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Energy Conservation in the Thin-Layer Approximation: V. The Surface Brightness in Supernova Remnants

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Abstract

Two new equations of motion for a supernova remnant (SNR) are derived in the framework of energy conservation for the thin-layer approximation. The first one is based on an inverse square law for the surrounding density and the second one on a non-cubic dependence of the swept mass. Under the assumption that the observed radio-flux scales as the flux of kinetic energy, two scaling laws are derived for the temporal evolution of the surface brightness of SNRs. The astrophysical applications cover two galactic samples of surface brightness and an extragalactic one.

Keywords

ISM: Supernova Remnants, Radio Continuum: Galaxies

1. Introduction

The surface brightness versus diameter, (Σ-D), for supernova remnants (SNRs) was initially analysed from a theoretical point of view in the framework of the initial conditions for the time evolution of SNRs [1] [2] [3] [4]. Some generic information on Σ-D can be found in reviews of SNRs [5] [6].

The astrophysical approach to the Σ-D has always mixed the observations with the theory and the statistics. We select some items among others: a catalog of 25 SNRs has been compiled by [7] with the conclusion that SNRs in the galactic halo have diameters greater than those in the disk, the evolutionary properties of SNRs have been deduced from observations with the Molonglo and Parkes radio telescopes [8], the Σ-D relationship was used to fix the scale for distances of SNRs [9], the diameters, luminosities, surface brightness, galactic heights for 231 SNRs were processed in the framework of the Sedov solution

[10], a new high-latitude SNR [11] was analysed, an updated radio Σ-D relationship was derived [12] and the phases of some supernova remnant have been determined from the observations [13]. In order to derive the Σ-D relationship, some hypotheses on the single equation of motion should be made, e.g., the conservation of the momentum [14]. Here conversely we will apply the energy conservation in the framework of the thin-layer approximation and we will derive an analytical expression for the Σ-D relationship. This paper derives two new equations of motion in the framework of energy conservation for the thin-layer approximation, see Section 2, applies the two analytical expressions for the surface brightness to two galactic samples, see Section 3, and to an extragalactic catalog, see Section 4.

2. The Equations of Motion

The conservation of kinetic energy in spherical coordinates in the framework of the thin-layer approximation, here taken to be an assumption, states that

$$\frac{1}{2}M_0(r_0)v_0^2 = \frac{1}{2}M(r)v^2, \tag{1}$$

where $M_0(r_0)$ and $M(r)$ are the swept masses at r_0 and r, while v_0 and v are the velocities of the thin layer at r_0 and r. We now present two equations of motion for SNRs and the back-reaction for one of the two.

2.1. The Inverse Square Law

The medium around the SN is assumed to scale as an inverse square law

$$\rho(r;r_0) = \begin{cases} \rho_c & \text{if } r \leq r_0 \\ \rho_c\left(\dfrac{r_0}{r}\right)^2 & \text{if } r > r_0 \end{cases} \tag{2}$$

where ρ_c is the density at $r = 0$ and r_0 is the radius after which the density starts to decrease. When the conservation of energy is applied, the velocity as a function of the radius is

$$v(r;r_0,v_0) = -\frac{\sqrt{-(2r_0-3r)r_0}\,v_0}{2r_0-3r}. \tag{3}$$

The trajectory, *i.e.* the radius as a function of time, is

$$r(t;t_0,r_0,v_0) = \frac{1}{6}\sqrt[3]{2}\sqrt[3]{r_0}\left((9t-9t_0)v_0+2r_0\right)^{2/3} + \frac{2}{3}r_0, \tag{4}$$

and the velocity as a function of time is

$$v(t;t_0,r_0,v_0) = \frac{\sqrt[3]{2}\sqrt[3]{r_0}\,v_0}{\sqrt[3]{(9t-9t_0)v_0+2r_0}}. \tag{5}$$

