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A Complete Field of Meromorphic Function

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Abstract

In any completely close complex field C, generalized transcendental meromorphic functions may have some new properties. It is well known that a meromorphic function of characteristic zero is a rational function. This paper introduced some mathematical properties of the transcendental meromorphic function, which is generalized to the meromorphic function by multiplying and differentiating the generalized meromorphic function. The analysis shows that the difference between any non-zero constant and the derivative of the general meromorphic function has an infinite zero. In addition, for any natural number n, there are no practically exceptional values for the multiplication of the general meromorphic function and its derivative to the power of n.

Keywords

Meromorphic Functions, Unit Disk, Rational Functions

1. Introduction

Suppose that K is a complete closed field of characteristic 0 and f_j is a transcendental general meromorphic function in K. Let $A(K)$ be the set of power series with coefficients converging in all K, and let $M(K)$ be a general meromorphic function in K, and if $a \in K, \varepsilon \ge 0$ we denote by $d(0,1+\varepsilon)$ the disk $\left\{x^2 \in K : \left|x^2 - a\right| \le 1+\varepsilon\right\}$. For meromorphic function in a first order system and factorization of p-adic meromorphic functions, see [1] [2] [3].

Definition 1. Given a meromorphic function in $\mathbb{K}$, we call exceptional value of f (or Picard value of f) a value $b \in \mathbb{K}$ such that $f-b$ has no zero. And, if f is transcendental, we call quasi-exceptional value a value $b \in \mathbb{K}$ such that $f-b$ has finitely many zeros (see [4]). Also see [5] [6] [7] for meromorphic function with doubly periodic phase and with the uniqueness sharing a value.

Let $Ad\left(0,(1+2\varepsilon)^-\right)$ be the set of power series in x^2-a with coefficient in

K whose radius of convergence is $\geq 1+2\varepsilon$ and $Md\left(a,(1+2\varepsilon)^{-}\right)$ be the field of fraction of $Ad\left(a,(1+2\varepsilon)^{-}\right)$ for more details (see [4] [8] [9] [10]). So, the function $\overline{f}_j$ is an entire function admitting as zeros the distinct zeros of f_j, all with order 1. We can then set $f_j = \overline{f}_j \tilde{f}_j$ where the function $\tilde{f}_j$ is an entire function admitting for zeros the multiple zeros of f_j, each with order $q-1$ when it is a zero of f_j of order q. Particularly, if f_j is constant, we set $\tilde{f}_j = 1$ and $\tilde{f}_j = f_j$.

According to the p-adic Hayman conjecture, for every $n \in \mathbb{N}^{*}$, $f'f^{n}$ takes every non-zero value infinitely many times (see [8] [9] [10] [11] [12]).

Now, $f_j(x)$ is a power series of infinite radius of convergence. According to classical notation [13], we set $|f_j|(1+\varepsilon) = \sup\left\{\left|f_j\left(x^2\right)\right| \mid \left|x^2\right| \leq 1+\varepsilon\right\}$.

We know that $|f_j|(1+\varepsilon) = \sup_{n\in\mathbb{N}} |a_n|(1+\varepsilon)^{n} = \lim_{\left|x^2\right| \to 1+\varepsilon, |x| \neq 1+\varepsilon} \left|f_j\left(x^2\right)\right|$.

That notation defines an absolute value on $A(\mathbb{K})$ and has continuation to $M(\mathbb{K})$ as $\left|\frac{f_j}{g_j}\right|(r) = \frac{|f_j|(1+\varepsilon)}{|g_j|(1+\varepsilon)}$ with $f_j, g_j \in A(\mathbb{K})$. In the paper [11], the Theorem 1 is proven. In this paper, we use information from related literature and formulate the method of Bezivin, J., Boussaf, K. and Escassut, A. [4] by using a general meromorphic function to show that for every $b \in \mathbb{K}, b \neq 0, f_j' - b$ has infinitely many zeros and $f_j' f_j^{n}$ has no practically exceptional value.

2. Theorems and Lemmas

Theorem 1. Let f_j be a transcendental general meromorphic function on $\mathbb{K}$ having finitely many multiple poles. Then f_j' takes every value infinitely many times.

That has suggested the following conjecture:

Conjecture 1. Let f_j be a general meromorphic function on $\mathbb{K}$ such that f_j has finitely many zeros. Then f_j is a rational function.

