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A Comparative Study of Synchronization Methods of Rucklidge Chaotic Systems with Design of Active Control and Backstepping Methods

Absana Tarammim*, Musammet Tahmina Akter

Department of Mathematics, Chittagong University of Engineering & Technology (CUET), Chattogram, Bangladesh
Email: *absana.tarammim@gmail.comm

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Abstract

The performance of two widely used chaos synchronization approaches, active control and backstepping control, is investigated in this study. These two methods are projected to synchronize two chaotic systems (Master/Drive of Rucklidge Systems) that are identical but have different initial conditions. The paper's significant feature is that based on error dynamics, controllers are designed using the appropriate variable and the time synchronization between master Rucklidge and drive Rucklidge systems using both methods. The control function of the active control method is designed on the proper selection of matrices. The chaotic behavior is controlled using a recursive backstepping design based on the Lyapunov stability theory with a validated Lyapunov function. The effectiveness of the controller in eradicating the chaotic behavior from the state trajectories is also revealed using numerical simulations with Matlab. The backstepping method is superior to the active control method for synchronization of the measured pair of systems, as it takes less time to synchronize while exhausting the first one than the second one with great performance, according to numerical simulation and graphical outcomes.

Keywords

Chaotic System, Synchronization, Active Control, Backstepping Control, Lyapunov Function

1. Introduction

Natural sciences are founded on the basis of chaos [1], so it is applied to various disciplines in mathematics, computer science, microbiology, meteorology, biology, engineering, geology, finance, economics, algorithmic trading, politics, population dynamics, psychology, philosophy, and robotics [2] [3] [4]. A variety of engineering and natural sciences problems are modeled using chaotic dynamical systems. Dynamic systems designated by non-linear differential equations can be sensitive to initial conditions [5]. This phenomenon is recognized as deterministic chaos to mean that, even if the system's mathematical description is deterministic, its activities demonstrate to be unpredictable. System instabilities and dynamic characteristics are generous in practice. Also, chaos is unpredictable and may lead to tremblings and exhaustion failures in mechanical systems; yet, chaos suppression is usually advantageous. Many researchers are interested in investigating mathematical models and the possibilities of chaos and its control mechanisms after discovering chaotic dynamics in the deterministic non-linear system [6] [7] [8]. The dynamic analysis of biologic and technological models has emerged as a significant research topic [9] [10] [11] [12] [13]. Many features of chaotic systems are studied, such as chaos control, chaos stability, amplitude death, chaos synchronization, pattern formation, etc. Because of its various applications in physics [14], control theory, biological networks, secure communication [15], artificial neural networks, chemical reactors [16], etc., chaos synchronization is a vital aspect in non-linear dynamical science.

Pecora and Carrol demonstrated that two chaotic systems could synchronize in 1990s [17] [18]. The concept of synchronizing two similar nonlinear chaotic systems that begin with different initial conditions and a first system (Master system) can follow the trajectories of a second one (Drive system) when a suitable control law is applied, and it appears that two chaotic systems cannot synchronize with one another. However, if the two systems share information on a regular basis, they will be able to synchronize. A wide variety of methods are projected to achieve chaos synchronization such as the linear state error feedback method [19], time-delay feedback method [20], active control approach [21] [22], impulsive method [23], backstepping approach [24] [25], and some other controlling methods are important in recent times [26] [27] [28] [29]. The above methods are applied to many practical systems such as Van der Pol Duffing oscillators [21], the Rikitake two-disc dynamo—a geophysical system [19], Chua's circuits [30], nonlinear Bloch equations modeling nuclear magnetic resonance [20], electric circuits modeling "jerk" equation [23], complex dynamos [31], nonlinear equations of acoustic gravity waves [32] and some other chaotic system are important in recent times [24] [29] [32] [33]. Backstepping design and active control, in particular, are regarded as two powerful strategies for controlling and synchronizing chaos.

Bai and Lonngren [34] projected the uses of chaos synchronization in the active control method. In this method, the synchronization speed is fast, and the amplitude of the oscillations is less. Because of the ease of implementation of chaotic synchronization, active control schemes have piqued the interest of researchers. Lately, we have used these methods to achieve synchronization and analyzed the effect of eigenvalues [35]. These eigenvalues of the chaotic system's coefficient matrix can be changed to obtain the required synchronization time. Using these studies, we can easily synchronize two identical chaos systems (*i.e.*, two of the same parameters) with active control. The active control process is a powerful algorithm for synchronizing two chaotic systems, both are identical or non-identical [36]. Most practical systems have non-identical components, as is widely known, for which the active control method is the more efficient procedure. If the chaotic system's non-linearity is known, then linear active control approaches can be designed to achieve global chaos control and synchronization based on the chaotic system's provided conditions. This approach has been used on a variety of real-world systems [34]; in the chaotic synchronization, the receptor tries to identify the chaotic signal sent from the emitter that means if two chaotic signals are asymptotically identical when the time goes to infinity so that two chaotic signals will be synchronized. There are no derivatives in the controller, and the Lyapunov exponents are not required for their implementation, and these properties provide active control methods an edge over other conventional control systems [27].

The backstepping control algorithm is a type of non-linear controller design. It is very existent for handling mismatched perturbation [31]. It operates on multiple chaotic systems regardless of whether they understand external stimulation or not, it requires just one controller to recognize synchronization between chaotic systems, and the controller has no derivatives [22]. Furthermore, the Backstepping design is a type of synthetic procedure to the controller that recursively connects the choice of a Lyapunov function [37]. The backstepping method has the advantage of being able to avoid cancellations of beneficial non-linearities acting in the system. So the backstepping method acts on stabilization and tracking other than the linearization method. It is reported [38] that, backstepping technique is employed for controlling, tracking, synchronizing many chaotic systems [39] and it can be guaranteed global stability and transient performance non-linear systems. The initiation of a non-linear function passed through by course error signal under the backstepping design framework is projected to simplify the non-linear controller design process, decrease the number of undetermined parameters, increase the strength, and reduce the energy consumption of the course-keeping controller and improved summarizing design method, while the design process of the controller is simplified to only one step. The Lyapunov function is to choose for stabilizing the system of different time steps with characterizations of the current approach and the control function is designed at the final step. Finally, this paper intends, schemes, and analyses a non-linear controller that comprehends the synchronized error fast and oscillation free convergence to zero.

To control chaos in the Rucklidge system, the Active control and Backstepping approaches are applied in this work. To provide global synchronization between two identical chaotic systems, we project active controllers and a recursive backstepping control and compare simulation results of the two strategies.

In Section 1, we give the theoretical foundation of our paper. The remainder of the paper will be written as follows: Dynamical Analysis of the Rucklidge System with two Parameters, Section 2; Synchronization via Active Control is presented in Section 3, with numerical results, whereas Backstepping control design for chaos synchronization is presented in Section 4, with numerical results and a comparative research description in Section 5. Finally, Section 6 brings the paper to a close.