More details can be found in [15]. The rate of transfer of mechanical energy, L_m, is

$$L_m(t) = \frac{1}{2}\rho(t)4\pi r(t)^2 v(t)^3, \tag{6}$$

where $\rho(t)$, $r(t)$ and $v(t)$ are the instantaneous density, radius and velocity of the SN. We now assume that the density in front of the advancing expansion scales as

$$\rho(t) = \rho_0 \left(\frac{r_0}{r(t)} \right)^d,\tag{7}$$

where r_0 is the radius at t_0 and d is a parameter which allows matching the observations. The mechanical luminosity is now

$$L_m(t) = \frac{1}{2} \rho_0 \left(\frac{r_0}{r(t)} \right)^d 4\pi r(t)^2 v(t)^3.\tag{8}$$

In the case here analysed of the inverse square profile for density we have

$$L_m(t; v_0, t_0, r_0) = \frac{DLM}{81 v_0 t - 81 v_0 t_0 + 18 r_0},\tag{9}$$

where

$$DLM = \rho_0 6^d \left(r_0 \left(\sqrt[3]{2} \sqrt[3]{r_0} \left((9t - 9t_0) v_0 + 2r_0 \right)^{\frac{2}{3}} + 4r_0 \right)^{-1} \right)^d$$
$$\times \pi \left(\sqrt[3]{2} \sqrt[3]{r_0} \left((9t - 9t_0) v_0 + 2r_0 \right)^{\frac{2}{3}} + 4r_0 \right)^2 r_0 v_0^3.\tag{10}$$

We now assume that the observed luminosity, L_ν, in a given band denoted by the frequency ν is proportional to the mechanical luminosity

$$L_\nu(t) = cost * L_m(t),\tag{11}$$

where L_ν is the observed radio luminosity in a given band and $cost$ a constant which resolves the mismatch between theory and observations. The surface brightness is the luminosity divided by the interested area

$$\Sigma = \frac{L_\nu(t)}{\pi r(t)^2},\tag{12}$$

which is

$$\Sigma(t; v_0, t_0, r_0) = cost \frac{4 \rho_0 6^d \left(\dfrac{r_0}{\sqrt[3]{2} \sqrt[3]{r_0} \left((9t - 9t_0) v_0 + 2r_0 \right)^{2/3} + 4r_0} \right)^d r_0 v_0^3}{9 v_0 t - 9 v_0 t_0 + 2 r_0}.\tag{13}$$

2.2. Non-Cubic Dependence

The swept mass is assumed to scale as

$$M(r; r_0, \delta) = \begin{cases} M_0 & \text{if } r \le r_0 \\ M_0 \left(\dfrac{r}{r_0} \right)^\delta & \text{if } r > r_0 \end{cases}\tag{14}$$

where M_0 is the swept mass at $r = r_0$, and δ is a regulating parameter less

than 3. The differential equation of the first order which regulates the motion is obtained by inserting the above $M(r)$ in Equation (1)

$$\frac{dr(t;r_0,v_0,\delta)}{dt} = \frac{v_0}{\sqrt{r^\delta r_0^{-\delta}}}, \tag{15}$$

which has as the solution

$$r(t;r_0,v_0,\delta) = \exp\left(\frac{ER}{\delta+2}\right) \tag{16}$$

where

$$
\begin{aligned}
ER = \ln\Big(& r_0^{\delta+1}\delta t v_0 - r_0^{\delta+1}\delta t_0 v_0 + 2r_0^{\delta+1}t v_0 - 2r_0^{\delta+1}t_0 v_0 + r_0^{\delta+2} \\
& + \frac{r_0^\delta \delta^2 t^2 v_0^2}{4} - \frac{r_0^\delta \delta^2 t t_0 v_0^2}{2} + \frac{r_0^\delta \delta^2 t_0^2 v_0^2}{4} + r_0^\delta \delta t^2 v_0^2 - 2r_0^\delta \delta t t_0 v_0^2 \\
& + r_0^\delta \delta t_0^2 v_0^2 + r_0^\delta t^2 v_0^2 - 2r_0^\delta t t_0 v_0^2 + r_0^\delta t_0^2 v_0^2 \Big).
\end{aligned} \tag{17}
$$