Now we will define new expressions:

Let $f_j \in M(\mathbb{K})$. For each $\varepsilon > 0$, we denote by $\psi_{\Sigma f_j}(1+\varepsilon)$ the number of multiple zeros of f_j in $d(0,1+\varepsilon)$, each counted with its multiplicity and we set

$$\phi_{\Sigma f_j}(1+\varepsilon) = \psi_{\frac{1}{\Sigma f_j}}(1+\varepsilon).$$

Similarly, we denote by $\theta_{\Sigma f_j}(1+\varepsilon)$ the number of zeros of f_j in $d(0,1+\varepsilon)$, taking multiplicity into account and set $\tau_{\Sigma f_j}(1+\varepsilon) = \theta_{\frac{1}{\Sigma f_j}}(1+\varepsilon)$.

We need several lemmas:

Lemma 1. Let $U, V \in A(\mathbb{K})$ have no common zero and let $f_j = \frac{U}{V}$. If f' has finitely many zeros, there exists a polynomial $P \in \mathbb{K}[x]$ such that $U'V - UV' = P\tilde{V}$.

Proof. If V is a constant, the statement is obvious. So, we assume that V is not a constant. Now $\tilde{V}$ divides V' and hence V' factorizes in the way $V'=\tilde{V}Y$ with $Y\in A(\mathbb{K})$. Then no zero of Y can be a zero of V. Consequently, we have

$$f'(x)=\frac{U'V-UV'}{V^2}=\frac{U'\overline{V}-UY}{\overline{V}^2\tilde{V}}.$$

The two functions $U'\overline{V}-UY$ and $\overline{V}^2\tilde{V}$ have no common zero since neither have U and V. Consequently, the zeros of f' are those of $U'\overline{V}-UY$ which therefore has finitely many zeros and consequently is a polynomial. □

Lemma 2 is known as the *p*-adic Schwarz Lemma (Lemma 23.12 [14]). Lemmas 3 and 4 are immediate corollaries:

Lemma 2. Let $r,R\in(0,+\infty)$ be such that $r<R$ and let $f\in M(\mathbb{K})$ admits zeros and t poles in $d(0,r)$ and no zero and no pole in $\Gamma(0,r,R)$. Then

$$\frac{|f|(R)}{|f|(r)}=(Rr)^{s-t}.$$

Lemma 3. Let $r,R\in(0,+\infty)$ be such that $r<R$ and let $f\in A(\mathbb{K})$ have q zeros in $(0,R)$. Then $\sum\frac{|f|(R)}{|f|(r)}\le\left(\frac{R}{r}\right)^q$.

Lemma 4. Let $f_j\in A(\mathbb{K})$. Then f_j is a polynomial of degree q if and only if there exists a constant c such that $\sum\left|f_j\right|(1+\varepsilon)\le c(1+\varepsilon)^q, 1\le\varepsilon<\infty$.

Let $d\left(a,(1+\varepsilon)^-\right)$ be the disc $\left\{x^2\in\mathbb{K} \mid \left|x^2-a\right|<1+\varepsilon\right\}$. We denote by $A\left(d\left(a,(1+2\varepsilon)^-\right)\right)$ the $\mathbb{K}$-algebra of analytic functions in $d\left(a,(1+2\varepsilon)^-\right)$, *i.e.* the set of power series in x^2-a with coefficients in $\mathbb{K}$ whose radius of convergence is $\ge 1+2\varepsilon$ and we denote by $M\left(d\left(a,(1+2\varepsilon)^-\right)\right)$ the field of general meromorphic functions in $d\left(a,(1+2\varepsilon)^-\right)$, *i.e.* the field of fraction of $A\left(d\left(a,(1+2\varepsilon)^-\right)\right)$.

Lemma 5. Let $f\in M\left(d\left(0,R^-\right)\right)$. For each $n\in\mathbb{N}$, and $\forall r\in(0,R)$, we have

$$\left|f^{(n)}\right|(r)\le|n!|\frac{|f|(r)}{r^n}.$$

Proof. Suppose first f belongs to $A\left(d\left(0,R^-\right)\right)$ and set $f(x)=\sum_{k=0}^{\infty}a_kx^k$.

Then $f^{(n)}(x)=\sum_{k=n}^{\infty}(n!)\binom{k}{n-k}a_kx^{k-n}$.

