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Very Original Proofs of Two Famous Problems: "Are There Any Odd Perfect Numbers?" (Unsolved until to Date) and "Fermat's Last Theorem: A New Proof of Theorem (Less than One and a Half Pages) and Its Generalization"

Demetrius Chr. Poulkas[1,2,3,4]

[1]Department of Mathematics, Aristotle University of Thessaloniki, Thessaloniki, Greece
[2]School of Electrical and Computer Engineering, National Technical University of Athens, Athens, Greece
[3]School of the Faculty of Engineering, Aristotle University of Thessaloniki, Thessaloniki, Greece
[4]Department of Mechanical Engineering and Aeronautics, Faculty of Engineering, University of Patras, Patra, Greece
Email: dimpoulkas@gmail.com

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Abstract

This article presents very original and relatively brief or very brief proofs about of two famous problems: 1) Are there any odd perfect numbers? and 2) "Fermat's last theorem: A new proof of theorem and its generalization". They are achieved with elementary mathematics. This is why these proofs can be easily understood by any mathematician or anyone who knows basic mathematics. Note that, in both problems, proof by contradiction was used as a method of proof. The first of the two problems to date has not been resolved. Its proof is completely original and was not based on the work of other researchers. On the contrary, it was based on a simple observation that all natural divisors of a positive integer appear in pairs. The aim of the first work is to solve one of the unsolved, for many years, problems of the mathematics which belong to the field of number theory. I believe that if the present proof is recognized by the mathematical community, it may signal a different way of solving unsolved problems. For the second problem, it is very important the fact that it is generalized to an arbitrarily large number of variables. This generalization is essentially a new theorem in the field of the number theory. To the classical problem, two solutions are given, which are presented in the chronological order in which they were achieved. *Note that the second solution is very short and does not exceed one and a half pages*. This leads me to believe that Fermat, as a great mathematician was not lying and that he had probably solved the problem, as he stated in his historic its letter, with a correspondingly brief solution. *To win the bet on the question of whether Fermat was telling truth or lying, go immediately to the end of this article before the General Conclusions*.

Keywords

Perfect Numbers, Odd Perfect Numbers, Fermat's Last Theorem, Generalization of the Fermat's Last Theorem, Prime Number Problems, Millennium Problems

1. Introduction

1.1. Are There Odd Perfect Numbers?

In number theory, a perfect number is a positive integer that is equal to the sum of its positive divisors, excluding the number itself. For instance, 6 has divisors 1, 2 and 3 (excluding itself), and $\sigma(6)=1+2+3=6$, so 6 is a perfect number.

This definition is ancient appearing as early as Euclid's Element's (VII.22) where it is called perfect number. Around 300BC, Euclid proved a rule of their formation (IX.36) according to which the number $q(q+1)/2$ is an even perfect number, when for p prime the number q is a prime of the form 2^p-1, what is now called a Mersenne prime. Two millennia later, Euler proved that all the even perfect numbers are of this form: $2^{p-1}\left(2^p-1\right)$. This is known as the "Euclid-Euler" theorem.

The first four perfect numbers 6, 28, 496 and 8128 were the only ones known in the first Greek mathematics, and the mathematician Nikomachos announced 8128 as early as 100 AD [1]. Later, to this day, many other researchers have been able to add other numbers to the list of perfect numbers.

Most numbers are not perfect. Although there are a surprisingly large number of results in terms of their form, two very simple questions, as to their wording, remain unresolved to this day: 1) Are there odd perfect numbers? 2) Are there infinite perfect numbers? Both of these questions-problems seem so simple, but have resisted thousands of attempts to answer with proofs or counter-examples. The first problem is one of the subject of present work while the second may be the subject of a subsequent our research.

It is unknown whether there is any odd perfect number, although various results have been obtained. In 1496, Jacques Lefebvre stated that Euclid's rule gives all the perfect numbers, thus implying that no odd perfect number exists. Euler stated: "Whether (…) there are any odd perfect numbers is a most difficult question". More recently, Carl Pomerance presented a heuristic argument suggesting that indeed no odd perfect number should exist. All perfect numbers are also Ore's harmonic numbers, and it has been conjectured as well that there are no odds. Also, numbers up to 10^{1500} have been checked without success, making the existence of odd perfect numbers appear unlikely. According to date research, if any odd perfect numbers exist, it must satisfy the following conditions: $N > 10^{1500}$ [3], N is not divisible by 105 [4], N is of the form $N \equiv 1(\bmod 12)$, $N \equiv 117(\bmod 468)$ or $N \equiv 81(\bmod 324)$ [5]. Euler showed that an odd perfect number, if it exists, must be of the form: $N = q^{4n+1} \cdot Q^2$, where p is a prime of the form $4\lambda+1$, (Fermat's $4n+1$ theorem). Also, many similar results have derived by other researchers.

These few, in order to motivate the reader to search for more information related to the perfect numbers, so that he can understand easily the originality of the proof that is presented in this article compared to the ones of other previous researchers.

Also we did not use the old definition of a perfect number, as described at the beginning, but the definition of the full sum of its natural divisors, that given later. Defined as $\sigma(N)$ the sum of all its natural divisors (including N itself). Note that when the sum of its natural divisors includes the number itself, the definition of a perfect number is expressed as follows: a number is perfect if $\sigma(N) = 2N$.

So, if the number d is a natural divisor of the of the odd positive integer N and the number $\frac{2n+1}{d}$ is a natural divisor of the number N and constitute a pair of his natural divisors. Also, if m the multitude of all pairs of the natural divisors of number N, besides of the trivial pair, it's applies:

$$\sum_{i=1}^{i=m}\left(d_i + \frac{2n+1}{d_i}\right) = 2n$$

We can distinguish the following cases:

A) $m=1$ and $\left(d, \frac{2n+1}{d}\right)$ the only non trivial pairs of the natural divisors of N.

B) $m \geq 1$ and $\left(d_1, \frac{2n+1}{d_1}\right)$ one pair of the m non trivial pairs of the natural divisors of N, such that, the divisor d_1 to be the smallest of all the divisors.

1.2. Fermat's Last Theorem: A New Proof of Theorem and Its Generalization

The Fermat's last theorem (known historically by this title) has been an unsolved puzzle in mathematics for over three centuries. The theorem itself is a deceptively simple formulation in mathematics, while Fermat is discovered 30 years after his death famously stated that the problem had been solved around 1637. His claim was a clear statement on the sidelines of a book, but Fermat died without to leaving any proof of his claim. This claim eventually became one of the most famous unsolved problems of mathematics. Efforts to prove it, led to an increased interest in number theory and over time Fermat's latest theorem gained top spot as one of the most popular unsolved problems in mathematics

[6] [7].

I was informed of its existence before it was solved by Professor Andrew Wiles [8]. For several years I never tried to solve it. At some point twelve years ago, I thought I would can to solve it in a different way from Professor Andrew Wiles, believing that he might have a brief solution. One morning in my office, I started to analyze the problem and within a short time I invented the double inequality (1.5) and at that moment with great enthusiasm I exclaimed like Archimedes: I solved Fermat's last theorem!

One of the most important steps was the invention of the variable λ. I thought that a tracker should be found that runs through all natural numbers from one to infinity with the speed of light, performing instantaneous checks on all possible combinations that can verify the Equations (1.1) and (2.1). As will be seen below, this is perfectly achieved with the ratios:

$\frac{y}{\lambda}$ for the problem 3.1 or $\frac{x_{m-1}}{\lambda}$ for the problem 3.2.

The variable λ is inserted with replacements of number z with the sum $y+\lambda$ or $z = y+\lambda$ for the problem 3.1 and number x_m with the sum $x_{m-1}+\lambda$ or $x_m = x_{m-1}+\lambda$ for the problem 3.2, where λ it is positive integer. It should be noted that when the Equations (1.1) and (2.1) are verify, it applies that $\lambda \geq 1$. *The double inequalities* (1.5) *and* (2.5) *are the keys to the solutions presented in this article*.

We distinguish the following Cases A). $\frac{y}{\lambda} \leq n$ for the problem 3.1 or $\frac{x_{m-1}}{\lambda} \leq \frac{n}{m-2}$ for the problem 3.2 and B). $\frac{y}{\lambda} > n$ for the problem 3.1 or $\frac{x_{m-1}}{\lambda} > \frac{n}{m-2}$ for the problem 3.2.

The proofs presented in this article starts from a zero basis. In other words, I consider that Equations (1.1) and (2.1) when the exponent n equals the number 1 have infinite integer solutions (trivial) and of course this case is ignored, whereas when the exponent n is greater than number one or $n > 1$ it's examined for whom of the exponents Equations (1.1) and (2.1) can have integer solutions and for whom of the exponents do not have, (as is clear, is ignored even the same the Fermat's assumption).

2. Are There Any Odd Perfect Numbers? (*Unsolved until to Date*)

2.1. Proof of the Problem

Every odd positive integer N is given by the formula $N = 2n+1$, $n \in N$. All natural divisors of the number N are positive odd integers and appear in pairs.

Thus, if the number d is a natural divisor of the number N, then the number $\frac{2n+1}{d}$ is also a natural divisor of *the* number N. The pair of numbers 1 and $2n+1$ is the trivial pair, from all pairs of the natural divisors of number N.

Therefore, for any positive odd integer N the sum $\sigma(N)$ of its natural divisors is equal to:

$$\sigma(N)=1+2n+1+\sum_{i=1}^{i=m}\left(d_i+\frac{2n+1}{d_i}\right)=2n+2+\sum_{i=1}^{i=m}\left(d_i+\frac{2n+1}{d_i}\right) \tag{1}$$

where, m the multitude of pairs of the natural divisors of number N, besides of the trivial pair $(1,N)=(1,2n+1)$.

If the number $N=2n+1$, $n\in N$ is perfect then the sum $\sigma(N)$ of its natural divisors is equal to:

$$\sigma(N)=\sigma(2n+1)=2(2n+1)=4n+2 \tag{2}$$

By combining the Conditions (1) and (2) we have:

$$2n+2+\sum_{i=1}^{i=m}\left(d_i+\frac{2n+1}{d_i}\right)=4n+2 \Leftrightarrow \sum_{i=1}^{i=m}\left(d_i+\frac{2n+1}{d_i}\right)=2n \tag{3}$$

Therefore, in order to be perfect a positive odd integer it must to be satisfied Condition (3).

We distinguish the following cases:

A). $m=1$ and $\left(d,\frac{2n+1}{d}\right)$ the only non trivial pair of the natural divisors of N.

Given Condition (3) we have the following sub-cases:

1) If $d\neq\frac{2n+1}{d}$, then due to Condition (3): $d+\frac{2n+1}{d}=2n \Leftrightarrow$

$$d^2-2nd+(2n+1)=0 \Leftrightarrow d=n\pm\sqrt{n^2-2n-1}=0, \text{ where } n\geq 3,\ n\in N \tag{4}$$

In order the root d (as shown in the Condition (4)) to be a positive integer, it must to be a positive integer and the mathematical expression: $\sqrt{n^2-2n-1}$.

We will prove that this is not possible, because it applies the below condition: $(n-2)^2<n^2-2n-1<(n-1)^2$. Indeed assuming that:
$(n-2)^2<n^2-2n-1<(n-1)^2$, then equivalently we have:
$n^2-4n+4<n^2-2n-1<n^2-2n+1 \Leftrightarrow -2n+5<0<2$. This condition, because $n\geq 3$, $n\in N$ (due to Condition (4)), it's true.

Therefore, $\forall n\geq 3$, $n\in N$, it's true and that:

$$(n-2)^2<n^2-2n-1<(n-1)^2 \Leftrightarrow$$
$$n-2<\sqrt{n^2-2n-1}<n-1 \tag{5}$$

It follows from Condition (5) that mathematical expression: $\sqrt{n^2-2n-1}$ cannot be a positive integer because it is contained between the consecutive positive integers ($n-2$) and ($n-1$).

So, in this sub-case, Condition (3) is not satisfied, $\forall n\geq 3$, $n\in N$.

2) If $d=\frac{2n+1}{d}$ or $d=\sqrt{2n+1}$, due to Condition (3): $d+\frac{2n+1}{d}=2n \Leftrightarrow$
$\sqrt{2n+1}+\sqrt{2n+1}=2n \Leftrightarrow n^2-2n-1=0 \Leftrightarrow n(n-2)=1$, which it's absurd, because $n(n-2)>1$, $\forall n\geq 3$, $n\in N$.

Therefore, in both sub-cases of case A, Condition (3) is not satisfied, $\forall n \geq 3$, $n \in N$.

B). Generalized proof of the problem

First (1st) generalized proof: We consider that $m \geq 1$ and $\left(d_1, \frac{2n+1}{d_1}\right)$ a pair from the *m* non trivial pairs of the natural divisors of *N*, such that the divisor d_1 to be the smallest of all the divisors of *N*, which are included in these pairs.

Given Condition (3) we have:

$$d_1 + \frac{2n+1}{d_1} + \sum_{i=2}^{i=m}\left(d_i + \frac{2n+1}{d_i}\right) = 2n \tag{6}$$

Also, if we set:

$$\sum_{i=2}^{i=m}\left(d_i + \frac{2n+1}{d_i}\right) = 2k \text{ ,}^{(1*)} \text{ where } k \text{ appropriate integer, } k \geq 0 \tag{7}$$

(1*) From now on by the expression, "*where k appropriate integer*", we will mean that *k* is given by the mathematical formula: $k = \frac{1}{2}\left(\sum_{i=2}^{i=m}\left(d_i + \frac{2n+1}{d_i}\right)\right)$.

By combining the Conditions (6) and (7) we have:

$$d_1 + \frac{2n+1}{d_1} + 2k = 2n \Leftrightarrow d_1^2 - 2(n-k)d_1 + (2n+1) = 0 \Leftrightarrow$$

$$d_1 = (n-k) \pm \sqrt{(n-k)^2 - 2n - 1}, \quad n \geq 3, \quad n \in N \tag{8}$$

where *k* appropriate integer, $k \geq 0$.

In order the root d_1 (as shown in the Condition (8)) to be a positive integer, it must to be a positive integer and the mathematical expression: $\sqrt{(n-k)^2 - 2n - 1}$.

We will prove that this is not possible, because it applies the below condition: $(n-k-2)^2 < (n-k)^2 - 2n - 1 < (n-k-1)^2$. Indeed, assuming that:

$(n-k-2)^2 < (n-k)^2 - 2n - 1 < (n-k-1)^2$, then equivalently we have:

$$(n-k)^2 - 4(n-k) + 4 < (n-k)^2 - 2n - 1 < (n-k)^2 - 2(n-k) + 1 \Leftrightarrow$$

$$-2n + 4k + 5 < 0 < 2k + 2 \tag{9}$$

Condition (9) from the right is satisfied. In order the same condition to be satisfied and from the left it must to be true that:

$$-2n + 4k + 5 < 0 \tag{10}$$

So, assuming that: $-2n + 4k + 5 < 0$ and taking into account divisor d_1 as is defined above and since $d_1 + \frac{2n+1}{d_1} + 2k = 2n$, where *k* appropriate integer, $k \geq 0$, we have: $-2n + 4k + 5 < 0 \Leftrightarrow 2n - 2k > 2k + 5 \Leftrightarrow 2n > 2n - 2k > 2k + 5$. Given the Condition (7) and because $d_1 \geq 3$ or $3 + \frac{2n+1}{3} + 2k \geq d_1 + \frac{2n+1}{d_1} + 2k = 2n$ (2*), we have equivalently the following continuation:

$$3+\frac{2n+1}{3}+2k \geq d_1+\frac{2n+1}{d_1}+2k=2n>2n-2k>2k+5 \Rightarrow$$

$3+\frac{2n+1}{3}+2k>2k+5 \Leftrightarrow \frac{2n+1}{3}>2 \Leftrightarrow 2n>5 \Leftrightarrow n>\frac{5}{2}$ or $n \geq 3$ (3*), which it's true, since due to Condition (8) it applies that, $n \geq 3$, $n \in N$.