2. Dynamical Analysis of the Rucklidge Chaotic System

In this work, the Rucklidge chaotic system (1) is described by [40]:

$$\begin{aligned} \dot{x} &= -bx + ay - yz \\ \dot{y} &= x \\ \dot{z} &= -z + y^2 \end{aligned} \tag{1}$$

where x, y, z and a, b are the state variables and positive constant parameters, respectively. We show in this paper that the system (1) is chaotic when the parameters are $a = 6.7, b = 2$. We use the chaotic system's (8) initial values for numerical simulations as: $x_1(0) = 0.001$, $y_1(0) = 0.001$, $z_1(0) = 0.001$.

And the chaotic system (9) as: $x_2(0) = 5$, $y_2(0) = 8$, $z_2(0) = 4$. Figures 1-3 demonstrate the Rucklidge System's (1) 2-D projections on (x, y), (y, z) and (z, x) space projections, respectively.

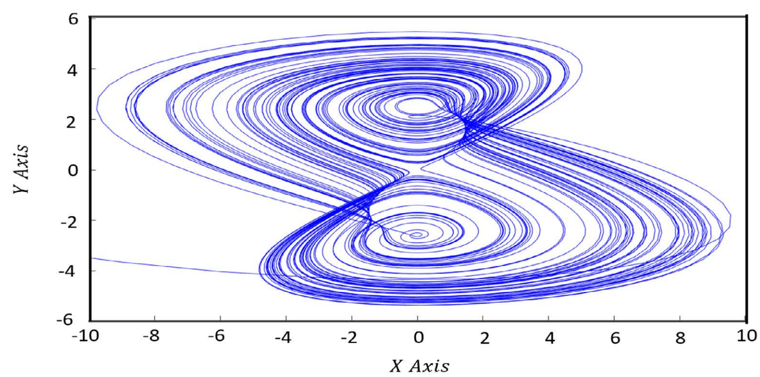

Figure 1. On the X-Y plane, a 2-D phase portrait of the Rucklidge chaotic system.

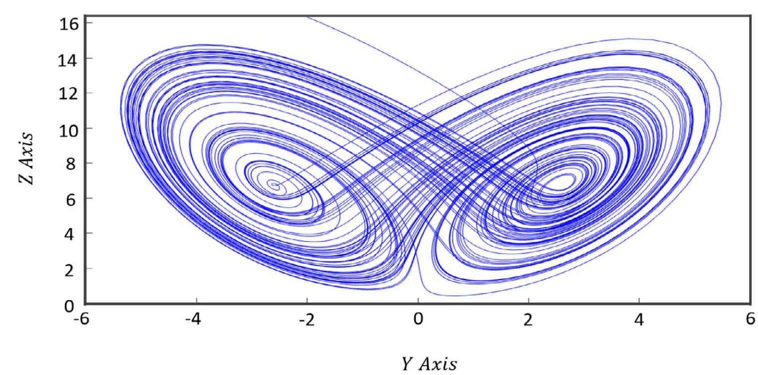

Figure 2. On the Y-Z plane, a 2-D phase portrait of the Rucklidge chaotic system.

The time response of before synchronization states for the Master system (x_1, y_1, z_1) and the Drive system (x_2, y_2, z_2) is shown in Figure 4.

The error dynamics in the uncontrolled state are shown in Figure 5, whereas the error dynamics in the controlled state are shown in Figure 6 and Figure 12.

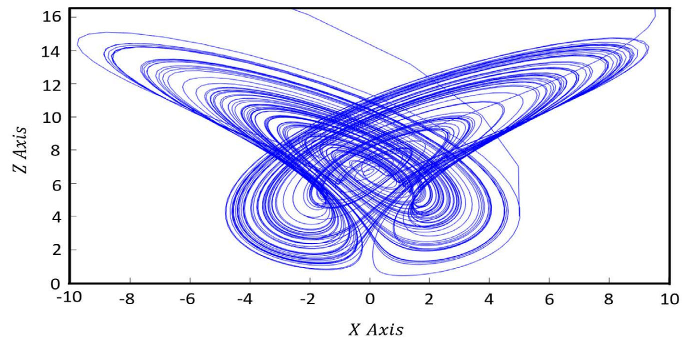

Figure 3. On the *X*-*Z* plane, a 2-D phase portrait of the Rucklidge chaotic system.

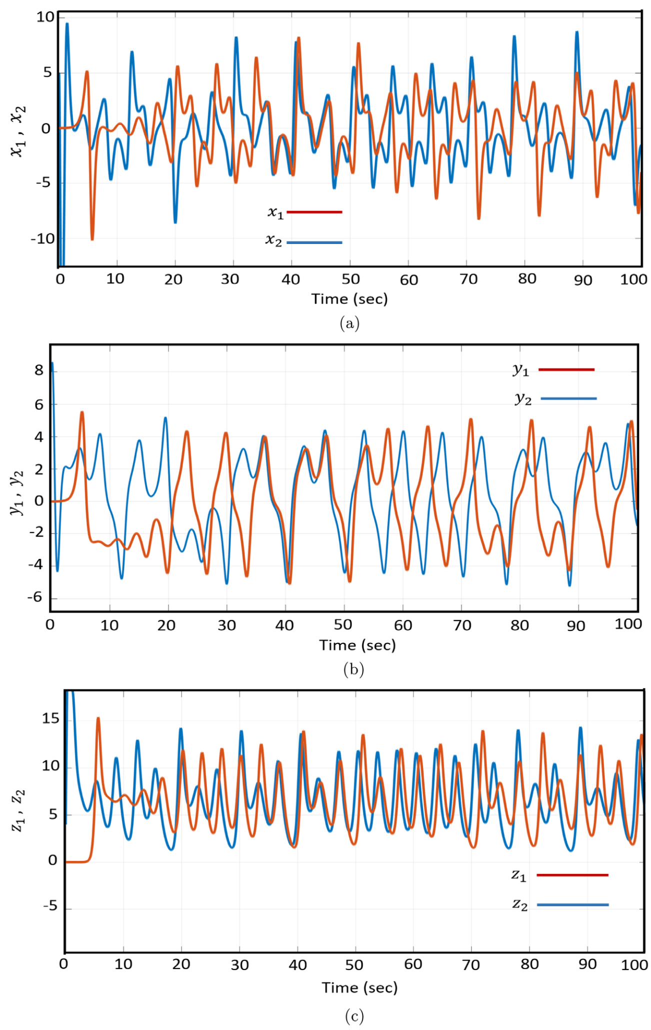

Figure 4. Time response of the states x, y, z before synchronization. (a) x_1, x_2; (b) y_1, y_2; (c) z_1, z_2.

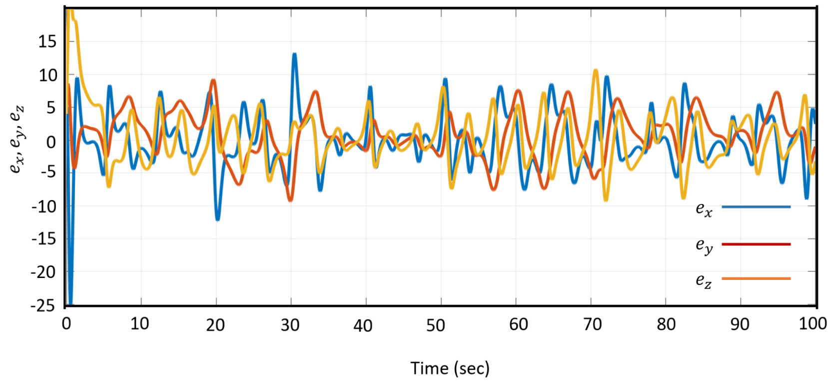

Figure 5. Time response of the error states before synchronization.

