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Boundary Control for Cooperative Elliptic Systems under Conjugation Conditions

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Abstract

The existence and uniqueness of the state for 2×2 Dirichlet cooperative elliptic systems under conjugation conditions are proved using Lax-Milgram lemma, then the boundary control for these systems is discussed. The set of equations and inequalities that characterizes this boundary control is found by theory of Lions, Sergienko and Deineka. The problem for cooperative Neumann elliptic systems under conjugation conditions is also considered. Finally, the problem for $n \times n$ cooperative elliptic systems under conjugation conditions is established.

Keywords

Cooperative Systems, Conjugation Conditions, Dirichlet and Neumann Conditions, Existence and Uniqueness of Solutions, Boundary Control

1. Introduction

In today's rapidly progressing science and technology, the field of control theory is at the forefront of the creative interplay of mathematics, engineering, and computer science.

Control theory has two objectives:

To understand the fundamental principle of control, and to characterize them mathematically.

The control problem is to choose the control space U to minimize an energy functional $J(u)$, subject to constraints on the control such as U_{ad} (set of admissible control) is a closed convex subset of U.

Various optimal control problems, of systems governed by finite order elliptic, parabolic and hyperbolic operators with finite number of variables have been in-

troduced by Lions [1]. These problems have been extended to non-cooperative systems in [2] [3] and cooperative systems in [4] [5] [6] [7]. The control problems for infinite order hyperbolic operators have been studied in [8] [9].

Some existence results have been established for nonlinear systems in [10] [11] [12] [13] [14].

Some applications for control problems were introduced for example in [2] [15].

New optimal control problems of distributed systems described by an elliptic, parabolic and hyperbolic operators with conjugation conditions and by a quadratic cost functional have been studied by Sergienko and Deineka [16] [17] [18].

In the present work, using the theory of Lions [1], Sergienko and Deineka [16] [17] [18], the boundary control for some cooperative elliptic systems of the form

$$-\Delta h_i = \sum_{j=1}^{n} a_{ij} h_j + f_i \qquad \text{in } \Omega, \ i, j = 1, 2, 3, \cdots, n, \tag{1}$$

under conjugation conditions is discussed, where

$$a_{ij} > 0 \quad \forall i \neq j. \tag{2}$$

System (1), where (2) is satisfied is called cooperative system. Such systems appear in some biological and physical problems [19].

Our paper is organized as follows: In Section 2, we first prove the existence and uniqueness of the state for 2×2 Dirichlet cooperative system under conjugation conditions, then we study the optimal boundary control of this system. Section 3 is devoted to discuss the boundary control for 2×2 Neumann cooperative elliptic system under conjugation conditions. In Section 4, we generalize the discussion which has been introduced in Section 2, to $n \times n$ Dirichlet cooperative system with conjugation conditions. Finally in Section 5, we generalize the problem which has been established in Section 3 to $n \times n$ Neumann cooperative elliptic system under conjugation conditions.

2. Boundary Control for 2 × 2 Dirichlet Elliptic Systems

In this section, we study the boundary control for the following 2×2 cooperative Dirichlet elliptic system

$$\begin{cases} -\Delta h_1 = a h_1 + b h_2 + f_1 & \text{in } \Omega, \\ -\Delta h_2 = c h_1 + d h_2 + f_2 & \text{in } \Omega, \\ h_1 = h_2 = 0 & \text{on } \Gamma, \end{cases} \tag{3}$$

under conjugation conditions:

$$\begin{cases} R_1 \left\{ \dfrac{\partial h_1}{\partial v_A} \right\}^- + R_2 \left\{ \dfrac{\partial h_1}{\partial v_A} \right\}^+ = [h_1] + \delta & \text{on } \gamma, \\[4mm] R_1 \left\{ \dfrac{\partial h_2}{\partial v_A} \right\}^- + R_2 \left\{ \dfrac{\partial h_2}{\partial v_A} \right\}^+ = [h_2] + \delta & \text{on } \gamma, \end{cases} \tag{4}$$

$$\begin{cases} \left[\dfrac{\partial h_1}{\partial v_A} \right] = \left[\displaystyle\sum_{i,j=1}^{n} \dfrac{\partial h_1}{\partial x_j} \cos\left(v, x_i\right) \right] = w_1 \qquad \text{on } \gamma, \\[4mm] \left[\dfrac{\partial h_2}{\partial v_A} \right] = \left[\displaystyle\sum_{i,j=1}^{n} \dfrac{\partial h_2}{\partial x_j} \cos\left(v, x_i\right) \right] = w_2 \qquad \text{on } \gamma, \end{cases} \tag{5}$$

where

a, b, c and d are given numbers such that $b, c > 0$,

and

$$R_1, R_2, w, \delta \in C\left(\gamma\right), \ R_1, R_2 \geq 0, \ R_1 + R_2 \geq R_0 > 0, \ R_0 = \text{constant}. \tag{6}$$

We first prove the existence of the state of systems (3) under the following conditions:

$$\begin{cases} a < \mu, \ d < \mu, \\ \left(\mu - a\right)\left(\mu - d\right) > bc, \end{cases} \tag{7}$$

where μ is a positive constant determined by Friedrich inequality:

$$\mu \int_{\Omega} |h|^2 \, \mathrm{d}x \leq \int_{\Omega} |\nabla h|^2 \, \mathrm{d}x. \tag{8}$$

Then, we prove the existence of boundary control for this system and we find the set of equations and inequalities that characterizes this boundary control.