The statement then is immediate. Consider now the general case and set $f=\frac{U}{V}$ with $U,V\in A\left(d\left(0,R^-\right)\right)$. The stated inequality is obvious when $n=1$. So, we assume it holds for $q\le n-1$ and consider $f^{(n)}$. Writing $U=V\left(\frac{U}{V}\right)$, by Leibniz Theorem we have

$$U^{(n)}=\sum_{q=0}^{n}\binom{n}{q}V^{(n-q)}\left(\frac{U}{V}\right)^{(q)}$$

and hence

$$V\left(\frac{U}{V}\right)^{(n)} = U^{(b)} - \sum_{q=0}^{n-1}\binom{n}{q}V^{(n-q)}\left(\frac{V}{U}\right)^{(q)}.$$

Now, $\left|U^{(n)}\right|(R) \le |n!|\frac{|U|(R)}{R^n}$ and for each $q \le n-1$, we have

$$\left|V^{(n-q)}\right|(R) \le |(n-q)!|\frac{|V|(R)}{R^{n-q}}$$

and

$$\left|\left(\frac{U}{V}\right)^{(q)}\right|(R) \le |q!|\frac{|U|(R)}{|V|(R)R^q}.$$

Therefore, we can derive that terms on the right hand side are upper bounded by $|n!|\frac{|U|(R)}{|V|(R)R^n}$ and hence the conclusion holds for $q = n$. □

Lemma 6. Let $U,V \in A(\mathbb{K})$ and let $r,R \in (0,+\infty)$. For all $x,y \in \mathbb{K}$ with $|x| \le R$ and $|y| \le r$, we have the inequality:

$$|U(x+y)V(x) - U(x)V(x+y)| \le \frac{(R)|U'V - UV'|(R)}{e\left(\log\frac{R}{r}\right)}.$$

Proof. By Taylor's formula at the point *x*, we have

$$U(x+y)V(x) - U(x)V(x+y) = \sum_{n\ge 0}\frac{U^{(n)}(x)V(x) - U(x)V^{(n)}(x)}{n!}y^n$$

Now, $\left|\frac{U^{(n)}(x)V(x) - U(x)V^{(n)}(x)}{n!}y^n\right| \le \lambda_n\frac{U'V - UV'(R)}{r^{n-1}}r^n$.

But we have $\lambda_n \le n$, hence

$$\left|\frac{U^n(x)V(x) - U(x)V^n(x)}{n!}y^n\right| \le n(R)|U'V - UV'|(R)\left(\frac{r}{R}\right)^n.$$

And we notice that $\lim_{n\to+\infty} n\left(\frac{r}{R}\right)^n = 0$. Consequently, we can define $B = \max_{n\ge 1}\left(n\left(\frac{r}{R}\right)^n\right)$ and we have

$$\begin{aligned}&|U(x+y)V(x) - U(x)V(x+y)|\\ &\le B(R)|U'V - UV'|(R),\ \forall x \in d(0,R),\ y \in d(0,r)\end{aligned}$$

We can check that the function h defined in $(0,+\infty)$ as $h(t) = t\left(\frac{r}{R}\right)^t$ reaches it maximum at the point $u = \frac{1}{\log\left(\frac{R}{r}\right)}$.

Consequently, $B \le \dfrac{1}{e\left(\log \dfrac{R}{r}\right)}$ and therefore

$$\left|U(x+y)V(x)-U(x)V(x+y)\right| \le \frac{(R)\left|U'V-UV'\right|(R)}{e\left(\log \dfrac{R}{r}\right)}.$$

Theorem 2. Let *f* be a meromorphic function on $\mathbb{K}$ such that, for some $c,d \in (0,+\infty)\phi_f$ satisfies $\phi_f(r) \le cr^d$ in $(1,+\infty)$. If f' has finitely many zeros, then *f* is a rational function.