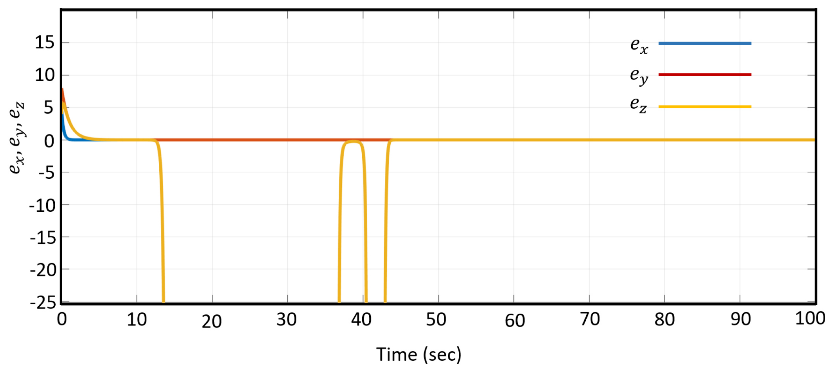

Figure 6. Time response of the synchronized states of error (e_x, e_y, e_z).

3. Synchronization via Active Control

When Master and Drive systems are in the same parameters, achieving synchronization using active control is appropriate and effective. Assume the existence of a Master system as

$$\dot{x} = Mx + g(x) \tag{2}$$

where $x = (x_1, y_1, \cdots, z_1)^{\mathrm{T}} \in R^n$, $M \in R^n \times R^n$, M is a constant system matrix and $g(x)$ is a sequence function that is nonlinear.

The Drive system is

$$\dot{y} = My + g(y) + u(t) \tag{3}$$

where $\dot{y} = (x_2, y_2, \cdots, z_2)^{\mathrm{T}} \in R^n$, $u(t) = (u_i(t)) \in R^n$, where $i = (1, 2, \cdots, n)$.

3.1. Definition

If an appropriate controller $u(t)$ exists that satisfies $\forall x, y, e \in R^n$, $\lim_{t \to 0} \|y - x\| = \lim_{t \to 0} \|e\| = 0$ the Master and Drive systems are then designed to be synchronized.

As a result, the error is defined as $e = y - x$.

Then the error dynamics is

$$\dot{e} = \dot{y} - \dot{x} = Me + G(x, y) + u(t) \tag{4}$$

where $G(x,y)=g(y)-g(x)$. Without e of system (4), controller $u(t)$ may eliminate non-linear section. That is

$$u(t)=v(t)-G(x,y) \tag{5}$$

where $v(t)=Ke$ represents a linear section with error variables. From Equation (5) and Equation (4) we get

$$\dot{e}=Me+v(t) \tag{6}$$

Equation (6) becomes when $v(t)$ is a linear section with error variables and $v(t)=ke$ is a constant matrix.

$$\dot{e}=(M+K)e \tag{7}$$

3.2. Proposition [7]

Satisfing the requirements needed for diagonal matrix $(M+K)$ is $\lambda_i \le 0$, where λ_i is the eigenvalue of matrix $(M+K)$, state vectors of system (7) is asymptotically converge to zero, as a result Master system (2) and Drive system (3) asymptotically synchronize. In this context, the Master and Drive systems are defined as follows:

The Master Rucklidge system is defined by

$$\begin{aligned} \dot{x}_1 &= -bx_1 + ay_1 - y_1z_1 \\ \dot{y}_1 &= x_1 \\ \dot{z}_1 &= -z_1 + y_1^2 \end{aligned} \tag{8}$$

where a and b are the system parameters.

The Drive Rucklidge system is defined by

$$\begin{aligned} \dot{x}_2 &= -bx_2 + ay_2 - y_2z_2 + u_1 \\ \dot{y}_2 &= x_2 + u_2 \\ \dot{z}_2 &= -z_2 + y_2^2 + u_3 \end{aligned} \tag{9}$$

$u_i=(1,2,3)$ are active control functions for the Master system. In Equation (9), we presented three control functions: $u_1(t)$, $u_2(t)$, and $u_3(t)$ which are must be regulated. We subtract Equation (8) from Equation (9) to estimate the control functions. Consider the state errors between the controlling system (9) and the master system (8) that needs to be controlled by using

$$e_x = x_2 - x_1; e_y = y_2 - y_1; e_z = z_2 - z_1; \tag{10}$$

Applying the active control design methods, we subtract Equation (8) from Equation (9) and use the definitions in Equation (10) to derive the error dynamics equation:

$$\begin{aligned} \dot{e}_x &= -be_x + ae_y - y_2z_2 + y_1z_1 + u_1 \\ \dot{e}_y &= e_x + u_2 \\ \dot{e}_z &= -e_z + e_y(y_2+y_1) + u_3 \end{aligned} \tag{11}$$

tracking the re-describing control functions as:

$$\begin{aligned} v_1 &= u_1 - y_2z_2 + y_1z_1 \\ v_2 &= u_2 \\ v_3 &= u_3 \end{aligned} \tag{12}$$

the error dynamics Equation (11) takes on a new state

$$\begin{aligned}\dot{e}_x &= -be_x + ae_y + v_1\\ \dot{e}_y &= e_x + v_2\\ \dot{e}_z &= -e_z + e_y\left(y_2 + y_1\right) + v_3\end{aligned} \tag{13}$$

The controllable state of error system (11) is a linear system with control effort $\left(v_1(t), v_2(t), v_3(t)\right)$ as a function of error states $\left(e_x, e_y, e_z\right)$. The error states $\left(e_x, e_y, e_z\right)$ converge to zero as time $t \to \infty$ passes if these stabilize the system. This means that active control is used to synchronize the Master and Drive systems. For the control $\left(v_1(t), v_2(t), v_3(t)\right)$, there are various possible sets. We created a constant matrix A that will govern the error dynamics (11) in accordance with the active control approach, such that

$$\left[v_1, v_2, v_3\right]^{\mathrm{T}} = A\left[e_x, e_y, e_z\right]^{\mathrm{T}} \tag{14}$$

where A is a constant matrix. To stabilize the state of the error system, the components of the matrix A must be chosen in such a way that the feedback system has all eigenvalues with negative real portions. In the tracking form, choose the matrix A:

$$A = \begin{pmatrix} -b & -a & 0\\ -1 & -1 & 0\\ 0 & -\left(y_2 + y_1\right) & 0 \end{pmatrix}$$

Bring Equation (14) into Equation (13), we may find

$$\dot{e} = Ke$$

$$\begin{pmatrix} \dot{e}_x\\ \dot{e}_y\\ \dot{e}_z \end{pmatrix} = \begin{pmatrix} -b & a & 0\\ 1 & 0 & 0\\ 0 & y_2 + y_1 & -1 \end{pmatrix}\begin{pmatrix} e_1\\ e_2\\ e_3 \end{pmatrix} + A\begin{pmatrix} e_1\\ e_2\\ e_3 \end{pmatrix}$$

$$K = \begin{pmatrix} -2b & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1 \end{pmatrix}$$