Existence and uniqueness of the state

By Cartesian product, we have the following chain of Sobolev spaces:

$$\left(H_0^1\left(\Omega\right)\right)^2 \subseteq \left(L^2\left(\Omega\right)\right)^2 \subseteq \left(H^{-1}\left(\Omega\right)\right)^2.$$

On $\left(H_0^1\left(\Omega\right)\right)^2$, we define the bilinear form:

$$\begin{aligned} a\left(h, \psi\right) = {} & \int_{\Omega} \nabla h_1 \nabla \psi_1 \mathrm{d}x + \int_{\Omega} \nabla h_2 \nabla \psi_2 \mathrm{d}x \\ & - \int_{\Omega} \left(ah_1\psi_1 + bh_2\psi_1 + ch_1\psi_2 + dh_2\psi_2\right) \mathrm{d}x \\ & + \int_{\gamma} \frac{[h_1][\psi_1]}{R_1 + R_2} \mathrm{d}\gamma + \int_{\gamma} \frac{[h_2][\psi_2]}{R_1 + R_2} \mathrm{d}\gamma. \end{aligned} \tag{9}$$

Then, we have

Lemma 2.1 *The bilinear form* (9) *is coercive on* $\left(H_0^1\left(\Omega\right)\right)^2$; *that is, there exists a positive constant C such that*

$$a\left(h, h\right) \geq C \|h\|^2_{\left[H_0^1(\Omega)\right]^2} \qquad \forall h = \{h_1, h_2\} \in \left(H_0^1\left(\Omega\right)\right)^2. \tag{10}$$

Proof.

As in [19], we choose m is large enough such that $a + m > 0$ and $d + m > 0$. Then,

$$\begin{aligned} a\left(h, h\right) = {} & \frac{1}{b} \int_{\Omega} \left(|\nabla h_1|^2 + m|h_1|^2\right) \mathrm{d}x + \frac{1}{c} \int_{\Omega} \left(|\nabla h_2|^2 + m|h_2|^2\right) \mathrm{d}x \\ & - \frac{a + m}{b} \int_{\Omega} |h_1|^2 \, \mathrm{d}x - \frac{d + m}{c} \int_{\Omega} |h_2|^2 \, \mathrm{d}x - 2\int_{\Omega} h_1 h_2 \mathrm{d}x \\ & + \int_{\gamma} \frac{[h_1][\psi_1]}{R_1 + R_2} \mathrm{d}\gamma + \int_{\gamma} \frac{[h_2][\psi_2]}{R_1 + R_2} \mathrm{d}\gamma. \end{aligned}$$

From (6), we get

$$a(h,h) \geq \frac{1}{b}\int_\Omega \left(|\nabla h_1|^2 + m|h_1|^2\right)dx + \frac{1}{c}\int_\Omega \left(|\nabla h_2|^2 + m|h_2|^2\right)dx$$
$$-\frac{a+m}{b}\int_\Omega |h_1|^2\,dx - \frac{d+m}{c}\int_\Omega |h_2|^2\,dx - 2\int_\Omega h_1 h_2 dx.$$

By Cauchy Schwartz inequality

$$a(h,h) \geq \frac{1}{b}\int_\Omega \left(|\nabla h_1|^2 + m|h_1|^2\right)dx + \frac{1}{c}\int_\Omega \left(|\nabla h_2|^2 + m|h_2|^2\right)dx$$
$$-\frac{a+m}{b}\int_\Omega |h_1|^2\,dx - \frac{d+m}{c}\int_\Omega |h_2|^2\,dx$$
$$-2\left(\int_\Omega |h_1|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |h_2|^2\,dx\right)^{\frac{1}{2}}.$$

From (8), we deduce

$$a(h,h) \geq \frac{1}{b}\left(1 - \frac{a+m}{\mu+m}\right)\|h_1\|^2 + \frac{1}{c}\left(1 - \frac{d+m}{\mu+m}\right)\|h_2\|^2 - \frac{2}{\mu+m}\|h_1\|\|h_2\|.$$

Therefore (7) implies

$$a(h,h) \geq C\left(\|h_1\|^2 + \|h_2\|^2\right) = C\|h\|^2_{\left[H_0^1(\Omega)\right]^2} \qquad \forall h \in \left(H_0^1(\Omega)\right)^2,$$

which proves the coerciveness condition of the bilinear form (2.7). Then using Lax-Milgram lemma, we can prove the following theorem:

Theorem 2.1 *For a given* $f = (f_1, f_2) \in \left(L^2(\Omega)\right)^2$ *there exists a unique solution* $h = (h_1, h_2) \in \left(H_0^1(\Omega)\right)^2$ *for systems* (3) *with conjugation conditions* (4) *and* (5) *if conditions* (7) *are satisfied.*

Formulation of the control problem

The space $U = \left(L^2(\Gamma)\right)^2$ is the space of controls. For a control $u = \{u_1, u_2\} \in \left(L^2(\Gamma)\right)^2$, the state $h(u) = \{h_1(u), h_2(u)\}$ of the system is given by the solution of

$$\begin{cases} -\Delta h_1(u) = ah_1(u) + bh_2(u) + f_1 & \text{in } \Omega, \\ -\Delta h_2(u) = ch_1(u) + dh_2(u) + f_2 & \text{in } \Omega, \\ h_1(u) = u_1, h_2(u) = u_2 & \text{on } \Gamma, \end{cases} \qquad (11)$$

under conjugation conditions

$$\begin{cases} R_1 \left\{\frac{\partial h_1(u)}{\partial v_A}\right\}^- + R_2\left\{\frac{\partial h_1(u)}{\partial v_A}\right\}^+ = \left[h_1(u)\right] + \delta & \text{on } \gamma, \\[3mm] R_1\left\{\frac{\partial h_2(u)}{\partial v_A}\right\}^- + R_2\left\{\frac{\partial h_2(u)}{\partial v_A}\right\}^+ = \left[h_2(u)\right] + \delta & \text{on } \gamma, \end{cases} \qquad (12)$$

$$\begin{cases} \left[\frac{\partial h_1(u)}{\partial v_A}\right] = \left[\sum_{i,j=1}^{n} \frac{\partial h_1(u)}{\partial x_j}\cos(v, x_i)\right] = w_1 & \text{on } \gamma, \\[3mm] \left[\frac{\partial h_2(u)}{\partial v_A}\right] = \left[\sum_{i,j=1}^{n} \frac{\partial h_2(u)}{\partial x_j}\cos(v, x_i)\right] = w_2 & \text{on } \gamma. \end{cases} \qquad (13)$$