Because Condition (10) or $-2n+4k+5<0$ implies Condition $n \geq 3$ (which is true) and since Condition $n \geq 3$ as true implies that: $-2n+4k+5<0$ either $-2n+4k+5 \geq 0$, we will continue the proof further, after the paragraph (3*), assuming that the second of the two previous conditions, is true, or:

$$-2n+4k+5 \geq 0.$$

(2*) Let $d_1<d_2<\cdots<d_k<\cdots<d_m$, where the natural numbers d_i, $1 \leq i \leq m$ are the first members of the pairs $\left(d_i, \frac{2n+1}{d_i}\right)$, $1 \leq i \leq m$, of the natural divisors of the number *N*, *i.e.* of all pairs besides of the trivial pair $(1,2n+1)$ and it applies that: $d_1+\frac{2n+1}{d_1}>d_k+\frac{2n+1}{d_k}$, $k \geq 2$, then we have: $d_1+\frac{2n+1}{d_1}>d_k+\frac{2n+1}{d_k} \Leftrightarrow (d_k-d_1)\left(\frac{2n+1}{d_1 d_k}-1\right)>0$, because $\frac{2n+1}{d_1 d_k}>1$, since $k \geq 2$ (see paragraph (4*)). From the last condition, equivalently we have: $(d_k-d_1)>0 \Leftrightarrow d_k>d_1$, which it's true, therefore it's true and that: $d_1+\frac{2n+1}{d_1}>d_k+\frac{2n+1}{d_k}$.

(3*) Condition $n \geq 3$ can be reached and as follows: Assuming that $-2n+4k+5<0$ and because $k \geq 0$, we have:

$$-2n+4k+5<0 \Leftrightarrow 0 \leq 4k<2n-5 \Rightarrow 0<2n-5 \Leftrightarrow 2n>5 \Leftrightarrow$$

$n>\frac{5}{2}$ or $n \geq 3$ which it's true, since due to Condition (8) it applies that $n \geq 3$, $n \in N$. Although this way is easier we preferred the previous way, which it is more extensive we to show that the choice, the divisor d_1, to be the smallest of all the non-trivial divisors of *N* is correct. As is easily understood, from the previous analysis and especially from the Condition (3), the smaller non-trivial divisor d_1 leads to the smallest possible value of *k*, which increases the probability the condition $-2n+4k+5<0$ to be, true (just as we would like).

Next assuming that: $-2n+4k+5 \geq 0$, where k appropriate integer, $k \geq 0$ and $n \geq 3$, $n \in N$, we have:

$$\{-2n+4k+5 \geq 0 \text{ and } n \geq 3 \text{ or } n>\frac{5}{2}\} \Leftrightarrow \{4k \geq 2n-5 \text{ and } n>\frac{5}{2}\} \Rightarrow$$

$4k \geq 2n-5>2\frac{5}{2}-5=0$ or $k>0$, which it's false, since $k \geq 0$.

Because there may be objections from some to this result, since the value $k=0$ is one while the distinct positive values of *k*, i.e. $k_1,k_2,\cdots,k_\infty$ are infinitely many we will consider that this result it's an indication that the Condition $-2n+4k+5 \geq 0$ is false and we will continue further our proof for $k>0$. So, given and Condition (3), we have: $-2n+4k+5\ge 0 \Leftrightarrow$

$$-2n+4\times\frac{1}{2}\times\left(2n-d_1-\frac{2n+1}{d_1}\right)+5\ge 0 \Leftrightarrow -2n+4n-2d_1-\frac{2(2n+1)}{d_1}+5\ge 0 \Leftrightarrow$$

$$2d_1^2-(2n+5)d_1+2(2n+1)\le 0,\ n\ge 31^{(4*)} \tag{11}$$

(4*) The positive odd numbers 9, 15, 21, 25, 27, 33, 35, 39, 45, 49, 51, 55, 57 are composite positive integers and all of them are less than the number 63 and have only a pair of non trivial natural divisors and (for all) it applies that: $m=1$ and $k=0$. The first odd positive integer which has more of one pairs of non trivial natural divisors is the number 63. For this odd positive integer number it applies that: $n=31$, $m=2$ and $k=\frac{1}{2}(7+9)=8$.

Now, we consider Inequality $2x^2-(2n+5)x+2(2n+1)\le 0$, where $x\in R$, $x\ge 2$, $n\ge 31$: Also, given equation $2x^2-(2n+5)x+2(2n+1)=0$, the following applies:

$D=\left[-(2n+5)\right]^2-4\times 2\times\left[2(2n+1)\right]=4n^2-12n+9=(2n-3)^2>0$ and so its roots are: $x_1=2$ and $x_2=n+\frac{1}{2}$.

Therefore, if where x we put d_1 and because d_1 takes values within the roots or $x_1=2<3\le d_1<\frac{2n+1}{3}<x_2=n+\frac{1}{2}$, then it applies that:

$2d_1^2-(2n+5)d_1+2(2n+1)<0$ and so Condition (11) is not satisfied. While, if we consider that $x_1=2\le d_1<\frac{2n+1}{3}<x_2=n+\frac{1}{2}$, then it applies that:

$2d_1^2-(2n+5)d_1+2(2n+1)\le 0$ and so Condition (11) is satisfied. But that's absurd because we consider that $d_1\ge 2$, while in fact it applies that: $d_1\ge 3$.

Therefore, Condition $-2n+4k+5\ge 0$ is false and it's true that:

$-2n+4k+5<0$ or $-2n+4k+5<0<2k+2$, where $n\ge 3$, $n\in N$ and k appropriate integer, $k\ge 0$.

So, consequently it applies that:

$$(n-k-2)^2<(n-k)^2-2n-1<(n-k-1)^2 \Leftrightarrow$$

$$n-k-2<\sqrt{(n-k)^2-2n-1}<n-k-1 \tag{12}$$

It follows from the Condition (12) that mathematical expression: $\sqrt{(n-k)^2-2n-1}$ cannot be a positive integer because it's contained between the consecutive integers $(n-k-2)$ and $(n-k-1)$.

Therefore, in the case B, Condition (8) and consequently Condition (3), is not satisfied $\forall n\ge 3$, $n\in N$.

Second (2^{nd}) generalized proof (continues from Condition (9) onwards):

The second generalized proof is presented here, because it's also original and undoubtedly confirms the correctness of the first proof.

According to Equation (8), the divisor d_1 to be a positive integer, the mathematical expression $(n-k)^2-2n-1$ must be perfect square of positive integer number. We remind you here, that k is an appropriate integer, where $k \geq 0$.

At first, we assume that: $(n-k)^2 - 2n - 1 = l^2$, where l a positive integer. For this equation to make sense, in conjunction with Equation (8), it must to be: a) the difference $(n-k)$ even integer and the l odd integer or b) the number l even integer and the difference $(n-k)$ odd integer.

a) If $n-k=2\rho$ and $l=2\mu+1$, where ρ,μ positive integers, then we have:

$$(n-k)^2 - 2n - 1 = l^2 \Leftrightarrow (2\rho)^2 - 2n - 1 = (2\mu+1)^2 \Leftrightarrow$$

$$4\rho^2 = 2n + 1 + 4\mu^2 + 4\mu + 1 \Leftrightarrow 4\rho^2 = 2(n+1) + 4\mu(\mu+1).$$

Because the number 4 divides the numbers $4\rho^2$ that is the first member of the last equality and $4\mu(\mu+1)$ which is term of the second member, it must also to divides and the number $2(n+1)$ which also is term of the same member. Therefore it should be valid that: $2(n+1) = 4\lambda$, where λ a positive integer and consequently: $n+1=2\lambda \Leftrightarrow n=2\lambda-1$.

Also, because $n-k=2\rho \Leftrightarrow n=2\rho+k$, so finally we have:

$n=2\rho+k=2\lambda-1$ or $2\rho+k=2\lambda-1$. From the last equality, if $k=0$ then $2\rho=2\lambda-1$, which it's absurd, because an even integer cannot be equal to an odd integer. While, if $k>0$ then $2\rho+k=2\lambda-1 \Leftrightarrow k=2(\lambda-\rho)-1$, that is, the number k is an odd positive integer.

b) If $n-k=2\rho+1$ and $l=2\mu$, where ρ,μ positive integers, then we have:

$$(n-k)^2 - 2n - 1 = l^2 \text{ then } (2\rho+1)^2 - 2n - 1 = (2\mu)^2 \Leftrightarrow$$

$$4\rho^2 + 4\rho + 1 - 2n - 1 = 4\mu^2 \Leftrightarrow 4\rho(\rho+1) = 2n + 4\mu^2.$$

Because the number 4 divides the numbers $4\rho(\rho+1)$ that is the first member of the last equality and $4\mu^2$ which is term of the second member, it must also to divides and the number $2n$ which also is term of the same member. Therefore it should be valid that: $2n=4\lambda$, where λ a positive integer and consequently $n=2\lambda$. Also, because $n-k=2\rho+1 \Leftrightarrow n=2\rho+k+1$, so finally we have: $n=2\rho+k+1=2\lambda$ or $2\rho+k+1=2\lambda$.

From the last equality, if $k=0$ then $2\rho+1=2\lambda$ which it's absurd, because an odd integer cannot be equal to an even integer. While, if $k>0$ then $2\rho+k+1=2\lambda \Leftrightarrow k=2(\lambda-\rho)-1$, *i.e.* the number k is again an odd positive integer.

Now, we will prove that the fact that k is always an odd integer leads us in absurd, as follows:

1) If $n=2n'+1$ and $k=2k'+1$, where n',k' positive integers, then we have:

$$(n-k)^2 - 2n - 1 = (2n'+1-2k'-1)^2 - 2(2n'+1) - 1 = 4(n'-k')^2 - 4n' - 3.$$

Because, according to our hypothesis, the mathematical expression

$(n-k)^2-2n-1$ is perfect square of a positive integer, then the mathematical expression $4(n'-k')^2-4n'-3$ which is its equivalent, as the difference $n-k$ is an even number, must be perfect square of a positive odd integer. Therefore, the following equality will apply: $4(n'-k')^2-4n'-3=(2\mu+1)^2$, where μ a positive integer. From the last condition equivalently, we have:

$$4(n'-k')^2-4n'-3=4\mu^2+4\mu+1 \Leftrightarrow (n'-k')^2-n'-1=\mu^2+\mu \Leftrightarrow$$

$$(n'-k')^2=\mu^2+\mu+1+n' \tag{13}$$

We will prove that Equality (13) is not possible. For this, we work as follows:

- If $\mu=n'$ then we have: $(n'-k')^2=(n')^2+n'+1+n' \Leftrightarrow$ $(n'-k')^2=(n'+1)^2 \Leftrightarrow -k'=1$ or $k'=-1$, which is absurd, since $k'>0$. So, it is $\mu\neq n'$.
- If $\mu>n'$ then we have: $(n'-k')^2>(n')^2+n'+1+n' \Leftrightarrow$ $(n'-k')^2>(n'+1)^2 \Leftrightarrow -k'>1$ or $k'<-1$, which is absurd, since $k'>0$. Therefore will be: $\mu<n'$.

Let a positive integer such that $\mu=n'-a$, then by substitution in the Equation (13) we have:

$(n'-k')^2=\mu^2+\mu+1+n'=(n'-a)^2+(n'-a)+1+n' \Leftrightarrow$
$(n')^2-(2k')n'+(k')^2=(n')^2-(2a)n'+a^2+n'-\alpha+1+n' \Leftrightarrow$
$(n')^2-(2k')n'+(k')^2=(n')^2-2(a-1)n'+(a^2-a+1)$. From the last equality of the two polynomials, if we equate the coefficients of the terms with the same degree, we have:

$$1=1,\ -2k'=-2(a-1) \text{ and } (k')^2=a^2-a+1.$$

From the second equation it arises that $\alpha=k'+1$, then by substitution in the third equation, we have:

$(k')^2=a^2-a+1 \Leftrightarrow (k')^2=(k'+1)^2-(k'+1)+1 \Leftrightarrow$
$(k')^2=(k')^2+2k'+1-k'-1+1 \Leftrightarrow k'+1=0$, which is absurd because $k'+1>0$, since $k'>0$.

Therefore, the equation: $(n'-k')^2=\mu^2+\mu+1+n'$, is not applies and consequently it applies that: $4(n'-k')^2-4n'-3\neq(2\mu+1)^2=l^2$.

This means that: $(n-k)^2-2n-1\neq l^2$, which is absurd because it contradicts our original assumption that: $(n-k)^2-2n-1=l^2$.

2) If $n=2n'$ and $k=2k'+1$, where n',k' positive integers, then we have:

$(n-k)^2-2n-1=(2n'-2k'-1)^2-2(2n')-1=(2(n'-k')-1)^2-4n'-1 \Leftrightarrow$
$(n-k)^2-2n-1=4(n'-k')^2-4(n'-k')+1-4n'-1=4(n'-k')^2-8n'+4k'$ or
$(n-k)^2-2n-1=4\left((n'-k')^2-2n'+k'\right)$. Because the mathematical expression $(n-k)^2-2n-1$ is perfect square of a positive integer, then the mathematical expression $4\left((n'-k')^2-2n'+k'\right)$ which is its equivalent, as the difference $(n-k)$ is an odd number, must be perfect square of a positive even integer. This means that and the mathematical expression: $(n'-k')^2-2n'+k'$, must be perfect square of a positive integer. We will prove that this is not possible, because it valid the below condition:

$$(n'-k'-2)^2 < (n'-k')^2 - 2n' + k' < (n'-k'-1)^2 .$$

Indeed, assuming that: $(n'-k'-2)^2 < (n'-k')^2 - 2n' + k' < (n'-k'-1)^2$, then equivalently we have:

$$(n'-k')^2 - 4(n'-k') + 4 < (n'-k')^2 - 2n' + k' < (n'-k')^2 - 2(n'-k') + 1 \Leftrightarrow$$

$$-2n' + 3k' + 4 < 0 < k' + 1 \tag{14}$$

Condition (14), from the right is satisfied. We will examine whether the same condition is satisfied and by the left.

At the first we observe that: $-2n + 4k + 5 \geq 0$, because according to our hypothesis it holds that $(n-k)^2 - 2n - 1 = l^2$, so condition (9) is not satisfied. Note, that if $-2n + 4k + 5 < 0$ Condition (9) is satisfied and consequently it applies that: $(n-k-2)^2 < (n-k)^2 - 2n - 1 < (n-k-1)^2$ and so the mathematical expression: $\sqrt{(n-k)^2 - 2n - 1}$ cannot be a positive integer, because it's contained between the consecutive integers $(n-k-2)$ and $(n-k-1)$. Therefore, if $-2n + 4k + 5 < 0$ it applies that $(n-k)^2 - 2n - 1 \neq l^2$, which is contrary to our hypothesis.

Then, given that $-2n + 4k + 5 \geq 0$, we have:

$-2n + 4k + 5 \geq 0 \Leftrightarrow -2(2n') + 4(2\kappa' + 1) + 5 \geq 0 \Leftrightarrow -4n' + 8\kappa' + 9 \geq 0 \Leftrightarrow$
$2 \times \left(-2n' + 3\kappa' + 4 + \kappa' + \frac{1}{2}\right) \geq 0$ or $-2n' + 3\kappa' + 4 + \kappa' + \frac{1}{2} \geq 0$.