In this specific selection, the eigenvalues of the system (13) are $\left(-2b, -1, -1\right)$. Due to the linearity system's stability theory, this option will result in the convergence of error states $\left(e_x, e_y, e_z\right)$ to zero as time t approaches infinity, and therefore the synchronization of the two systems is achieved under the control system (15)

$$\begin{aligned}u_1 &= -be_x - ae_y + y_2 z_2 - y_1 z_1\\ u_2 &= -e_x - e_y\\ u_3 &= -\left(y_2 + y_1\right)e_y\end{aligned} \tag{15}$$

3.3. Simulation and Results

To get numerical results, we apply MatLab Simulink and the Runge-Kutta algorithm of 4th order with a time grid of 0.05. We continue with the initial conditions of the Master system Equation (8) as follows:

$$x_1(0) = 0.001, y_1(0) = 0.001, z_1(0) = 0.001 \tag{16}$$

and the Drive system's initial conditions (9) as follows:

$$x_2(0) = 5, y_2(0) = 8, z_2(0) = 4 \tag{17}$$

The time response of the error states obtained from numerical simulations of synchronization systems (11) is shown in **Figure 6** under the controller (15) stimulated at time t = 100 seconds, thereby guaranteeing the synchronization of system (8) and (9). It is clear that after control signals are activated, the error vectors converge to zero quickly. We calculate the synchronization measure for verifying the synchronization act from **Figure 7**, the average error on the system state variables are given by

$$e = \sqrt{e_x^2 + e_y^2 + e_z^2}$$

and **Figures 8-10** demonstrate the response time of the synchronization of the driven system (9) to the driver (8), control signals are activated at the time t = 0, and **Figure 11** show the response time of action control (15) for achieve chaos synchronization between the two chaotic systems.

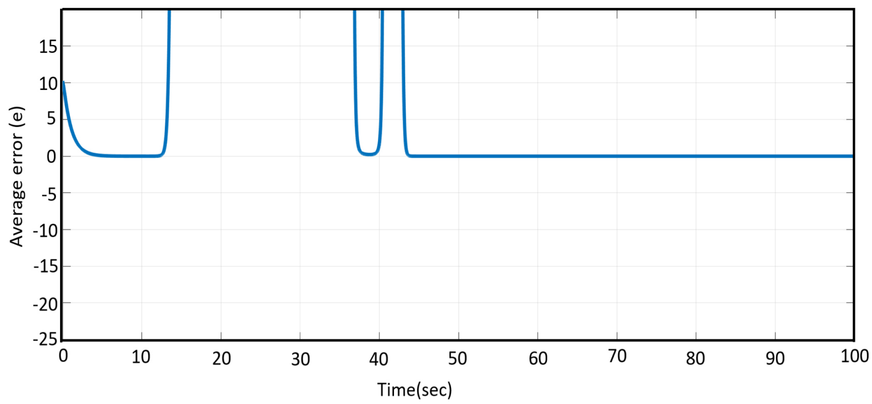

Figure 7. The response time of the synchronized states of average error.

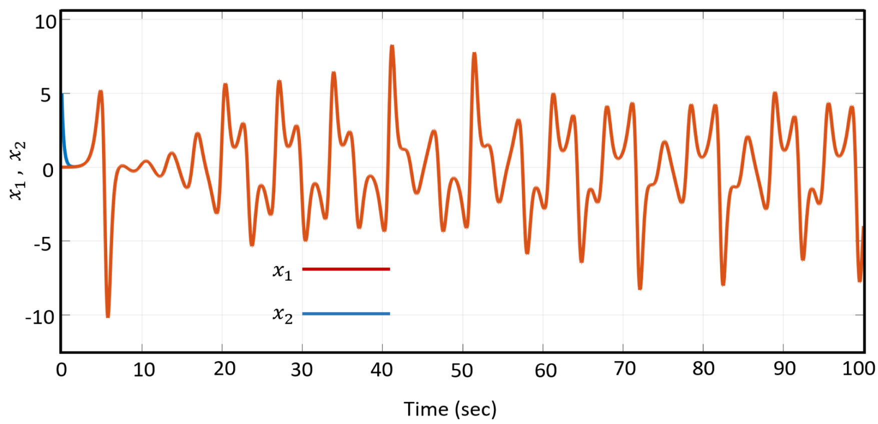

Figure 8. The response time of the synchronized states (x_1, x_2).

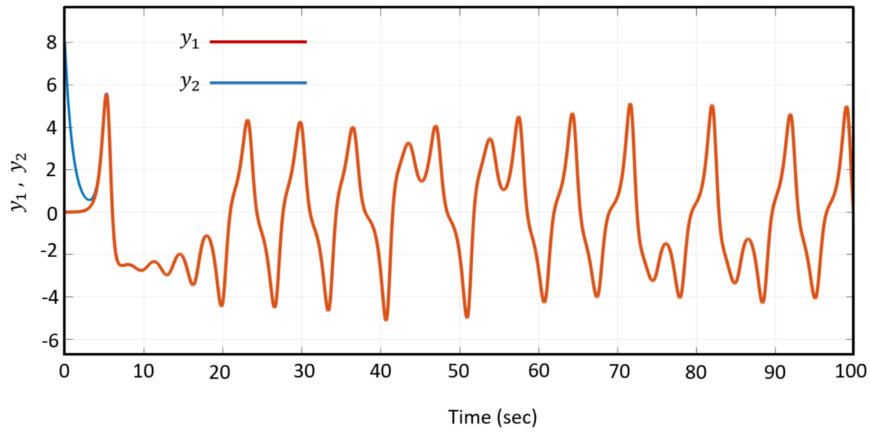

Figure 9. Time response of the synchronized states (y_1, y_2).

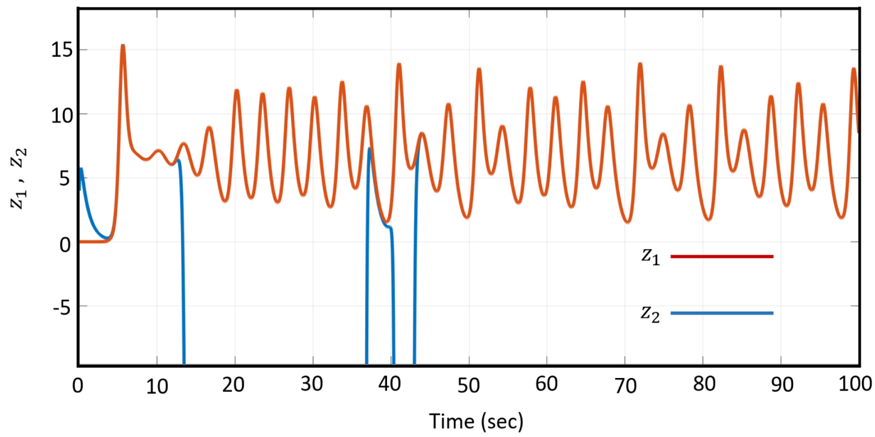

Figure 10. Time response of the synchronized states (z_1, z_2).