The observation equation is given by:

$$z(u) = \{z_1(u), z_2(u)\} = Ch(u) = C\{h_1(u), h_2(u)\} = \{h_1(u), h_2(u)\}$$

For a given $z_d = \{z_{1d}, z_{2d}\} \in \left(L^2(\Gamma)\right)^2$, the cost functional is given by

$$J(v) = \left\|\frac{\partial h_1(v)}{\partial v_A} - z_{1d}\right\|_{L^2(\Gamma)}^2 + \left\|\frac{\partial h_2(v)}{\partial v_A} - z_{2d}\right\|_{L^2(\Gamma)}^2 + (Nv, v)_{\left(L^2(\Gamma)\right)^2}, \qquad (14)$$

where N is a hermitian positive definite operator such that:

$$(Nv, v)_{\left(L^2(\Gamma)\right)^2} \geq Q\|v\|_{\left(L^2(\Gamma)\right)^2}^2, \quad Q > 0. \qquad (15)$$

The control problem then is to:

$$\begin{cases} \text{Find } u = \{u_1, u_2\} \in U_{ad} \left(\text{closed convex subset of } \left(L^2(\Gamma)\right)^2\right) \text{ such that :} \\ J(u) = \inf J(v) \quad \forall v \in U_{ad}. \end{cases} \qquad (16)$$

The cost functional (14) can be written as

$$J(v) = \left\|\frac{\partial h_1(v)}{\partial v_A} - \frac{\partial h_1(0)}{\partial v_A} + \frac{\partial h_1(0)}{\partial v_A} - z_{1d}\right\|_{L^2(\Gamma)}^2$$

$$+ \left\|\frac{\partial h_2(v)}{\partial v_A} - \frac{\partial h_2(0)}{\partial v_A} + \frac{\partial h_2(0)}{\partial v_A} - z_{2d}\right\|_{L^2(\Gamma)}^2$$

$$+ (Nv, v)_{\left(L^2(\Gamma)\right)^2}.$$

If we let:

$$\pi(u, v) = \left(\frac{\partial h_1(u)}{\partial v_A} - \frac{\partial h_1(0)}{\partial v_A}, \frac{\partial h_1(v)}{\partial v_A} - \frac{\partial h_1(0)}{\partial v_A}\right)_{L^2(\Gamma)}$$

$$+ \left(\frac{\partial h_2(u)}{\partial v_A} - \frac{\partial h_2(0)}{\partial v_A}, \frac{\partial h_2(v)}{\partial v_A} - \frac{\partial h_2(0)}{\partial v_A}\right)_{L^2(\Gamma)}$$

$$+ (Nu, v)_{\left(L^2(\Gamma)\right)^2}.$$

and

$$f(v) = \left(z_{1d} - \frac{\partial h_1(0)}{\partial v_A}, \frac{\partial h_1(v)}{\partial v_A} - \frac{\partial h_1(0)}{\partial v_A}\right)_{L^2(\Gamma)}$$

$$+ \left(z_{2d} - \frac{\partial h_2(0)}{\partial v_A}, \frac{\partial h_2(v)}{\partial v_A} - \frac{\partial h_2(0)}{\partial v_A}\right)_{L^2(\Gamma)}.$$

Then

$$J(v) = \pi(v, v) - 2f(v) + \left\|z_{1d} - \frac{\partial h_1(0)}{\partial v_A}\right\|_{L^2(\Gamma)}^2 + \left\|z_{2d} - \frac{\partial h_2(0)}{\partial v_A}\right\|_{L^2(\Gamma)}^2.$$

From (15), $\pi(v, v) \geq M\|v\|_{\left(L^2(\Gamma)\right)^2}^2$. Using the theory of Lions [1], there exists a unique optimal control of problem (16), moreover it is characterized by

Theorem 2.2 *Let us suppose that* (10) *holds and the cost functional is given*

---

by (14), *then the boundary control u is characterized by*

$$
\begin{cases}
-\Delta p_1(u) - a p_1(u) - c p_2(u) = 0 & \text{in } \Omega, \\[4pt]
-\Delta p_2(u) - b p_1(u) - d p_2(u) = 0 & \text{in } \Omega, \\[6pt]
p_1(u) = -\left(\dfrac{\partial h_1(u)}{\partial v_A} - z_{1d} \right), \quad p_2(u) = -\left(\dfrac{\partial h_2(u)}{\partial v_A} - z_{2d} \right) & \text{on } \Gamma, \\[10pt]
\left[\displaystyle\sum_{i,j}^{n} \dfrac{\partial p_1(u)}{\partial x_j} \cos(v, x_i) \right] = \left[\displaystyle\sum_{i,j}^{n} \dfrac{\partial p_2(u)}{\partial x_j} \cos(v, x_i) \right] = 0 & \text{on } \gamma, \\[12pt]
\left\{ \dfrac{\partial p_1(u)}{\partial v_A} \right\}^{\pm} = \dfrac{1}{R_1 + R_2}\big[p_1(u) \big], \quad \left\{ \dfrac{\partial p_2(u)}{\partial v_A} \right\}^{\pm} = \dfrac{1}{R_1 + R_2}\big[p_2(u) \big] & \text{on } \gamma, \\[12pt]
\left(p_1(u), \dfrac{\partial h_1(v-u)}{\partial v_A} - \dfrac{\partial h_1(0)}{\partial v_A} \right)_{L^2(\Gamma)} \\[12pt]
+ \left(p_2(u), \dfrac{\partial h_2(v-u)}{\partial v_A} - \dfrac{\partial h_2(0)}{\partial v_A} \right)_{L^2(\Gamma)} + \left(Nu, v-u \right)_{(L^2(\Gamma))^2} \geq 0,
\end{cases}
$$

together with (11), (12) and (13), where $p(u) = \{ p_1(u), p_2(u) \}$ is the adjoint state.