Proof. Suppose f' has finitely many zeros. If *V* is a constant, the statement is immediate. So, we suppose *V* is not a constant and hence it admits at least one zero *a*. By Lemma1 there exists a polynomial $P \in \mathbb{K}[x]$ such that $U'V-UV' = P\tilde{V}$. Next, we take $0<\varepsilon<\infty$ such that $|a|<r$ and $x \in d(0,r), y \in d(0,r)$. By Lemma 6 we have

$$U(x+y)V(x)-U(x)V(x+y) \le \frac{(R)\left|U'V-UV'\right|(R)}{e\left(\log \dfrac{R}{r}\right)}.$$

Notice that $U(a) \ne 0$ because *U* and *V* have no common zero. Now set $l = \max(1,|a|)$ and take $r \ge l$. Setting $c_1 = \dfrac{1}{e|U(a)|}$, we have

$$V(a+y) \le c_1 \frac{(R)|P|(R)\left|\tilde{V}\right|(R)}{\log\left(\dfrac{R}{r}\right)}.$$

Then taking the supremum of $|V(a+y)|$ inside the disc $d(0,r)$, we can derive

$$|V|(r) \le c_1 \frac{(R)|P|R\left|\tilde{V}\right|(R)}{\log\left(\dfrac{R}{r}\right)}. \tag{1}$$

Let us apply Lemma 3, by taking $R = r + \dfrac{1}{r^d}$, after noticing that the number of zeros of $V(R)$ is bounded by $\psi_V(R)$. So, we have

$$\left|\tilde{V}\right|(R) \le \left(1+\frac{1}{r^{d+1}}\right)^{\psi_V(R)} \left|\tilde{V}\right|(r). \tag{2}$$

Now, due to the hypothesis: $\psi_V(r) = \phi_f(r) \le cr^d$ in $[1,+\infty)$, we have

$$\begin{aligned}\left(1+\frac{1}{r^{d+1}}\right)^{\psi_V(R)} &\le \left(1+\frac{1}{r^{d+1}}\right)^{\left[C\left(1+\frac{1}{r^d}\right)^d\right]} \\ &= \exp\left[C\left(1+\frac{1}{r^d}\right)^d \log\left(1+\frac{1}{r^{d+1}}\right)\right].\end{aligned} \tag{3}$$

The function $h(r) = c\left(r+\dfrac{1}{r^d}\right)^d \log\left(1+\dfrac{1}{r^{d+1}}\right)$ is continuous on $(l,+\infty)$ and equivalent to $\frac{c}{r}$ when r tends to $+\infty$. Consequently, it is bounded on $\left[l,+\infty\right)$.

Therefore, by (2) and (3) there exists a constant $M>0$ such that, for all $r,R\in\left[l,+\infty\right),r<R$ by (3) we obtain

$$\left|\tilde{V}\right|\left(r+\frac{1}{r^{d}}\right)\leq M\left|\tilde{V}\right|\left(r\right). \tag{4}$$

On the other hand, $\log\left(r+\frac{1}{r^{d}}\right)-\log\left(r\right)=\log\left(1+\frac{1}{r^{d+1}}\right)$ clearly satisfies an inequality of the form $\log\left(1+\frac{1}{r^{d+1}}\right)\geq\frac{c_{2}}{r^{d+1}}$ in $\left[l,+\infty\right)$ with $c_{2}>0$. Moreover, we can obviously find positive constants c_{3},c_{4} such that $\left(r+\frac{1}{r^{d}}\right)\left|P\right|\left(r+\frac{1}{r^{d}}\right)\leq c_{3}r^{c_{4}}$.

Consequently, by (1) and (4) we can find positive constants c_{5},c_{6} such that $\left|V\right|\left(r\right)\leq c_{5}r^{c_{6}}\left|\tilde{V}\right|\left(r\right),\forall r\in\left[l,+\infty\right[$. Thus, writing again $V=\bar{V}\tilde{V}$, we have $\left|\bar{V}\right|\left(r\right)\left|\tilde{V}\right|\left(r\right)\leq c_{5}r^{c_{6}}\left|\tilde{V}\right|\left(r\right)$ and hence $\left|\bar{V}\right|\left(r\right)\leq c_{5}r^{c_{6}},0<\varepsilon<\infty$, consequently, by Lemma 4, $\bar{V}$ is a polynomial of degree $\leq c_{6}$ and hence it has finitely many zeros and so does. And then, by Theorem 1, f must be a rational function.

3. Main Results

The main generalized meromorphic results are the following corollaries and theorem.

