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Verifications of the Scattering Theory on Manifolds

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Abstract

Scattering theory plays the main role in the study of manifolds and the Laplacian spectrum. In this article, we process justifying the continuous Laplacian spectrum Δ_{g_i} and Δ_{h_i} on a complete Riemannian manifold. (M, g_i) is categorized by the use of bounded curvature of the metric. In particular, the covariant derivative is limitedly considered as an application in the geodesic distance from a fixed point.

Keywords

Manifolds, Scattering theory, Spectrum

1. Introduction

A great number of researchers referred to the connection between, time-dependent, time-independent, Laplacian, manifold, wave operators, matrices, Riemannian metric, and Schrödinger equation linked to the theory of scattering.

For example, Itoa, K. and Skibsted, E. in [1] included time-dependent scattering theory along with allowed range perturbation and scattering by obstacles. The "independent" and "dependent" scattering by particles has been studied in appropriate single-particle, and examples of independent scattering are described by Michael I. Mishchenko, see [2]. The scattering theory for the Laplacian on symmetric spaces of a non-compact type in the frame work of Agmon-Hörmander has been updated by Koichi Kaizuka in [3]. Thierry Cazenave and Ivan Naumk in [4] modified scattering for the critical nonlinear Schrödinger equation. The exhibited conditions under which the stationary wave operators and the strong wave operators exist and coincide have been discussed by R. Tiedra de Aldecoa [5]. The scattering matrices for dissipative quantum system and

Neumann maps have been studied by many authors see [6] [7]. Subsequently, Rainer Hempel, Olaf Post, and Ricardo Weder [8] obtained the existence and completeness of the wave operators for perturbations of the Riemannian metric for the Laplacian on a complete manifold of dimension.

In this paper, we follow the exact reviews and approaches of Werner Muller and Corm Salomonsen in [9] with a slight change. The current study contributes to the expansion of the knowledge in this field by addressing the scattering theory for the Laplacian spectrum (Δ_{g_i} and Δ_{h_i}) on the manifold with bounded curvature comparison dynamics.

Definition 1. Let $\beta:[0,\infty)\to\mathbb{R}$ be a positive, continuous, non-increasing function. Then β is called a function of moderate decay, if it satisfies the following condition:

(i) $\sup_{x\in[1,\infty)} x\beta(x)<\infty$;

(ii) $\exists C_\beta>0:\beta(x+y)\ge C_\beta\beta(x)\beta(y), x,y\ge 1$ (1)

Further β is called of sub-exponential decay if for any $c>0$, $\mathrm{e}^{cx}\beta(x)\to\infty$. As $x\to\infty$.

Definition 2. Let β be a function of moderate decay. Two metrics $g,h\in M$ are said to be β-equivalent up to order k if There exist $q\in M$ and $C>0$ such that for all $x\in M$ we have ${}^{k}|g-h|_g(x)\le C\beta(1+d_g(x,q))$ holds.

In this case, we write $g\sim_\beta^k h$.

Definition 3. Let $s>0$. For $s>\varepsilon\ge 0$ let $K_\varepsilon(M,g;s)\in\mathbb{N}\cup\{\infty\}$ be the smallest number such that there exists a sequence $\{x_i\}_{i=1}^{\infty}$ such that $\sup_{x\in M}\#\{i\in\mathbb{N}|x\in B_{3S+\varepsilon}(x_i)\}\le K_\varepsilon(M,g;s)$ Further, let $K(M,g;s)=K_0(M,g;s)$ put

$$k(M,g;s)=1.$$

Definition 4. Let (M,g) be a complete. Then $\Delta:C_c^\infty(M)\to L^2(M)$ is essentially self-adjoint and function $f(\sqrt{\Delta})$ can be defined by the spectral theorem for unbounded self-adjoint operators by $f(\sqrt{\Delta})=\int_0^\infty f(\lambda)\mathrm{d}E_{(\lambda)}$, where $\mathrm{d}E_{(\lambda)}$ is the projection spectral measure associated with $\sqrt{\Delta}$. Let $f\in L^1(\mathbb{R})$ be even and let $\hat{f}(\lambda)=\int_{-\infty}^{\infty} f(x)\cos(\lambda x)\mathrm{d}x$. Then $f(\sqrt{\Delta})$ can also be defined by

$$f(\sqrt{\Delta})=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(\lambda)\cos(\lambda\sqrt{\Delta})\mathrm{d}\lambda. \tag{2}$$

Eichhorn, Proposition 2.1 in [10] has shown that M can be endowed with a canonical topology given by a metrizable uniform structure. For a given Riemannian metric g_i on M, denote by ∇^{g_i} the Levi-Civita connection 2.5 in [11] of g and by $\|0\|_g$ the norm induced by g in the fibers of $\oplus_{p,q\ge 0}(TM\otimes T^*M^{\otimes q})$. Let h be any other Riemannian metric on M. For $k\ge 0$ set

$$\begin{aligned}&\sum_{i=1}^{\infty}{}^{k}|g_i-h_i|_{g_i}(x)\\&=\sum_{i=1}^{\infty}\left(|g_i-h_i|_g(x)\right)+\sum_{j=0}^{k-1}\left(\left|(\nabla^g)^j(\nabla^g-\nabla^h)\right|(x)\right), x\in M,\end{aligned} \tag{3}$$

and $\sum_{i=1}^{\infty}\left({}^{k}\|g_i-h_i\|_{g_i}\right)=\sup_{x\in M}{}^{k}\sum_{i=1}^{\infty}\left(|g_i-h_i|_{g_i}(x)\right)$. Recall that two metrics g_i,h_i are said to be quasi-isometric if there exist $C_1,C_2>0$ such that

$$C_{1g}(x)\le\sum_{i=1}^{\infty}h_i(x)\le C_{2g}(x),\ \text{for all}\ \ x\in M \tag{4}$$

in the sense of positive definite quadratic forms. We shall write $g_i\sim h_i$ for quasi-isometric metrics g_i and h_i. If g and h are quasi-isometric, then (4) implies that for all $p,q\ge 0$, there exist $A_{p,q}B_{p,q}>0$ such that for every tensor field T on M of bidegree (p,q) we have

$$A_{p,q}|T|(x)\le\sum_{i=1}^{\infty}|T|_{h_i}(x)\le B_{p,q}|T|_g(x),x\in M \tag{5}$$