Proof.

The optimal control u is characterized by [1]

$$
\pi(u, v-u) - L(v-u) \geq 0 \qquad \forall u \in U_{ad}.
$$

Then

$$
\left(\frac{\partial h_1(u)}{\partial v_A} - \frac{\partial h_1(0)}{\partial v_A}, \frac{\partial h_1(v-u)}{\partial v_A} - \frac{\partial h_1(0)}{\partial v_A} \right)_{L^2(\Gamma)}
$$

$$
+ \left(\frac{\partial h_2(u)}{\partial v_A} - \frac{\partial h_2(0)}{\partial v_A}, \frac{\partial h_2(v-u)}{\partial v_A} - \frac{\partial h_2(0)}{\partial v_A} \right)_{L^2(\Gamma)} + \left(Nu, v-u \right)_{(L^2(\Gamma))^2}
$$

$$
- \left(z_{1d} - \frac{\partial h_1(0)}{\partial v_A}, \frac{\partial h_1(v-u)}{\partial v_A} - \frac{\partial h_1(0)}{\partial v_A} \right)_{L^2(\Gamma)}
$$

$$
- \left(z_{2d} - \frac{\partial h_2(0)}{\partial v_A}, \frac{\partial h_2(v-u)}{\partial v_A} - \frac{\partial h_2(0)}{\partial v_A} \right)_{L^2(\Gamma)} \geq 0.
$$

So

$$
\left(\frac{\partial h_1(u)}{\partial v_A} - z_{1d}, \frac{\partial h_1(v-u)}{\partial v_A} - \frac{\partial h_1(0)}{\partial v_A} \right)_{L^2(\Gamma)}
$$

$$
+ \left(\frac{\partial h_2(u)}{\partial v_A} - z_{2d}, \frac{\partial h_2(v-u)}{\partial v_A} - \frac{\partial h_2(0)}{\partial v_A} \right)_{L^2(\Gamma)} + \left(Nu, v-u \right)_{(L^2(\Gamma))^2} \geq 0. \tag{17}
$$

Since the model A of the system is given by

$$
Ah(x) = A(h_1, h_2) = \left(-\Delta h_1 - a h_1 - b h_2,\ -\Delta h_2 - c h_1 - d h_2 \right),
$$

and since

$$
\left(A^* p, h \right) = \left(p, Ah \right),
$$

then

$$\left(p(u), Ah(u) \right)$$
$$= \left(p_1(u), -\Delta h_1(u) - ah_1(u) - bh_2(u) \right) + \left(p_2(u), -\Delta h_2(u) - ch_1(u) - dh_2(u) \right)$$
$$= \left(-\Delta p_1(u) - ap_1(u) - cp_2(u), h_1(u) \right) + \left(-\Delta p_2(u) - bp_1(u) - dp_2(u), h_2(u) \right)$$

and since the adjoint state is defined by:

$$\begin{cases} -\Delta p_1(u) - ap_1(u) - cp_2(u) = 0 & \text{in } \Omega, \\ -\Delta p_2(u) - bp_1(u) - dp_2(u) = 0 & \text{in } \Omega, \\ p_1(u) = -\left(\dfrac{\partial h_1(u)}{\partial v_A} - z_{1d} \right) & \text{on } \Gamma, \\ p_2(u) = -\left(\dfrac{\partial h_2(u)}{\partial v_A} - z_{2d} \right) & \text{on } \Gamma, \\ \left[\dfrac{\partial p_1(u)}{\partial v_{A^*}} \right] = \left[\dfrac{\partial p_2(u)}{\partial v_{A^*}} \right] = 0 & \text{on } \gamma, \\ \left\{ \dfrac{\partial p_1(u)}{\partial v_{A^*}} \right\}^{\pm} = \dfrac{1}{R_1 + R_2} \left[p_1(u) \right], \ \left\{ \dfrac{\partial p_2(u)}{\partial v_{A^*}} \right\}^{\pm} = \dfrac{1}{R_1 + R_2} \left[p_2(u) \right] & \text{on } \gamma, \end{cases}$$

hence (17) implies

$$\left(p_1(u), \frac{\partial h_1(v-u)}{\partial v_A} - \frac{\partial h_1(0)}{\partial v_A} \right)_{L^2(\Gamma)}$$
$$+ \left(p_2(u), \frac{\partial h_2(v-u)}{\partial v_A} - \frac{\partial h_2(0)}{\partial v_A} \right)_{L^2(\Gamma)} + \left(Nu, v-u \right)_{(L^2(\Gamma))^2} \geq 0.$$

3. Boundary Control for 2 × 2 Neumann Elliptic Systems

In this section, we study the boundary control for 2×2 cooperative Neumann elliptic system in the form

$$\begin{cases} -\Delta h_1 = ah_1 + bh_1 + f_1 & \text{in } \Omega \subset R^n, \\ -\Delta h_2 = ch_1 + dh_2 + f_2 & \text{in } \Omega \subset R^n, \\ \dfrac{\partial h_1}{\partial v_A} = g_1, \dfrac{\partial h_2}{\partial v_A} = g_2 & \text{on } \Gamma, \end{cases} \tag{18}$$

with conjugation conditions (4) and (5), where $g_i \in \left(L^2(\Gamma) \right)^2$, $i = 1, 2$. For this, we introduce again the bilinear form (9) which is coercive on $\left(H^1(\Omega) \right)^2$, since

$$\left(H_0^1(\Omega) \right)^2 \subseteq \left(H^1(\Omega) \right)^2.$$

Then, by Lax-Milgram lemma, there exists a unique solution $h = (h_1, h_2)$ for system (18) such that:

$$a(h, \psi) = L(\psi) \quad \forall \psi \in \left(H^1(\Omega) \right)^2, \tag{19}$$

