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Projective Changes between Generalized (α, β)-Metric and Randers Metric

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Abstract

Projective change between two Finsler metrics arises from Information Geometry. Such metrics have special geometric properties and will play an important role in Finsler geometry. The purpose of the present paper is to find a relation to characterize the projective change between generalized (α,β)-metric $F = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$ (μ_1, μ_2 and $\mu_3 \neq 0$ are constants) and Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are 1-forms. Further, we study such projective change when generalized (α,β)-metric F has some curvature property.

Keywords

Finsler Space with (α,β)-Metric, Projective Change, Locally Projectively Flat, Randers Metric

1. Introduction

The concept of projective change between two Finsler spaces has been studied by many geometers [1]-[6]. An interesting result concerned with the theory of projective change was given by Rapscak [7]. He proved necessary and sufficient conditions for projective change. S. Bacso and M. Matsumoto [8] discussed the projective change between Finsler spaces with (α,β)-metric. H. S. Park and Y. Lee have studied on projective changes between a Finsler space with (α,β)-metric and the associated Riemannian metric.

In Riemannian geometry, two Riemannian metrics α and $\bar{\alpha}$ on a manifold M are projectively related if and only if their spray coefficients have the relation $G^i_\alpha = \bar{G}^i_{\bar{\alpha}} + P_0 y^i$, where $P = P(x)$ is a scalar function on M and $P_0 = P_{x^k} y^k$. In Finsler geometry, two Finsler metrics F and $\bar{F}$ on a manifold M are called projectively related if $G^i = \bar{G}^i + P y^i$, where G^i and $\bar{G}^i$ are the geodesic coefficients of F and $\bar{F}$, respectively and $P = P(x, y)$ is a scalar function on the slit tangent bundle TM_0.

In [9], we introduced the generalized (α, β)-metric

$$F = \mu_1 \alpha + \mu_2 \beta + \mu_3 \frac{\beta^2}{\alpha} \quad (\mu_1, \mu_2 \text{ and } \mu_3 \neq 0 \text{ are constants}) \tag{1.1}$$

where α is a Riemannian metric, β is a 1-form.

We know from [4], that two Finsler metrics F and $\bar{F} = \bar{\alpha} + \bar{\beta}$ are projectively related if and only if their spray coefficients have the following relation:

$$G^i = \bar{G}^i + P y^i \tag{1.2}$$

where $P(y)$ is a scalar function on $TM - \{0\}$ and homogeneous of degree one in y.

Also, from [1] we know that a Finsler metric is called a projectively flat metric if it is projectively related to a Minkowskian metric. From [4], we know that the Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$ is projectively flat if and only if $\bar{\alpha}$ is projectively flat and $\bar{\beta}$ is closed.

The purpose of the present paper is to continue the study on the generalized (α, β)-metric $F = \mu_1 \alpha + \mu_2 \beta + \mu_3 \frac{\beta^2}{\alpha}$ and to investigate the locally projective flatness. Also, the projective change between between generalized (α, β)-metric $F = \mu_1 \alpha + \mu_2 \beta + \mu_3 \frac{\beta^2}{\alpha}$ and Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are 1-forms. Further, we characterized such projective change. Precisely, we have the following

Theorem 1.1. *Let* $F = \mu_1 \alpha + \mu_2 \beta + \mu_3 \frac{\beta^2}{\alpha}$ *and* $\bar{F} = \bar{\alpha} + \bar{\beta}$, *be two* (α, β)-*metrics, where* α *and* $\bar{\alpha}$ *are two Riemannian metrics;* β *and* $\bar{\beta}$ *are* 1-*forms. Then F is projectively related to* $\bar{F}$, *if and only if the following equations, holds*

$$G^i = G^i_{\bar{\alpha}} + \theta y^i - \tau \frac{\mu_3 \alpha^2}{\mu_1} b^i$$

$$b_{i|j} = \tau \frac{1}{\mu_1} \left[\left(\mu_1 + 2b^2 \right) a_{ij} - 3\mu_3 b_i b_j \right]$$

$$d\bar{\beta} = 0$$

where $b^i = a^{ij} b_j ; b = \|\beta\|_\alpha$ and $b_{i|j}$ are the coefficients of the covariant derivative of β with respect to α; $\tau = \tau(x)$ is a scalar function and $\theta = \theta_i y^i$ is a 1-form on M.