2. Theorems and Lemmas

Lemma 1. Let β be of moderate decay. Then there exist a constants $C>0$ and $c>0$ such that,

$$\beta(x)\ge C\mathrm{e}^{-cx},x\in[1,\infty) \tag{6}$$

Lemma 2. Let $g,h\in CM$ be quasi-isometric. For every $k\ge 0$, there exists a polynomial $P_k(X_1,\cdots,X_k)$ depending on the quasi-isometry constants, with nonnegative coefficients and vanishing constant term, such that

$$\begin{aligned}&{}^{k}|g-h|_h(x)\\&\le P_k\left(|g-h|_g(x),\left|\nabla^g-\nabla^h\right|_g(x),\cdots,\left|\left(\nabla^g\right)^{k-1}\left(\nabla^g-\nabla^h\right)\right|_g(x)\right),x\in M\end{aligned}$$

Proof. From (4) follows that $|g-h|_h(x)\le C_3|g-h|_h(x)$ and

$$\left|\nabla^g-\nabla^h\right|_h(x)\le C_4\left|\nabla^g-\nabla^h\right|_g(x),x\in M\ . \tag{7}$$

This is as important as the first two terms in (3) and deals with the question for $k=0,1$. Now we shall proceed by induction. Let $k\ge 2$ and suppose that the lemma holds for $l\le k-1$. For each, $p\le 0$ we have

$$\left(\nabla^h\right)^p\left(\nabla^h-\nabla^g\right)=\nabla^g\left(\nabla^h\right)^{p-1}\left(\nabla^h-\nabla^g\right)+\left(\nabla^h-\nabla^g\right)\left(\nabla^h\right)^{p-1}\left(\nabla^h-\nabla^g\right) \tag{8}$$

Let $p\le k$ using (7), (6) and the hypothesis, we can estimate the point wise h norm the second term on the right-hand side of (8) in desired way deal with the first term. We use the formula

$$\begin{aligned}&\left(\nabla^g\right)^p\left(\nabla^h\right)^l\left(\nabla^h-\nabla^g\right)\\&=\left(\nabla^g\right)^{(p+1)}\left(\nabla^h\right)^{(l-1)}\left(\nabla^h-\nabla^g\right)+\left(\nabla^g\right)^p\left(\nabla^h-\nabla^g\right)\left(\nabla^h\right)^{(l-1)}\left(\nabla^h-\nabla^g\right).\end{aligned}$$

Applying the Leibniz rule, we get

$$\begin{aligned}&\left|\left(\nabla^g\right)^p\left(\nabla^h-\nabla^g\right)\left(\nabla^h\right)^{(l-1)}\left(\nabla^h-\nabla^g\right)\right|_g(x)\\&\le C\sum_{i=0}^{p}\left|\left(\left(\nabla^g\right)^i\left(\nabla^h-\nabla^g\right)\right)\right|_g(x)\cdot\left|\left(\left(\nabla^g\right)^{(p-i)}\left(\nabla^h\right)^{(l-1)}\left(\nabla^h-\nabla^g\right)\right)\right|_g(x)\end{aligned}$$

for some $C>0$ and all $x\in M$ .Inserting (8) and iterating these formulas reduces everything to the induction hypothesis.

Lemma 3. Let β be a function of moderate decay. Then for all $x,y,q\in M$, we have

$$C_\beta \beta\left(1+d\left(x,y\right)\right)\leq \frac{\beta\left(1+d\left(x,q\right)\right)}{\beta\left(1+d\left(y,q\right)\right)}\leq \frac{1}{C_\beta \beta\left(1+d\left(x,y\right)\right)} \tag{9}$$

Moreover, for every $q'\in M$ there exists a constant $C>0$, depending only on q and q' such that

$$C^{-1}\beta\left(1+d\left(x,q'\right)\right)\leq \beta\left(1+d\left(x,q\right)\right)\leq \beta\left(1+d\left(x,q'\right)\right).$$

Lemma 4. There exists a constant $C>0$ depend only on K such that

$$\tilde{i}\left(x\right)\geq C\tilde{i}\left(p\right)^n \mathrm{e}^{-(n-1)\sqrt{K}d(x,p)} \tag{10}$$

for all $x,p\in M$.

Lemma 5. For $r\leq \tilde{i}\left(x_0\right)$,

$$\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\int_0^r\left(\frac{\sin t\sqrt{K}}{\sqrt{K}}\right)^{(n-1)}\mathrm{d}t\leq VOL\left(B_r\left(x_0\right)\right)\leq \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\int_0^r\left(\frac{\sin h\sqrt{K}}{\sqrt{K}}\right)^{(n-1)}\mathrm{d}t$$

We note that the inequality on the right-hand side holds for all $r\in\mathcal{R}$. In particular $Vol\left(B_r\left(x_0\right)\right)=0\left(\mathrm{e}^{(n-1)\sqrt{Kr}}\right)$ as $r\to\infty$.

It is also important to know the maximal possible decay of the injectivity radius.

Lemma 6. $k_\varepsilon\left(M,g;s\right)$ finite for all $s>\varepsilon$. Moreover, there exist constants $C,c>0$, which depend only on K, such that for $s>\frac{2\pi}{\sqrt{K}}+\varepsilon$, we have $k_\varepsilon\left(M,g;s\right)\leq C\mathrm{e}^{cs}$.

Lemma 7. Let $k\geq 1$ be even. Assume that M has bounded curvature of order k. Let $k>0$ be such that $\sup_{x\in M}\sum_{l=0}^{2k}\left|\nabla^l R\left(x\right)\right|\leq k$, there exist constants $r_0=r_0\left(k\right)>0$ and $C=C\left(k\right)>0$ such that for all $x_0\in M$ and $r_i\leq\min\left\{r_0,\tilde{r}\left(x_0\right)\right\}$ one has $\sum_{i=1}^\infty\left\|u_i\right\|_{W^{2K}\left(B_{r_i}\left(x_0\right)\right)}\leq C\sum_{i=1}^\infty\left\|u_i\right\|_{H^{2k}\left(B_{r_i}\left(x_0\right)\right)}$ for all $u_i\in C_0^\infty\left(B_{r_i}\left(x_0\right)\right)$.