where

$$L_g(\psi) = \int_\Omega f_1(x)\psi_1(x)dx + \int_\Omega f_2(x)\psi_2(x)dx$$

$$+ \int_\Gamma g_1(x)\psi_1(x)d\Gamma + \int_\Gamma g_2(x)\psi_2(x)d\Gamma + \int_\gamma \frac{(R_2 w_1 - \delta)[\psi_1]}{R_1 + R_2}d\gamma$$

$$+ \int_\gamma \frac{(R_2 w_2 - \delta)[\psi_2]}{R_1 + R_2}d\gamma - \int_\gamma w_1 \psi_1^+ d\gamma - \int_\gamma w_2 \psi_2^+ d\gamma,$$

is a continuous linear form defined on $\left(H^1(\Omega)\right)^2$. Then, applying Green's formula, we get

$$\int_\Omega (-\Delta h_1 - ah_1 - bh_2)\psi_1(x)dx + \int_\Omega (-\Delta h_2 - ch_1 - dh_2)\psi_2(x)dx$$

$$= -\int_\Gamma \frac{\partial h_1}{\partial v_A}\psi_1(x)d\Gamma - \int_\Gamma \frac{\partial h_2}{\partial v_A}\psi_2(x)d\Gamma - \int_\gamma \frac{\partial h_1}{\partial v_A}\psi_1(x)d\gamma$$

$$-\int_\gamma \frac{\partial h_2}{\partial v_A}\psi_2(x)d\gamma + a(h,\psi) = \int_\Omega f_1\psi_1 dx + \int_\Omega f_2\psi_2 dx,$$

using (19), the state $h(u) = (h_1(u), h_2(u))$ of the system is given by the solution of

$$(20) \quad \begin{cases} -\Delta h_1(u) = ah_1(u) + bh_2(u) + f_1 & \text{in } \Omega = \Omega_1 \cup \Omega_2, \\ -\Delta h_2(u) = ch_1(u) + dh_2(u) + f_2 & \text{in } \Omega = \Omega_1 \cup \Omega_2, \\ \dfrac{\partial h_1(u)}{\partial v_A} = g_1 + u_1 & \text{on } \Gamma, \\ \dfrac{\partial h_2(u)}{\partial v_A} = g_2 + u_2 & \text{on } \Gamma, \end{cases}$$

under conjugation conditions (12), (13). For a given $z_d = (z_{1d}, z_{2d}) \in \left(L^2(\Gamma)\right)^2$, the cost functional is again given by (14), then there exists a unique optimal control $u \in U_{ad}$ for (16) and we deduce:

Theorem 3.1 *If the cost functional is given by* (14), *there exists a unique boundary control*
$u = (u_1, u_2) \in \left(L^2(\Gamma)\right)^2$, such that:

$$J(u) \leq J(v) \quad \forall v \in U_{ad} \subset \left(L^2(\Gamma)\right)^2,$$

moreover it is characterized by the following equations and inequalities

$$\begin{cases} -\Delta p_1(u) - ap_1(u) - cp_2(u) = 0 & \text{in } \Omega, \\ -\Delta p_2(u) - bp_1(u) - dp_2(u) = 0 & \text{in } \Omega, \\ \dfrac{\partial p_1(u)}{\partial v_{A^*}} = h_1(u) - z_{1d} & \text{on } \Gamma, \\ \dfrac{\partial p_2(u)}{\partial v_{A^*}} = h_2(u) - z_{2d} & \text{on } \Gamma, \\ \left[\dfrac{\partial p_1(u)}{\partial v_{A^*}}\right] = \left[\dfrac{\partial p_2(u)}{\partial v_{A^*}}\right] = 0 & \text{on } \gamma, \\ \left\{\dfrac{\partial p_1}{\partial v_{A^*}}\right\}^\pm = \dfrac{1}{R_1 + R_2}[p_1(u)], \; \left\{\dfrac{\partial p_2(u)}{\partial v_{A^*}}\right\}^\pm = \dfrac{1}{R_1 + R_2}[p_2(u)] & \text{on } \gamma, \end{cases}$$

$$\int_\Gamma \left(p_1(u) + Nu_1 \right)\left(v_1 - u_1 \right) d\Gamma + \int_\Gamma \left(p_2(u) + Nu_2 \right)\left(v_2 - u_2 \right) d\Gamma \geq 0.$$

Together with (20), (12) and (13).

4. Boundary Control for $n \times n$ Cooperative Dirichlet Systems

In this section, we generalize the discussion which has been introduced in section 2 to $n \times n$ cooperative Dirichlet system of the form

$$\begin{cases} -\Delta h_i + \sum_{j=1}^n a_{ij} h_j = f_i & \text{in } \Omega, \\ h_i = 0 & \text{on } \Gamma, \ i = 1, 2, \cdots, n, \end{cases} \tag{21}$$

under conjugation conditions

$$R_1 \left\{ \frac{\partial h_i}{\partial v_A} \right\}^- + R_2 \left\{ \frac{\partial h_i}{\partial v_A} \right\}^+ = [h_i] + \delta \quad \text{on } \gamma, \tag{22}$$

and

$$\left[\frac{\partial h_i}{\partial v_A} \right] = \left[\sum_{i,j=1}^n \frac{\partial h_i}{\partial x_j} \cos(v, x_i) \right] = w_i \quad \text{on } \gamma, \quad i = 1, 2, \cdots, n. \tag{23}$$

To prove the existence of the state of system (21), we assume that:

$$\begin{cases} \text{the matrex } (\mu - MI) \text{ is a non-singular } M\text{-matrix which means} \\ \text{that all the principal minors extracted from it are positive,} \end{cases} \tag{24}$$

where, I is identity matrix and μ is a positive constant determined by Friedrich inequality(8).