From the last condition if $-2n' + 3k' + 4 < 0$, equivalently we have:

$0 + \kappa' + \frac{1}{2} > -2n' + 3\kappa' + 4 + \kappa' + \frac{1}{2} \geq 0$ or $\kappa' + \frac{1}{2} > 0$, which is true. While on the contrary if $-2n' + 3k' + 4 \geq 0$, we have:

$-2n' + 3\kappa' + 4 + \kappa' + \frac{1}{2} \geq 0 + \kappa' + \frac{1}{2} > 0$ or $-2n' + 3\kappa' + 4 + \kappa' + \frac{1}{2} > 0$, which is false, since $-2n' + 3\kappa' + 4 + \kappa' + \frac{1}{2} \geq 0$. So it's true that $-2n' + 3k' + 4 < 0$.

Therefore, Condition (14) or $-2n' + 3k' + 4 < 0 < k' + 1$ it's true and consequently it applies that:

$(n'-k'-2)^2 < (n'-k')^2 - 2n' + k' < (n'-k'-1)^2 \Leftrightarrow$
$n'-k'-2 < \sqrt{(n'-k')^2 - 2n' + k'} < n'-k'-1$. It follows from the last condition that mathematical expression: $\sqrt{(n'-k')^2 - 2n' + k'}$ cannot be a positive integer because it's contained between the consecutive integers $(n'-k'-2)$ and $(n'-k'-1)$.

So, it applies that: $2^2\left((n'-k')^2 - 2n' + k'\right) \neq (2\mu)^2 = l^2$. This means that:

$(n-k)^2 - 2n - 1 \neq l^2$, which is absurd because it contradicts our original assumption that: $(n-k)^2 - 2n - 1 = l^2$.

Therefore, in all cases it applies that: $(n-k)^2 - 2n - 1 \neq l^2$, so Condition (8) and consequently Condition (3) is not satisfied, $\forall n \geq 3$, $n \in N$.

Important comment (a posteriori that is after the completion of the proof):

The fact that the condition $-2n + 4k + 5 \geq 0$ is false is also substantiated as follows:

a) If $k = 0$ and $-2n + 4k + 5 \geq 0$, since $n \geq 3$, we have:

$4k \geq 2n-5 \geq 2\times 3-5=1$ or $4\times 0=0\geq 1$, which is absurd.

b) If $k>0$ and $-2n+4k+5\geq 0$, because $n\geq 31$ and $k\geq 8$ (see paragraph (4*) a little above), then $4k\geq 2n-5\geq 2\times 31-5=57$ or $4k\geq 57$. So, since $k\geq 8$, it applies that: $4k\geq 4\times 8=32\geq 57$, which is absurd.

Therefore, in all the cases, condition $-2n+4k+5\geq 0$ is false and it's true that: $-2n+4k+5<0$.

2.2. Comments-Remarks

1) According to the above when, $d=\frac{2n+1}{d}$ or $d^2=2n+1$, in the sum of non trivial pairs of the natural divisors of *N*, divisor *d* is calculated twice. If *d*, is calculated once, we will have:

i) If $m=1$ and $\left(d,\frac{2n+1}{d}\right)$ the only non trivial pair of the natural divisors of N and $\frac{2n+1}{d}=d$, then due to Condition (3): $d=2n$. That is absurd because an odd integer cannot be equal to an even integer (since number d, is odd integer).

ii) If $m>1$ and $\left(d_\lambda,\frac{2n+1}{d_\lambda}\right)$ one pair of the m non trivial pairs of divisors of N and $\frac{2n+1}{d_\lambda}=d_\lambda$, then due to Condition (3): $d_\lambda+\sum_{i=1,i\neq\lambda}^{i=m}\left(d_i+\frac{2n+1}{d_i}\right)=2n$.

Given Condition (7) equivalently, we have: $d_\lambda+2k'=2n$, where,

$\sum_{i=1,i\neq\lambda}^{i=m}\left(d_i+\frac{2n+1}{d_i}\right)=2k'$ and k' appropriate integer, $k'>0$ or $d_\lambda=2(n-k')$.

That is absurd because an odd integer cannot be equal to an even integer (since number d_λ, is odd integer).

2) Condition (5) has proved for $n\geq 3$. So, if $n=1$ or $n=2$ we have:

For $n=1$, $N=2\times 1+1=3$ and $\sigma(3)=2\times 3=6\neq 4=1+3$

For $n=2$, $N=2\times 2+1=5$ and $\sigma(5)=2\times 5=10\neq 6=1+5$

3) If $n=0$, $N=2\times 0+1=1$, the number 1 can be considered as an odd perfect number if in the sum $\sigma(1)$, divisor 1is calculated twice. So, we have: $\sigma(1)=2\times 1=2=1+1$.

Conclusion: From the above it is concluded that there are not positive odd perfect numbers. Maybe only the number 1, if in the sum $\sigma(1)$, divisor 1 is calculated twice, can be considered as the only odd perfect number.

3. "Fermat's Last Theorem: A New Proof of Theorem and Its Generalization"

3.1. A New Proof of Fermat's Last Theorem (Classical Problem)

If x,y,z are positive integers that differ from each other then the following equation:

$$x^n + y^n = z^n, \text{ where } n \in N, \; n > 1 \tag{1.1}$$

when $n \geq 3$ have no positive integer solutions.

Proof of Theorem

We consider positive integers x, y, z that differ from each other and assume that they verify the Equation (1.1) for a natural number $n > 1$. Also, we assume without loss of the generality, that:

$$x < y < z \Leftrightarrow x^n < y^n < z^n \tag{1.2}$$

Given Equation (1.1) and condition (1.2), based on the above hypothesis, we have:

$$x^n + x^n < x^n + y^n = z^n \Leftrightarrow 2x^n < z^n \Leftrightarrow 2 < \left(\frac{z}{x}\right)^n \tag{1.3}$$

$$y^n + y^n > x^n + y^n = z^n \Leftrightarrow 2y^n > z^n \Leftrightarrow \left(\frac{z}{y}\right)^n < 2 \tag{1.4}$$

By combining the conditions (1.3) and (1.4) we have:

$$\left(\frac{z}{y}\right)^n < 2 < \left(\frac{z}{x}\right)^n \tag{1.5}$$

Note: The double inequality (1.5) *is necessary but not sufficient*, *i.e.* the converse is not always the case. For example, we consider that $x = 3, y = 4, z = 5$ and $n = 3$. We have:

$$\left(\frac{5}{4}\right)^3 \cong 1.96 < 2 < 4.63 \cong \left(\frac{5}{3}\right)^3 \text{ and } 3^3 + 4^3 = 91 \neq 125 = 5^3.$$

If we replace the number z with the sum $y + \lambda$ or $z = y + \lambda$, where λ is a positive integer, then for the positive integers x, y, z that according to hypothesis we originally made, verify Equation (1.1) for a natural number $n > 1$, it is true that:

$$x^n + y^n = z^n = (y + \lambda)^n \text{ or } \left(\frac{z}{y}\right)^n = \left(1 + \frac{\lambda}{y}\right)^n \tag{1.6}$$

$$(1.6) \Leftrightarrow x^n + y^n = z^n = (y + \lambda)^n = y^n + ny^{n-1}\lambda + \cdots + ny\lambda^{n-1} + \lambda^n \Leftrightarrow$$

$$x^n - \lambda^n = ny^{n-1}\lambda + \cdots + ny\lambda^{n-1} > 0 \Rightarrow x^n > \lambda^n \text{ or } x > \lambda \tag{1.7}$$

Given Condition (1.7) and the original hypothesis it's true that:

$$1 \leq \lambda < x < y < z \tag{1.8}$$

We distinguish the following cases:

A) $\dfrac{y}{\lambda} \leq n$

If $n \geq \dfrac{y}{\lambda} \Leftrightarrow \dfrac{1}{n} \leq \dfrac{\lambda}{y} \Leftrightarrow 1 + \dfrac{1}{n} \leq 1 + \dfrac{\lambda}{y} \Leftrightarrow \left(1 + \dfrac{1}{n}\right)^n \leq \left(1 + \dfrac{\lambda}{y}\right)^n$. Due to (1.6)

$$\left(1 + \frac{1}{n}\right)^n \leq \left(1 + \frac{\lambda}{y}\right)^n = \left(\frac{z}{y}\right)^n \tag{1.9}$$

Considering Bernoulli's inequality for $n>1$, it applies:

$$\left(1+\frac{1}{n}\right)^n > 1+n\frac{1}{n} = 2 \tag{1.10}$$

By combining the Conditions (1.9) and (1.10) we have:

$$\left(\frac{z}{y}\right)^n \geq \left(1+\frac{1}{n}\right)^n > 2 \tag{1.11}$$

Due to condition (1.11) the double inequality (1.5) is not satisfied and therefore in this case Equation (1.1) has no positive integer solutions, for any natural number n greater than the number one or $\forall n>1$.

B) $\frac{y}{\lambda} > n$

Since in case A. Equation (1.1) doesn't have positive integer solutions, obviously if they exist, this will be the case in case B when condition

$\frac{y}{\lambda} > n$ is applied. So, we have:

$$\frac{y}{\lambda} > n \Leftrightarrow y > \lambda n \text{ or } \frac{\lambda}{y} < \frac{1}{n} \tag{1.12}$$

Now, we will prove that when positive integers x, y, z verify Equation (1.1) for a natural number $n>1$, number y is greater than the product of number 3 by the number λ or $y > 3\lambda$.

We work as follows:

a) If $1<n<3$ or $n=2$, assuming that $y \leq 3\lambda$, we distinguish two areas: $3 < y \leq 3\lambda$ and $y \leq 3 = 3\lambda, \lambda = 1$. Thus, given the latter areas, we have.

a.1) If $3 < y \leq 3\lambda$, assuming that the trinity x, y, z verify Equation (1.1) for a natural number $n>1$, then because from $3<y \Leftrightarrow y \geq 4$ and from $3 < 3\lambda \Leftrightarrow 1 < \lambda$ or $\lambda \geq 2$, it applies the following:

Since $y \leq 3\lambda$ or $\frac{\lambda}{y} \geq \frac{1}{3} \Leftrightarrow \frac{\lambda}{y} \geq \frac{2}{y} \geq \frac{1}{3}$ or $\frac{2}{y} \geq \frac{1}{3} \Leftrightarrow y \leq 6$. Now, assuming that the trinity $x, y, y+\lambda$ verify Equation (1.1) for a natural number $n>1$, such that $4 \leq y \leq 6$ and $\lambda \geq 2$, we have: $x^2 + y^2 = (y+\lambda)^2 \Leftrightarrow$ $x^2 + y^2 = (y+\lambda)^2 \geq (y+2)^2 \Leftrightarrow x^2 + y^2 \geq y^2 + 4y + 4 \Leftrightarrow x^2 \geq 4y + 4 \Leftrightarrow$ $(y-1)^2 \geq x^2 \geq 4y+4 \Leftrightarrow y^2 - 6y - 3 \geq 0$ or $y(y-6) - 3 \geq 0$, which is absurd, since if $4 \leq y \leq 6$ it is $y(y-6) - 3 < 0$. So, it applies that $x^2 + y^2 \neq (y+\lambda)^2$, $\forall y \geq 4$. Thus, the area $3 < y \leq 3\lambda$ is not valid and therefore condition $y \leq 3\lambda$ is false.

a.2) If $y \leq 3 = 3\lambda, \lambda = 1$, assuming also that the integers $x, y, y+1$, verify Equation (1.1), it applies that: $(y+1)^2 = y^2 + x^2 \Leftrightarrow (y+1)^2 - y^2 = x^2 \Leftrightarrow 2y + 1 = x^2$. This means that the number x is an odd integer and because due to Condition (1.8) it applies that: $1 \leq \lambda < x < y < z$, it follows that $x > \lambda = 1$. So, $x = 3$ and in this sub-case no trinity $x, y, y+1$ that satisfying Equation (1.1) can be formed. Thus, the area $y \leq 3 = 3\lambda, \lambda = 1$ is not valid and therefore condition $y \leq 3\lambda$ is false.

b) If $n \geq 3$, assuming again that $y \leq 3\lambda$ or $\frac{\lambda}{y} \geq \frac{1}{3}$, we have:

$$\left(\frac{z}{y}\right)^n = \left(1+\frac{\lambda}{y}\right)^n \geq \left(1+\frac{1}{3}\right)^n \geq \left(1+\frac{1}{3}\right)^3 = 1+3\frac{1}{3}+3\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^3 > 2 \text{ or } \left(\frac{z}{y}\right)^n > 2$$

So, Condition (1.5) is not satisfied and Equation (1.1) has not positive integer solutions. Thus and in this case condition $y \leq 3\lambda$ is false.

Therefore, in all cases condition $y \leq 3\lambda$ is false and it's true that:

$$y > 3\lambda \geq 3 \Leftrightarrow \frac{\lambda}{y} < \frac{1}{3} \text{ or } \frac{y}{\lambda} > 3 \tag{1.13}$$

Next, we will prove that when positive integers x, y, z verify Equation (1.1) for a natural number $n > 1$, number λ is less than the difference $y-2$ or $\lambda < y-2$.

Indeed, given the Conditions (1.12), (1.13) and considering separately the intervals $1 < n < 3$ and $n \geq 3$, we have:

1) If $1 < n < 3$ or $n\lambda < 3\lambda$, assuming that $\lambda \geq y-2$ and since $y > 3\lambda$, we have:

$n\lambda < 3\lambda \Leftrightarrow n\lambda - 2 < 3\lambda - 2 < y - 2 \leq \lambda \Rightarrow 3\lambda - 2 < \lambda \Leftrightarrow 2\lambda < 2$ or $\lambda < 1$, which it is absurd, since $\lambda \geq 1$. So, condition $\lambda \geq y-2$ is false.

2) If $n \geq 3$ assuming again that $\lambda \geq y-2$ and since $y > \lambda n$, we have:

$\lambda \geq y-2 > n\lambda - 2 \Rightarrow \lambda > n\lambda - 2 \Leftrightarrow 2 > n\lambda - \lambda \Leftrightarrow 2 > \lambda(n-1)$. From the last condition, because $n \geq 3$, equivalently we have:

$2 > \lambda(n-1) \geq \lambda(3-1) = 2\lambda \Rightarrow 2 > 2\lambda$ or $\lambda < 1$, which is absurd, since $\lambda \geq 1$. So, condition $\lambda \geq y-2$ is false.

Therefore, in all cases, condition $\lambda \geq y-2$ is false and it's true that:

$$\lambda < y-2 \tag{1.14}$$

Given Condition (1.14), we have:

$$\lambda < y-2 \Leftrightarrow \frac{1}{\lambda} > \frac{1}{y-2} \Leftrightarrow \frac{y}{\lambda} > \frac{y}{y-2}^{(5*)} \tag{1.15}$$

(5*) Be careful not to get confused. Condition (1.15), which corresponding to Condition (1.15), has nothing to do with the present proof, it's simply written here to exist link to the continuation of proof of problem 3.1, as this recently given by the author of the present article in a previous its paper [9] and is now listed on the Appendix 2 (the old proof continues from this point onwards after, as it was in its original publication, adding some clarifications).

We distinguish the following sub-cases:

B1. $1 < n < 3$ or $n = 2$ and **B2.** $n \geq 3$

In the first sub-case we will prove that Equation (1.1) can have positive integer solutions while in the second sub-case we will prove that Equation (1.1) has no positive integer solutions.