Lemma 8. Let $k\in\mathbb{N}$ be even. Suppose that $\left(M,g_i\right)$ has bounded curvature of order $2k$ Let $\beta:M\to\mathbb{R}^+$ be a function of moderate decay. Then there exists a canonical bounded inclusions $H^k_{\beta_\tau^{\sim-2kn}}\left(M\right)\to W^k_\beta\left(M\right)$ and $H^k_\beta\left(M\right)\to W^k_{\beta\tau^{\sim 2kn}}\left(M\right)$

Proof. By Theorem (2.6) in [9] in M there exist a covering $\sum_{i=1}^\infty B_{\frac{\tilde{r}}{2^k}(x_i)}\left(x_i\right)$ of M by balls and a constant $C>0$ such that

$$\forall x\in M:\left\{x_i\,\middle|\,x\in\sum_{i=1}^\infty B_{\tilde{r}\left(x^i\right)}\left(x_i\right)\right\}\leq C \tag{11}$$

Let $\varphi\in C^\infty\left(\mathbb{R}\right)$ go be such that $\varphi=1$ on $\left[0,1\right]$ and $\varphi=0$ on $\left[2,\infty\right)$ for

$x \in M$ and $1 \le j \le k$, we define

$$\sum_{j=1}^{k} \varphi_{j,x}(y) = \begin{cases} \sum_{j=1}^{k} \varphi\left(2^{j} \frac{d(x,y)}{\tilde{\tau}(x)}\right), & y \in B_{\tilde{\tau}(x)}(x); \\ 0, & \text{otherwise.} \end{cases}$$

then $\sum_{j=1}^{k} \varphi_{j,x} \in C_0^{\infty}(M)$. Let $f \in H^k(M)$. Using Lemma 6, it follows that $\varphi_{j,x} f \in H^k(B_{\tilde{\tau}x}(x))$. Then by Lemma 7, we get $\varphi_{j,x} f \in W^k(B_{\tilde{\tau}x}(x))$ and by the Leibniz rule there is $C > 0$ such that

$$\left|\sum_{j=1}^{k} \nabla^{j}(\varphi_{k,x} f)\right|_g (y) \le C \sum_{p=0}^{j} \left|\nabla^{p} \varphi_{k,x}\right|_g (y) \cdot \left|\nabla^{j-p} f\right|_g (y), y \in M .$$

By estimating the supremum-norm of the derivatives of $\varphi_{k,x}$ and using Lemma 7, we get

$$\begin{aligned} \left\|\varphi_{k,x} f\right\|_{W^k} &\le C \|f\|_{W^k}\left(B_{\frac{\tau}{2^{k-1}}(x)}(x)\right) + C' \sum_{p=1}^{k} \binom{k}{p} (x) \left\|\varphi_{k-1,x} f\right\|_{W^{k-p}} \\ &\le C \|f\|_{H^k}\left(B_{\frac{\tilde{\tau}}{2^{k-1}}(x)}(x)\right) + C'' \sum_{p=1}^{k} \binom{k}{p} \tilde{\tau}^{-p}(x) \left\|\varphi_{k-1,x} f\right\|_{H^{k-p}} \end{aligned}$$

By induction, this yields

$$\left\|\varphi_{k,x_i} f\right\|_{W^k} \le C \tilde{\tau}^{-K}(x_i) \|f\|_{H^k}\left(B_{\frac{\tilde{\tau}}{2^{k-1}}(x)}(x)\right) + C' \sum_{p=1}^{k} \binom{k}{p} \tilde{\tau}^{-P}(x) \left\|\varphi_{k-1,x} f\right\|_{W^{k-p}} \quad (12)$$

Let $f \in H_{\beta}^{k}$. By Lemma 7, (11) and (12) we get

$$\begin{aligned} \|f\|_{W_{\beta}^{k}} &\le C \sum_{i=1}^{\infty} \beta^{\frac{1}{2}}(x_i) \left\|\varphi_{k,x_i} f\right\|_{W^k} \tilde{\tau}^{k}(x_i) \le C \sum_{i=1}^{\infty} \beta^{\frac{1}{2}}(x_i) \left\|\varphi_{k,x_i} f\right\|_{H^k} \\ &\le C \sum_{i=1}^{\infty} \beta^{\frac{1}{2}}(x_i) \tilde{\tau}^{-k}(x_i) \|f\|_{H^k\left(\beta_{\tilde{\tau}(x_i)}(x)\right)} \end{aligned}$$

By (10) there exists $C_1 > 0$ such that $\tilde{\tau}(x_i)^{-k} \tilde{\tau}(x)^{kn} \le C_1$ for all $i \in \mathbb{N}$ and $x \in B_{\tau(x_i)}(x_i)$. This implies $\sum_{i=1}^{\infty} \beta^{\frac{1}{2}}(x_i) \tilde{\tau}^{-k}(x_i) \|f\|_{H^k\left(B_{\tau(x_i)}(x_i)\right)} \le C_2 \|f\|_{H^k_{\tilde{\tau}^{-2kn\beta}}}$. Assume that (M, g) is complete. Then $\Delta : C_0^{\infty}(M) \to L^2(M)$ is essentially self-ad joint and function $f\left(\sqrt{\Delta}\right)$ can be defined by the spectral theorem for unbounded self-ad joint operators by $f\left(\sqrt{\Delta}\right) = \int_0^{\infty} f(\lambda) \mathrm{d}E_{\lambda}$, where $\mathrm{d}E_{\lambda}$ is the projection spectral measure associate with $\sqrt{\Delta}$. Let $f \in L^1(\mathbb{R})$ be even and let $\tilde{f}(\lambda) = \int_{-\infty}^{\infty} f(x) \cos(\lambda x) \mathrm{d}x$, then $f\left(\sqrt{\Delta}\right)$ can also be defined by

$$f\left(\sqrt{\Delta}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\lambda) \cos\left(\lambda \sqrt{\Delta}\right) \mathrm{d}\lambda \quad (13)$$

This representation has been used in [12] to study the kernel of $f\left(\sqrt{\Delta}\right)$ we will used (13) to study $f\left(\sqrt{\Delta}\right)$ as operator in weighted L^2-spaces. To this end we need to study $\cos\left(\lambda \sqrt{\Delta}\right)$ as operator in $L_{\beta}^{2}(M)$ given $s > 0$, let $K(M, g; s)$ be the constant introduced in Definition (1.3).

Theorem 1. Assume that (M,g) has bounded curvature. Let β be a function of moderate decay. Then $\cos(\lambda\sqrt{\Delta})$ extends to a bounded operator in $L^2_\beta(M)$ for all $s\in\mathbb{R}$ and there exist $C,c>0$, such that

$$\left\|\cos\left(s\sqrt{\Delta}\right)\right\|_{L^2_\beta,L^2_\beta}\le C\mathrm{e}^{c|s|},\ s\in\mathbb{R}. \tag{14}$$

Moreover $\cos\left(s\sqrt{\Delta}\right):L^2_\beta(M)\to L^2_\beta(M)$ is strongly continuous in *S*.