The velocity is

$$v(t;r_0,v_0,\delta) = \frac{NV}{(\delta+2)(\delta t v_0 - \delta t_0 v_0 + 2t v_0 - 2t_0 v_0 + 2r_0)^2} \tag{18}$$

where

$$
\begin{aligned}
NV = v_0 \Big(& r_0^\delta \left(v_0(\delta+2)(t-t_0) + 2r_0\right)^2\Big)^{(\delta+2)^{-1}} \times \Big(24^{-(\delta+2)^{-1}} \delta^2 t v_0 \\
& - 24^{-(\delta+2)^{-1}} \delta^2 t_0 v_0 + 84^{-(\delta+2)^{-1}} \delta t v_0 - 84^{-(\delta+2)^{-1}} \delta t_0 v_0 \\
& + 4^{\frac{\delta+1}{\delta+2}} \delta r_0 + 84^{-(\delta+2)^{-1}} t v_0 - 84^{-(\delta+2)^{-1}} t_0 v_0 + 84^{-(\delta+2)^{-1}} r_0 \Big).
\end{aligned} \tag{19}
$$

Figure 1 reports the analytical solution for the NCD case as given by Equation (16) for SN 1993J.
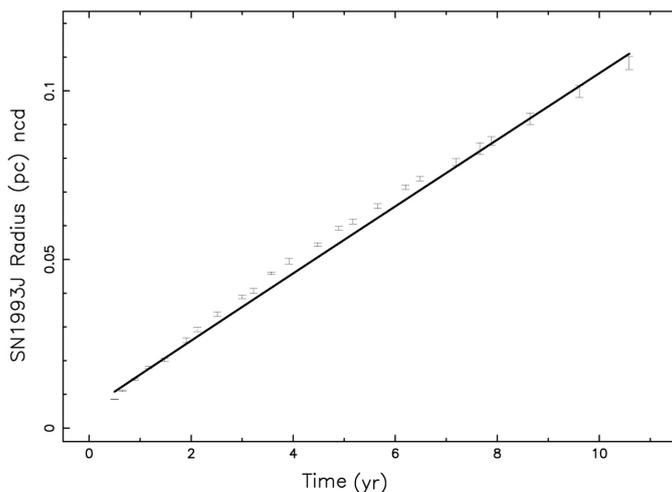
Figure 1. Analytical solution for the NCD case with parameters $r_0 = 0.006$ pc, $v_0 = 10000$ km/s, $t_0 = 0.026$ yr and $\delta = 0.2$, which gives $\chi^2 = 3751$. The astronomical data of SN 1993J are represented with vertical error bars.

The mechanical luminosity is assumed to scale as in Equation (8) and therefore in the NCD case is

$$L_m(t; r_0, v_0, t_0, \delta) = 16 \frac{e^{EA} v_0^3 \left(r_0 e^{EB} \right)^d \pi \rho_0}{(\delta+2)^3 \left(v_0(\delta+2)(t-t_0) + 2r_0 \right)^6} S_2^{3(\delta+2)^{-1}}$$

$$\times \left(S_1 4^{\frac{\delta+1}{\delta+2}} + 4^{-(\delta+2)^{-1}} v_0 \delta^2 (t-t_0) \right)^3, \quad (20)$$

where

$$EA = \frac{1}{\delta+2} \Big(-4\ln(2) + 2\ln\Big(4v_0(\delta+2)(t-t_0) r_0^{\delta+1} + 4r_0^{\delta+2}$$

$$+ \Big(\big(t^2 - 2tt_0 + t_0^2 \big) \delta^2 + \big(4t^2 - 8tt_0 + 4t_0^2 \big) \delta + 4t^2 - 8tt_0 + 4t_0^2 \big) v_0^2 r_0^{\delta} \Big) \Big), \quad (21)$$

and

$$EB = \frac{2\ln(2) - \ln\left(\left(v_0(\delta+2)(t-t_0) + 2r_0 \right)^2 r_0^{\delta} \right)}{\delta+2}, \quad (22)$$