B1. $1 < n < 3$ or $n = 2$ (Equation (1.1) can have positive integer solutions)

Proof: Given the Conditions (1.12) and (1.13), we have:

$1+\frac{1}{2} > 2^{\frac{1}{2}} > 1+\frac{1}{3} > 1+\frac{\lambda}{y}$. So, because $n=2$, equivalently we have:

$1+\frac{1}{n} > 2^{\frac{1}{n}} > 1+\frac{1}{3} > 1+\frac{\lambda}{y} \Rightarrow 1+\frac{\lambda}{y} < 2^{\frac{1}{n}}$ or $\left(\frac{z}{y}\right)^n = \left(1+\frac{\lambda}{y}\right)^n < 2$. It applies also and that: $\left(\frac{z}{x}\right)^2 = \left(\frac{y}{x}\right)^2 + 1 > 1+1 = 2 \Rightarrow \left(\frac{z}{x}\right)^2 > 2$ or $\left(\frac{z}{x}\right)^n > 2$ (because $n=2$).

Therefore, in this case, due to the last conditions double inequality (1.5) is satisfied, so Equation (1.1) can have positive integer solutions (which indeed exist, known since the Greek antiquity as Pythagorean Triads).

B2. $n \ge 3$ (Equation (1.1) has no positive integer solutions)

Proof: If $n \ge 3 \Leftrightarrow \frac{1}{n} \le \frac{1}{3} \Leftrightarrow 2^{\frac{1}{n}} \le 2^{\frac{1}{3}} \Leftrightarrow -2^{\frac{1}{n}} \ge -2^{\frac{1}{3}} \Leftrightarrow$

$$1+\frac{\lambda}{y} - 2^{\frac{1}{n}} \ge 1+\frac{\lambda}{y} - 2^{\frac{1}{3}} \tag{1.16}$$

Also, if $n \ge 3$ due to Condition (1.12), we have:

$$\frac{\lambda}{y} < \frac{1}{n} \le \frac{1}{3} \Leftrightarrow y > 3\lambda \ge 3 \text{ or } y \ge 4 \tag{1.17}$$

Next, we will prove and we that: $x^3 + y^3 \ne z^3$, despite the fact that, Euler was the first which proved it. For this, assuming that the positive integers x, y, z verify Equation (1.1) for $n=3$, then it applies that:

$\left(\frac{z}{y}\right)^3 = \left(1+\frac{\lambda}{y}\right)^3 < 2$. From the last condition, equivalently, we have:

$\left(1+\frac{\lambda}{y}\right)^3 = 1+3\frac{\lambda}{y}+3\left(\frac{\lambda}{y}\right)^2+\left(\frac{\lambda}{y}\right)^3 < 2 \Leftrightarrow 3\frac{\lambda}{y}+3\left(\frac{\lambda}{y}\right)^2+\left(\frac{\lambda}{y}\right)^3 < 1$ or $3\frac{\lambda}{y} < 1$ and $3\left(\frac{\lambda}{y}\right)^2+\left(\frac{\lambda}{y}\right)^3 < 1$. Because the area $\frac{4}{3} < \lambda < \frac{y}{3}$ is *excluded*(6*) and given from the last condition that $3\frac{\lambda}{y} < 1$ and due to the conditions (1.13), (1.17) it applies that $y \ge 4$, also equivalently we have: $\lambda < \frac{y}{3}$ or $\lambda < \frac{4}{3} \le \frac{y}{3} \Leftrightarrow \lambda < \frac{4}{3}$ or $\lambda \le 1$, which is *false*(7*), because it is $\lambda \ge 1$ and is allowed only the value $\lambda = 1$, for which Equation (1.1) has no integer solutions. Thus condition $\left(1+\frac{\lambda}{y}\right)^3 < 2$ is false, so it's true that: $\left(1+\frac{\lambda}{y}\right)^3 \ge 2$ or $1+\frac{\lambda}{y} - 2^{\frac{1}{3}} \ge 0$.

Consequently due to Condition (1.16), it is true that:

$$1+\frac{\lambda}{y} - 2^{\frac{1}{n}} \ge 1+\frac{\lambda}{y} - 2^{\frac{1}{3}} \ge 0 \tag{1.18}$$

(6*) Proof why the area $\frac{4}{3} < 2 \le \lambda < \frac{y}{3}$ is excluded: If $\lambda \ge 2$, it applies that:

$\left(\frac{z}{y}\right)^3 = \left(1+\frac{\lambda}{y}\right)^3 = 1+3\frac{\lambda}{y}+S_2 \geq 1+3\frac{2}{y}+S_2$, where S_2 is a positive real number which, if we exclude the first two terms, is equal to the sum of the remaining 2 terms of the binomial: $\left(1+\frac{\lambda}{y}\right)^3$. At first we will prove that $3\frac{2}{y}>1$. Indeed, assuming that $3\frac{2}{y}>1$ and due to (1.13), (1.17) it's $y \geq 4$, we have: $3\frac{2}{y}>1 \Leftrightarrow 3>\frac{y}{2}\geq\frac{4}{2}=2$, which it's true. So, if $n=3$ and $\lambda \geq 2$, it's true that $3\frac{2}{y}>1$ and consequently it applies that: $\left(1+\frac{\lambda}{y}\right)^3 \geq 1+3\frac{2}{y}+S_2 > 1+1+S_2 > 2$ or $\left(\frac{z}{y}\right)^3 = \left(1+\frac{\lambda}{y}\right)^3 > 2$. Thus, condition (1.5) is not satisfied and Equation (1.1) has no positive integer solutions. Therefore, the area $\frac{4}{3}<2\leq\lambda<\frac{y}{3}$ is not valid and is excluded.

(7*) Proof that Condition $\lambda \leq 1$ is false: Let the positive integers x, y, z verify Equation (1.1) for $n=3$ and $\lambda=1$, then we have:

$$(y+1)^3 = x^3+y^3 \Leftrightarrow (y+1)^3 - y^3 = x^3 \Leftrightarrow 3y^2+3y+1 = x^3 \Leftrightarrow$$
$$3y(y+1) = x^3-1 \tag{1.19}$$

- If $x=2\rho$, where ρ a positive integer, given Condition (1.19), we have: $3y(y+1)=(2\rho)^3-1$ or $3y(y+1)=2(4\rho^3)-1$, which is absurd, because one of the consecutive integers y and $y+1$ is even and divides the odd integer $(2\rho)^3-1=2(4\rho^3)-1$.
- If $x=2\rho+1$, where ρ a positive integer, then again from condition (1.19), we have: $3y(y+1)=(2\rho+1)^3-1$ or
$3y(y+1)=8\rho^3+12\rho^2+6\rho=2\rho(4\rho^2+6\rho+3) \Leftrightarrow$
$3y(y+1)=2\rho\left[2(2\rho^2+3\rho+1)+1\right]$. From the last equality, because it applies that $y\geq 4$ and $y>x>\rho$ and one of the consecutive integers y and $y+1$ is even and divides the number $2\rho\left[2(2\rho^2+3\rho+1)+1\right]$, that is the second member of the last equality, it will also divides and the odd integer $2(2\rho^2+3\rho+1)+1$ (this number, is the largest of the factors of number $2\rho\left[2(2\rho^2+3\rho+1)+1\right]$), which that is also absurd. So, in all sub-cases, condition $\lambda\leq 1$ is false because it implies that: $x^3+y^3\neq z^3$.

Please, at this point, pay special attention to the following remark: What about the trinities x, y, z for which, it applies that:
$\left(\frac{z}{y}\right)^3<2$, $x^3+y^3\neq z^3$ and $n>3$. An example of such a trinity is the following: $(x,y,z)=(3,4,5)$, $\left(\frac{5}{4}\right)^3=\frac{125}{64}<2$ and $3^3+4^3\neq 5^3$.

Thus, the question arises whether in this case Equation (1.1) can have positive integer solutions for $n>3$. To this question we answer as follows: Because $n\geq 4$, we have: $\left(1+\frac{\lambda}{y}\right)^n \geq \left(1+\frac{\lambda}{y}\right)^4 = 1+4\frac{\lambda}{y}+S_3$, where S_3 is positive real number which, if we exclude the first two terms, is equal to the sum of the remaining 3 terms of the binomial: $\left(1+\frac{\lambda}{y}\right)^4$. At first we will prove that $4\frac{\lambda}{y} \geq 1$.

Indeed, assuming that $4\frac{\lambda}{y} \geq 1$ due to (1.13), (1.17) it's $y \geq 4$, we have: $4\frac{\lambda}{y} \geq 1 \Leftrightarrow \lambda \geq \frac{y}{4} \geq \frac{4}{4} = 1$, which it's true, since $\lambda \geq 1$. Therefore, if $n > 3$ it's true that $4\frac{\lambda}{y} \geq 1$ and consequently it applies that:

$$\left(\frac{z}{y}\right)^n = \left(1+\frac{\lambda}{y}\right)^n \geq 1+4\frac{\lambda}{y}+S_3 \geq 1+1+S_3 > 2 \text{ or } \left(\frac{z}{y}\right)^n > 2, \ \forall n > 3.$$

So, in this case, condition (1.5) is not satisfied and Equation (1.1) has no positive integer solutions.

Thus, all these x, y, z trinities, for which it is true that: $\left(\frac{z}{y}\right)^3 < 2$ and $x^3 + y^3 \neq z^3$, when $n > 3$ are ignored. So, condition (1.18) it applies for all the rest x, y, z trinities.

Finally, given the Conditions (1.12), (1.13), (1.17), (1.18), $n \geq 3$ and all the trinities x, y, z except of the trinities for which it's true that: $\left(\frac{z}{y}\right)^3 < 2$ and $x^3 + y^3 \neq z^3$, we have: $1+\frac{\lambda}{y} - 2^{\frac{1}{n}} \geq 0$ or $1+\frac{1}{3} \geq 1+\frac{1}{n} > 1+\frac{\lambda}{y} \geq 2^{\frac{1}{n}}$ and $\left(\frac{z}{y}\right)^n = \left(1+\frac{\lambda}{y}\right)^n \geq 2$.

Therefore, in sub-case B2, due to the last condition the double inequality (1.5) is not satisfied and also Equation (1.1) has no positive integer solutions.

Conclusion 3.1: From the above it is concluded that Equation (1.1), when $n < 3$ can have positive integer solutions, while when $n \geq 3$ doesn't have positive integer solutions. In the second case, Fermat's last theorem is verified.

3.2. Generalization of Fermat's Last Theorem (*New Theorem*)

If $x_1, x_2, x_3, \cdots, x_{m-1}, x_m$ are positive integers that differ from each other (m finite number), then for $m \geq 3$ the following equation:

$$(x_1)^n + (x_2)^n + (x_3)^n + \cdots + (x_{m-1})^n = (x_m)^n, \text{ where } n \in N, \ n > 1 \tag{2.1}$$

when $n \geq m^2 - 2m$, have no integer solutions. For $m = 3$, Fermat's last theorem taking place.

Proof of New Theorem

We consider positive integers $x_1, x_2, x_3, \cdots, x_{m-1}, x_m$ that differ from each other (m finite number) and assume that verify Equation (2.1) for a natural number $n > 1$. Also, we assume without loss of the generality, that:

$$x_1 < x_2 < \cdots < x_{m-1} < x_m \Leftrightarrow (x_1)^n < (x_2)^n < \cdots < (x_{m-1})^n < (x_m)^n \tag{2.2}$$

Taking into account Equation (2.1) and Condition (2.2), on the basis of the above hypothesis we have:

$$(x_1)^n + (x_1)^n + \cdots + (x_1)^n < (x_1)^n + (x_2)^n + \cdots + (x_{m-1})^n = (x_m)^n \text{ or}$$

$$(m-1)(x_1)^n < (x_m)^n \Leftrightarrow (m-1) < \left(\frac{x_m}{x_1}\right)^n \tag{2.3}$$

$$(x_{m-1})^n + (x_{m-1})^n + \cdots + (x_{m-1})^n > (x_1)^n + (x_2)^n + \cdots + (x_{m-1})^n = (x_m)^n \text{ or}$$

$$(m-1)(x_{m-1})^n > (x_m)^n \Leftrightarrow \left(\frac{x_m}{x_{m-1}}\right)^n < m-1 \tag{2.4}$$

By combining Conditions (2.3) and (2.4) we have:

$$\left(\frac{x_m}{x_{m-1}}\right)^n < m-1 < \left(\frac{x_m}{x_1}\right)^n \tag{2.5}$$

Note: The double inequality (2.5) is *necessary but not sufficient*, *i.e.* the converse is not always the case. For example we consider that $(x_1, x_2, x_3, x_4) = (3,4,5,6)$, $m = 4$ and $n = 5$. We have,

$3^5 + 4^5 + 5^5 = 4392 \neq 7776 = 6^5$ and $\left(\frac{6}{5}\right)^5 < 4-1 = 3 < \left(\frac{6}{3}\right)^5$.

If we replace the number x_m with the sum $x_{m-1} + \lambda$ or $x_m = x_{m-1} + \lambda$, where λ is a positive integer, for the positive integers $x_1, x_2, x_3, \cdots, x_{m-1}, x_m$ that according to the hypothesis we originally made, verify Equation (2.1) for a natural number $n > 1$, it is true that:

$$(x_1)^n + (x_2)^n + \cdots + (x_{m-1})^n = (x_m)^n = (x_{m-1} + \lambda)^n \tag{2.6}$$

We distinguish the following cases:

A) $\frac{x_{m-1}}{\lambda} \leq \frac{n}{m-2}$

If $\frac{x_{m-1}}{\lambda} \leq \frac{n}{m-2} \Leftrightarrow \frac{m-2}{n} \leq \frac{\lambda}{x_{m-1}} \Leftrightarrow 1 + \frac{m-2}{n} \leq 1 + \frac{\lambda}{x_{m-1}}$. So, due to (2.6) we have: $\left(1 + \frac{m-2}{n}\right)^n \leq \left(1 + \frac{\lambda}{x_{m-1}}\right)^n = \left(\frac{x_{m-1} + \lambda}{x_{m-1}}\right)^n = \left(\frac{x_m}{x_{m-1}}\right)^n \Leftrightarrow$

$$\left(\frac{x_m}{x_{m-1}}\right)^n \geq \left(1 + \frac{m-2}{n}\right)^n \tag{2.7}$$

Considering Bernoulli's inequality for $n > 1$, it applies:

$$\left(1 + \frac{m-2}{n}\right)^n > 1 + n\frac{m-2}{n} = 1 + m - 2 = m - 1 \tag{2.8}$$

By combining the Conditions (2.7) and (2.8) we have,

$$\left(\frac{x_m}{x_{m-1}}\right)^n \geq \left(1 + \frac{m-2}{n}\right)^n > m - 1 \tag{2.9}$$

Due to the Condition (2.9), the double inequality (2.5) is not satisfied and therefore in this case Equation (2.1) has no positive integer solutions, for any natural number n greater than the number one or $\forall n > 1$.

B) $\frac{x_{m-1}}{\lambda} > \frac{n}{m-2}$

Because in case A. Equation (2.1) doesn't have positive integer solutions, obviously if they exist, this will be in case B when condition B is applied. So, we have:

$$\frac{x_{m-1}}{\lambda} > \frac{n}{m-2} \Leftrightarrow x_{m-1} > \frac{n}{m-2}\lambda \text{ or } \frac{\lambda}{x_{m-1}} < \frac{m-2}{n} \tag{2.10}$$

Now, we will prove that when the positive integers $x_1, x_2, x_3, \cdots, x_{m-1}, x_m$ verify Equation (2.1) for a natural number $n > 1$, the number x_{m-1} is greater than the product of number m by the number λ or $x_{m-1} > m\lambda$.