Proof. Let $s>0$ Choose a sequence $\{X_k\}_{k=1}^{\infty}\subset M$ which minimizes. $\kappa(M,g_i;s)$. For $k\in\mathbb{N}$ let P_k denote the multiplication by the characteristic function of $B_s(x_k)\setminus\bigcup_{i=0}^{k-1}B_s(x_i)$. Then each P_k is an orthogonal projection in $L^2(M)$ and $L^2_\beta(M)$ respectively. Moreover the projections satisfy $P_kP_{k'}=0$ for $k\ne k'$ and $\sum_{k=1}^{\infty}P_k=1$ where the series is strongly convergent. Obviously the image of P_k consists of functions with support in $B_s(x_k)$. Now recall that $\cos\left(\tau\sqrt{\Delta}\right)$ has unit propagation speed [13], *i.e.*, $\sup p\cos\left(\tau\sqrt{\Delta}\right)\delta_s\subset\overline{B_{|\tau|}(x)}$ for all $x\in M$ and $\tau\in\mathbb{R}$. Let $f\in L^2(M)$. Then it follows that

$\sup p\cos\left(s\sqrt{\Delta}\right)P_kf\subset B_{2s}(x_k)$ and

$\sup p\cos\left(s\sqrt{\Delta}\right)\left(\left(1-\chi_{B_{3s}(x_k)}\right)f\right)\subset M-B_{2s}(x_k)$ Hence

$$\begin{aligned}\left\|\cos\left(s\sqrt{\Delta}\right)f\right\|_\beta^2&=\sum_{k=1}^{\infty}\left\langle\cos\left(s\sqrt{\Delta}\right)P_kf,\cos\left(s\sqrt{\Delta}\right)f\right\rangle_\beta\\&=\sum_{k=1}^{\infty}\left\langle\cos\left(s\sqrt{\Delta}\right)P_kf,\cos\left(s\sqrt{\Delta}\right)\right\rangle\end{aligned} \tag{15}$$

Now observe that the norm of $\left(s\sqrt{\Delta}\right)$ as an operation in $L^2(M)$ is bounded by 1. This implies

$$\left|\left\langle\cos\left(s\sqrt{\Delta}\right)P_kf,\cos\left(s\sqrt{\Delta}\right)\left(\chi_{B_{3s}(x_k)}f\right)\right\rangle\right|\le\sup_{y\in B_{3s}(x_k)}\beta(y)\|P_kf\|_{L^2}\cdot\left\|\chi_{B_{3s}(x_k)}f\right\|_{L^2}$$

To estimate the right-hand side, we write

$$\sup_{y\in B_{3s}(x_k)}\beta(y)\|P_kf\|_{L^2}^2\le C_\beta^{-1}\frac{1}{\beta(1+4s)}\|P_kf\|_{L^2_\beta}^2$$

Since the support of P_kf is contained in $B_s(x_k)$ we can use (9) to estimate the right-hand side. This gives $\sup_{y\in B_{3s}(x_k)}\beta(y)\|P_kf\|_{L^2}^2\le C_\beta^{-1}\frac{1}{\beta(1+4s)}\|P_kf\|_{L^2_\beta}^2$. A similar inequality holds with respect to $\left\|\chi_{B_{3s}}(x_k)f\right\|_{L^2}$ putting the estimations together, we get

$$\left|\left\langle\cos\left(s\sqrt{\Delta}\right)P_kf,\cos\left(s\sqrt{\Delta}\right)\left(\chi_{B_{3s}(x_k)}f\right)\right\rangle\right|\le C_\beta^{-1}\frac{1}{\beta(1+6S)}\|P_kf\|_{L^2_\beta}\left\|\chi_{B_{3s}(x_k)}f\right\|_{L^2_\beta}$$

Now recall that by Lemma 6, we have $\kappa(M,g;s)<\infty$. Hence together with (14) and (15) we obtain

$$\left\|\cos\left(s\sqrt{\Delta}\right)f\right\|_{L^2_\beta}^2\le C_\beta^{-1}\frac{1}{\beta(1+6s)}\|f\|_\beta^2\sum_{k=1}^{\infty}\left\|\chi_{B_{3s}(x_k)}f\right\|_{L^2_\beta}\le C_\beta^{-1}\frac{1}{\beta(1+6s)}\kappa(M,g,s)^{\frac{1}{2}}\|f\|_{L^2_\beta}^2$$

Recall that by (1) we have $\beta(x) \le C(1+d(x,p))^{-1}, x \in M$. Therefore, $L^2(M) \subset L^2_\beta(M)$, and $L^2(M)$ is a dense subspace of $L^2_\beta(M)$. This implies that $\cos(s\sqrt{\Delta})$ extends to a bounded operator in $L^2_\beta(M)$. Moreover by (7) and Lemma 6, it follows that there exist constants $C, c > 0$ such that $\left\|\cos(s\sqrt{\Delta})\right\|^2_{L^2_\beta, L^2_\beta} \le C\mathrm{e}^{cs}, s \in [0, \infty)$. Since $\cos(-s\sqrt{\Delta}) = \cos(s\sqrt{\Delta})$ this extends to all $s \in \mathbb{R}$ such that holds. The strong continuity is a consequence of the local bound of the norm and the strong continuity on the dense subspace $\cos(-s\sqrt{\Delta})L^2(M) \subseteq L^2_\beta(M)$. Using Theorem 1, we can study $f(\sqrt{\Delta})$ as an operator in $L^2_\beta(M)$ given $c \ge 0$, let $\mathcal{F}'(c) = \left\{ f \in L^1(\mathbb{R}) : \int_{-\infty}^{\infty} \left|\tilde{f}(\lambda)\right| \mathrm{e}^{c|\lambda|_{\mathrm{d}\lambda<\infty}} \right\}$.