$$S_1 = (\delta+1)(t-t_0)v_0 + \frac{r_0(\delta+2)}{2}, \quad (23)$$

and

$$S_2 = \left(v_0(\delta+2)(t-t_0) + 2r_0 \right)^2 r_0^{\delta}. \quad (24)$$

The surface brightness is derived according to Equation (12) and in the NCD case is

$$\Sigma(t; r_0, v_0, t_0, \delta)$$

$$= cost \frac{16\rho_0 \left(4^{-(\delta+2)^{-1}} v_0 \delta^2 t - 4^{-(\delta+2)^{-1}} v_0 \delta^2 t_0 + SC 4^{\frac{\delta+1}{\delta+2}} \right)^3 e^{SA} v_0^3 r_0^{\frac{(d+3)\delta+2d}{\delta+2}} SB^{\frac{-6-6\delta}{\delta+2}}}{(\delta+2)^3}, \quad (25)$$

where

$$SA = -\frac{d \left(-2\ln(2) + 2\ln\left(v_0(\delta+2)(t-t_0) + 2r_0 \right) + \delta \ln(r_0) \right)}{\delta+2}, \quad (26)$$

$$SB = v_0(\delta+2)(t-t_0) + 2r_0 \quad (27)$$

and

$$SC = (\delta+1)(t-t_0)v_0 + \frac{r_0(\delta+2)}{2}. \quad (28)$$

A comparison of the two models here implemented for the trajectory is reported in **Figure 2**.

2.3. Non-Cubic Dependence and Back Reaction

The radiative losses per unit length are assumed to be proportional to the flux of momentum as an assumption
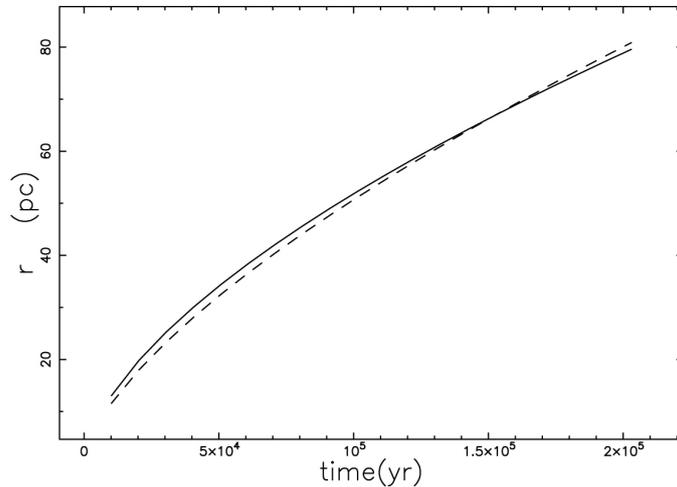
Figure 2. Analytical solution for the inverse square model (dashed line) when $r_0 = 1\,\text{pc}$, $v_0 = 4000\,\text{km/s}$, $t_0 = 10\,\text{yr}$, and NCD model (full line) with the same parameters and $\delta = 1.3$.

$$-\varepsilon \rho_s v^2 4\pi r^2, \tag{29}$$

where ε is a constant and ρ_s is the density in the thin advancing layer.

The volume, V, of the advancing layer is

$$V = 4\pi r^2 \Delta r \tag{30}$$

with $\Delta r = r/12$, therefore the above density for the advancing layer is

$$\rho_s = \frac{M(r; r_0, \delta)}{V}. \tag{31}$$

Inserting in the above equation the velocity to the first order as given by Equation (18) the radiative losses, $Q(r; r_0, v_0, \delta, \varepsilon)$, are

$$Q(r; r_0, v_0, \delta, \varepsilon) = -12 \frac{\varepsilon M_0 v_0^2}{r}. \tag{32}$$

