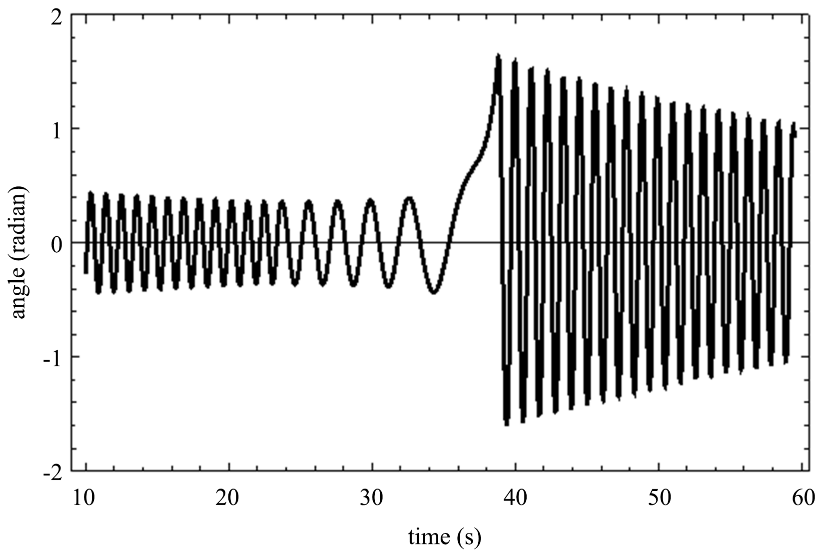


Figure 5. The one-periodic solution to a thermomechanical drinking bird defined by Equation (3.1), and it is the motion after initialization at $\theta \simeq \pi/2$ [6].

It is remarkable that the independent solution is constructed by changing constant coefficients into time-dependent coefficients given by physical requirements. The other solutions, $\theta_{0\to2\pi}(t)$, $\theta_{0\to3\pi}(t),\cdots$, could be created by supplying appropriate values of masses, lengths and initial conditions, but it becomes difficult to search parameter values corresponding to other independent solutions. The new independent solution, $\theta_{0\to\pi}(t)$, appears to be constructed as if $\theta_0(t)$ and $\theta_\pi(t)$ were integrated. Therefore, we termed the new solution, $\theta_{0\to\pi}(t)$, as the *bifurcation-integration solution*, and it is fundamental to recognize that the solution corresponds to a nonequilibrium irreversible state.

A drinking bird's simple motion signifies a fundamental physical event for phase transitions from a thermomechanical equilibrium state to a nonequilibrium irreversible state, and the transition of states can be expressed mathematically by the change from NDE-CC to NDE-TC, when physically consistent time-dependent coefficients are found. The NDE-CC and NDE-TC must be categorized respectively as the independent class of nonlinear differential equations. The fundamental properties of the bifurcation-integration solutions should be investigated in the field of nonlinear differential equations.

4. Conclusions

The bifurcation solutions were first found by Henri Poincaré in 1885 in the nonlinear differential equations with constant coefficients (NDE-CC), and we found the independent bifurcation-integration solutions in nonlinear differential equations with time-dependent coefficients (NDE-TC). It is found in the mechanism of transitions from thermodynamic equilibrium states to NISs in a drinking bird's motion. The emergence of bifurcation-integration solutions corresponds to a phase transition from thermodynamic equilibrium to a nonequilibrium irreversible state in physical terms. In addition, the discovery of a new solution in NDE-TC is helpful for studying nonequilibrium irreversible states. The thermomechanical dynamics (TMD) is self-consistent, and the dissipative equation of motion, time-dependent internal energy, thermodynamic work, entropy and temperature of nonequilibrium irreversible states are consistently applied for the first time to the analysis of a drinking bird [7] and a low temperature Stirling engine [11]. A thermoelectric converter device of a drinking bird is recently constructed [12]. We hope that researchers in mathematics, physics and sciences in general would investigate the fundamental properties of bifurcation-integration solutions. The nonlinear differential equations will have far more fundamental contributions to sciences.

The dissipative equation of motion and physical quantities of a drinking bird and a low temperature Stirling engine are defined self-consistently and solved by the method of TMD, and the results enabled us to investigate applications to other types of heat engines, such as α-, β-, γ-type heat engines [13]. The consistent applicability of TMD indicates that reproducibility, testability and self-consistency for a scientific theory are maintained [7]. We are thinking of possible applications to other types of heat engines, as well as internal combustion engines [14], and a low-temperature, low-speed thermoelectric generator, for example, a thermoelectric generation (TEG)-rotary engine, or a (TEG)-diesel engine by employing hydrogen-fuel.

The TMD analysis proves that thermoelectric generations from a low temperature heat source is possible. It contributes to thermoelectric energy conversion devices for low revolutions of heat engines, resulting in thermoelectric generation Stirling engine (TEG-Stirling engine) by employing the axial flux generation mechanism [15] [16]. The radial flux generators (AFGs) are commonly used for electric power generators, such as the wind, geothermal and thermal turbines and nuclear plants [17] [18] [19]. The radial flux generators are qualified for producing high electric power, but the axial flux generation TEG-Stirling engine is better qualified for reactivating electric power from a low temperature heat source ($40^\circ C < T < 100^\circ C$), such as boiled water, waste heat from industries, hot springs, geothermal and thermal plants.

The TMD analyses helped us obtain time-dependent physical quantities in NISs and study varieties of technological possibility for energy activations and conversions of a low temperature waste heat. The method of TMD is self-consistent and applicable from vast fields of macroscopic phenomena to microscopic thermal phenomena occurring at small length and short time scales [20]. Microscopic descriptions and approaches, as well as their applications in thermal science, engineering, and energy transport, radiation and convection could be investigated. The theoretical analyses of TMD for nonequilibrium irreversible states contribute to sustainable environmental technologies (SETs). The produced electricity can be collected and used for electrolysis, for example, to produce basic chemicals [21] (H_2, O_2, C, COOH, CH_3COOH, *etc.*), to support sustainable development goals.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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