For this we work as follows:

a) $1 < n < m(m-2)$. In this case, assuming that $x_{m-1} \le m\lambda$, we distinguish two areas: $m < x_{m-1} \le m\lambda$ and $x_{m-1} \le m = m\lambda$, $\lambda = 1$. So, given the latter areas, we have:

a.1) If $m < x_{m-1} \le m\lambda$, assuming that the positive integers $x_1, x_2, x_3, \cdots, x_{m-1}, x_m$ verify Equation (2.1) for a natural number $n > 1$, then since from $m < x_{m-1} \Leftrightarrow x_{m-1} \ge m+1$ and from $m < m\lambda \Leftrightarrow 1 < \lambda$ or $\lambda \ge 2$, it applies the following:

a.1.1) $n = 2$ and $m < x_{m-1} \le m\lambda$.

Let the positive integers $x_1, x_2, \cdots, x_{m-1}, x_{m-1} + \lambda$ verify Equation (2.1) $n = 2$, $\lambda \ge 2$ and $\left(1 + \frac{\lambda}{x_{m-1}}\right)^2 = 1 + 2\frac{\lambda}{x_{m-1}} + \left(\frac{\lambda}{x_{m-1}}\right)^2$. At the first, we will prove that: $\frac{(m+1)(m-2)}{2} \ge 2$. Indeed, if $\frac{(m+1)(m-2)}{2} \ge 2 \Leftrightarrow m^2 - m - 6 \ge 0$, which it applies $\forall m \ge 3$, so is true and that: $\frac{(m+1)(m-2)}{2} \ge 2$. Then, we will prove that: $2\frac{\lambda}{x_{m-1}} \ge m-2$. Now, assuming that: $2\frac{\lambda}{x_{m-1}} \ge m-2$, given that $\frac{(m+1)(m-2)}{2} \ge 2$ and due to the conditions (2.11), (2.17) it applies that $x_{m-1} \ge m+1$ we have: $2\frac{\lambda}{x_{m-1}} \ge m-2 \Leftrightarrow \lambda \ge \frac{x_{m-1}}{2}(m-2) \ge \frac{(m+1)(m-2)}{2} \ge 2$ which it's true, since $\lambda \ge 2$. So, condition $2\frac{\lambda}{x_{m-1}} \ge m-2$ it's true. Therefore, it applies that: $\left(1 + \frac{\lambda}{x_{m-1}}\right)^2 = 1 + 2\frac{\lambda}{x_{m-1}} + \left(\frac{\lambda}{x_{m-1}}\right)^2 \ge 1 + (m-2) + \left(\frac{\lambda}{x_{m-1}}\right)^2 > m-1$ or $\left(\frac{x_m}{x_{m-1}}\right)^n > m-1$ (because $n = 2$). Thus, in this case, condition (2.5) is not satisfied and Equation (2.1) has no positive integer solutions. So, the area $m < x_{m-1} \le m\lambda$ is not valid.

a.1.2) $n \ge 3$ and $m < x_{m-1} \le m\lambda$.

Since $x_{m-1} \le m\lambda$ or $\frac{\lambda}{x_{m-1}} \ge \frac{1}{m} \Leftrightarrow \frac{\lambda}{x_{m-1}} \ge \frac{2}{x_{m-1}} \ge \frac{1}{m} \Leftrightarrow x_{m-1} \le 2m$ or

$\frac{1}{x_{m-1}} \geq \frac{1}{2m}$. Then, if the positive integers $x_1, x_2, x_3, \cdots, x_{m-1}, x_m$ verify Equation (2.1) for a natural number $n \geq 3$ and given that $m < x_{m-1} \leq 2m$ and $\lambda \geq 2$, we have:

$\left(1+\frac{\lambda}{x_{m-1}}\right)^n \geq \left(1+\frac{\lambda}{2m}\right)^n = 1+n\frac{\lambda}{2m}+\frac{n(n-1)}{2}\left(\frac{\lambda}{2m}\right)^2 + S_{n-2}$, where S_{n-2} is a positive real number which, if we exclude the first three terms, is equal to the sum of the remaining $(n-2)$ terms of the binomial: $\left(1+\frac{\lambda}{2m}\right)^n$.

Now, assuming that $n\frac{\lambda}{2m}+\frac{n(n-1)}{2}\left(\frac{\lambda}{2m}\right)^2 < m-2$ or $n\frac{\lambda}{2}+\frac{n(n-1)}{8m}\lambda^2 < m(m-2)$ since $n \leq m(m-2)-1$ or $m(m-2) \geq n+1$ and $\lambda \geq 2$, equivalently we have:

$$n+\frac{n(n-1)}{2m} = n\frac{2}{2}+\frac{n(n-1)}{8m}2^2 \leq n\frac{\lambda}{2}+\frac{n(n-1)}{8m}\lambda^2 < n+1 \leq m(m-2) \Rightarrow$$

$n+\frac{n(n-1)}{2m} < n+1 \Leftrightarrow \frac{n(n-1)}{2m} < 1 \Leftrightarrow n^2-n-2m < 0 \Leftrightarrow$

$n^2-n-2m \leq n^2-n-6 < 0$ or $n^2-n-6 < 0$, which is false since

$n^2-n-6 \geq 0^{(8*)}$, $\forall n \geq 3$.

Therefore, condition $n\frac{\lambda}{2m}+\frac{n(n-1)}{2}\left(\frac{\lambda}{2m}\right)^2 < m-2$ is false and it's true that: $n\frac{\lambda}{2m}+\frac{n(n-1)}{2}\left(\frac{\lambda}{2m}\right)^2 \geq m-2$. So, consequently it that:

$\left(1+\frac{\lambda}{x_{m-1}}\right)^n \geq 1+n\frac{\lambda}{2m}+\frac{n(n-1)}{2}\left(\frac{\lambda}{2m}\right)^2 + S_{n-2} \geq 1+(m-2)+S_{n-2} > m-1$ or

$\left(\frac{x_m}{x_{m-1}}\right)^n = \left(1+\frac{\lambda}{x_{m-1}}\right)^n > m-1$. Therefore, due to the last condition the double inequality (2.5) is not satisfied and so Equation (2.1) has not positive integer solutions and the area $m < x_{m-1} \leq m\lambda$ also is not valid.

a.2) If $x_{m-1} \leq m = m\lambda$, $\lambda = 1$, assuming that the positive integers $x_1, x_2, x_3, \cdots, x_{m-1}, x_m$ verify Equation (2.1) for a natural number $n > 1$, it applies the following:

a.2.1) $n = 2$ and $x_{m-1} \leq m = m\lambda$. At first, because when $m = 3$, as proved in problem 3.1, condition $x_{m-1} \leq m\lambda$ is false, we consider for the continuation in this paragraph that $m \geq 4$. So, in this case, we have:

$$(x_{m-1}+1)^2 = (x_{m-1})^2+(x_{m-2})^2+\cdots+(x_1)^2 \Leftrightarrow \quad (x_{m-1}+1)^2 = (x_{m-1})^2+\sum_{i=1}^{m-2}(x_i)^2$$

$\Leftrightarrow (x_{m-1}+1)^2-(x_{m-1})^2 = \sum_{i=1}^{m-2}(x_i)^2 \Leftrightarrow 2x_{m-1}+1 = \sum_{i=1}^{m-2}(x_i)^2$. From the last condition it follows that the sum $\sum_{i=1}^{m-2}(x_i)^2$ is an odd positive integer or $\sum_{i=1}^{m-2}(x_i)^2 = 2\rho+1$, where ρ a positive integer. Also since $x_{m-1} \leq m$, equivalently we have:

$2m+1 \geq 2x_{m-1}+1=\sum_{i=1}^{m-2}(x_i)^2$. The sets of positive integers $x_1, x_2, \cdots, x_m$ that can to satisfy the conditions $2m+1 \geq \sum_{i=1}^{m-2}(x_i)^2$ and $\sum_{i=1}^{m-2}(x_i)^2=2\rho+1$ have, as the case may be, in the positions (x_{m-2}, x_{m-3}) the integers: $(m-1, m-2)$ or $(m-1, m-3)$ or $(m-2, m-3)$, so the following shall apply respectively:

- $2m+1 \geq \sum_{i=1}^{m-2}(x_i)^2 \geq (m-1)^2+(m-2)^2 \Leftrightarrow m^2-4m+2 \leq 0 \Leftrightarrow m(m-4)+2 \leq 0$, which is absurd because $m(m-4)+2>0$, $\forall m \geq 4$.
- $2m+1 \geq \sum_{i=1}^{m-2}(x_i)^2 \geq (m-1)^2+(m-3)^2 \Leftrightarrow 2m^2-10m+9 \leq 0 \Leftrightarrow$ $2m(m-5)+9 \leq 0$, which is absurd because $2m(m-5)+9>0$, $\forall m \geq 4$.
- $2m+1 \geq \sum_{i=1}^{m-2}(x_i)^2 \geq (m-2)^2+(m-3)^2 \Leftrightarrow m^2-6m+6 \leq 0 \Leftrightarrow m(m-6)+6 \leq 0$, which is absurd $\forall m \geq 5$, because in this case it applies that $m(m-6)+6>0$, while when $m=4$ it applies that $m(m-6)+6<0$ and the inequality is satisfied. But, if $m=4$ the only set of positive integers $x_1, x_2, \cdots, x_m$ that can be formed is: $(1,2,4,5)$ and it applies that: $5^2=(4+1)^2 \neq 2^2+1^2$. Thus, in all sub-cases, condition $2x_{m-1}+1=\sum_{i=1}^{m-2}(x_i)^2$ does not apply.

From the above analysis, is inferred that when $x_{m-1} \leq m \leq m\lambda$ and $n=2$, Equation (2.1) is not verified. Therefore, in this case condition $x_{m-1} \leq m\lambda$ is false.

a.2.2) $3 \leq n < m(m-2)$ and $x_{m-1} \leq m = m\lambda$, $\lambda=1$. In this sub-case, we have:

$$\left(1+\frac{1}{x_{m-1}}\right)^n = 1+n\frac{1}{m}+\frac{n(n-1)}{2!}\left(\frac{1}{m}\right)^2+S \text{, where } S=\sum_{i=3}^{i=n}\frac{(n)_i}{i!}\left(\frac{1}{m}\right)^i>0.$$

Now, we will prove that: $n\frac{1}{m}+\frac{n(n-1)}{2}\left(\frac{1}{m}\right)^2 \geq m-2$. So, assuming that:

$n\frac{1}{m}+\frac{n(n-1)}{2}\left(\frac{1}{m}\right)^2 < m-2$ or $n+\frac{n(n-1)}{2m} < m(m-2)$, since $n \leq m(m-2)-1$ or $m(m-2) \geq n+1$, equivalently we have:

$$n+\frac{n(n-1)}{2m} < n+1 \leq m(m-2) \Leftrightarrow n+\frac{n(n-1)}{2m} < n+1 \Leftrightarrow \frac{n(n-1)}{2m} < 1 \Leftrightarrow$$

$n^2-n-2m<0 \Leftrightarrow n^2-n-2m \leq n^2-n-6<0$ or $n^2-n-6<0$, which it is false because $n^2-n-6 \geq 0$ (8*), $\forall n \geq 3$.

Therefore, Condition $n\frac{1}{m}+\frac{n(n-1)}{2}\left(\frac{1}{m}\right)^2 < m-2$ is false and it's true that: $n\frac{1}{m}+\frac{n(n-1)}{2}\left(\frac{1}{m}\right)^2 \geq m-2$. So, consequently it applies that:

$$\left(\frac{x_m}{x_{m-1}}\right)^n = \left(1+\frac{\lambda}{x_{m-1}}\right)^n \geq 1+n\frac{1}{m}+\frac{n(n-1)}{2}\left(\frac{1}{m}\right)^2+S \geq 1+(m-2)+S > m-1 \text{ or}$$

$\left(\frac{x_m}{x_{m-1}}\right)^n > m-1$, $\forall n \ge 3$. Therefore, Condition (2.5) is not satisfied and Equation (2.1) has not positive integer solutions and in this sub-case condition $x_{m-1} \le m\lambda$ also is false.

(8*) Proof that $n^2 - n - 6 \ge 0$: Given Equation $x^2 - x - 6 = 0$, the following applies: $D = (-1)^2 - 4 \times 1 \times (-6) = 25 > 0$ and so its roots are:

$x_{1,2} = \frac{1 \pm \sqrt{25}}{2 \times 1}$. So, if $x \ge x_2 = \frac{1+\sqrt{25}}{2} = 3$ is $f(x) = x^2 - x - 6 \ge 0$ and consequently $n^2 - n - 6 \ge 0$, $\forall n \ge 3$.

b) If $n \ge m(m-2)$, assuming again that $x_{m-1} \le m\lambda$ or $\frac{\lambda}{x_{m-1}} \ge \frac{1}{m}$, we have:

$$\left(1+\frac{\lambda}{x_{m-1}}\right)^n \ge \left(1+\frac{1}{m}\right)^n \ge \left(1+\frac{1}{m}\right)^{m(m-2)} = 1 + m(m-2)\frac{1}{m} + \cdots + \left(\frac{1}{m}\right)^{m(m-2)} > m-1$$

or $\left(\frac{x_m}{x_{m-1}}\right)^n = \left(1+\frac{\lambda}{x_{m-1}}\right)^n \ge 1 + (m-2) + \cdots + \left(\frac{1}{m}\right)^{m(m-2)} > m-1$, so condition (2.5) is not satisfied and Equation (2.1) has not positive integer solutions. Thus, in this case, condition $x_{m-1} \le m\lambda$ is false.

Therefore, in all cases condition $x_{m-1} \le m\lambda$ it's false and so it's true that:

$$x_{m-1} > m\lambda \ge m \Leftrightarrow \frac{\lambda}{x_{m-1}} < \frac{1}{m} \text{ or } x_{m-1} \ge m+1 \tag{2.11}$$

Then, we will prove that when the positive integers $x_1, x_2, x_3, \cdots, x_{m-1}, x_m$ verify Equation (2.1) for a natural number $n > 1$, the number λ is less than the difference $x_{m-1} - (m-1)$ or $\lambda < x_{m-1} - (m-1)$.

Indeed, taking into account Conditions (2.10), (2.11) and considering separately the intervals $1 < n < m(m-2)$ and $n \ge m(m-2)$, we have:

1) If $1 < n < m(m-2)$, given the conditions (2.11) or $x_{m-1} > m\lambda$, $n < m(m-2)$ or $\frac{n\lambda}{m-2} < m\lambda$ and assuming that $\lambda \ge x_{m-1} - (m-1)$, we have:

$\frac{n\lambda}{m-2} < m\lambda \Leftrightarrow \frac{n\lambda}{m-2} - (m-1) < m\lambda - (m-1) < x_{m-1} - (m-1) \le \lambda \Rightarrow$ $m\lambda - (m-1) < \lambda \Leftrightarrow m\lambda - \lambda < m-1 \Leftrightarrow \lambda(m-1) < m-1$ or $\lambda < 1$, which is absurd since $\lambda \ge 1$. So, condition $\lambda \ge x_{m-1} - (m-1)$ is false.