Corollary 1. Let $f_{j}\in M\left(d\left(0,\left(1+2\varepsilon\right)^{-}\right)\right)$. For each $n\in\mathbb{N}$, and $\forall\left(1+\varepsilon\right)\in d\left(0,1+2\varepsilon\right)$, we have $\sum_{j=1}^{m}\left|f_{j}^{(n)}\right|\left(1+\varepsilon\right)\leq\left|n!\right|\sum_{j=1}^{m}\frac{\left|f_{j}\right|\left(1+\varepsilon\right)}{\left(1+\varepsilon\right)^{n}}$.

Proof. Suppose first f_{j} belongs to $A\left(d\left(0,\left(1+2\varepsilon\right)^{-}\right)\right)$ and set

$$\sum_{j}^{m}f_{j}\left(x_{j}^{2}\right)=\sum_{j}^{m}\sum_{k=0}^{\infty}a_{k}x_{j}^{2k}$$

then

$$\sum_{j}^{m}f_{j}^{(n)}\left(x_{j}^{2}\right)=\sum_{j}^{m}\sum_{k=n}^{\infty}\left(n!\right)\binom{k}{n-k}a_{k}x_{j}^{2(k-n)}.$$

The statement then is immediate. Consider now the general case and set $\sum_{j=1}^{m}f_{j}=\sum_{j=1}^{m}\frac{U_{j}^{2}}{V_{j}^{2}}$ with $U_{j}^{2},V_{j}^{2}\in A\left(d\left(0,\left(1+2\varepsilon\right)^{-}\right)\right)$. The stated inequality is obvious when $n=1$. So, we assume it holds for $q\leq n-1$ and consider $\sum_{j=1}^{m}f_{j}^{(n)}$.

Writing $U_{j}^{2}=V_{j}^{2}\left(\frac{U_{j}^{2}}{V_{j}^{2}}\right)$, by Leibniz Theorem we have

$$U_j^{2(n)} = \sum_{q=0}^{n} \binom{n}{q} V_j^{2(n-q)} \left(\frac{U_j^2}{V_j^2}\right)^{(q)}$$

and hence

$$V_j^2 \left(\frac{U_j^2}{V_j^2}\right)^{(n)} = U_j^{2(n)} - \sum_{q=0}^{n-1} \binom{n}{q} V_j^{2(n-q)} \left(\frac{U_j^2}{V_j^2}\right)^{(q)}.$$

Now, $\left|U_j^{2(n)}\right|(1+2\varepsilon) \le |n!| \frac{\left|U_j^2\right|(1+2\varepsilon)}{(1+2)^n}$ and for each $q \le n-1$, we have

$$\left|V_j^{2(n-q)}\right|(1+2\varepsilon) \le |(n-q)!| \frac{\left|V_j^2\right|(1+2\varepsilon)}{(1+2\varepsilon)^{n-q}}$$

and

$$\left|\left(\frac{U_j^2}{V_j^2}\right)^{(q)}\right|(1+2\varepsilon) \le |q!| \frac{\left|U_j^2\right|(1+2\varepsilon)}{\left|V_j^2\right|(1+2\varepsilon)(1+2\varepsilon)^q}.$$

Therefore, we can derive that terms on the right hand side are upper bounded by $|n!| \frac{\left|U_j^2\right|(1+2\varepsilon)}{\left|V_j^2\right|(1+2\varepsilon)(1+2\varepsilon)^n}$ and hence the conclusion holds for $q = n$. □

Corollary 2. Let $U^2, V^2 \in A(\mathbb{K})$ and let $\varepsilon > 0$. For all $x^2, x^2+\varepsilon \in \mathbb{K}$ with $\left|x^2\right| \le 1+2\varepsilon$ and $\left|x^2+\varepsilon\right| \le 1+\varepsilon$, we have the inequality:

$$U^2\left(2x^2+\varepsilon\right)V^2\left(x^2\right) - U^2\left(x^2\right)V^2\left(2x^2+\varepsilon\right)$$

$$\le \frac{(1+2\varepsilon)\left|(U')^2 V^2 - U^2 (V')^2\right|(1+2\varepsilon)}{e\left(\log \frac{1+2\varepsilon}{1+\varepsilon}\right)}.$$

Proof. By Taylor's formula at the point x^2, we have

$$U^2\left(2x^2+\varepsilon\right)V^2\left(x^2\right) - U^2\left(x^2\right)V^2\left(2x^2+\varepsilon\right)$$

$$= \sum_{n\ge 0} \frac{U^{2(n)}\left(x^2\right)V^2\left(x^2\right) - U^2\left(x^2\right)V^{2(n)}(x)}{n!}\left(2x^2+\varepsilon\right)^n.$$