Lemma 9. Let β a function of moderate decay. If λ and $\bar{\lambda}$ satisfy conditions (b) of Corollary 4.3 in [9] then

$$H^2_\beta(M) = (\Delta-\lambda)^{-1}\left(L^2_\beta(M)\right).$$

Proof. First: note that $C_0^\infty(M)$ is dense in $L^2_\beta(M)$. Indeed $C_0^\infty(M)$ is dense in $L^2(M)$ and $L^2(M)$ is dense in $L^2_\beta(M)$. Let $f = \sum_{i=1}^{\infty}(\Delta-\lambda)^{-1} g_i, g_i \in L^2_\beta(M)$. Then there exists a sequence $\{\varphi_i\}_{i\in\mathbb{N}} \subset C_0^\infty(M)$ which converges to $\sum_{i=1}^{\infty} g_i$ in $L^2_\beta(M)$ and $(\Delta-\lambda)^{-1}\varphi_i$ converges to f in $L^2(M)$. Let $\varphi \in L_0^\infty(M)$. Then

$$\langle f, \Delta\varphi\rangle = \lim_{i\to\infty}\left\langle (\Delta-\lambda)^{-1}, \Delta\varphi\right\rangle = \lim_{i\to\infty}\left\langle \varphi_i + \lambda(\Delta-\lambda)^{-1}\varphi_i, \varphi\right\rangle = \langle g+\lambda f, \varphi\rangle.$$

Thus $\Delta f = \sum_{i=1}^{\infty}(g_i + \lambda f) \in L^2_\beta(M)$ and hence $f \in L^2_\beta(M)$ now suppose that $f \in L^2_\beta(M)$ and set $g = (\Delta-\lambda)f$. Then $g \in L^2_\beta(M)$ and we need to show that $f = \sum_{i=1}^{\infty}(\Delta-\lambda)^{-1} g_i$. Let $\varphi \in C_0^\infty(M)$. By definition of $\sum_{i=1}^{\infty}(\Delta-\lambda)^{-1} g_i$, there exists a sequence $\{g\}_{i\in\mathbb{N}} \subset L^2(M)$ such that $(\Delta-\lambda)^{-1} g_i$ converges to $(\Delta-\lambda)^{-1} g$ in $L^2_\beta(M)$ as $i \to \infty$. Using this fact, we get

$$\left\langle \sum_{i=1}^{\infty}(\Delta-\lambda)^{-1} g_i, \sum_{i=1}^{\infty}\varphi_i\right\rangle = \left\langle g_i, (\Delta-\bar{\lambda})^{-1}\sum_{i=1}^{\infty}\varphi_i\right\rangle = \left\langle (\Delta-\lambda)f, (\Delta-\bar{\lambda})^{-1}\varphi\right\rangle. \quad (16)$$

Now, observe that $(\Delta-\lambda)^{-1}\sum_{i=1}^{\infty}\varphi_i$ belongs to $H^2(M)$. By Lemma (3.1) in [9] there exists a sequence $\sum_{i=1}^{\infty}\varphi_i \subset (M)$ which converges to $(\Delta-\bar{\lambda})^{-1}$ in $H^2(M)$. Thus

$$\left\langle (\Delta-\lambda)f, (\Delta-\bar{\lambda})^{-1}\varphi\right\rangle = \lim_{i\to\infty}\left\langle (\Delta-\lambda)f, \sum_{i=1}^{\infty}\varphi_i\right\rangle$$
$$= \left\langle f, (\Delta-\bar{\lambda})\sum_{i=1}^{\infty}\varphi_i\right\rangle = \langle f, \varphi\rangle.$$

Together with (16) this implies that $(\Delta-\lambda)^{-1} g_i$.

Lemma 10. Let β be of moderate decay. Assume that $g_i \sim_\beta^k h_i$ then the Sobolev spaces $W_\xi^k(M; g_i)$ and $W_\xi^k(M; h_i)$ are equivalent.

Proof. First note that by Lemma 1.7 in [9] the metrics g and h are quasi-isometric. This implies that $L^2_\xi(M; g_i)$ and $L^2_\xi(M; h_i)$ are equivalent. So the statement of the lemma holds for $k = 0$. Let $f \in C^\infty(M)$ and $k \in \mathbb{N}$ by induction we will prove that for $l \le k$ there exists $C_1 > 0$ such that for

$a, b \in \mathbb{N}_0$, $a + b = l$,

$$\left|\left(\nabla^{g_i}\right)^a \left(\nabla^{h_i}\right)^b f\right|_{h_i}(x) \le C_l \sum_{i=0}^{(a+b)} \left|\left(\nabla^{g}\right)^i f\right|_{(x)}(x), x \in M. \quad (17)$$

Let $l = 1$. Since on functions the connections equal, (17) follows from quasi-isometry of g and h_i. Next suppose that (17) holds for $1 \le l < k$. To establish (17) for $l+1$, we proceed by induction with respect to a. Let $a, b \in \mathbb{N}_0$ with $a + b = l + 1$. We may assume that $a < l + 1$. Using

$$\sum_{i=1}^{\infty}\left(\nabla^{g_i}\right)^a \left(\nabla^{h_i}\right)^b f = \sum_{i=1}^{\infty}\left(\nabla^{g_i}\right)^a \left(\nabla^{h_i} - \nabla^{g_i}\right)\left(\nabla^{h_i}\right)^{(b-1)} f + \left(\nabla^{g_i}\right)^{(a+1)}\left(\nabla^{h_i}\right)^{(b-1)} f,$$

and $g \sim_{\beta}^{k} h$, it follows that (17) holds for $l+1$. Especially, putting $a = 0$ we get

$$\sum_{i=1}^{\infty}\left|\left(\nabla^{h_i}\right)^l f\right|_{h_i}(x) \le C_l \sum_{i=0}^{l}\left|\left(\nabla^{g}\right)^i f\right|_{g_i}(x), x \in M, l \le k. \quad (18)$$

Suppose that $f \in C^{\infty}(M) \cap W_{\xi}^{k}(M; g_i)$ then (18) implies that
$f \in C^{\infty}(M) \cap W_{\xi}^{k}(M; h_i)$ and $\|f\|_{W_{\xi}^{k}(M;h_i)} \le C\|f\|_{W_{\xi}^{k}(M;g_i)}$.

By Lemma (3.1) in [9] $C^{\infty}(M) \cap W_{\xi}^{k}(M; g_i)$ is dense in $W_{\xi}^{k}(M; g_i)$. Therefore this inequality holds for all $f \in C^{\infty}(M, g_i)$. By symmetry, a similar inequality holds with the roles of g_i and h_i inter-changed. This concludes the proof.