The purpose of this section **Figures 8-10** is to exhaust active control procedure to synchronize two chaotic chaotic systems (8) and (9) by defining a controller that the Drive system capability to trail the Master System and the states of two chaotic systems (8) and (9) show similar activities for all future states [21] [41].

4. Backstepping Control Design for Chaos Synchronization

This section focuses on Backstepping Control systems because this class offers various examples of chaotic circuits and systems [42] we explore the backstepping control design for a general system of the form:

$$\dot{x} = f(x) + g(x)\mu_1 \tag{18}$$

be a responsive affine nonlinear system, with $\mu_1 \in R$ as the control input, $x \in R^n$ as the state, and f and g as nonlinear functions, with $f(0) = 0$. The control system 18 can use the usual backstepping design. To stabilize systems that are in strict feedback form, we can use recursive application of backstepping control design as follows.

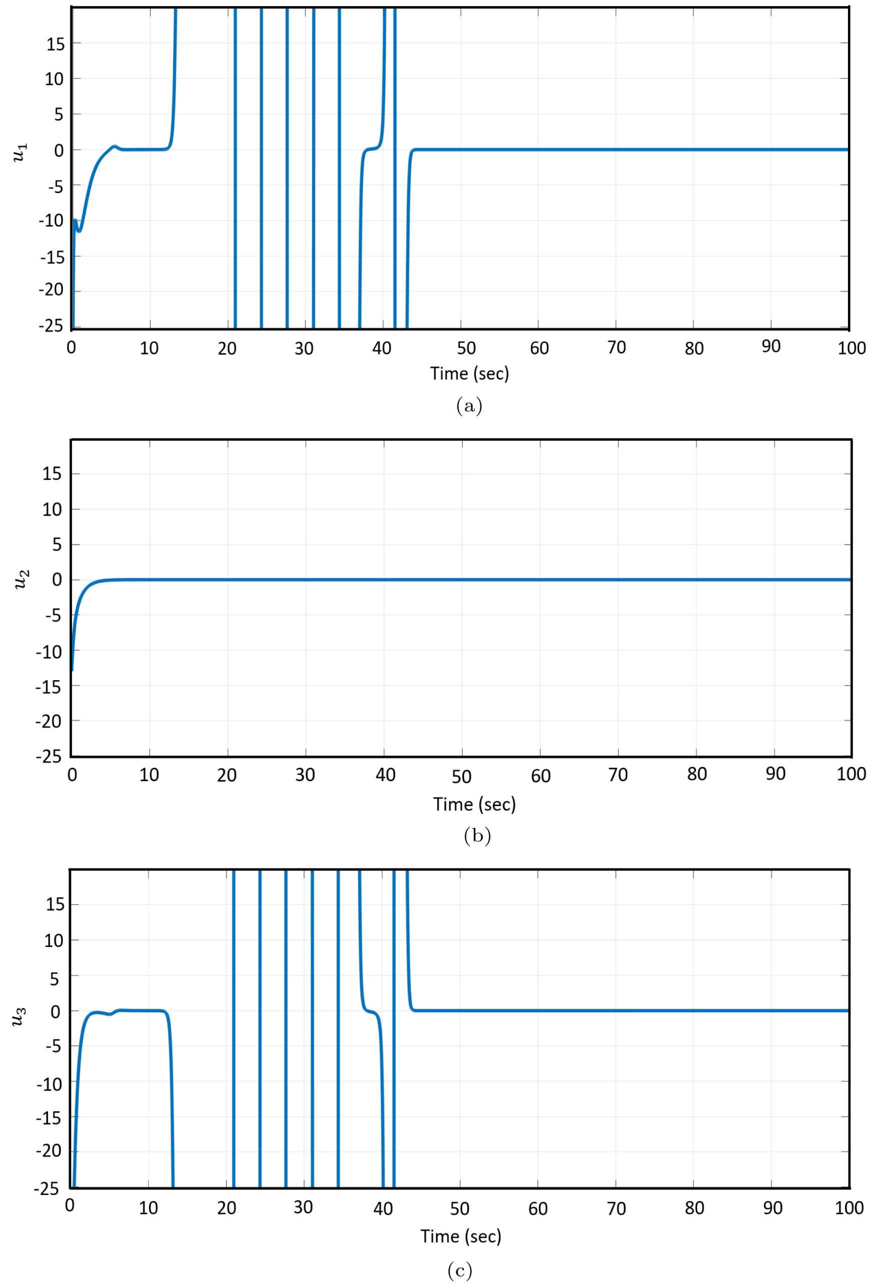

Figure 11. Time response of the synchronized states u_i. (a) u_1; (b) u_2; (c) u_3.

$$\begin{aligned} \dot{x} &= f(x) + g(x)\mu_1 \\ \dot{\mu}_1 &= f_1(x,\mu_1) + g_1(x,\mu_1)\mu_2 \\ &\vdots \\ \dot{\mu}_{k-1} &= f_{k-1}(x,\mu_1,\cdots,\mu_{k-1}) + g_{k-1}(x,\mu_1,\cdots,\mu_{k-1})\mu_k \\ \dot{\mu}_k &= f_k(x,\mu_1,\cdots,\mu_k) + g_k(x,\mu_1,\cdots,\mu_k)u \end{aligned} \tag{19}$$

where $\mu_1, \mu_2, \cdots, \mu_k$ and u are scalars [43].

Backstepping design is a Lyapunov-based control approach that is well-organized. Backstepping control design of systems (18) is a recursive strategy that ensures the system's global asymptotic stability. The fundamental concept is to extend Lyapunov's method by breaking down the full system design model (19) into a series of design challenges for lower-order systems. Depending on the state variables, control parameters, and stabilizing functions, incorporate new variables into transformation processes. The *i*th subsystem may be stabilized with respect to a specific Lyapunov function $V_i, i = 1, 2, \cdots, n$ by utilizing the backstepping control design at the *i*th step. By treating the variable μ_1 as a virtual control input, the approach is used to stabilize the first equation. Similarly, the second equation is stabilized by continuing to use the variable μ_2 as the virtual control, when μ_1 is designed and so on. So the design of the final control input *u*, which generally depends on *x* and $\mu_1, \mu_2, \cdots, \mu_k$, is steadily achieved in *n* steps [43] [44]. So it is proved that backstepping design is suitable for controlling chaos, stabilization, and tracking problems.