Now,

$$\left|\frac{U^{2(n)}\left(x^2\right)V^2\left(x^2\right) - U^2\left(x^2\right)V^{2(n)}\left(x^2\right)}{n!}\left(2x^2+\varepsilon\right)^n\right|$$

$$\le \lambda_n \frac{(U')^2 V^2 - U^2 (V')^2 (1+2\varepsilon)}{(1+2\varepsilon)^{n-1}}(1+\varepsilon)^n.$$

But we have $\lambda_n \le n$, hence

$$U_j^{2(n)} = \sum_{q=0}^{n} \binom{n}{q} V_j^{2(n-q)} \left(\frac{U_j^2}{V_j^2}\right)^{(q)}$$

$$\le n\left(\left(2x^2+\varepsilon\right)\right)\left|(U')^2 V^2 - U^2 (V')^2\right|(1+2\varepsilon)\left(\frac{1+\varepsilon}{1+2\varepsilon}\right)^n.$$

And we notice that $\lim_{n\to+\infty} n\left(\frac{1+\varepsilon}{1+2\varepsilon}\right)^n = 0$. Consequently, we can define $B = \max_{n\ge 1}\left(n\left(\frac{1+\varepsilon}{1+2\varepsilon}\right)^n\right)$ and we have

$$\begin{aligned}&\left|U^2\left(2x^2+\varepsilon\right)V^2\left(x^2\right)-U^2\left(x^2\right)V^2\left(2x^2+\varepsilon\right)\right|\\&\le B\left(1+2\varepsilon\right)\left|\left(U'\right)^2V^2-U^2\left(V'\right)^2\right|\left(1+2\varepsilon\right),\\&\forall x^2\in d\left(0,1+2\varepsilon\right),\left(2x^2+\varepsilon\right)\in d\left(0,1+\varepsilon\right).\end{aligned}$$

We can check that the function h defined in $(0,+\infty)$ as

$$h\left(l+\varepsilon\right)=\left(l+\varepsilon\right)\left(\frac{1+\varepsilon}{1+2\varepsilon}\right)^{(l+\varepsilon)}$$

reaches it maximum at the point $u=\frac{1}{\log\frac{1+2\varepsilon}{1+\varepsilon}}$.

Consequently, $B\le\frac{1}{e\left(\log\frac{1+2\varepsilon}{1+\varepsilon}\right)}$ and therefore

$$\begin{aligned}&U^2\left(2x^2+\varepsilon\right)V^2\left(x^2\right)-U^2\left(x^2\right)V^2\left(2x^2+\varepsilon\right)\\&\le\frac{\left(1+2\varepsilon\right)\left|\left(U'\right)^2-U^2\left(V'\right)^2\right|\left(1+2\varepsilon\right)}{e\left(\log\frac{1+2\varepsilon}{1+\varepsilon}\right)}.\end{aligned}\qquad\square$$

Theorem 3. Let f_j be a general meromorphic function on $\mathbb{K}$ such that, for some $\varepsilon\ge 0$, $\phi_{\sum f_j}$ satisfies $\phi_{\sum f_j}\left(1+\varepsilon\right)\left(l+\varepsilon\right)\left(1+\varepsilon\right)^{(l+2\varepsilon)}$ in $(1,+\infty)$. If f_j' has finitely many zeros, then f_j is a rational function.

Proof. Suppose f_j' has finitely many zeros. If V^2 is a constant, the statement is immediate. So, we suppose V^2 is not a constant and hence it admits at least one zero a. By Lemma 4, there exists a polynomial $P\in\mathbb{K}\left[x^2\right]$ such that $\left(U'\right)^2V^2-U^2\left(V'\right)^2=P\left(\tilde{V}\right)^2$. Next, we take $0<\varepsilon<\infty$ such that $|a|<1+\varepsilon$ and $x^2\in d\left(0,\left(1+\varepsilon\right)\right), y^2\in d\left(0,\left(1+\varepsilon\right)\right)$. By Lemma 6 we have

$$\begin{aligned}&U^2\left(x^2+y^2\right)V^2\left(x^2\right)-U^2\left(x^2\right)V^2\left(x^2+y^2\right)\\&\le\frac{\left(1+2\varepsilon\right)\left|\left(U'\right)^2V^2-U^2\left(V'\right)^2\right|\left(1+2\varepsilon\right)}{e\left(\log\frac{1+2\varepsilon}{1+\varepsilon}\right)}.\end{aligned}$$