The sum of the radiative losses between r_0 and r is given by the following integral, L,

$$L(r; r_0, v_0, \delta, \varepsilon) = \int_{r_0}^{r} Q(r; r_0, v_0, \delta, \varepsilon) \, \mathrm{d}r = -12\varepsilon M_0 v_0^2 \ln(r) + 12\varepsilon M_0 v_0^2 \ln(r_0). \tag{33}$$

The conservation of energy in the presence of the back reaction due to the radiative losses is

$$\frac{2\pi r_0^3 v^2}{3} \left(\frac{r}{r_0} \right)^{\delta} + 16\varepsilon \pi r_0^3 v_0^2 \ln(r) - 16\varepsilon \pi r_0^3 v_0^2 \ln(r_0) = \frac{2\pi r_0^3 v_0^2}{3}. \tag{34}$$

An analytical solution for the velocity to second order, $v_c(r; r_0, c_0, \delta, \varepsilon)$, is

$$v_c(r; r_0, v_0, \delta, \varepsilon) = r^{-\frac{\delta}{2}} r_0^{\frac{\delta}{2}} \sqrt{24\ln(r_0)\varepsilon - 24\ln(r)\varepsilon + 1} v_0. \tag{35}$$

The inclusion of the back reaction allows the evaluation of the SRS's maximum length, $r_{back}(r_0, \delta, \varepsilon)$, which can be derived by setting the above velocity

equal to zero.

$$r_{back}\left(r_0, \delta, \varepsilon\right) = \mathrm{e}^{\frac{24\ln(r_0)\varepsilon + 1}{24\varepsilon}}. \tag{36}$$

Figure 3 reports the finite radius of the advancing SNR as a function of ε.

3. Galactic Application

In the following we will process a sample of data x_i, y_i with i varying between 1 and N by a power law fit of the type

$$y(x) = Cx^\alpha, \tag{37}$$

where C and α are two constants to be found from the sample.

The *first* source for the observed Σ-*D* relationship for the SNRs of our galaxy can be found in Figure 3 of [10] which is now digitized, see **Figure 4**.

Table 1 reports the minimum diameter, $D_{\min}$, the average diameter, $\bar{D}$, the maximum diameter, $D_{\max}$, the minimum Σ-*D*, Σ-$D_{\min}$, the average Σ-*D*, $\overline{Σ\text{-}D}$ and the maximum Σ-*D*, Σ-$D_{\max}$ as well as the two parameters of the power law fit.

A *second* source for the Σ-D relationship is the Green's catalog [16] where the flux density in Jy at 1 GHz, S_1, and the mayor and minor angular size in arcmin, θ, are reported for 295 SNRs. The surface-brightness in SI is
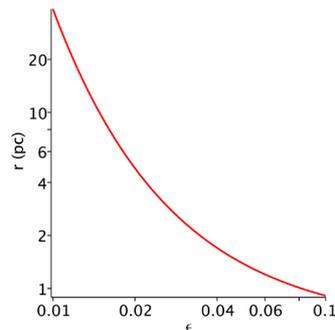
Figure 3. Length of the SNR, r_{back}, when $r_0 = 0.6\,\mathrm{pc}$ as function of ε.
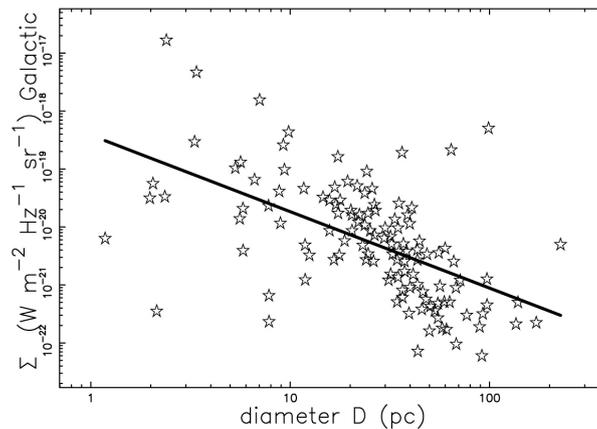
Figure 4. Observed Σ-*D* relationship (empty stars) and fitted power law (full line) with parameters as in **Table 1**.