Consider the following Master system of Rucklidge systems:

$$\begin{aligned} \dot{x}_1 &= -bx_1 + ay_1 - y_1 z_1 \\ \dot{y}_1 &= x_1 \\ \dot{z}_1 &= -z_1 + y_1^2 \end{aligned} \tag{20}$$

in relation to the Drive system as

$$\begin{aligned} \dot{x}_2 &= -bx_2 + ay_2 - y_2 z_2 + u \\ \dot{y}_2 &= x_2 \\ \dot{z}_2 &= -z_2 + y_2^2 \end{aligned} \tag{21}$$

where u is a to-be-defined control function. We required only one controller in backstepping method. By using the error states definition (10) and subtracting Equation (20) from (21), we obtain

$$\begin{aligned} \dot{e}_x &= -be_x + ae_y - z_2 e_y - y_1 e_z + u \\ \dot{e}_y &= e_x \\ \dot{e}_z &= -e_z + e_y \left(y_2 + y_1 \right) \end{aligned} \tag{22}$$

to re-arrange error system (22) into system (23) according to sequence of full system (19)

$$\begin{aligned} \dot{e}_z &= -e_z + e_y \left(y_2 + y_1 \right) \\ \dot{e}_y &= e_x \\ \dot{e}_x &= -be_x + ae_y - z_2 e_y - y_1 e_z + u \end{aligned} \tag{23}$$

The previous system (23) is an equilibrium point $(0,0,0)$ in the absence of all control u. If u is chosen, then the equilibrium point residues are unchanged, the challenge of synchronization between the Master and Drive systems can be transformed into a problem of asymptotical system stabilization (23). The goal of this project is to find a control law u that will stabilize the system's error variables (23) at the origin. Since the system comprises three states or three non-linear differential equations, a recursive strategy will have three steps. We split down the system (23) into three subsystems, each with a single input and output. We initiate the formation with the first subsystem of the first non-linear differential equation, and one continues until the last subsystem. In the procedure of formation a change of coordinates $w_i = (e_x, e_y, e_z)$ is done.

4.1. First Step

To begin, we use e_y as a virtual controller to stabilize the first equation in (23), the object of the control is to drive $(e_x, e_y, e_z) = (0,0,0)$ *i.e.* $(x_1; y_1; z_1) = (x_2; y_2; z_2)$, and define the first new virtual variable of the Backstepping design as $w_1 = e_z$ and Considering another new virtual variable as $w_2 = e_y - \alpha_1$. This last term will not be used in the first step, but it is needs to w_2 to join the first subsystem on w_1 to the next subsystem on w_2 which will be considered in the second step, α_1 is the stabilization function. To find the stabilization function α_1, consider the Lyapunov function [37], given by:

$$V_1(w_1) = \frac{1}{2}(w_1)^2 \tag{24}$$

substituting the derivative of w_1 for the time derivative of Equation (24) yields:

$$\dot{V}_1 = -(w_1)^2 + w_1 w_2 (y_2 + y_1) + w_1 (y_2 + y_1) \alpha_1 \tag{25}$$

If the estimated stabilisation function α_1 is chosen as $\alpha_1 = 0$, as a result, $\dot{V}_1$ is indefinitely negative. The second term in Equation (25) will be eradicated in the next step.

4.2. Second Step

We stabilize the second equation in (23) for the stability of the second subsystem and its dynamics are calculated by putting the second equation of the system (23) into the second virtual variable with derivatives:

$$\dot{w}_2 = \dot{e}_y - \dot{\alpha}_1$$

The stabilization function α_2 for the second subsystem is preferred such as the new virtual state variable $w_3 = e_x - \alpha_2$. Let Lyapunov function for the second subsystem as:

$$V_2(w_1, w_2) = V_1(w_1) + \frac{1}{2}(w_2)^2 \tag{26}$$

Substituting the derivative of w_2 in time derivative of $\dot{V}_2$ then we get:

$$\dot{V}_2 = -w_1^2 - w_2^2 + w_2 w_3 + w_2 (w_2 + w_1 (y_2 + y_1) + \alpha_2) \tag{27}$$

The stabilization function α_2 is chosen as:

$$\alpha_2 = -w_2 - w_1 (y_2 + y_1) \tag{28}$$

And by substituting $\alpha_2, \dot{V}_2$ converted:

$$\dot{V}_2 = -w_1^2 - w_2^2 + w_2 w_3 \tag{29}$$

then $\dot{V}_2$ is negative definite, the third term in Equation (29) will be eradicated in the next step.

4.3. Third Step

To get the third virtual variable state, the time derivative of both the error on the third coordinate state and the stabilization function α_2. The equation of the third subsystem become:

$$\dot{w}_3 = \dot{e}_x - \dot{\alpha}_2$$
$$\dot{w}_3 = -be_x + ae_y - z_2 e_y - y_1 e_z + u - \dot{\alpha}_2 \quad (30)$$

We choose the Lyapunov function for the third subsystem is as:

$$V_3(w_2, w_3) = V_2(w_2) + \frac{1}{2}(w_3)^2 \quad (31)$$

It's time derivative gives:

$$\begin{aligned}\dot{V}_3 &= -w_1^2 - w_2^2 + w_2 w_3 + w_3 \dot{w}_3 \\ \dot{V}_3 &= -w_1^2 - w_2^2 - bw_3^2 + w_3\left(w_2 - b\alpha_2 + (a - z_2)e_y - y_1 e_z - \dot{\alpha}_2 + u\right)\end{aligned} \quad (32)$$

If the control law u is preferred as follows

$$u = -w_2 + b\alpha_2 - (a - z_2)e_y + y_1 e_z + \dot{\alpha}_2 \quad (33)$$

Then,

$$\dot{V}_3 = -w_1^2 - w_2^2 - bw_3^2 \quad (34)$$

is negative definite. The final system of the time derivative of the error model in (w_1, w_2, w_3) coordinates in (35), the error dynamics (e_x, e_y, e_z) will converge to zero as $t \to \infty$, whereas the equilibrium $(0,0,0)$ residues are asymptotically stable, according to the LaSalle-Yoshizawa theorem [43]. As a result, the master-drive system has been synchronized. We now study the complete space of $(\dot{w}_1, \dot{w}_2, \dot{w}_3)$:

$$\begin{aligned}\dot{w}_1 &= -w_1 + w_2(y_2 + y_1) \\ \dot{w}_2 &= e_x \\ \dot{w}_3 &= -be_x + ae_y - z_2 e_y - y_1 e_z + u - \dot{\alpha}_2\end{aligned} \quad (35)$$

The equilibrium $(0,0,0)$ of system (35) is universally asymptotically stable, as illustrated in **Figure 13**, according to LaSalle-Yoshizawa theorem [26] [27]. That is, the control law u has no effect on the system's equilibrium of (22), *i.e.* $(0,0,0)$ is still the equilibrium. As a result, the new chaotic system (35) is stabilized at the origin under the direction of the controller (33), and the two Rucklidge chaotic systems are synchronized.

4.4. Simulation and Results

We solved system (22) with the controllers defined in (33) using the fourth-order Runge-Kutta algorithm with initial conditions $x_1(0) = 0.001$, $y_1(0) = 0.001$, $z_1(0) = 0.001$ and $x_2(0) = 5$, $y_2(0) = 8$, $z_2(0) = 4$ with a time step of 0.01, and fixing the parameter values of a and b as shown in **Figures 1-3**. To demonstrate the efficiency of the control law, we show numerical findings. The time response of the error state (e_x, e_y, e_z) is shown in **Figure 12**, which was