By Cartesian product, we have the following chain of Sobolev spaces:

$$\left(H_0^1(\Omega) \right)^n \subseteq \left(L^2(\Omega) \right)^n \subseteq \left(H^{-1}(\Omega) \right)^n.$$

On $\left(H_0^1(\Omega) \right)^n$, the bilinear form is defined by:

$$a(h, \psi) = \sum_{i=1}^n \int_\Omega \nabla h_i \nabla \psi_i \, dx + \sum_{i,j=1}^n \int_\Omega a_{ij} h_i \psi_i \, dx + \sum_{i=1}^n \int_\gamma \frac{[h_i][\psi_i]}{R_1 + R_2} d\gamma. \tag{25}$$

As in lemma 2.1, (24), implies

$$a(h, h) \geq C \sum_{i=1}^n \|h_i\|^2 = C \|h\|^2_{\left(H_0^1(\Omega) \right)^n}, \ \forall h \in \left(H_0^1(\Omega) \right)^n. \tag{26}$$

Now, let

$$L(\psi) = \sum_{i=1}^n \int_\Omega f_i(x) \psi_i(x) \, dx + \sum_{i=1}^n \int_\gamma \frac{(R_2 w - \delta)[\psi_i]}{R_1 + R_2} d\gamma - \sum_{i=1}^n \int_\gamma w_i \psi_i^+ d\gamma, \tag{27}$$

be a continuous linear form on $\left(H_0^1(\Omega) \right)^n$, then using Lax-Milgram lemma, there exists a unique solution $h \in \left(H_0^1(\Omega) \right)^n$ such that:

$$a(h, \psi) = L(\psi) \quad \forall \psi = \left(\psi_i \right)_{i=1}^n \in \left(H_0^1(\Omega) \right)^n.$$

Then, we have

Theorem 4.1 *For* $f = \{f_i\}_{i=1}^n \in \left(L^2(\Omega) \right)^n$ *there exists a unique solution*

$h \in \left(H_0^1 (\Omega) \right)^n$ *for cooperative Dirichlet system* (21) *with conjugation conditions* (22) *and* (23) *if condition* (24) *is satisfied.*

So, we can formulate the corresponding control problem:

The space $U = \left(L^2 (\Gamma) \right)^n$ is the space of controls. For a control

$u = \{ u_1, u_2, \cdots, u_n \} \in \left(L^2 (\Gamma) \right)^n$, the state $h(u) = \{ h_1(u), h_2(u), \cdots, h_n(u) \}$ of the system is given by the solution of

$$\begin{cases} -\Delta h_i(u) + \sum_{j=1}^{n} a_{ij} h_j(u) = f_i(u) & \text{in } \Omega, \\ h_i(u) = u_i & \text{on } \Gamma, \; i = 1, 2, \cdots, n, \end{cases} \tag{28}$$

under conjugation conditions:

$$\begin{cases} R_1 \left\{ \dfrac{\partial h_i(u)}{\partial v_A} \right\}^- + R_2 \left\{ \dfrac{\partial h_i(u)}{\partial v_A} \right\}^+ = \left[h_i(u) \right] + \delta & \text{on } \gamma, \\ \left[\left[\dfrac{\partial h_i(u)}{\partial v_A} \right] = \left[\sum_{i,j=1}^{n} \dfrac{\partial h_i(u)}{\partial x_j} \cos(v, x_i) \right] = w_i \right. & \text{on } \gamma, \; i = 1, 2, \cdots, n. \end{cases} \tag{29}$$

The observation may be takes as

$$z(u) = \{ z_1(u), z_2(u), \cdots, z_n(u) \} = Ch(u) = C \{ h_1(u), h_2(u), \cdots, h_n(u) \}$$
$$= \{ h_1(u), h_2(u), \cdots, h_n(u) \},$$

the cost functional is given by

$$J(v) = \sum_{i=1}^{n} \left\| \dfrac{\partial h_i(v)}{\partial v_A} - z_{id} \right\|_{L^2(\Gamma)}^2 + \left(Nv, v \right)_{(L^2(\Gamma))^n}, \tag{30}$$

where N is a hermitian positive definite operator such that:

$$\left(Nv, v \right)_{(L^2(\Gamma))^n} \ge M \| v \|_{(L^2(\Gamma))^n}^2, \quad M > 0, \quad \forall v \in U. \tag{31}$$

The control problem then is to find:

$$\begin{cases} u = \{ u_1, u_2, \cdots, u_n \} \in U_{ad} \text{ such that :} \\ J(u) = \inf J(v) \quad \forall v \in U_{ad}, \end{cases} \tag{32}$$

where U_{ad} is closed convex subset of $\left(L^2(\Gamma) \right)^n$. The cost functional (30) can be written as

$$J(v) = \sum_{i=1}^{n} \left\| \dfrac{\partial h_i(v)}{\partial v_A} - \dfrac{\partial h_i(0)}{\partial v_A} + \dfrac{\partial h_i(0)}{\partial v_A} - z_{id} \right\|_{L^2(\Gamma)}^2 + \left(Nv, v \right)_{(L^2(\Gamma))^n},$$

if we let

$$\pi(u, v) = \sum_{i=1}^{n} \left(\dfrac{\partial h_i(u)}{\partial v_A} - \dfrac{\partial h_i(0)}{\partial v_A}, \dfrac{\partial h_i(v)}{\partial v_A} - \dfrac{\partial h_i(0)}{\partial v_A} \right)_{L^2(\Gamma)} + \left(Nv, v \right)_{(L^2(\Gamma))^n},$$

and

$$L(v) = \sum_{i=1}^{n} \left(z_{id} - \dfrac{\partial h_i(0)}{\partial v_A}, \dfrac{\partial h_i(v)}{\partial v_A} - \dfrac{\partial h_i(0)}{\partial v_A} \right)_{L^2(\Gamma)},$$

then

$$J(v) = \pi(v,v) - 2L(v) + \sum_{i=1}^{n} \left\| z_{id} - \frac{\partial h_i(0)}{\partial v_A} \right\|_{L^2(\Gamma)}^2.$$