2) If $n \ge m(m-2)$, assuming again that $\lambda \ge x_{m-1} - (m-1)$ and taking into account Condition (2.10), we have: $\lambda \ge x_{m-1} - (m-1) > \frac{n}{m-2}\lambda - (m-1) \Rightarrow$

$\lambda > \frac{n}{m-2}\lambda - (m-1) \Leftrightarrow m-1 > \frac{n}{m-2}\lambda - \lambda \Leftrightarrow m-1 > \lambda\left(\frac{n}{m-2} - 1\right)$. From the last condition since $n \ge m(m-2)$, equivalently we have:

$m-1 > \lambda\left(\frac{n}{m-2} - 1\right) \ge \lambda\left(\frac{m(m-2)}{m-2} - 1\right) = \lambda(m-1) \Rightarrow m-1 > \lambda(m-1)$ or $\lambda < 1$, which is absurd, since $\lambda \ge 1$. So, condition $\lambda \ge x_{m-1} - (m-1)$ is false and it's true that:

$$\lambda < x_{m-1} - (m-1).$$

Therefore, when Equation (2.1) is verified for a natural number $n > 1$ it's true that:

$$\lambda < x_{m-1} - (m-1) \tag{2.12}$$

Given condition (2.12) we have:

$$\lambda < x_{m-1} - (m-1) \Leftrightarrow \frac{1}{\lambda} > \frac{1}{x_{m-1} - (m-1)} \Leftrightarrow \frac{x_{m-1}}{\lambda} > \frac{x_{m-1}}{x_{m-1} - (m-1)}^{(9*)} \tag{2.13}$$

(9*) Be careful not to get confused. Condition (2.13), which corresponding to the Condition (2.20), has nothing to do with the present proof, it's simply written here to exist link to the continuation of proof of problem 3.2, as this recently given by the author of the present article in a previous its paper [9] and is now listed on the Appendix 2 (the old proof continues from this point onwards after, as it was in its original publication, adding some clarifications).

We distinguish the following sub-cases:

B1. $1 < n < m(m-2)$ and **B2.** $n \geq m(m-2)$

In the first sub-case we will prove that Equation (2.1) can have positive integer solutions while in the second sub-case we will prove that Equation (2.1) has no positive integer solutions.

B1. $1 < n < m(m-2)$ (Equation (2.1) can have positive integer solutions)

Proof: At the first we will prove that: $\left(1+\frac{\lambda}{x_{m-1}}\right)^n < \left(1+\frac{1}{m}\right)^n < m-1$. Indeed, since due to condition (2.11) it is $\frac{\lambda}{x_{m-1}} < \frac{1}{m}$, we have: $\left(1+\frac{\lambda}{x_{m-1}}\right)^n < \left(1+\frac{1}{m}\right)^n$.

Developing the binomial $\left(1+\frac{1}{m}\right)^n$ it applies: $\left(1+\frac{1}{m}\right)^n = 1 + n\frac{1}{m} + S$, where $S = \sum_{i=2}^{i=n} \frac{(n)_i}{i!}\left(\frac{1}{m}\right)^i > 0$. Next, we will prove that: $m-1 > \left(1+\frac{1}{m}\right)^n = 1 + \frac{n}{m} + S$.

So, assuming that: $1 + \frac{n}{m} + S \geq m-1$ or $\frac{n}{m} + S \geq m-2$ and considering separately the sub-intervals: a) $1 < n \leq m$ and b) $m < n < m(m-2)$, we have:

a) $1 < n \leq m$. Because $\frac{n}{m} \leq 1$, given condition $\frac{n}{m} + S \geq m-2$, we have equivalently: $S \geq (m-2) - \frac{n}{m} \Leftrightarrow S \geq (m-2) - \frac{n}{m} \geq (m-2) - 1 \Leftrightarrow$ $S \geq m-3 \geq 3-3 = 0$ or $S \geq 0$, which is absurd, since $S > 0$.

b) $m < n < m(m-2)$. Because $m < n \Leftrightarrow \frac{n}{m} > 1$ or $\frac{n}{m} - 1 > 0$ so, given again condition $\frac{n}{m} + S \geq m-2$, equivalently we have: $\frac{n}{m} + S \geq m-2 \geq 3-2 = 1 \Leftrightarrow$ $\frac{n}{m} + S \geq 1$ or $\left(\frac{n}{m} - 1\right) + S \geq 0$, which it's absurd, because $\left(\frac{n}{m} - 1\right) + S > 0$

since $\frac{n}{m}-1>0$ and $S>0$. Therefore, condition $1+\frac{n}{m}+S\geq m-1$ or $\left(1+\frac{1}{m}\right)^n \geq m-1$ is false and it's true that: $m-1>\left(1+\frac{1}{m}\right)^n$. So, based on the previous analysis it's true that:

$$m-1>\left(1+\frac{1}{m}\right)^n>\left(1+\frac{\lambda}{x_{m-1}}\right)^n \Leftrightarrow (m-1)^{\frac{1}{n}}>1+\frac{1}{m}>1+\frac{\lambda}{x_{m-1}} \tag{2.14}$$

Considering Bernoulli's inequality for $n>1$, it follows that:

$$\left(1+\frac{m-2}{n}\right)^n>1+n\frac{m-2}{n}=m-1 \text{ or } 1+\frac{m-2}{n}>(m-1)^{\frac{1}{n}} \tag{2.15}$$

Finally, by combining the Conditions (2.14) and (2.15) it's true that:

$$1+\frac{m-2}{n}>(m-1)^{\frac{1}{n}}>1+\frac{1}{m}>1+\frac{\lambda}{x_{m-1}} \text{ and } \left(1+\frac{\lambda}{x_{m-1}}\right)^n<m-1 \text{ or}$$

$\left(\frac{x_m}{x_{m-1}}\right)^n=\left(1+\frac{\lambda}{x_{m-1}}\right)^n<m-1$. Also, it is true and the follows condition:

$$\left(\frac{x_m}{x_1}\right)^n=\left(\frac{x_{m-1}}{x_1}\right)^n+\cdots+\left(\frac{x_2}{x_1}\right)^n+1>1+\cdots+1+1=m-1 \text{ or } \left(\frac{x_m}{x_1}\right)^n>m-1$$

Therefore, in this sub-case, due to the last conditions the double inequality (2.5) always is satisfied and so Equation (2.1) can have positive integer solutions.

B2. $n\geq m(m-2)$ (Proof that Equation (2.1) has no positive integer solutions)

Proof: If $n\geq m(m-2) \Leftrightarrow \frac{1}{n}\leq\frac{1}{m(m-2)} \Leftrightarrow (m-1)^{\frac{1}{n}}\leq(m-1)^{\frac{1}{m(m-2)}} \Leftrightarrow$

$$-(m-1)^{\frac{1}{n}}\geq-(m-1)^{\frac{1}{m(m-2)}} \Leftrightarrow 1+\frac{\lambda}{x_{m-1}}-(m-1)^{\frac{1}{n}}\geq 1+\frac{\lambda}{x_{m-1}}-(m-1)^{\frac{1}{m(m-2)}} \tag{2.16}$$

Also if $n\geq m(m-2)$, due to Condition (2.10), we have:

$$\frac{\lambda}{x_{m-1}}<\frac{m-2}{n}\leq\frac{m-2}{m(m-2)}=\frac{1}{m} \Rightarrow \frac{\lambda}{x_{m-1}}<\frac{1}{m} \Leftrightarrow x_{m-1}>m\lambda\geq m \text{ or}$$

$$x_{m-1}\geq m+1 \tag{2.17}$$

Now, assuming that the positive numbers $x_1, x_2, x_3, \cdots, x_{m-1}, x_m$ verify Equation (2.1) for $n=m(m-2)$, then it's applied that: $\left(\frac{x_m}{x_{m-1}}\right)^{m(m-2)}=\left(1+\frac{x_m}{x_{m-1}}\right)^{m(m-2)}<m-1$. From last condition, equivalently we have: $1+m(m-2)\frac{\lambda}{x_{m-1}}+\theta<m-1$, where $\theta=\sum_{i=2}^{i=n}\frac{(n)_i}{i!}\left(\frac{\lambda}{x_{m-1}}\right)^i>0$, $n=m(m-2)$. Next, also equivalently we have: $m(m-2)\frac{\lambda}{x_{m-1}}+\theta<m-2$ or

$m(m-2)\frac{\lambda}{x_{m-1}} < m-2$ and $\theta < m-2$. Given that: $m(m-2)\frac{\lambda}{x_{m-1}} < m-2$, it applies that: $m\frac{\lambda}{x_{m-1}} < 1$ or $\lambda < \frac{x_{m-1}}{m}$. So, because the area $\frac{m+1}{m} < 2 \le \lambda < \frac{x_{m-1}}{m}$ is *excluded*(10*), given that $\lambda < \frac{x_{m-1}}{m}$ and due to the Conditions (2.11), (2.17) it is $x_{m-1} \ge m+1$, equivalently we have: $\lambda < \frac{m+1}{m} \le \frac{x_{m-1}}{m} \Leftrightarrow \lambda < 1+\frac{1}{m}$ or $\lambda \le 1$, which is *false*(11*), because it is $\lambda \ge 1$ and is allowed only the value $\lambda = 1$ for which Equation (2.1) is not verified. Therefore, condition

$\left(1+\frac{\lambda}{x_{m-1}}\right)^{m(m-2)} < m-1$ is false, so it's true that:

$\left(1+\frac{\lambda}{x_{m-1}}\right)^{m(m-2)} \ge m-1 \Leftrightarrow 1+\frac{\lambda}{x_{m-1}}-(m-1)^{\frac{1}{m(m-2)}} \ge 0$.

Consequently due to Condition (2.16), is true that:

$$1+\frac{\lambda}{x_{m-1}}-(m-1)^{\frac{1}{n}} \ge 1+\frac{\lambda}{x_{m-1}}-(m-1)^{\frac{1}{m(m-2)}} \ge 0 \tag{2.18}$$

(10*) Proof why the area $\frac{m+1}{m} < 2 \le \lambda < \frac{x_{m-1}}{m}$ is excluded: If $\lambda \ge 2$, it applies that:

$\left(1+\frac{\lambda}{x_{m-1}}\right)^{m(m-2)} = 1+m(m-2)\frac{\lambda}{x_{m-1}}+s_{n-1} \ge 1+m(m-2)\frac{2}{x_{m-1}}+s_{n-1}$, where s_{n-1} is a positive real number which, if we exclude the first two terms, is equal to the sum of the remaining $(n-1)$ terms of the binomial: $\left(1+\frac{\lambda}{x_{m-1}}\right)^{m(m-2)}$. At the first, we will prove that: $m(m-2) > \frac{m+1}{2}$. So, assuming that: $m(m-2) > \frac{m+1}{2}$, equivalently we have: $2m^2-4m > m+1 \Leftrightarrow 2m^2-5m-1 = m(2m-5)-1 > 0$, which obviously it's true $\forall m \ge 3$, so it's true and that: $m(m-2) > \frac{m+1}{2}$. Next, we will prove that: $m(m-2)\frac{2}{x_{m-1}} \ge m-2$.

Indeed, given condition: $m(m-2) > \frac{m+1}{2}$ and due to the conditions (2.11), (2.17) it's $\frac{m+1}{x_{m-1}} \le 1$, we have: $m(m-2) > \frac{m+1}{2} \Leftrightarrow$

$m(m-2)\frac{2}{x_{m-1}}-(m-2) > \frac{m+1}{2}\times\frac{2}{x_{m-1}}-(m-2) = \frac{m+1}{x_{m-1}}-(m-2)$. Also, it applies that: $\frac{m+1}{x_{m-1}}-(m-2) \le 1-(m-2) = 3-m \le 0$. Based on the previous analysis we have: $m(m-2)\frac{2}{x_{m-1}}-(m-2) > 0 \ge \frac{m+1}{x_{m-1}}-(m-2)$. So, it's true that:

$m(m-2)\frac{2}{x_{m-1}}-(m-2)>0$ or $m(m-2)\frac{2}{x_{m-1}}>(m-2)$. Consequently, we have:

$$\left(1+\frac{\lambda}{x_{m-1}}\right)^{m(m-2)} \geq 1+m(m-2)\frac{2}{x_{m-1}}+s_{n-1}>1+(m-2)+s_{n-1}>m-1 \text{ or}$$

$\left(\frac{x_m}{x_{m-1}}\right)^{m(m-2)}>m-1$. Therefore, condition (2.5) is not satisfied and Equation (2.1) has no positive integer solutions. So, the area $\frac{m+1}{m}<\lambda<\frac{x_{m-1}}{m}$ is not valid and is excluded.

(11*) Proof that Condition $\lambda \leq 1$ is false:

Let the positive integers $x_1, x_2, x_2, \cdots, x_m$ verify Equation (2.1) for a natural $n \geq m(m-2) \geq 3$ and $\lambda = 1$, then it applies that:

$$(x_{m-1}+1)^n=(x_{m-1})^n+\sum_{i=1}^{m-2}(x_i)^n \Leftrightarrow (x_{m-1}+1)^n-(x_{m-1})^n=\sum_{i=1}^{m-2}(x_i)^n.$$

Given the last condition, because the numbers x_{m-1} and $x_{m-1}+1$ are consecutive integers it arises that one will be odd and the other even integer, so the difference $(x_{m-1}+1)^n-(x_{m-1})^n$ will be an odd integer and therefore the sum $\sum_{i=1}^{m-2}(x_i)^n$ will also be an odd integer or $\sum_{i=1}^{m-2}(x_i)^n=2\rho+1$, where ρ a positive integer. Therefore, if for convenience we set: $x_{m-1}=x$, the previous one equation is written as follows:

$$(x+1)^n-x^n=2\rho+1 \tag{2.19}$$

- If $n=3$, from Condition (2.19), we have: $(x+1)^3-x^3=2\rho+1 \Leftrightarrow$ $3x^2+3x+1=2\rho+1 \Leftrightarrow 3x(x+1)=2\rho \Leftrightarrow 3=2\times\frac{\rho}{x(x+1)}=2\xi$, which is absurd, since an odd integer cannot be equal to an even integer ($\xi=\frac{\rho}{x(x+1)}$, as since $x>2$ the product $x(x+1)$ divides the integer ρ), so if $n=3$ it applies that: $(x+1)^3-x^3 \neq 2\rho+1$.

Given Condition (2.19), equivalently we have:

$$(x+1-x)\left[(x+1)^{n-1}+(x+1)^{n-2}x+\cdots+(x+1)x^{n-2}+x^{n-1}\right]=2\rho+1 \Leftrightarrow$$
$$x(x+1)\left[(x+1)^{n-3}+(x+1)^{n-4}x+\cdots+(x+1)x^{n-4}+x^{n-3}\right]=2\rho+1-(x+1)^{n-1}-x^{n-1}.$$

Because the second member of the last equation is an even positive integer or $(2\rho+1)-(x+1)^{n-1}-x^{n-1}=2k$, where k a positive integer and since $x>2$ the product $x(x+1)$ divides the integer k or $\frac{k}{x(x+1)}=\mu$, also equivalently we have:

$$(x+1)^{n-3}+(x+1)^{n-4}x+\cdots+(x+1)x^{n-4}+x^{n-3}=2\times\frac{k}{x(x+1)}=2\mu \tag{2.20}$$

- If $n=4$, from (2.20) $\Leftrightarrow x(x+1)\left[x+1+(1\times x)+(x+1)\times 1+x\right]=2k \Leftrightarrow$

$x(x+1)\left[2(2x+1)\right]=2k$ or $x(x+1)(2x+1)=k \Rightarrow x(2x+1)$ divides k. So, because $2x+1$ divides $k \Rightarrow 2x$ or x ($x \geq 4$) divides $k-1$, which is absurd because, as the number x divides k which is term of the number $k-1$, it must also to divides and the number 1. So, if $n=4$ it applies that: $(x+1)^4 - x^4 \neq 2\rho+1$.

- If $n \geq 5$, again from (2.20) equivalently, we have:
  $x(x+1)\left[(x+1)^{n-5}+(x+1)^{n-6}x+\cdots+(x+1)x^{n-6}+x^{n-5}\right]=2\mu-(x+1)^{n-3}-x^{n-3}$,
  which is absurd because the second member of the last equality is an odd integer or $2\mu-(x+1)^{n-3}-x^{n-3}=2\nu+1$, where ν a positive integer and one of the consecutive integers x and $x+1$ is even and divides the odd integer $2\mu-(x+1)^{n-3}-x^{n-3}=2\nu+1$.