Table 1. Statistics of the observed Σ-D galactic relationship and the two parameters of the power law fit. The theoretical parameters of the inverse square solution are $r_0 = 1\,\text{pc}$, $v_0 = 4000\,\text{km/s}$, $d = 1.1$, $t_0 = 10\,\text{yr}$, $t_{\min} = 2t_0$ and $t_{\max} = 2.05 \times 10^5\,\text{yr}$.

parameter	observed	theoretical
$D_{\min}\,(\text{pc})$	1.18	3.44
$\bar{D}\,(\text{pc})$	37.17	37.4
$D_{\max}\,(\text{pc})$	227.17	86.708
$\Sigma_{\min}\,(\text{pc})$	5.96×10^{-23}	3×10^{-20}
$\bar{\Sigma}\,(\text{pc})$	2.16×10^{-19}	2.24×10^{-19}
$\Sigma_{\max}\,(\text{pc})$	1.66×10^{-17}	2.54×10^{-19}
C	3.83×10^{-19}	1.32×10^{-17}
α	-1.32	-1.33

$$\Sigma = 1.181 \times 10^{-19}\, \frac{S_1}{\theta}\, \frac{\text{W}}{\text{m}^2 \cdot \text{Hz} \cdot \text{sr}} \quad \text{SI,} \tag{38}$$

but after [8] the radio astronomers use the following conversion

$$\Sigma = 1.505 \times 10^{-19}\, \frac{S_1}{\theta}\, \frac{\text{W}}{\text{m}^2 \cdot \text{Hz} \cdot \text{sr}} \quad \text{astronomy,} \tag{39}$$

which will also be adopted here.

The probability density function (PDF), $p(z)$, to have an SNR as a function of the galactic height z is characterized by an exponential PDF

$$p(z) = \frac{1}{b}\exp{-\frac{z}{b}}, \tag{40}$$

with $b = 83\,\text{pc}$ [10]. The linear diameter (D) of an SNR increases with the galactic height see Figure 2 in [10] and in the framework of the model with an inverse square law model for density the parameter r_0 is chosen to have the following dependence with the galactic height

$$r_0 = 0.01 + 0.5\left(\frac{z}{z_{\max}}\right), \tag{41}$$

where $z_{\max} = 687$ is the maximum galactic height which belongs to an SNR. The above assumption coupled with an initial velocity for the inverse square law model for density of $v_0 = 4000\,\text{km/s}$ for all the SNRs will ensure a longer radius for SNRs with higher galactic heights, see **Figure 5**.

We now simulate the galactic Σ-D relationship for a number of theoretical SNRs, N, equal to that observed according to the following rules

1) We randomly generate N parameters z according to the exponential PDF (40).

2) At each random value of z we associate an initial parameter r_0 according to the empirical Equation (41).

3) We randomly generate N times, t, according to the uniform distribution between a minimum value of time, t_{min} and a maximum value of time, t_{max}.

4) Given the parameters r_0, v_0, t_0 and t we evaluate the radius according to Equation (4) for an inverse power law profile for density. The diameter is obtained doubling the above result.

5) Σ is now generated according to Equation (13) once the regulating parameter d is provided.

The theoretical display of the results is reported in **Figure 6**, were we matched the three main parameters ($\bar{D}(\text{pc}), \overline{\Sigma\text{-}D}, \alpha$) which for the observations are $(37.17, 2.16 \times 10^{-19}, -1.32)$ and for our simulation are $(37.4, 2.24 \times 10^{-19}, -1.33)$.

The surface brightness of the Green's catalog is reported in **Figure 7** and our simulation in **Figure 8**; see also **Table 2** for a comparison between the data of the catalog and the simulation.