Corollary 1.1. *Let* $F = \mu_1 \alpha + \mu_2 \beta + \mu_3 \frac{\beta^2}{\alpha}$ *and* $\bar{F} = \bar{\alpha} + \bar{\beta}$, *be two* (α, β)-

metrics, where α *and* $\bar{\alpha}$ *are two Riemannian metrics;* β and $\bar{\beta}$ *are* 1-*forms. Then F is projectively flat if the following relation holds:*

$$G^i = G^i_{\bar{\alpha}} + \theta y^i - \tau \frac{\mu_3 \alpha^2}{\mu_1} b^i \tag{1.3}$$

where $b^i = a^{ij} b_j ; b = \|\beta\|_\alpha$ and $b_{i|j}$ are the coefficients of the covariant derivative of β with respect to α; $\tau = \tau(x)$ is a scalar function and $\theta = \theta_i y^i$ is a 1-form on *M*.

Theorem 1.2. *Let* $F = \mu_1 \alpha + \mu_2 \beta + \mu_3 \frac{\beta^2}{\alpha}$ *the* (α, β) *-metric an n-dimensional manifold M, with* α *is a Riemannian metric;* β *is a* 1*-form. Then F is locally projectively flat if and only if*

$$\begin{aligned} &2\mu_3 \frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)\left[\frac{\partial \beta}{\partial x^k} - \frac{\beta}{\alpha}\frac{\partial \alpha}{\partial x^k}\right] y^k + \left(\mu_2 + 2\mu_3 \frac{\beta}{\alpha}\right)\left(\frac{\partial b_i}{\partial x^k} - \frac{\partial b_k}{\partial x^i}\right) y^k \\ &+ \left(\mu_1 - \mu_3 \frac{\beta^2}{\alpha^2}\right)\left[\frac{\partial}{\partial y^i}\left(\frac{\partial \alpha}{\partial x^k}\right) y^k - \frac{\partial \alpha}{\partial x^i}\right] = 0. \end{aligned} \tag{1.4}$$

Finally, we have shown that the generalized (α, β)-metric satisfy the sign property.

2. Preliminaries

Definition 2.1. [1] *Let*

$$D^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i\right) \tag{2.1}$$

where G^i are the spray coefficients of *F*. The tensor $D = D^i_{jkl} \partial_i \otimes dx^j \otimes dx^k \otimes dx^l$ is called the Douglas tensor. If Douglas tensor vanishes then Finsler metric is called Douglas metric.

Some interesting results concerning Douglas metrics are recently obtained in [10] & [11].

The function $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$ and it satisfies the following condition:

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \le b < b_0. \tag{2.2}$$

Also, *F* is a Finsler metric if and only if $\|\beta_x\|_\alpha < b_0$ for any $x \in M$.

In general, the (α, β)-metrics are defined as follows:

Definition 2.2. [1] *For a given Riemannian metric* $\alpha = \sqrt{a_{ij} y^i y^j}$ *and one form* $\beta = b_i y^i$, *satisfying* $\|\beta_x\|_\alpha < b_0$ *for* $\forall x \in M$, *then:* $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, *is called* (α, β) *-metric.*

The covariant derivative of β with respect to α is $\nabla\beta = b_{i|j} dx^i \otimes dx^j$. Also, in [1], the following notations are given:

$$r_{ij} = \frac{1}{2}\left(b_{i|j} + b_{j|i}\right); s_{ij} = \frac{1}{2}\left(b_{i|j} - b_{j|i}\right). \tag{2.3}$$