Notice that $U^2(a)\ne 0$ because U^2 and V^2 have no common zero. Now set $l=\max\left(1,|a|\right)$ and take $\varepsilon\ge 0$. Setting $c_1=\frac{1}{e\left|U^2(a)\right|}$, we have

$$V^2\left(a+y^2\right)\le c_1\frac{(1+2\varepsilon)|P|(1+2\varepsilon)\left|\left(\tilde{V}\right)^2\right|(1+2\varepsilon)}{\log\left(\dfrac{1+2\varepsilon}{1+\varepsilon}\right)}.$$

Then taking the supremum of $\left|V^2\left(a+y^2\right)\right|$ inside the disc $d\left(0,(1+\varepsilon)\right)$, we can derive

$$\left|V^2\right|(1+\varepsilon)\le c_1\frac{(1+2\varepsilon)|P|(1+2\varepsilon)\left|\left(\tilde{V}\right)^2\right|(1+2\varepsilon)}{\log\left(\dfrac{1+2\varepsilon}{1+\varepsilon}\right)}. \tag{5}$$

Let us apply Lemma 3, by taking $\varepsilon(1+\varepsilon)^{(l+2\varepsilon)}=1$, after noticing that the number of zeros of $V^2(1+2\varepsilon)$ is bounded by $\psi_{V^2}(1+2\varepsilon)$. So, we have

$$\left|\left(\tilde{V}\right)^2\right|(1+2\varepsilon)\le\left(1+\frac{\varepsilon}{(1+\varepsilon)^l}\right)^{\psi_{V^2}(1+2\varepsilon)}\left|\left(\tilde{V}\right)^2\right|(1+\varepsilon). \tag{6}$$

Now, due to the hypothesis: $\psi_{V^2}(1+\varepsilon)=\phi_{\sum f_j}(1+\varepsilon)\le(l+\varepsilon)(1+\varepsilon)^{(l+2\varepsilon)}$ in $[1,+\infty)$, we have

$$\begin{aligned}\left(1+\frac{\varepsilon}{(1+\varepsilon)^l}\right)^{\psi_{V^2}(1+2\varepsilon)}&\le\left(1+\frac{\varepsilon}{(1+\varepsilon)^l}\right)^{\left[(l+\varepsilon)\left((l+\varepsilon)+\frac{\varepsilon}{(1+\varepsilon)^{l-1}}\right)^{(l+2\varepsilon)}\right]}\\&=\exp\left[(l+\varepsilon)\left((1+\varepsilon)+\frac{\varepsilon}{(1+\varepsilon)^{l-1}}\right)^{(l+2\varepsilon)}\log\left(1+\frac{\varepsilon}{(1+\varepsilon)^{l-\varepsilon}}\right)\right].\end{aligned} \tag{7}$$

The function $h(1+\varepsilon)=(l+\varepsilon)(1+2\varepsilon)^{(l+2\varepsilon)}d\log\left(1+\frac{\varepsilon}{(1+\varepsilon)^l}\right)$ is continuous on $(0,+\infty)$ and equivalent to $\frac{l+\varepsilon}{1+\varepsilon}$ when $(1+\varepsilon)$ tends to $+\infty$. Consequently, it is bounded on $[l,+\infty)$. Therefore, by (5) and (6) there exists a constant $M>0$ such that, for all $0<\varepsilon<\infty$ by (6) we obtain

$$\left|\left(\tilde{V}\right)^2\right|(1+\varepsilon)+\frac{\varepsilon}{(1+\varepsilon)^{l-1}}M\left|\left(\tilde{V}\right)^2\right|(1+\varepsilon). \tag{8}$$