4. Extragalactic Application

A sample of SNRs in nearby galaxies with diameter in pc and flux at 1.4 GHz in mJy has been collected [17]. The connected catalog is available at http://cdsweb.u-strasbg.fr/. The statistics of the Σ-D relationship for this extragalactic sample is reported in **Table 3** and displayed in **Figure 9**.
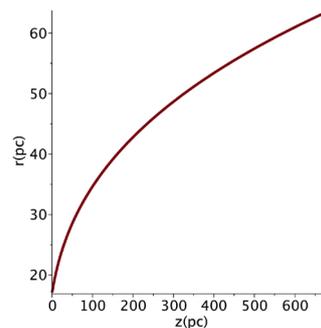
Figure 5. Theoretical radius as given by the inverse square model, see Equation (2), as function of the galactic height; $v_0 = 4000 \text{ km/s}$, $t_0 = 10 \text{ yr}$ and $t = 5 \times 10^4 \text{ yr}$.
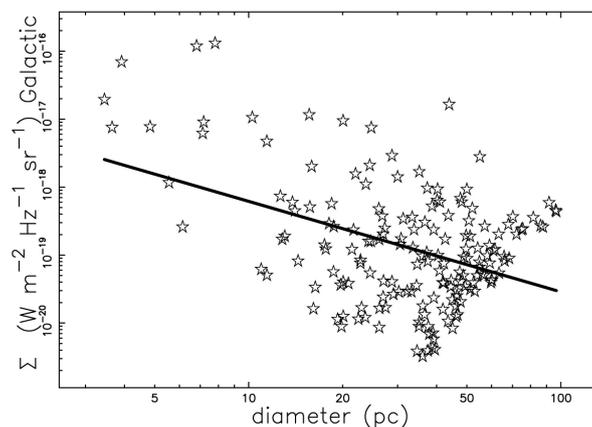
Figure 6. Theoretical Σ-D relationship (empty stars) and fitted power law (full line) with parameters as in **Table 1**.
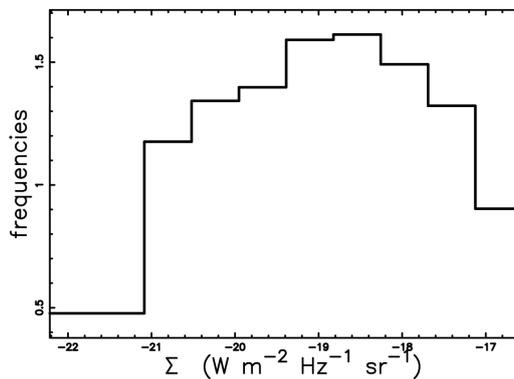
Figure 7. Histogram of the Σ-D for 294 SNR as derived from the Green's catalog [16].
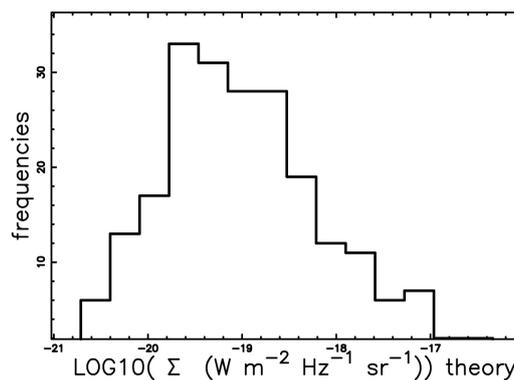
Figure 8. Histogram for the Σ-D of our simulation in the framework of the inverse square model.
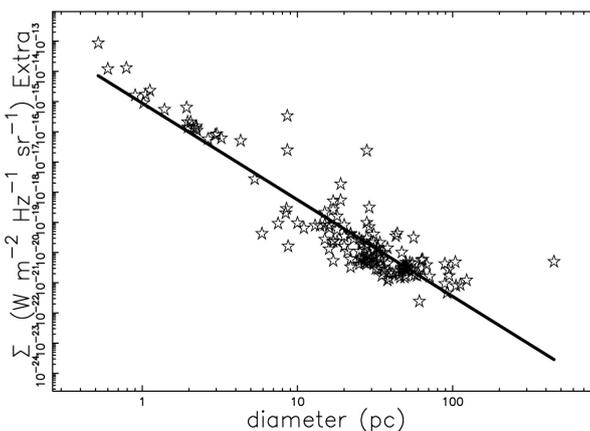
Figure 9. Observed Σ-D relationship for the extragalactic case (empty stars) and fitting power law (full line) with parameters as in **Table 3**.