So, if $n \geq 5$ it applies that: $(x+1)^n - x^n \neq 2\rho+1$. Therefore, in all cases or for $n \geq 3$ it applies that: $(x+1)^n - x^n \neq 2\rho+1$. Thus, and for $n=m(m-2)$ and $m \geq 3$ it applies that: $\sum_{i=1}^{m-1}(x_i)^{m(m-2)} \neq (x_m)^{m(m-2)}$. So, condition $\lambda \leq 1$ it's also false.

Please, at this point, pay special attention to the following: What about the sets of the positive numbers $x_1, x_2, x_3, \cdots, x_{m-1}, x_m$ for which it applies that: $\left(\frac{x_m}{x_{m-1}}\right)^{m(m-2)} < m-1$, $\sum_{i=1}^{m-1}(x_i)^{m(m-2)} \neq (x_m)^{m(m-2)}$ and $n > m(m-2)$. Thus, the question arises whether in this case Equation (2.1) can have positive integer solutions for $n > m(m-2)$. To this question we answer as follows:

Because $n \geq m(m-2)+1$, we have the following:

Now, at the first we will prove that: $\frac{m+1}{m(m-2)+1}(m-2) \geq 1$. Indeed, if $\frac{m+1}{m(m-2)+1}(m-2) \geq 1$, equivalently we have: $m^2-m-2 \geq m^2-2m+1 \Leftrightarrow m \geq 3$, which it's true, so it's true and that: $\frac{m+1}{m(m-2)+1}(m-2) \geq 1$. Then, assuming that $\frac{m(m-2)+1}{x_{m-1}}\lambda \geq m-2$, given condition $\frac{m+1}{m(m-2)+1}(m-2) \geq 1$ and due to the conditions (2.11), (2.17) it's $x_{m-1} \geq m+1$, we have:

$$\frac{m(m-2)+1}{x_{m-1}}\lambda \geq m-2 \Leftrightarrow \lambda \geq \frac{x_{m-1}}{m(m-2)+1}(m-2) \geq \frac{m+1}{m(m-2)+1}(m-2) \geq 1,$$

which it's true, since $\lambda \geq 1$. So, it's true and that: $\frac{m(m-2)+1}{x_{m-1}}\lambda \geq m-2$.

Therefore, consequently it applies that:

$$\left(\frac{x_m}{x_{m-1}}\right)^n \geq \left(\frac{x_m}{x_{m-1}}\right)^{m(m-2)+1} = 1+\frac{m(m-2)+1}{x_{m-1}}\lambda + s_{n-1} \geq 1+(m-2)+s_{n-1} > m-1 \text{ or}$$

$\left(\frac{x_m}{x_{m-1}}\right)^n > m-1$, $\forall n > m(m-2)$. So, in this case, condition (2.5) is not satisfied and Equation (2.1) has no positive integer solutions.

Thus, all these sets of the positive numbers $x_1, x_2, x_3, \cdots, x_{m-1}, x_m$ for which it's true that: $\sum_{i=1}^{m-1}(x_i)^{m(m-2)} \neq (x_m)^{m(m-2)}$ and $\left(\frac{x_m}{x_{m-1}}\right)^{m(m-2)} < m-1$, if $n > m(m-2)$ are ignored.

So, Condition (2.18), it applies for all the rest sets of the positive numbers $x_1, x_2, x_3, \cdots, x_{m-1}, x_m$.

Finally, given the Conditions (2.10), (2.11), (2.15), (2.17), (2.18) and that $n \geq m(m-2)$, we have: $1+\frac{\lambda}{x_{m-1}}-(m-1)^{\frac{1}{n}} \geq 0$ or

$1+\frac{1}{m} \geq 1+\frac{m-2}{n} > 1+\frac{\lambda}{x_{m-1}} \geq (m-1)^{\frac{1}{n}}$ and $\left(\frac{x_m}{x_{m-1}}\right)^n = \left(1+\frac{\lambda}{x_{m-1}}\right)^n \geq m-1$.

Therefore, in this sub-case double inequality (2.5) is not satisfied and so Equation (2.1) has no positive integer solutions.

Conclusion 3.2: From the above it is concluded that Equation (2.1) when $n < m^2 - 2m$ can have integer solutions, while when $n \geq m^2 - 2m$ have no integer solutions. In the second case for $m = 3$, answer to Fermat's Last Theorem is given.

4. A Very Brief Solution to "Fermat's Last Theorem"

Here we present to you the shortest solution that has ever been achieved to "Fermat's last theorem", the most famous mathematical problem of the world.

The problem:

If x, y, z positive integers, $x \neq y \neq z$, the following Equation:

$$x^n + y^n = z^n, \quad n \in N, \quad n > 1 \tag{3.1}$$

when $n \geq 3$ have no positive integer solutions.

Proof: Let x, y, z positive integers, $x \neq y \neq z$, that verify Equation (3.1) for a natural number $n > 1$. Then, assuming that $x < y < z$, without loss of the generality, we have:

$$z^n = x^n + y^n \Leftrightarrow \left(\frac{z}{y}\right)^n = \left(\frac{x}{y}\right)^n + 1 \Rightarrow \left(\frac{z}{y}\right)^n < 1+1 = 2 \quad \text{or} \quad \left(\frac{z}{y}\right)^n < 2 \tag{3.2}$$

Note: Condition (3.2) is *necessary but not sufficient i.e.* the converse is not always the case.

If we set: $\lambda = z - y \geq 1$, where λ is a positive integer, it's true that:

$$x^n + y^n = z^n = (y+\lambda)^n \quad \text{or} \quad \left(\frac{z}{y}\right)^n = \left(1+\frac{\lambda}{y}\right)^n \tag{3.3}$$

From (3.3) $\Leftrightarrow x^n + y^n = z^n = (y+\lambda)^n = y^n + ny^{n-1}\lambda + \cdots + ny\lambda^{n-1} + \lambda^n \Leftrightarrow$

$$x^n - \lambda^n = ny^{n-1}\lambda + \cdots + ny\lambda^{n-1} > 0 \Rightarrow x^n > \lambda^n \quad \text{or} \quad x > \lambda \tag{3.4}$$

Given Condition (3.4) and the original hypothesis it's true that:

$$1 \leq \lambda < x < y < z \tag{3.5}$$

Now, we will prove that if $n \geq 3$, the Condition (3.2) never is satisfied. Indeed, assuming that the positive integers x, y, z verify Equation (3.1) for a natural number $n \geq 3$, then by combining the Conditions (3.2) and (3.3), we have:

$$\left(\frac{z}{y}\right)^n = \left(1+\frac{\lambda}{y}\right)^n = 1 + n\frac{\lambda}{y} + s_{n-1} < 2 \Leftrightarrow n\frac{\lambda}{y} + s_{n-1} < 1 \Rightarrow n\frac{\lambda}{y} < 1 \tag{3.6}$$

where s_{n-1} is a positive real number which, if we exclude the first two terms, is equal to the sum of the remaining $(n-1)$ terms of the development of the binomial: $\left(1+\frac{\lambda}{y}\right)^n$. Given Condition (3.6), because $n \geq 3$, we have:

$$n\frac{\lambda}{y} < 1 \Leftrightarrow \lambda < \frac{y}{n} \leq \frac{y}{3} \Rightarrow \lambda < \frac{y}{3} \Leftrightarrow y > 3\lambda \geq 3 \text{ or } y \geq 4 \tag{3.7}$$

Also, given Condition (3.7), because the area $\frac{4}{3} < 2 \leq \lambda < \frac{y}{3}$ is *excluded*[12*], equivalently we have: $\lambda < \frac{4}{3} \leq \frac{y}{3} \Leftrightarrow \lambda < \frac{4}{3}$ or $\lambda \leq 1$, which is *false*[13*], since $\lambda \geq 1$, so is allowed only the value $\lambda = 1$, for which Equation (3.1) is not verified. Therefore, if $n \geq 3$, the Condition (3.2) never is satisfied and so Equation (3.1) has no positive integer solutions or $x^n + y^n \neq z^n$, $\forall n \geq 3$.

(12*) Proof why the area $\frac{4}{3} < 2 \leq \lambda < \frac{y}{3}$ is excluded: If $\lambda \geq 2$, then it applies that: $\left(\frac{z}{y}\right)^n = \left(1+\frac{\lambda}{y}\right)^n = 1 + n\frac{\lambda}{y} + S_{n-1} \geq 1 + n\frac{2}{y} + S_{n-1}$, where s_{n-1} as previous defined. At the first, we will prove that $n\frac{2}{y} > 1$. Indeed, if $n\frac{2}{y} > 1$ and due to Condition (3.7) it's $y \geq 4$, we have: $n\frac{2}{y} > 1 \Leftrightarrow n > \frac{y}{2} \geq \frac{4}{2} = 2$ or $n > 2$, which it's true, since $n \geq 3$. Therefore, $\forall n \geq 3$ and $\lambda \geq 2$ it's true that $n\frac{2}{y} > 1$. Consequently it applies that:

$\left(\frac{z}{y}\right)^n = \left(1+\frac{\lambda}{y}\right)^n \geq 1 + n\frac{2}{y} + S_{n-1} > 1 + 1 + S_{n-1} > 2$ or $\left(\frac{z}{y}\right)^n > 2$. Therefore, Condition (3.2) is not satisfied and Equation (3.1) has no integer solutions. Thus, the area $\frac{4}{3} < 2 \leq \lambda < \frac{y}{3}$ is not valid and is excluded.

(13*) Proof that Condition $\lambda \leq 1$ is false:

Let the positive integers x, y, z verify Equation (3.1) for $n \geq 3$ and $\lambda = 1$, then it applies that:

$0 = (y+1)^n - y^n - x^n \Leftrightarrow 0 = y^n + ny^{n-1} + \theta - y^n - x^n = ny^{n-1} - x^n + \theta$,

where $\theta = \sum_{i=2}^{i=n} \frac{(n)_i}{i!} y^i > 0$. Now, we will prove that: $3y^{n-1} - x^n > 0$. Indeed, if $3y^{n-1} - x^n > 0$, because $x \geq 3$ [14*] or $\frac{x}{3} \geq 1$, then we have: $3y^{n-1} - x^n > 0 \Leftrightarrow$

$\frac{3}{x}\times\left(\frac{y}{x}\right)^{n-1}>1 \Leftrightarrow \left(\frac{y}{x}\right)^{n-1}>\frac{x}{3}\geq 1$, which is true, since $\left(\frac{y}{x}\right)^{n-1}>1$. So, it's true and that: $3y^{n-1}-x^n>0$. Therefore, $\forall n\geq 3$ is $ny^{n-1}-x^n\geq 3y^{n-1}-x^n>0$. Then, from the equation: $0=ny^{n-1}-x^n+\theta$, since $ny^{n-1}-x^n>0$ and $\theta>0$, it follows that: $0=ny^{n-1}-x^n+\theta>0$ or $0>0$, which is absurd. So, $0\neq(y+1)^n-y^n-x^n, \forall n\geq 3$ and Condition $\lambda\leq 1$ it's false.

(14*) We consider that $x\geq 3$, because due to the Conditions (3.4) and (3.5) it's valid that $x>\lambda\geq 1$ and the x is an odd integer, as arises from equation: $(y+1)^n-y^n=x^n$, since the numbers y and $y+1$ are consecutive integers, so one will be odd and the other even integer and consequently the difference $(y+1)^n-y^n$ will be an odd integer. So and the numbers x and x^n will also are odd integers. Therefore, it applies that: $x\neq 1$, $x\neq 2$ and $x\geq 3$.

Note: A wonderful second proof that the condition $\lambda\leq 1$ it's false, is given in the case B2) of the problem 3.2, a little above. Also, to the Annex 1 is presented a second proof why the area $\frac{4}{3}<2\leq\lambda<\frac{y}{3}$ is excluded.

5. General Conclusions

The proofs of the two problems which presented in this article are brief and very simple as their wording. They are achieved without the use of abstract algebra or elements from other fields of modern mathematics of the twentieth century. For this reason, they can be easily understood by any mathematician or anyone who knows basic mathematics. That means they have pedagogical value.

It is also worth mentioning that the proofs of the two problems were achieved without the use of a computer. When it comes to the conjecture of the odd perfect numbers, modern researchers have been trying to prove it by using computer-assisted methods. Although computers have changed the way we approach Mathematics, their overuse at the expense of mathematical thinking is an abuse and this should be seriously taken into consideration by the scientific community.

Also, it's very important, that Fermat's last theorem is generalized to an arbitrarily large number of variables. This generalization is essentially a new theorem in the field of number theory very useful for researchers of this field, because it can give answers to many open problems of the number theory.

Finally, the proofs presented here are completely original and were not based on the work of other researchers.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix 1

A second proof why the area $\frac{4}{3}<2\le\lambda<\frac{y}{3}$ *is excluded, which is concerned the problem* "*A very brief solution to Fermat's last theorem*".

If $\lambda\ge 2$, it applies that: $\left(\frac{z}{y}\right)^n=\left(1+\frac{\lambda}{y}\right)^n=1+n\frac{\lambda}{y}+S_{n-1}\ge 1+n\frac{2}{y}+S_{n-1}$, where s_{n-1} is a positive real number which, if we exclude the first two terms, is equal to the sum of the remaining $(n-1)$ terms of the development of the binomial: $\left(1+\frac{\lambda}{y}\right)^n$. Also, since $n>2$ and due to Condition (3.7) it applies that: $y\ge 4$ or $\frac{4}{y}-1\le 0$, we have: $n>2 \Leftrightarrow n\frac{2}{y}-1>2\frac{2}{y}-1 \Leftrightarrow$ $n\frac{2}{y}-1>0\ge\frac{4}{y}-1$ or $n\frac{2}{y}-1>0$. On the contrary, if $n\frac{2}{y}-1\le 0$ then equivalently we have: $\frac{4}{y}-1<n\frac{2}{y}-1\le 0$ or $\frac{4}{y}-1<0$, which is false, since $\frac{4}{y}\le 1$. So, it's true that: $n\frac{2}{y}>1$, $\forall n\ge 3$ and consequently:

$$\left(\frac{z}{y}\right)^n=\left(1+\frac{\lambda}{y}\right)^n\ge 1+n\frac{2}{y}+S_{n-1}>1+1+S_{n-1}>2 \text{ or } \left(\frac{z}{y}\right)^n>2.$$

Therefore, condition (3.2) is not satisfied and Equation (3.1) has no positive integer solutions. Thus, the area $\frac{4}{3}<2\le\lambda<\frac{y}{3}$ is not valid and is excluded.

Appendix 2

Problem 1. Fermat's Last Theorem (Classical Problem)

Continuity of the proof, as it is had given recently by the author of this article and now it is set out here, adding some clarifications.

Based on Condition (1.15), we distinguish the following sub cases:

B₁. $\frac{y}{\lambda}>\frac{y}{y-2}\ge n$

We have: $\frac{y}{y-2}\ge n \Leftrightarrow y\ge yn-2n \Leftrightarrow 2n\ge yn-y$ (because $n>1$) $\Leftrightarrow$ $y\le\frac{2n}{n-1}$ (due to (1.12)), we have: $\lambda n<y\le\frac{2n}{n-1}$ or $\lambda<\frac{2}{n-1}$. Equation (1.1) has positive integer solutions when $1\le\lambda<\frac{2}{n-1}$. So, $1<\frac{2}{n-1}\Leftrightarrow n<3$. While on the contrary, Equation (1.1), has no positive integer solutions when $\lambda=0$ or $\lambda<\frac{2}{n-1}\le 1$. So, $\frac{2}{n-1}\le 1\Leftrightarrow n\ge 3$.