It is clear that $s_{ij}=0$ if and only if β is closed. Also, we can take:

$$s_j=b^i s_{ij};s_j^i=a^{il}s_{lj};s_0^i=s_j^i y^i;r_{00}=r_{ij}y^i y^j.$$

If we consider the fundamental tensor of Randers space $g_{ij}=\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$, then we have the following formulae

$$p^i=\frac{1}{\alpha}y^i=a_{ij}\frac{\partial\alpha}{\partial y^j};\ p_i=a_{ij}p^j=\frac{\partial\alpha}{\partial y^i};$$

$$l^i=\frac{1}{L}y^i=g_{ij}\frac{\partial L}{\partial y^j};l_i=g^{ij}\frac{\partial L}{\partial y^j}=p_i+b_i;$$

$$l_i=\frac{1}{L}p^i;l^i l_j=p^i p_i=1;l^i p_i=\frac{\alpha}{L};$$

$$p^i l_i=\frac{L}{\alpha};b_i p^i=\frac{\beta}{\alpha};b_i l^i=\frac{\beta}{L}.$$

The geodesic coefficients G^i of F and the geodesic coefficients G_α^i of α, are related as follows (see [1]):

$$G^i=G_\alpha^i+\alpha Q s_0^i+\left\{-2Q\alpha s_0+r_{00}\right\}\left\{\Psi b^i+\Theta\alpha^{-1}y^i\right\} \tag{2.4}$$

where

$$\begin{aligned} Q&=\frac{\phi'}{\phi-s\phi'}\\ \Theta&=\frac{\phi\phi'-s\left(\phi\phi'+\phi'\phi'\right)}{2\phi\left(\phi-s\phi'+\left(b^2-s^2\right)\phi''\right)}\\ \Psi&=\frac{\phi''}{2\left(\phi-s\phi'+\left(b^2-s^2\right)\phi''\right)}. \end{aligned} \tag{2.5}$$

In [2] and [4], the condition for an (α,β)-metric to be locally projectively flat is presented as follows:

Lemma 2.1. *A Finsler space $F^n=(M,F)$ is locally projectively flat if and only if*

$$\frac{\partial F}{\partial x^j}-\frac{\partial^2 F}{\partial x^k \partial y^i}y^k=0. \tag{2.6}$$

In [12], we have the following condition for an (α,β)-metric to be a Douglas metric

$$\begin{aligned} &\alpha Q\left(s_0^i y^j-s_0^j y^i\right)+\Psi\left(-2\alpha Q s_0+r_{00}\right)\left(b^i y^j-b^j y^i\right)\\ &=\frac{1}{2}\left(G_{kl}^i y^j-G_{kl}^j y^i\right)y^k y^l \end{aligned} \tag{2.7}$$

where $G_{kl}^i=\Gamma_{kl}^i-\gamma_{kl}^i$ and $\gamma_{kl}^i=\frac{\partial^2 G_\alpha^i}{\partial y^k \partial y^l}$.

Theorem 2.3. [12] *Let $F=\alpha\phi(s),s=\frac{\beta}{\alpha}$ be an (α,β)-metric on an open subset $U\subset R^n(n\geq 3)$, where $\alpha=\sqrt{a_{ij}(x)y^i y^j}$ and one form $\beta=b_i y^i\neq 0$. Let*

$b = \|\beta_x\|_\alpha$. *Suppose that the following conditions holds*

a) β is not parallel with respect to α;

b) F is not of Randers type;

c) $db \neq 0$ everywhere or $b = constant$ on U. Then F is a Douglas metric on U if and only if the function $\phi = \phi(s)$ satisfies the following ODE

$$\left\{1+\left(k_1+k_2s^2\right)s^2+k_3s^2\right\}\phi''(s)=\left(k_1+k_2s^2\right)\left\{\phi(s)-s\phi'(s)\right\} \tag{2.8}$$

and the covariant derivative $\nabla\beta = b_{i|j}y^i dx^j$ of β with respect to α satisfies the following equation

$$b_{i|j} = 2\tau\left\{\left(1+k_1b^2\right)a_{ij}+\left(k_2b^2+k_3\right)b_ib_j\right\} \tag{2.9}$$

where $\tau = \tau(x)$ is a scalar function on U and k_1, k_2, k_3 are constants with $(k_2, k_3) \neq (0,0)$.