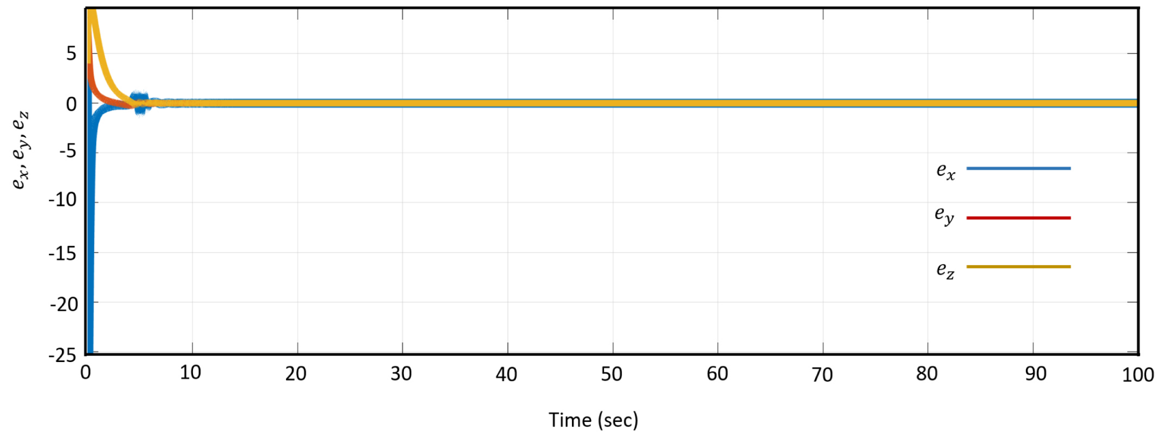

Figure 12. Tracking error e_x, e_y, e_z.

obtained from numerical simulations of the synchronization systems (20) and (21) under the controller (33).

From **Figure 13** to examine the equilibrium $(0,0,0)$ of system (35) and the average error propagation on the system state variables is used to calculate the synchronization quantity this are shown in **Figure 14** given by

$$e = \sqrt{e_x^2 + e_y^2 + e_z^2}$$

After an initial transcience of around $t = 6\text{ s}$, the Master-Drive system is globally synchronized. The exponential convergence of the synchronization quality described by error propagation on average error states also confirms this in **Figure 17(b)**.

Figure 15 shows the time response of the control law.

The persistence of this part **Figure 16** is to synchronize two chaotic systems (20) and (21) shattering Backstepping control process by defining a controller, and for all future states, the effectiveness of the Drive system to track the Master System and the states of two chaotic systems (20) and (21) indicate similar characteristics.

5. Active Control and Backstepping Approaches Are Compared

We obtained a relationship between synchronization time to investigate and compare the synchronization performance of the two strategies. **Figure 17** illustrates the synchronization error (e) for the two techniques when controls are activated at $t = 0$. So, for backstepping control, synchronization was achieved at $t = 6\text{ s}$, and for active control, synchronization was achieved at ($t = 22\text{ s}$), with a time delay of 16s. It is demonstrated that the error signals converge to the origin extremely smoothly with a low decay rate and sufficient synchronization speed, indicating that the researched controllers.

The active control and backstepping control techniques are used to create a controller for a three-dimensional autonomous chaotic system, with backstepping control reducing the number of controllers from three to one, and active control works according to dimension. So Backstepping control performs significantly, reducing the controller complexity and cost than the active control method.

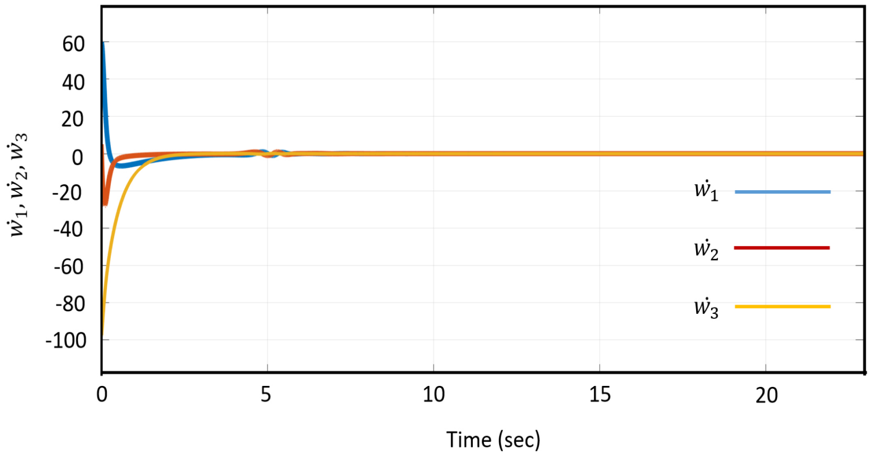

Figure 13. Final system of the error model after synchronization.

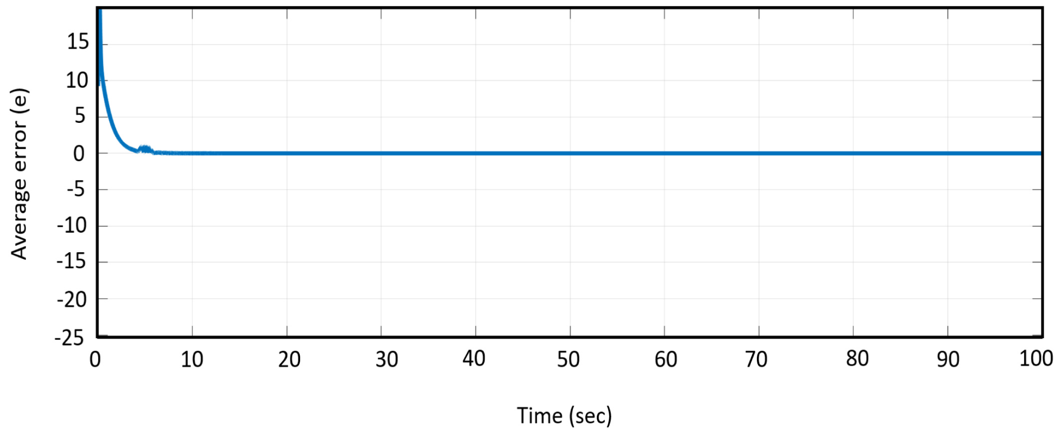

Figure 14. Average error *e*.

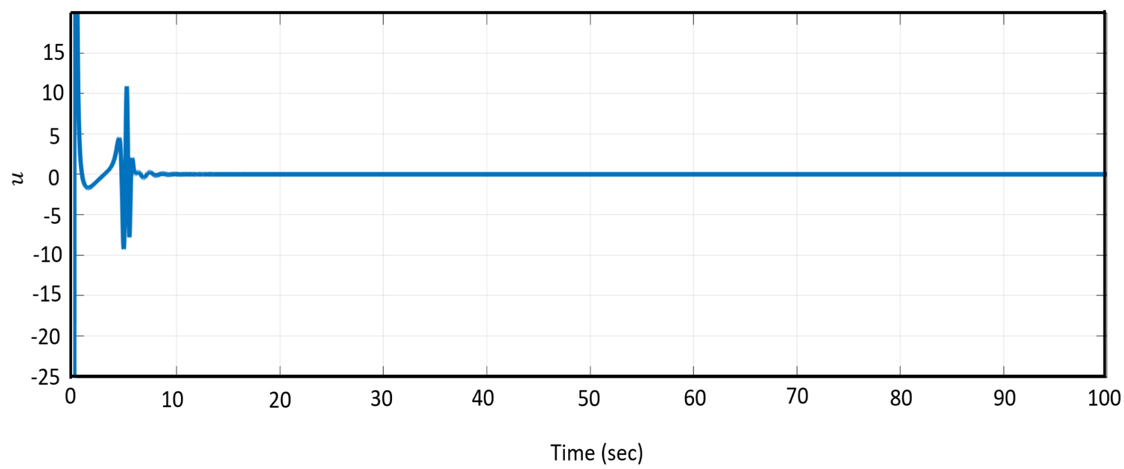

Figure 15. Control effort *u*.