Next we compare the Sobolev spaces $H_{\xi}^{2k}(M; g_i)$ and $H_{\xi}^{2k}(M; h_i)$. Let Δ_{g_i} denote the Laplace operator with respect to the metric g. Recall, that $\sum_{i=1}^{\infty}\Delta_{g_i} = \sum_{i=1}^{\infty}\left(\nabla^{g_i}\right)^* \nabla^{g_i}$, and that the formal ad joint $\left(\nabla^{g_i}\right)^*$ of $\left(\nabla^{g_i}\right)$ is given by $\left(\nabla^{g}\right)^* = -Tr\left(g^{-1}\nabla^{g_i}\right)$. Where $\sum_{i=1}^{\infty}\Delta_{g_i} = \sum_{i=1}^{\infty}\left(\nabla^{g_i}\right)^* \nabla^{g_i}$ is the isomorphism induced by the metric and $Tr: T^*M \otimes TM \to \mathbb{R}$ denotes $\sum_{i=1}^{\infty}\Delta_{g_i} = \sum_{i=1}^{\infty}\left(\nabla^{g_i}\right)^* \nabla^{g_i}$ contraction. Since contraction commutes with covariant differentiation and $\nabla^{g_i} g_i^{-1} = 0$, we get the well-known formula
$\Delta = -Tr\left(g^{-1}\nabla^2\right)$. This can be iterated. For $\omega_1 \otimes \cdots \otimes \omega_k \in \left(T^*M\right)^{\otimes k}$ define $g_j^{-1}\left(\omega_1 \otimes \cdots \otimes \omega_k\right): \omega_1 \otimes \cdots \otimes \omega_{j-1} \otimes g^{-1}\left(\omega_j\right) \otimes \omega_{j+1} \otimes \cdots \otimes \omega_k$, and let $Tr_{i.j}\left(g_j^{-1}\right)$ denote, $\left(g_j^{-1}\right)$ followed by the contraction of the *i*th and *j*th component using. That contraction commutes with covariant differentiation and $\nabla^{g_i} g_i^{-1} = 0$, we get

$$\Delta_g^k = (-1)^k Tr_{1,2}\left(g_2^{-1}\right) \circ \cdots \circ Tr_{2k-1,2k}\left(g_{2k}^{-1}\right)\left(\nabla^{g}\right)^{2k}. \quad (19)$$

In more traditional notation this mean $\Delta_{g_i}^k f = (-1)^k \sum_{i_1,\cdots,i_k} f_{;i_1 i_1 i_2 i_2 \cdots i_k i_k}$. For short notation we will write $Tr\left(\left(g^{-1}\right)^{\otimes k}\right) := Tr_{1,2}\left(g_2^{-1}\right) \circ \cdots \circ Tr_{2k-1,2k}\left(g_{2k}^{-1}\right)$.

Lemma 11. Assume that $g_i \sim_{\beta}^{2k} \beta$. Then for each $l, 0 \le l \le 2k$ and

$j, 0 \le j \le 2l$, there exist section $\xi_{jl}^g, \xi_{jl}^h \in C^\infty\left(Hom\left(T^*M\right)^{\otimes j}, \mathbb{R}\right)$ such that $\sum_{l=0}^{2k}\left(\Delta_g^l - \Delta_h^l\right) = \sum_{j=0}^{2l} \xi_{jl}^g \circ \left(\nabla^g\right)^j = \sum_{j=0}^{2l} \xi_{jl}^h \circ \left(\nabla^h\right)^j$ and there exists $C < 0$ such that for $0 \le p \le l$, $\sum_{j=0}^{2l}\left(\left|\left(\nabla^g\right)^p \xi_{jl}^g\right|(x)\right) \le C\beta(x)$, $\sum_{j=0}^{2l}\left(\left|\left(\nabla^h\right)^p \xi_{jl}^h\right|_h(x)\right) \le C\beta(x)$, $x \in M$.

Lemma 12. Assume that β is a function of moderate decay and there exist real numbers a, b such that

(i) $b \ge 1$, and $a + b = 2$,

(ii) $\beta^{\frac{b}{3}} \in L^1(M)$,

(iii) $\beta^{\frac{a}{3}} \tilde{\tau}^{-n(n+2)} \in L^\infty(M)$.

Let M_β be the operator of multiplication by β. Then the operator all $M_{\tau-2n} M_\beta \Delta^p e^{-t\Delta}$ is a trace-class operator for $\beta \in \mathbb{N}$ and t in a compact interval, the trace-class norm is bounded.

3. Main Results

The main verification results are the following corollaries and lemma.

Corollary 1. Let $K, \lambda > 0$ be given. There exists $r_0 = r_0(K, \lambda) > 0$ and $C = C(\lambda) > 0$ such that for all $r_i \le r_0$, $p \in \varepsilon ll^m(r_i, K, \lambda)$ and $x_0 \in B_{r_i}$. $\sum_{i=1}^n \|u_i\|_{W^m(B_{r_i})} \le C \sum_{i=1}^n \left(\|P_{u_i}\|_{L^2(B_{r_i})} + \|u_i\|_{L^2(B_{r_i})}\right)$ for all $\sum_{i=1}^n u_i \in C_0^\infty\left(\sum_{i=1}^n B_{r_i}\right)$.

Proof. Let $1 \ge r_i > 0$ and let $P \in \varepsilon ll^m(r_i, K, \lambda)$. Put $P_0 = \sum_{|\alpha|=m} a_\alpha(0) D^\alpha$ .By lemma 17.1.2 in [14] there exists $C_1 > 0$ which depends only on λ such that for all $\sum_{i=1}^n u_i \in C_0^\infty(B_{r_i})$:

$$\sum_{i=1}^n \|u_i\|_{W^m(B_{r_i})} \le C \sum_{i=1}^n \left(\|P_0 u_i\|_{L^2(B_{r_i})} + \|u_i\|_{L^2(B_{r_i})}\right). \tag{20}$$

Now $\sum_{i=1}^n p u_i = \sum_{i=1}^n P_0 u_i + \sum_{i=1}^n (P - P_0) u_i$. Thus $\sum_{i=1}^n \|u_i\|_{W^m(B_{r_i})} \le C \sum_{i=1}^n \left(\|P u_i\|_{L^2(B_{r_i})} + \|(P - P_0) u_i\|_{L^2(B_{r_i})} + \|u_i\|_{L^2(B_{r_i})}\right)$. Next observe that

$$\sum_{i=1}^n (P - P_0) u_i = \sum_{i=1}^n \sum_{|\alpha|=m} \left(a_\alpha(x) - a_\alpha(0)\right) D^\alpha u_i + \sum_{i=1}^n \sum_{|\alpha|<m} a_\alpha(x) D^\alpha u_i$$

Hence by lemma 17.1.2 in [14]:

$$\begin{aligned} &\sum_{i=1}^n \|(P - P_0) u_i\|_{L^2(B_{r_i})} \\ &\le \sum_{i=1}^n r_i \sum_{|\alpha|=m} \|a_\alpha\|_{C^1(B_{r_i})} \|u_i\|_{W^m(B_{r_i})} + \sum_{i=1}^n \sum_{|\alpha|<m} \|a_\alpha\|_{C^O(B_{r_i})} \|u_i\|_{W^{m-1}(B_{r_i})} \\ &\le K \sum_{i=1}^n \left(r_i \|u_i\|_{W^m(B_{r_i})} + \|u_i\|_{W^{m-1}(B_{r_i})}\right) \end{aligned} \tag{21}$$