Remark: The above equation holds good in dimension $n \geq 3$.

3. Main Results

By the Theorem 2.1, we compute the coefficients $b_{i|j}$ for $F = \mu_1\alpha + \mu_2\beta + \mu_3\dfrac{\beta^2}{\alpha}$, taking into account that $F = \alpha\phi(s)$, where $\phi(s) = \mu_1 + \mu_2 s + \mu_3 s^2$, using Equation (2.9), we get

$$b_{i|j} = \tau\left[\left(1+\frac{2\mu_3}{\mu_1}b^2\right)a_{ij} - \frac{3\mu_3}{\mu_1}b_ib_j\right]. \tag{3.1}$$

Next, we obtain

$$r_{00} = \tau\left[\left(1+\frac{2\mu_3}{\mu_1}b^2\right)\alpha^2 - \frac{3\mu_3}{\mu_1}\beta^2\right] \tag{3.2}$$

Make use of (2.5) for $\phi(s) = \mu_1 + \mu_2 s + \mu_3 s^2$, we get

$$\begin{aligned} Q &= \frac{2\mu_3 s+\mu_2}{\mu_1-\mu_3 s^2}, \\ \Theta &= \frac{\mu_1\mu_2 - s^2\left(4\mu_3^2 s+3\mu_2\mu_3\right)}{2\left(\mu_1+\mu_2 s+\mu_3 s^2\right)\left(\mu_1-3\mu_3 s^2+2\mu_3 b^2\right)}, \\ \Psi &= \frac{\mu_3}{\mu_1-3\mu_3 s^2+2\mu_3 b^2}. \end{aligned} \tag{3.3}$$

Plugging (3.3) in (2.4), we get

$$\begin{aligned} G^i = G^i_\alpha &+ \frac{\alpha^2\left(2\mu_3\beta+\mu_2\alpha\right)}{\mu_1\alpha^2-\mu_3\beta^2}s^i_0 + \left\{\frac{-2\alpha\left(2\mu_3\beta+\mu_2\alpha\right)}{\mu_1\alpha^2-\mu_3\beta^2}s_0 + r_{00}\right\} \\ &\times\left\{\frac{\mu_3\alpha^2}{\mu_1\alpha^2-3\mu_3\beta^2+2\mu_3b^2\alpha^2}b^i\right. \\ &\left.+\frac{\mu_1\mu_2\alpha^3-4\mu_3^2\beta^3-3\mu_2\mu_3\alpha\beta^2}{\left(2\mu_1\alpha^2+2\mu_2\alpha\beta+2\mu_3\beta^2\right)\left(\mu_1\alpha^2-3\mu_3\beta^2+2\mu_3b^2\alpha^2\right)}y^i\right\}, \end{aligned} \tag{3.4}$$

where r_{00} is given in (3.2).

Now, we can formulate the first result:

Remark. The (α,β)-metric $F=\mu_1\alpha+\mu_2\beta+\mu_3\frac{\beta^2}{\alpha}$ is a Douglas metric with respect to Theorem 2.1, if and only if (3.1) is of the form

$$b_{i|j}=\tau\left[\left(1+\frac{2\mu_3}{\mu_1}b^2\right)a_{ij}-\frac{3\mu_3}{\mu_1}b_ib_j\right].$$

for some scalar function $\tau=\tau(x)$, where $b_{i|j}$ represents the coefficients of the covariant derivative $\beta=b_iy^i$ with respect to α. In this case β is closed.