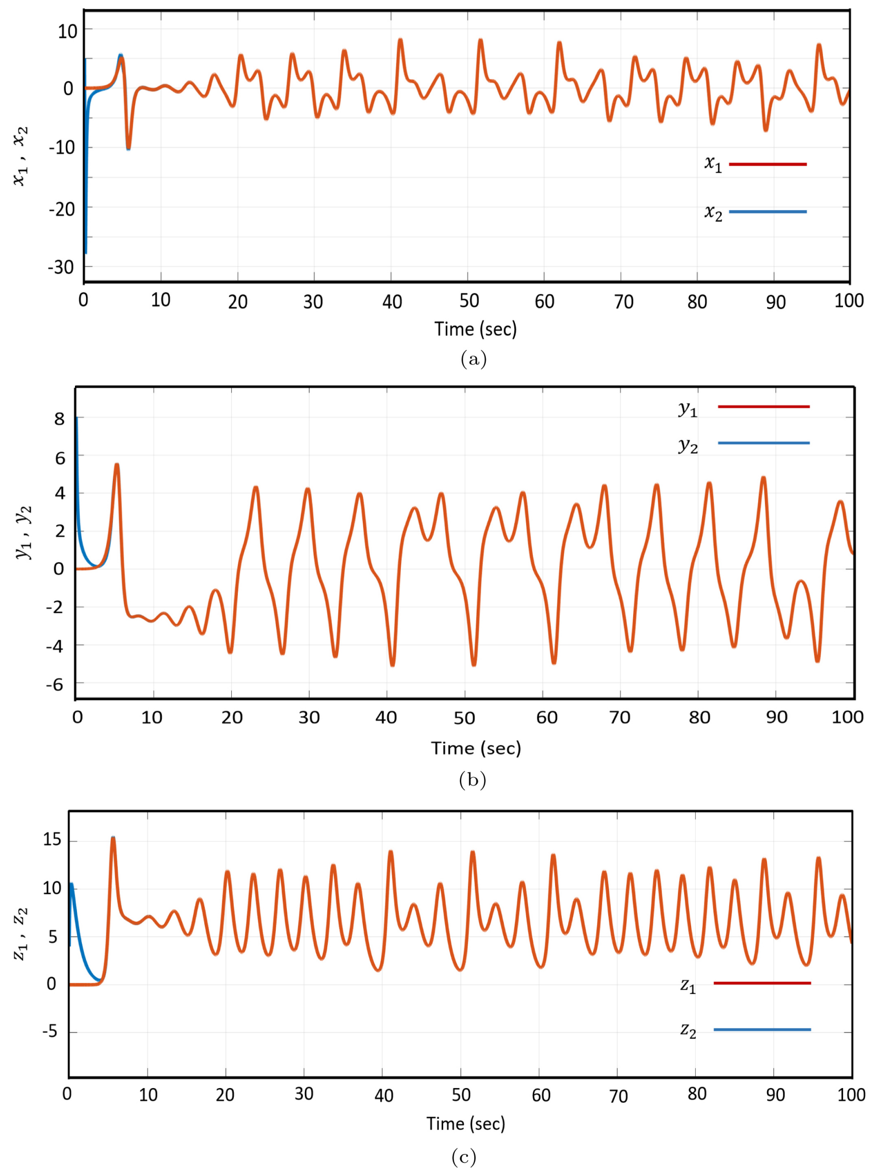

Figure 16. Time response of the states x, y, z after synchronization. (a) Time response of (x_1, x_2); (b) Time response of (y_1, y_2); (c) Time response of (z_1, z_2).

6. Conclusions

The performance of two control techniques for chaos synchronization, active control and recursive backstepping control, was investigated in this work. The two techniques are proven to have outstanding synchronization performance, with the active control slightly outperforming the backstepping. The performance of theoretically designed nonlinear controllers was verified by numerical simulations that confirmed the proposed controller's effectiveness. The results are presented in graphical style, together with a time history (Figures 1-16). The summaries are as follows:

- Three controllers are found in active control design, and one controller is originated for the Backstepping design;

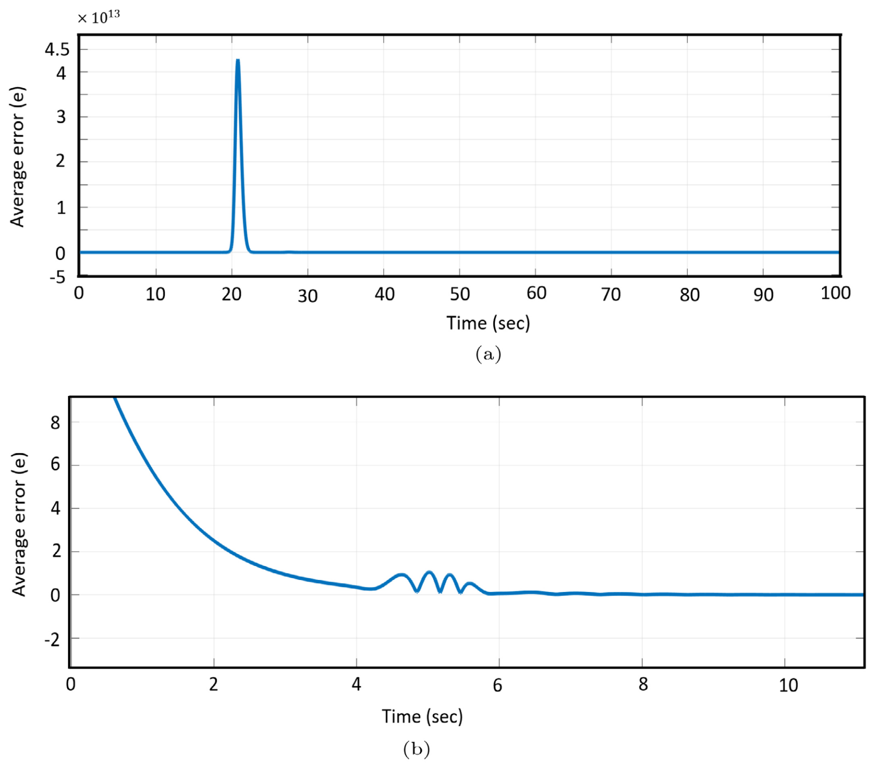

Figure 17. Comparison of synchronization times for ($0 \le t \le 100$) between (a) active control and (b) backstepping approach with controller activated. (a) Zoom of the time response of average error e in Active control; (b) Zoom of the time response of average error e in backstepping control.

- In two techniques, the error dynamics converge to zero as $t \to \infty$, hence the equilibrium point $(0,0,0)$ remains asymptotically stable;
- Similar activities are shown in both ways for the states of two chaotic systems (Master and Drive);
- The results reveal that the backstepping strategy converges to zero faster than the active control technique for momentary error dynamics.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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