The theoretical Σ-D is now evaluated in the framework of the NCD model, see Equation (25), with a numerical procedure which is similar to that of the galactic case but with the difference that there is no dependence of r_0 on z. The numerical value of r_0 in the extragalactic case is now randomly generated according to the uniform distribution between a minimum value, $r_{0,\min}$, and a maximum value, $r_{0,\max}$, see **Table 3** and **Figure 10**.
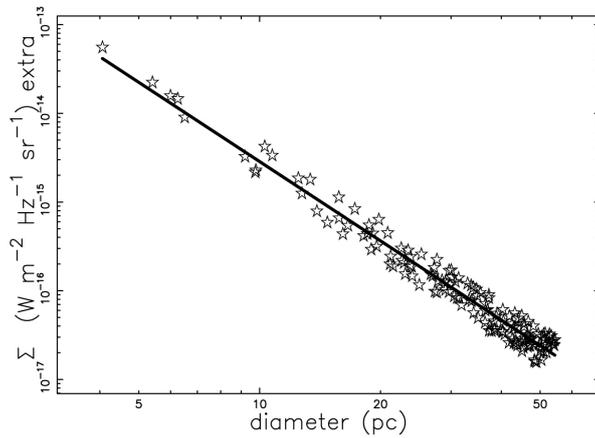
Figure 10. Theoretical Σ-*D* relationship for the extragalactic case in the framework of the NCD model (empty stars) and fitting power law (full line) with parameters as in **Table 3**.

Table 2. Statistics of the observed [16] and simulated Σ in the framework of the inverse square model.

parameter	observed	theoretical
Σ_{min} (pc)	1.42×10^{-25}	1.27×10^{-21}
$\overline{\Sigma}$ (pc)	1.13×10^{-18}	1.13×10^{-18}
Σ_{max} (pc)	4.34×10^{-17}	6.17×10^{-17}

Table 3. Statistics of the observed Σ-*D* extragalactic relationship and the two parameters of the power law fit. The theoretical parameters for the NCD case are $r_0 = 1\,\mathrm{pc}$, $r_{0,min} = 2r_0$, $r_{0,max} = 1.4r_{0,min}$, $v_0 = 4000\,\mathrm{km/s}$, $\delta = 1.3$, $d = 1.1$, $t_0 = 10\,\mathrm{yr}$, $t_{min} = 2t_0$ and $t_{max} = 2.9 \times 10^4\,\mathrm{yr}$.

parameter	observed	theoretical
D_{min} (pc)	0.51	4.05
$\overline{D}$ (pc)	34.6	34.23
D_{max} (pc)	450	54.39
Σ_{min} (pc)	2.4×10^{-22}	1.87×10^{-17}
$\overline{\Sigma}$ (pc)	6.13×10^{-16}	8.82×10^{-16}
Σ_{max} (pc)	8.6×10^{-14}	4.14×10^{-14}
C	8.85×10^{-16}	2.64×10^{-12}
α	-3.1	-2.96

5. Conclusions

When an analytical law of motion is available, a theoretical Σ-*D* relationship can be derived as a function of time. Here, in the framework of the energy conservation in the thin-layer approximation, we have derived one formula for Σ-*D* when the density of the surrounding ISM scales as an inverse square law, see Equation

(13), and another formula for the NCD case, see Equation (25). The two formulae allow simulating:

1) The galactic Σ-D relationship as given by the data of [10], see **Figure 6** and **Table 1**.

2) The galactic Σ-D relationship as given by a catalog of SNRs [16], see **Figure 8** and **Table 2**.

3) An extragalactic Σ-D catalog [17], see **Figure 10** and **Table 3**.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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