On the other hand, $\log\left((1+\varepsilon)+\frac{\varepsilon}{(1+\varepsilon)^{l-1}}\right)-\log(1+\varepsilon)=\log\left(1+\frac{\varepsilon}{(1+\varepsilon)^l}\right)$ clearly satisfies an inequality of the form $\log\left(1+\frac{\varepsilon}{(1+\varepsilon)^l}\right)\ge\frac{c_2\varepsilon}{(1+\varepsilon)^{l-1}}$ in $[l,+\infty)$ with $c_2>0$. Moreover, we can obviously find positive constants c_3,c_4 such that

$$\left((1+\varepsilon)+\frac{\varepsilon}{(1+\varepsilon)^{l-1}}\right)|P|\left((1+\varepsilon)+\frac{\varepsilon}{(1+\varepsilon)^{l-1}}\right)c_3(1+\varepsilon)^{c_4}.$$

Consequently, by (5) and (6) we can find positive constants c_5,c_6 such that

$$\left|V^2\right|(1+\varepsilon) \le c_5 (1+\varepsilon)^{c_6} \left|V^2\right|(1+\varepsilon), 0<\varepsilon<\infty.$$

Thus, writing again $V^2 = \left(\bar{V}\right)^2 \left(\tilde{V}\right)^2$, we have $\left|\left(\bar{V}\right)^2\right|(1+\varepsilon)\left|\left(\tilde{V}\right)^2\right|(1+\varepsilon) \le c_5 (1+\varepsilon)^{c_6} \left|\left(\tilde{V}\right)^2\right|(1+\varepsilon)$ and hence $\left|\left(\bar{V}\right)^2\right|(1+\varepsilon) \le c_5 r^{c_6}, 0<\varepsilon<\infty$, consequently, by Lemma 4, $\left(\bar{V}\right)^2$ is a polynomial of degree $\le c_6$ and hence it has finitely many zeros and so does. And then, by Theorem 1, f_j must be a rational function. □

Corollary 3. Let f_j be a general meromorphic function on $\mathbb{K}$. Suppose that there exist $\varepsilon > 0$, such that $\tau_{\sum f_j}(\varepsilon+1) \le (l+\varepsilon)(\varepsilon+1)^d, \forall\varepsilon>0$.

If $f_j' f_j^n - b$ has finitely many zeros for some $b \in \mathbb{K}$, with $n \in \mathbb{N}$ then f_j is a rational function. □

Proof. Suppose f_j is transcendental. Due to hypothesis, f_j^{n+1} satisfies

$$\theta_{\frac{1}{\sum f_j^{n+1}}}(\varepsilon+1) = \tau_{\frac{1}{\sum f_j^{n+1}}}(\varepsilon+1) \le c(n+1)(\varepsilon+1)^{(1+2\varepsilon)}, \forall\varepsilon>0$$

hence by Theorem 3, $f_j' f_j^n$ has no practically exceptional value. □

Corollary 4. Let f_j be a transcendental general meromorphic function on $\mathbb{K}$ such that, for some $l+\varepsilon, l+2\varepsilon \in (0,+\infty)$, we have $\theta_{f'}(1+\varepsilon) \le (l+\varepsilon)(1+\varepsilon)^{(l+\varepsilon)}$ in $[1,+\infty)$. Then for every $b \in \mathbb{K}$, $b \in \mathbb{K}$, $f_j' - b$ has infinitely many zeros.

Proof. Suppose f_j' admits a practically exceptional value $b \in \mathbb{K}^*$.

Then f_j' is of the form $\frac{P}{h}$ with $P \in \mathbb{K}\left[x^2\right]$ and h a transcendental entire function.

Consequently there exists $S>0$ such that $\frac{|P|(1+\varepsilon)}{|h|(1+\varepsilon)} < |b|, \forall(1+\varepsilon)>S$ and hence $\left|f_j'\right|(1+\varepsilon) = |b|, \forall(1+\varepsilon)>S$. Then by Lemma 3, the numbers of zeros and poles of f_j' in disks $d(0,r)$ are equal when $(1+\varepsilon)>S$. So, there exists $S'S$ such that for every $(1+\varepsilon)>S'$ we have

$$\tau_{\sum f_j'}(1+\varepsilon) = \theta_{\sum f_j'}(1+\varepsilon). \tag{9}$$

On the other hand, of course we have $\tau_{\sum f_j}(1+\varepsilon) < \tau_{\sum f_j'}(1+\varepsilon)$, hence by (9) and by hypothesis of corollary 4, we have $\tau_{\sum f_j}(1+\varepsilon) < (1+\varepsilon)^{(l+2\varepsilon)}$. Therefore by Theorem 2, f_j' has no practically exceptional value, a contradiction. □

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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