From (31),

$$\pi(v,v) \ge N \|v\|_{(L^2(\Gamma))^n}^2,$$

using the theory of Lions [1], there exists a unique optimal control of problem(32); moreover it is characterized by

Theorem 4.2 *Let us suppose that* (26) *holds and the cost functional is given by* (13), *then the boundary control u is characterized by*

$$
\begin{cases}
-\Delta p_i(u) + \sum_{j=1}^{n} a_{ij} p_i(u) = 0 & \text{in } \Omega, \\[2mm]
p_i(u) = -\left(\dfrac{\partial h_i(u)}{\partial v_A} - z_{id} \right) & \text{on } \Gamma, \\[2mm]
\left[\dfrac{\partial p_i(u)}{\partial v_{A^*}} \right] = 0, & \text{on } \gamma, \\[2mm]
\left\{ \dfrac{\partial p_i(u)}{\partial v_{A^*}} \right\}^{\pm} = \dfrac{1}{R_1 + R_2} \left[p_i(u) \right], & \text{on } \gamma, \\[2mm]
\sum_{i=1}^{n} \int_{\Gamma} \dfrac{\partial p_i(u)}{\partial v_{A^*}} N u_i (v_i - u_i) \, \mathrm{d}\Gamma \ge 0,
\end{cases}
\tag{33}
$$

together with (28) and (29) where $p(u) = \{ p_1(u), p_2(u), \cdots, p_n(u) \}$ is the adjoint state.

Proof. The optimal control $u = \{u_i\}_{i=1}^{n} \in \left(L^2(\Gamma) \right)^n$ is characterized by:

$$\pi(u, v-u) \ge L(v-u) \qquad \forall v = \{v_1, v_2, \cdots, v_n\} \in U_{ad},$$

$$\pi(u, v-u) - L(v-u)$$
$$= \sum_{i=1}^{n} \left(\frac{\partial h_i(u)}{\partial v_A} - \frac{\partial h_i(0)}{\partial v_A}, \frac{\partial h_i(v-u)}{\partial v_A} - \frac{\partial h_i(0)}{\partial v_A} \right)_{L^2(\Gamma)}$$
$$+ \sum_{i=1}^{n} \left(N u_i, v_i - u_i \right)_{L^2(\Gamma)} - \left(z_{id} - \frac{\partial h_i(0)}{\partial v_A}, \frac{\partial h_i(v-u)}{\partial v_A} - \frac{\partial h_i(0)}{\partial v_A} \right)_{L^2(L^2(\Gamma))}$$

$$\ge 0,$$

then

$$\sum_{i=1}^{n} \left(\frac{\partial h_i(u)}{\partial v_A} - z_{id}, \frac{\partial h_i(v-u)}{\partial v_A} - \frac{\partial h_i(0)}{\partial v_A} \right)_{L^2(\Gamma)} + \sum_{i=1}^{n} \left(N u_i, v_i - u_i \right)_{L^2(\Gamma)} \ge 0, \tag{34}$$

since the adjoint state is defined by (33), (34) implies

$$\sum_{i=1}^{n} \left(p_i(u), \frac{\partial h_i(v-u)}{\partial v_A} - \frac{\partial h_i(0)}{\partial v_A} \right)_{L^2(\Gamma)} + \sum_{i=1}^{n} \left(N u_i, v_i - u_i \right)_{L^2(\Gamma)} \ge 0.$$

Applying Green's formula, we obtain

$$\sum_{i=1}^{n} \left(A^* p_i(u), h_i(v-u) \right)_{L^2(\Omega)} + \sum_{i=1}^{n} \left(\frac{\partial p_i(u)}{\partial v_{A^*}}, h_i(v-u) \right)_{L^2(\Gamma)}$$

$$+ a\left(p(u), h(v-u) \right) + \sum_{i=1}^{n} \left(Nu_i, v_i - u_i \right)_{L^2(\Gamma)} \geq 0.$$

Since

$$\left(p, Ah \right) = \left(A^* p, h \right) = a(p, h) \quad \text{and} \quad A^* p = 0,$$

we obtain by using equation (28),

$$\sum_{i=1}^{n} \left(\frac{\partial p_i(u)}{\partial v_{A^*}}, v_i - u_i \right)_{L^2(\Gamma)} + \sum_{i=1}^{n} \left(Nu_i, v_i - u_i \right)_{L^2(\Gamma)} \geq 0,$$

hence

$$\sum_{i=1}^{n} \int_{\Gamma} \frac{\partial p_i(u)}{\partial v_{A^*}} Nu_i \left(v_i - u_i \right) \mathrm{d}\Gamma \geq 0.$$

5. Boundary Control for *n* × *n* Cooperative Neumann Systems

We generalize here, the results which have been established in section (3) to the following $n \times n$ Neumann elliptic system

$$\begin{cases} -\Delta h_i + \sum_{j=1}^{n} a_{ij} h_j = f_i & \text{in } \Omega, \\ \dfrac{\partial h_i}{\partial v_A} = g_i & \text{on } \Gamma, \ i = 1, 2, \cdots, n, \end{cases} \tag{35}$$

with conjugation conditions (22) and (23), where $g = \{g_1, g_2, \cdots, g_n\} \in \left(L^2(\Gamma) \right)^n$ are given functions. Since

$$\left(H_0^1(\Omega) \right)^n \subseteq \left(H^1(\Omega) \right)^n,$$

the bilinear form (25) is coercive on $\left(H^1(\Omega) \right)^n$.