B₂. $\frac{y}{\lambda}>n\ge\frac{y}{y-2}$. We have: $n\ge\frac{y}{y-2}\Leftrightarrow yn-2n\ge y\Leftrightarrow yn-y\ge 2n$ (because $n>1$) $\Leftrightarrow y \geq \frac{2n}{n-1}$ (due to (1.12)), we have: $y \geq \frac{2n}{n-1} > \lambda n$ *** or $\frac{2n}{n-1} > \lambda n \Leftrightarrow \lambda < \frac{2}{n-1}$. So we're being led to the same conclusion as **B**$_1$.

***The inequality $y \geq \frac{2n}{n-1} > \lambda n$, was written this way, with the following reasoning:

Assuming that: $\lambda n \geq \frac{2n}{n-1} \Leftrightarrow \lambda \geq \frac{2}{n-1} > 0$ or $\lambda > 0$, $\forall n \geq 2$. This means, because λ is a positive integer, that $\lambda \geq 1$ for all natural numbers *n* that are greater than or equal to 2 or $n \geq 2$. So Equation (1.1) has solutions for every $n \geq 2$, therefore and when $n = 3$, which is absurd, because $z^3 \neq x^3 + y^3$ and as known Euler was the first to prove it. Also, the same we prove and we, at the problems 3.1 and 3.2, in the case B2, in my article that is published in AMP magazine. Even, in the same article there is more comprehensible and strong proof for the condition $\lambda < y - 2$ or (1.14).

For this, condition $\lambda n \geq \frac{2n}{n-1}$ is false. (Honestly, I wonder why it is not understood. Is there, a stronger absurd than this?)

Thus, for the same, at this point, we present a second proof that is perhaps more rigorous mathematically and non-disputable.

Assuming that $\lambda n \geq \frac{2n}{n-1}$ or $\lambda \geq \frac{2}{n-1}$ are apply the following: Let $x, y, y+\lambda$ positive integers that verify Equation (1.1) for a natural number $n>1$, $y \geq \frac{2n}{n-1}$ and $\left(\frac{z}{y}\right)^n = \left(1+\frac{\lambda}{y}\right)^n = 1 + n\frac{\lambda}{y} + S_{n-1}$, where S_{n-1} is positive real number which, if we exclude the first two terms, is equal to the sum of the remaining $(n-1)$ terms of development of the binomial: $\left(1+\frac{\lambda}{y}\right)^n$.

Then, assuming that $n\frac{\lambda}{y} \geq 1$, because $y \geq \frac{2n}{n-1}$, we have: $n\frac{\lambda}{y} \geq 1 \Leftrightarrow$ $\lambda \geq \frac{y}{n} \geq \frac{2n}{n(n-1)} = \frac{2}{(n-1)}$, which it's true, since $\lambda \geq \frac{2}{n-1}$. So, and the condition $n\frac{\lambda}{y} \geq 1$ it's true. Therefore, it applies that:

$\left(\frac{z}{y}\right)^n = 1 + n\frac{\lambda}{y} + S_{n-1} \geq 1 + 1 + S_{n-1} > 2$ or $\left(\frac{z}{y}\right)^n > 2$. Thus, condition (1.5) is not satisfied and Equation (1.1) has no positive integer solutions. So, condition $\lambda n \geq \frac{2n}{n-1}$ is false.

That is why inequality $\lambda n \geq \frac{2n}{n-1}$ is rejected. Therefore we consider the inequality $\frac{2n}{n-1} > \lambda n$ is acceptable and so we ended up in the inequality:

$y \geq \frac{2n}{n-1} > \lambda n$.

Conclusion 1: From the above it is concluded that Equation (1.1), when $n < 3$ have positive integer solutions, whereas when $n \geq 3$ does not have positive integer solutions. In the second case, Fermat's last theorem is verified.

Problem 2. Generalization of the "Fermat's Last Theorem" (New Theorem)

Continuity of the proof, as it is had given recently by the author of this article and now it is set out here, adding some clarifications.

Based on Condition (2.20) we distinguish the following sub cases:

B₁. $\frac{x_{m-1}}{\lambda} > \frac{x_{m-1}}{x_{m-1}-(m-1)} \geq \frac{n}{m-2}$

We have: $\frac{x_{m-1}}{x_{m-1}-(m-1)} \geq \frac{n}{m-2} \Leftrightarrow (m-2)x_{m-1} \geq nx_{m-1} - n(m-1) \Leftrightarrow$

$n(m-1) \geq (n-m+2)x_{m-1}$ (if $n-m+2>0$ or $n \geq m-1$) $\Leftrightarrow$

$x_{m-1} \leq \frac{n(m-1)}{(n-m+2)}$ (due to (2.10)) we have: $\lambda \frac{n}{m-2} < x_{m-1} \leq \frac{n(m-1)}{n-m+2}$ or

$$\lambda < \frac{(m-1)(m-2)}{n-m+2} \tag{2.21}$$

Based on Condition (2.21) we have:

- When $\lambda \geq 1$, Equation (2.1) has positive integer solutions. So, $1 \leq \lambda < \frac{(m-1)(m-2)}{n-m+2}$ or $1 < \frac{(m-1)(m-2)}{n-m+2} \Leftrightarrow n < m^2 - 2m$. Because $n-m+2>0$ or $n \geq m-1$, finally it applies that $m-1 \leq n < m^2 - 2m$.
- While on the contrary, $\lambda = 0 < 1$ the Equation (1.1) has no positive integer solutions.

  So, $\lambda < \frac{(m-1)(m-2)}{n-m+2} \leq 1$ or $\frac{(m-1)(m-2)}{n-m+2} \leq 1 \Leftrightarrow n \geq m^2 - 2m$.

B₂. $\frac{x_{m-1}}{\lambda} > \frac{n}{m-2} \geq \frac{x_{m-1}}{x_{m-1}-(m-1)}$

We have: $\frac{n}{m-2} \geq \frac{x_{m-1}}{x_{m-1}-(m-1)} \Leftrightarrow nx_{m-1} - n(m-1) \geq (m-2)x_{m-1} \Leftrightarrow$

$(n-m+2)x_{m-1} \geq n(m-1)$ (if, $n-m+2>0$ or $n \geq m-1$) $\Leftrightarrow$

$x_{m-1} \geq \frac{n(m-1)}{n-m+2}$ (due to (2.10)) we have: $x_{m-1} \geq \frac{n(m-1)}{n-m+2} > \frac{n}{m-2}\lambda$ *** or

$\frac{(m-1)(m-2)}{n-m+2} > \lambda$. So we're being led to the same conclusion as **B₁**.

***The inequality $x_{m-1} \geq \frac{n(m-1)}{(n-m+2)} > \frac{n}{m-2}\lambda$, was written this way, with the following reasoning:

Assuming that: $\frac{n}{m-2}\lambda \geq \frac{n(m-1)}{(n-m+2)} \Leftrightarrow \lambda \geq \frac{(m-1)(m-2)}{(n-m+2)} > 0$ or $\lambda > 0$,

$\forall n \geq 2$ and $\forall m \geq 3$. This means, because λ is a positive integer, that $\lambda \geq 1$ for all natural numbers n that are greater than or equal to 2 or $n \geq 2$ and for all the positive integers m that are greater than or equal to 3 or $m \geq 3$. So Equation (2.1) has solutions for every $n \geq 2$ and $m \geq 3$, therefore and when $n = 3$ and $m = 3$, which is absurd, because $(x_3)^3 \neq (x_2)^3 + (x_1)^3$ and as known Euler was the first to prove it. Also, the same we prove and we, at the problems 3.1 and 3.2, in the case B2, in my article that this published in AMP magazine. Even, in the same article there is more comprehensible and strong proof for the condition $\lambda < x_{m-1} - (m-1)$ or (2.19).

For this, condition $\frac{n}{m-2}\lambda \geq \frac{n(m-1)}{(n-m+2)}$ is false. (Honestly, I wonder why it is not understood. Is there, a stronger absurd than this?)

Thus, at this point for the same, we present a second proof that is perhaps more rigorous mathematically and non-disputable.

Assuming that $\frac{n}{m-2}\lambda \geq \frac{n(m-1)}{(n-m+2)}$ or $\lambda \geq \frac{(m-1)(m-2)}{(n-m+2)}$, then are apply the following: Let $x_1, x_2, x_3, \cdots, x_{m-1}, x_{m-1}+1$ positive integers that verify Equation (2.1) for a natural number $n > 1$, $x_{m-1} \geq \frac{n(m-1)}{n-m+2}$ and

$\left(\frac{x_m}{x_{m-1}}\right)^n = \left(1 + \frac{\lambda}{x_{m-1}}\right)^n = 1 + n\frac{\lambda}{x_{m-1}} + S_{n-1}$, where S_{n-1} is positive real number which, if we exclude the first two terms, is equal to the sum of the remaining $(n-1)$ terms of development of binomial: $\left(1 + \frac{\lambda}{x_{m-1}}\right)^n$.

Then, assuming that $n\frac{\lambda}{x_{m-1}} \geq m-2$, because $x_{m-1} \geq \frac{n(m-1)}{n-m+2}$, we have:

$$n\frac{\lambda}{x_{m-1}} \geq m-2 \iff \lambda \geq \frac{x_{m-1}}{n}(m-2) \geq \frac{n(m-1)}{n(n-m+2)}(m-2) = \frac{(m-1)(m-2)}{n-m+2},$$

which it's true, since $\lambda \geq \frac{(m-1)(m-2)}{n-m+2}$. Therefore, condition $n\frac{\lambda}{x_{m-1}} \geq m-2$ it's true and consequently it applies that:

$\left(\frac{x_m}{x_{m-1}}\right)^n = \left(1 + \frac{\lambda}{x_{m-1}}\right)^n = 1 + n\frac{\lambda}{x_{m-1}} + S_{n-1} \geq 1 + (m-2) + S_{n-1} > m-1$. So, because $\left(\frac{x_m}{x_{m-1}}\right)^n > m-1$, condition (2.5) is not satisfied and Equation (2.1) has no positive integer solutions. Thus, condition $\frac{n}{m-2}\lambda \geq \frac{n(m-1)}{(n-m+2)}$ it's false.

That is why inequality $\frac{n}{m-2}\lambda \geq \frac{n(m-1)}{(n-m+2)}$ is rejected. Therefore we consider the inequality $\frac{n(m-1)}{(n-m+2)} > \frac{n}{m-2}\lambda$ is acceptable and so we ended up in the inequality $x_{m-1} \ge \frac{n(m-1)}{(n-m+2)} > \frac{n}{m-2}\lambda$.

Conclusion 2: From the above it is concluded that the Equation (2.1) when $n < m^2 - 2m$ have integer solutions, whereas when $n \ge m^2 - 2m$ have no integer solutions. In the second case, for $m = 3$, answer to Fermat's Last Theorem is given.

Analysis of Results

1) From condition $n \ge m^2 - 2m$, if $m = 3$ we have: $n \ge 3^2 - 2\times 3 = 3$. We observe that the proof of Fermat's last theorem is taking place. This is a very strong indication that its generalization is correct.

2) According to the previous analysis is $\lambda < \frac{(m-1)(m-2)}{n-m+2}$, because if the opposite is true or $\lambda \ge \frac{(m-1)(m-2)}{n-m+2}$, Equation (2.1) do not have integer positive solutions.

Therefore, in this case we have:

i) If $\lambda < \frac{(m-1)(m-2)}{n-m+2} \le 1$ or $\frac{(m-1)(m-2)}{n-m+2} \le 1 \Leftrightarrow n \ge m^2 - 2m$ and Equation (2.1) has no positive integer solutions.

ii) Whereas, if $\frac{(m-1)(m-2)}{n-m+2} > \lambda \ge 1$ it is $m-1 \le n < m^2 - 2m$ and so Equation (2.1) can have positive integer solutions.

3) What happens if $n-m+2 \le 0$ or $1 < n \le m-2 < m-1$ and $m \ge 4$?

3.1) $n-m+2 \le 0$ and $n \ge 3$, $m \ge n+2 \ge 3+2 = 5$ or $3 \le n \le m-2 < m-1$, $m \ge 5$.

First we will prove that if $n \ge 3$ and n a natural number, it is true that: $1+\frac{1}{n} < n^{\frac{1}{n}}$.

Proof: If $1+\frac{1}{n} < n^{\frac{1}{n}} \Leftrightarrow \ln(n+1) - \ln(n) < \frac{1}{n}\ln n \Leftrightarrow \frac{\ln(n+1)}{n+1} < \frac{\ln n}{n}$.

Considering the real function $f(x) = \frac{\ln x}{x}$, $x \ge 3$, we have:

$$f'(x) = \frac{1-\ln x}{x} < 0, \quad x \ge 3.$$

So the function $f(x)$ is genuine decreasing. Therefore, it's true that: $\frac{\ln(x+1)}{x+1} < \frac{\ln x}{x}$, $\forall x \ge 3$ and consequently: $\frac{\ln(n+1)}{n+1} < \frac{\ln n}{n}$, $\forall n \ge 3$. Finally, as a consequence of the last condition, it is also true and that: $1+\frac{1}{n} < n^{\frac{1}{n}}$, where $n \ge 3$, $n \in N$.

Then, given that $1+\frac{1}{n} < n^{\frac{1}{n}}$, $3 \le n < m-1$ and Condition (2.10), we have:

$$\frac{\lambda}{x_{m-1}} < \frac{1}{n} < n^{\frac{1}{n}} - 1 < (m-1)^{\frac{1}{n}} - 1 \quad \text{or} \quad \frac{\lambda}{x_{m-1}} < (m-1)^{\frac{1}{n}} - 1 \Leftrightarrow$$

$$1+\frac{\lambda}{x_{m-1}}<(m-1)^{\frac{1}{n}} \text{ or } \left(\frac{x_m}{x_{m-1}}\right)^n<m-1.$$

3.2) When $n-m+2\leq 0$ and $n=2$, $m\geq n+2=2+2=4$ or $2\leq n\leq m-2<m-1$ and $m\geq 4$ taking account condition (2.10) or $x_{m-1}>\lambda n\geq 1\times 2=2$ or $\frac{\lambda}{x_{m-1}}<\frac{1}{2}$, we have:

$\left(1+\frac{\lambda}{x_{m-1}}\right)^2=1+2\frac{\lambda}{x_{m-1}}+\left(\frac{\lambda}{x_{m-1}}\right)^2<1+2\frac{1}{2}+\left(\frac{1}{2}\right)^2=\frac{9}{4}<3$ or $\left(1+\frac{\lambda}{x_{m-1}}\right)^2<3$ or (because $n=2$ and $m-1\geq 3$): $\left(1+\frac{\lambda}{x_{m-1}}\right)^n<m-1$. Also, it's true and the condition: $\left(\frac{x_m}{x_1}\right)^n=\left(\frac{x_{m-1}}{x_1}\right)^n+\cdots+\left(\frac{x_2}{x_1}\right)^n+1>1+\cdots+1+1=m-1$ or $\left(\frac{x_m}{x_1}\right)^n>m-1$.

Therefore, due to the last conditions when $n-m+2\leq 0$ the double inequality (2.5) can be satisfied and so Equation (2.1) can have positive integer solutions.

For example if $x_1=3$, $x_2=4$, $x_3=12$, $x_4=13$, $\lambda=1$, $n=2$ and $m=4$. We have, $n-m+2=2-4+2=0$, $3^2+4^2+12^2=13^2$ and $\left(\frac{13}{12}\right)^2\cong 1.18<3<18.78=\left(\frac{13}{3}\right)^2$.