By the Poincare inequality there exists $C_2 > 0$ which is independent of $\sum_{i=1}^n r_i \le 1$ such that for all $\sum_{i=1}^n u_i \in C_0^\infty \sum_{i=1}^n (B_{r_i})$:

$\sum_{i=1}^{n}\|u_i\|_{W^{m-1}(B_{r_i})} \le \sum_{i=1}^{n} r_i C_2 \|u_i\|_{W^m(B_{r_i})}$. Using this inequality, it's follows from (21) that $\sum_{i=1}^{n}\|(P-P_0)u_i\|_{L^2(B_{r_i})} \le \sum_{i=1}^{n} r_i C(K)\|u_i\|_{W^m(B_{r_i})}$. Together with (20) we get

$$\sum_{i=1}^{n}(1-r_i CC(K))\|u_i\|_{W^m(B_{r_i})} \le C\sum_{i=1}^{n}\left(\|Pu_i\|_{L^2(B_{r_i})}+\|u_i\|_{L^2(B_{r_i})}\right)$$

Set $r_0=\min\left\{1,\frac{1}{2CC(K)}\right\}$ then it follows that for all $\sum_{i=1}^{n} r_i \le r_0$ and

$$\sum_{i=1}^{n} u_i \in C_0^\infty \sum_{i=1}^{n}(B_{r_i}) : \sum_{i=1}^{n}\|u_i\|_{W^m(B_{r_i})} \le 2C\sum_{i=1}^{n}\left(\|Pu_i\|_{L^2(B_{r_i})}+\|u_i\|_{L^2(B_{r_i})}\right).$$

Corollary 2. Assume (M,g_i) has bounded curvature and let β be functions of moderate decay. Then there exists a constant $C=C(M,g_i,\beta)$ such that for all functions $f_i \in \mathcal{F}'(c)$, the operator $f_i(\sqrt{\Delta})$ extends to abounded operator in $L^2_\beta(M)$. Moreover, there exists a constant $C_1=C_1(M,g_i,\beta)>0$ such that $\sum_{i=1}^{n}\left\|f_i(\sqrt{\Delta})\right\|_{L^2_\beta,L^2_\beta} \le C_1\sum_{i=1}^{n}\left\|\hat{f}\right\|_{L^1_{e^{c|\cdot|}}}$ for all f_i as above. If $\kappa(M,g_i;s)$ is at most sub-exponentially increasing, then $c(M,g_i;\beta)>0$ can be chosen arbitrarily.

Proof. By Theorem 1, there exist constants $C,c>0$, depending on (M,g_i,β) such that $\left\|\cos(\sqrt{\Delta})\right\|_{L^2_\beta,L^2_\beta} \le Ce^{c|\cdot|}$, for all $s\in\mathbb{R}$. Let $\varphi\in L^2(M)$ using (15), it follows that $\sum_{j=1}^{n}\left\|f_j(\sqrt{\Delta})\varphi\right\|_{L^2_\beta} \le \frac{C}{\sqrt{2\pi}}\sum_{i=1}^{n}\left\|\hat{f}_i\right\|_{L^1_{e^{c|\cdot|}}}$. Since $L^2(M)=L^2_\beta(M)$, it follows from (2) that $f(\sqrt{\Delta})$ extends to a bounded operator in $L^2_\beta(M)$. The last statement is obvious.

Corollary 3. Let β be a function of moderate decay. Assume that there exist real numbers a,b such that:

(i) $a+b=2$,

(ii) $\beta^b \in L^1(M)$,

(iii) $\beta^a \tilde{t}^{-\frac{1}{2}n(n+1)} \in L^\infty(M)$.

Let M_β the operator of multiplication by β. Then for every $p\in\mathbb{N}_0$ the operator $M_\beta\left(\sum_{i=1}^{n}\Delta_{g_i}^p\right)e^{-t\Delta_{g_i}}$ is Hilbert-Schmidt. For $e^{-t\left(\sum_{i=1}^{n} g_i\right)}$ in a compact interval in $\mathbb{R}^+$ the Hilbert-Schmidt norm is bounded.

Proof. We have $M_\beta\Delta^p e^{-t\Delta}=\left(M_\beta e^{-\frac{1}{2}\Delta}\right)\left(\Delta^p e^{\frac{-1}{2}\Delta}\right)$. Note that the operator norm of $\Delta^p e^{-\frac{1}{2}\Delta}$ is bounded on compact subsets of $\mathbb{R}^+$. Hence we assume that $p=0$. Lemma 11, (i) implies that $e^{-t\Delta}I \in L^2_{\beta^b}(M)$. Let $e^{-t\Delta}(x,y)$ be the kernel $e^{-t\Delta}$ then $\left\langle I, e^{-t\Delta}\right\rangle_{L^2} = \int_M\int_M \prod_{i=1}^{n}\beta^b(x)e^{-t\Delta_{g_i}}(x,y)\,dy\,dx$. The integral converges since $e^{-t\Delta}(x,y)\ge 0$ we get

$$\int_M \int_M \prod_{i=1}^{n} \left|\beta(x) \mathrm{e}^{-t\Delta_{g_i}}\right|^2 (x,y)\,\mathrm{d}y\mathrm{d}x$$
$$= \int_M \int_M \prod_{i=1}^{n} \beta^2(x)\left(\mathrm{e}^{-t\Delta_{g_i}}(x,y)\right)^2 \mathrm{d}y\mathrm{d}x$$
$$\le \sup_{z,w\in M}\left|\prod_{i=1}^{n}\beta^a(z)\mathrm{e}^{-t\Delta_{g_i}}(z,w)\right|\int_M\int_M \prod_{i=1}^{n}\beta^b(x)\mathrm{e}^{-t\Delta_{g_j}}(x,y)\,\mathrm{d}y\mathrm{d}x$$
$$\le C\sup_{z\in M}\left|\beta^a(z)\tilde{t}^{-\frac{n(n+1)}{2}}(z)\right|\int_M \beta^b(x)\left(\mathrm{e}^{-t\Delta}(1)\right)(x)\,\mathrm{d}x$$
$$\le C_1\left\|\mathrm{e}^{-t\Delta}(1)\right\|_{L^2_{\beta^b}}.$$

This proves the corollary.