Then using Lax-Milgram lemma, there exists a unique solution y for system (35) such that:

$$a(h, \psi) = L_N(\psi), \quad \forall \psi \in \left(H^1(\Omega) \right)^n,$$

where

$$L_N(\psi) = \sum_{i=1}^{n} \int_{\Omega} f_i(x) \psi_i(x) \mathrm{d}x + \sum_{i=1}^{n} \int_{\Gamma} g_i(x) \psi_i(x) \mathrm{d}x$$

$$+ \sum_{i=1}^{n} \int_{\gamma} \frac{(R_2 w - \delta)[\psi_i]}{R_1 + R_2} \mathrm{d}\gamma - \sum_{i=1}^{n} \int_{\gamma} w \psi_i^+ \mathrm{d}\gamma,$$

is a continuous linear form defined on $\left(H^1(\Omega) \right)^n$. Let us multiply both sides of first equation of (35) by $\psi \in \left(H^1(\Omega) \right)^n$ and integrate over Ω, we get

$$\sum_{i=1}^{n} \int_{\Omega} \left(-\Delta h_i + \sum_{j=1}^{n} a_{ij} h_j \right) \psi_i(x) \mathrm{d}x = \sum_{i=1}^{n} \int_{\Omega} f_i \psi_i \mathrm{d}x.$$

Applying Green's formula,

$$\sum_{i=1}^{n} \int_{\Omega} (-\Delta h_i) \psi_i \mathrm{d}x - \sum_{i=1}^{n} \int_{\Omega} a_{ij} h_i \psi_i \mathrm{d}x - \sum_{i=1}^{n} \int_{\Gamma} \frac{\partial h_i}{\partial v_A} \psi_i \mathrm{d}\Gamma$$

$$- \sum_{i=1}^{n} \int_{\gamma} \frac{\partial h_i}{\partial v_A} \psi_i \mathrm{d}\gamma + \sum_{i=1}^{n} \int_{\gamma} \frac{[h_i][\psi_i]}{R_1 + R_2} \mathrm{d}\gamma$$

$$= \sum_{i=1}^{n} \int_{\Gamma} g_i \psi_i \mathrm{d}\Gamma + \sum_{i=1}^{n} \int_{\Omega} f_i \psi_i \mathrm{d}x$$

$$\sum_{i=1}^{n} \int_{\Omega} \left(-\Delta h_i - \sum_{j=1}^{n} a_{ij} h_j \right) \psi_i(x) \mathrm{d}x - \sum_{i=1}^{n} \int_{\Gamma} \left(\frac{\partial h_i}{\partial v_A} \right) \psi_i(x) \mathrm{d}\Gamma$$

$$- \sum_{i=1}^{n} \int_{\gamma} \left(\frac{\partial h_i}{\partial v_A} \right) \psi_i(x) \mathrm{d}\gamma + \sum_{i=1}^{n} \int_{\gamma} \frac{[h_i][\psi_i]}{R_1 + R_2} \mathrm{d}\gamma$$

$$= \sum_{i=1}^{n} \int_{\Gamma} g_i \psi_i \mathrm{d}\Gamma + \sum_{i=1}^{n} \int_{\Omega} f_i \psi_i \mathrm{d}x,$$

then from

$$a(h, \psi) = L_N(\psi),$$

we obtain the Neumann conditions

$$\frac{\partial h_i}{\partial v_A} = g_i \quad \text{on } \Gamma.$$

Then, we have the corresponding control problem:

The space $U = (L^2(\Gamma))^n$ is the space of controls, the state $h(u) = \{h_1(u), h_2(u), \cdots, h_n(u)\}$ of the system is given by the solution of

$$\begin{cases} -\Delta h_i(u) + \sum_{j=1}^{n} a_{ij} h_j(u) = f_i(u) & \text{in } \Omega, \\ \dfrac{\partial h_i(u)}{\partial v_A} = g_i + u_i & \text{on } \Gamma, \end{cases} \tag{36}$$

under conjugation conditions (29), where $u = \{u_1, u_2, \cdots, u_n\}$ is a given control in $(L^2(\Gamma))^n$. For a given $z_d = \{z_{1d}, z_{2d}, \cdots, z_{nd}\} \in (L^2(\Gamma))^n$, the cost functional is again given by (30). As in theorem (4.2), we can prove:

Theorem 5.1 *Let us suppose that* (26) *holds and the cost functional is given by* (30), *there exists a unique optimal control u, such that*:

$$J(u) = \inf J(v) \qquad \forall v \in U_{ad} \subset (L^2(\Gamma))^n,$$

moreover it is characterized by the following equations and inequalities

$$\begin{cases} -\Delta p_i(u) + \sum_{j=1}^{n} a_{ij} p_i(u) = 0 & \text{in } \Omega, \\ \dfrac{\partial p_i(u)}{\partial v_{A^*}} = \left(\dfrac{\partial h_i(u)}{\partial v_A} - z_{id} \right) & \text{on } \Gamma, \\ \left[\dfrac{\partial p_i(u)}{\partial v_{A^*}} \right] = 0 & \text{on } \gamma, \\ \left\{ \dfrac{\partial p_i(u)}{\partial v_{A^*}} \right\}^{\pm} = \dfrac{1}{R_1 + R_2} [p_i(u)] & \text{on } \gamma, \end{cases}$$

$$\sum_{i=1}^{n} \left(p_i(u), v_i - u_i \right)_{\left(L^2(\Gamma)\right)^n} + \left(Nu, v - u \right)_{\left(L^2(\Gamma)\right)^n} \geq 0, \ i = 1, 2, \cdots, n,$$

together with(36) and (29), where $p(u) = \left\{ p_1(u), p_2(u), \cdots, p_n(u) \right\}$ is the adjoint state.

6. Conclusion

In the present work, we concentrated on optimal control problems for cooperative elliptic systems under conjugation conditions. We proved the existence and uniqueness of the state for 2×2 Dirichlet cooperative elliptic system. Then we discussed the existence and uniqueness of the optimal control of boundary type for this system and we gave the set of equations and inequalities that characterizes this control. Also, we studied the problem with Neumann condition. At last, we generalized the discussion to $n \times n$ cooperative elliptic systems under conjugation conditions.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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