Lemma 13. Let β be a function of moderate decay, satisfying the conditions of Lemma 11. Let g_i, h_i be two complete metrics on M such that $g_i \sim_\beta^2 h_i$. Let Δ_{g_i} and Δ_{h_i} be the Laplacians of g_i and h_i, respectively. Then $\sum_{i=1}^{\infty}\left(\Delta_{g_i}-\Delta_{h_i}\right)\mathrm{e}^{-t\Delta_{g_i}}$ and $\sum_{i=1}^{\infty}\mathrm{e}^{-t\Delta_{g_i}}\left(\Delta_{g_i}-\Delta_{h_i}\right)$ are trace class operators, and the trace norm is uniformly bounded for τ in a compact subset of $(0,\infty)$.

Proof. We decompose $\mathrm{e}^{-\tau\Delta_{g_i}}$ as $\mathrm{e}^{-\tau\Delta_{g_i}} = \sum_{i=1}^{\infty}\left(\mathrm{e}^{-t\Delta_{g_i}} M_{\beta^{\frac{-1}{3}}}\right)\cdot\left(M_{\beta^{\frac{1}{3}}}\mathrm{e}^{\frac{-t}{2}\Delta_{g_i}}\right)$. By Lemma 11, the second factor is a Hilbert-Schmidt operator and it suffices to show that $\left(\Delta_{g_i}-\Delta_{h_i}\right)\mathrm{e}^{-t\Delta_g}M_{\beta^{\frac{-1}{3}}}$ is Hilbert-Schmidt and that the Hilbert-Schmidt norm is bounded for t in a compact interval, using Lemmas 8, and Lemmas 10, it follows that the Hilbert-Schmidt norm can be estimated by

$$\sum_{i=1}^{\infty}\left(\left\|\left(\Delta_{g_i}-\Delta_{h_i}\right)\mathrm{e}^{-t\Delta_g}M_{\beta^{\frac{-1}{3}}}\right\|_2^2\right)$$
$$\le C\sum_{i=0}^{2}\int_M\int_M\left|\left(\nabla^g\right)^i \mathrm{e}^{-t\Delta_{g_i}}(x,y)\beta^{\frac{-1}{3}}(y)\right|_{g_i}^2 \beta^2(x)\,\mathrm{d}x\mathrm{d}y$$
$$= C\sum_{i=0}^{\infty}\int_M\left\|\mathrm{e}^{t\Delta_{g_i}}(.,y)\beta^{\frac{-1}{3}}(y)\right\|_{W^2_{\beta^2}}^2 \mathrm{d}y$$
$$\le C_1\sum_{i=0}^{\infty}\int_M\left\|\mathrm{e}^{t\Delta_{g_i}}(.,y)\beta^{\frac{1}{3}}(y)\right\|_{H^2_{\beta\tilde{t}^{2-4n}}}^2 \mathrm{d}y$$
$$\le C_2\sum_{q=0}^{1}\int_M\left\|\beta(.)\tilde{t}^{-2n}(.)\Delta_g^q\mathrm{e}^{-t\Delta_g}(.,y)\beta^{\frac{-1}{3}}(y)\right\|_2^2 \mathrm{d}y$$
$$= C_2\sum_{q=0}^{1}\int_M\left\|M_\beta M_{\tilde{t}^{-2n}}\Delta_{g_i}^q\mathrm{e}^{-t\Delta_g}M_{\beta^{\frac{-1}{3}}}\right\|_2^2 \mathrm{d}y.$$

By Lemma 13, the right-hand side is finite and bounded for t in a compact interval of $\mathbb{R}^+$ prove that $\sum_{i=1}^{\infty}\mathrm{e}^{-t\Delta_{g_i}}\left(\Delta_{g_i}-\Delta_{h_i}\right)$ is a trace class operator, it suffices to establish it for its adjoint $\sum_{i=1}^{\infty}\left(\Delta_{g_i}-\left(\Delta_{h_i}\right)^{*g_i}\right)\mathrm{e}^{-t\Delta_{g_i}}$ with respect to t. By

(19) and (18) we have $\sum_{i=1}^{\infty}\Delta_{g_i}\left(\Delta_{h_i}\right)^{*g}=\sum_{i=1}^{\infty}\left(\left(\xi_{01}^{g_i}\right)^{*g}+\left(\nabla\right)^{*g}\circ\left(\xi_{11}^{g_i}\right)^{*g}+\left[\left(\nabla^{g_i}\right)^{*g}\right]^2\circ\left(\xi_{21}^{g_i}\right)^{*g}\right)$ using (14) and (16), it follows that there exists $\eta_j\in C^{\infty}\left(Hom\left(\left(T^*M\right)^{*j}\mathbb{R}\right)\right)$ such that $\sum_{i=1}^{\infty}\left(\Delta_{g_i}-\left(\Delta_{h_i}\right)^{*g_i}\right)=\sum_{i=1}^{\infty}\left(\eta_0+\eta_1\circ\nabla^{g_i}+\eta_2\circ\left(\nabla^{g_i}\right)^2\right)$ and these sections satisfy

$$\sum_{i=1}^{\infty}\left|\eta_j\right|_{g_i}(x)\le C\beta(x),\ 0\le j\le 2,\ x\in M. \tag{22}$$

By principle we have

$$\begin{aligned}\sum_{i=1}^{\infty}\left(\mathrm{e}^{-t\Delta_{g_i}}-\mathrm{e}^{-t\Delta_{h_i}}\right)&=\sum_{i=1}^{\infty}\left(\int_0^t\mathrm{e}^{-s\Delta_{g_i}}\left(\Delta_{h_i}-\Delta_{g_i}\right)\mathrm{e}^{-(t-s)\Delta_{h_i}}\mathrm{d}s\right)\\&=\sum_{i=1}^{\infty}\left(\int_0^{\frac{t}{2}}\mathrm{e}^{-s\Delta_{g_i}}\left(\Delta_{h_i}-\Delta_{g_i}\right)\mathrm{e}^{-(t-s)\Delta_{h_i}}\mathrm{d}s+\int_{\frac{t}{2}}^{t}\mathrm{e}^{-s\Delta_{g_i}}\left(\Delta_{h_i}-\Delta_{g_i}\right)\mathrm{e}^{-(t-s)\Delta_{h_i}}\mathrm{d}s\right)\end{aligned} \tag{23}$$

Using (22) and (23) we can proceed as above and prove that $\sum_{i=1}^{\infty}\left(\Delta_{g_i}-\left(\Delta_{h_i}\right)^{*g_i}\right)\mathrm{e}^{-t\Delta_{g_i}}$ is a trace class operator.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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