If β is closed, then $s_{ij}=0\Rightarrow b_{i|j}=b_{j|i}$ and $s_0^i=0:s_0=0$.

Replace (3.2) in (3.4), we get:

$$G^i=G^i_\alpha-\tau\left[\frac{-\mu_1\mu_2\alpha^3+4\mu_3^2\beta^3-3\mu_2\mu_3\alpha\beta^2}{\mu_1\left(2\mu_1\alpha^2+2\mu_2\alpha\beta+2\mu_3\beta^2\right)}\right]y^i+\tau\frac{\mu_3\alpha^2}{\mu_1}b^i. \tag{3.5}$$

We consider a scalar function $P=P(y)$ on $TM-\{0\}$, *i.e.*,

$$G^i=G^i_{\bar\alpha}+Py^i. \tag{3.6}$$

From (3.5) and (3.6), we get

$$P+\tau\left[\frac{-\mu_1\mu_2\alpha^3+4\mu_3^2\beta^3-3\mu_2\mu_3\alpha\beta^2}{\mu_1\left(2\mu_1\alpha^2+2\mu_2\alpha\beta+2\mu_3\beta^2\right)}\right]y^i=G^i_\alpha-G^i_{\bar\alpha}+\tau\frac{\mu_3\alpha^2}{\mu_1}b^i. \tag{3.7}$$

Since RHS of above equation is in quadratic form, thus there must be a 1-form $\theta=\theta_iy^i$, such that

$$P+\tau\left[\frac{-\mu_1\mu_2\alpha^3+4\mu_3^2\beta^3-3\mu_2\mu_3\alpha\beta^2}{\mu_1\left(2\mu_1\alpha^2+2\mu_2\alpha\beta+2\mu_3\beta^2\right)}\right]=\theta$$

Then, we get

$$G^i=G^i_{\bar\alpha}+\theta y^i-\tau\frac{\mu_3\alpha^2}{\mu_1}b^i. \tag{3.8}$$

Using (3.1) and (3.8) and also the above remark, we can conclude the following result

Theorem 3.4. *Let* $F=\mu_1\alpha+\mu_2\beta+\mu_3\frac{\beta^2}{\alpha}$ *and* $\bar F=\bar\alpha+\bar\beta$, *be two* (α,β)-*metrics, where* α *and* $\bar\alpha$ *are two Riemannian metrics,* β *and* $\bar\beta$ *are* 1-*forms. Then F is projectively related to* $\bar F$, *if and only if the following equations, holds*

$$G^i=G^i_{\bar\alpha}+\theta y^i-\tau\frac{\mu_3\alpha^2}{\mu_1}b^i$$

$$b_{i|j}=\tau\frac{1}{\mu_1}\left[\left(\mu_1+2b^2\right)a_{ij}-3\mu_3b_ib_j\right]$$

$$d\bar\beta=0$$

where $b^i=a^{ij}b_j;b=\|\beta\|_\alpha$ and $b_{i|j}$ are the coefficients of the covariant derivative of β with respect to α; $\tau=\tau(x)$ is a scalar function and $\theta=\theta_iy^i$ is a

1-form on *M*.

The proof is obtained using (3.1) and (3.8). Also, we can now formulate the following corollary:

Corollary 3.2. *Let* $F=\mu_1\alpha+\mu_2\beta+\mu_3\frac{\beta^2}{\alpha}$ *and* $\bar{F}=\bar{\alpha}+\bar{\beta}$, *be two* (α,β)-*metrics, where* α *and* $\bar{\alpha}$ *are two Riemannian metrics*; β and $\bar{\beta}$ *are* 1-*forms. Then F is projectively flat if the following relation holds*:

$$G^i=G^i_{\bar{\alpha}}+\theta y^i-\tau\frac{\mu_3\alpha^2}{\mu_1}b^i \tag{3.9}$$

where $b^i=a^{ij}b_j;b=\|\beta\|_\alpha$ and $b_{i|j}$ are the coefficients of the covariant derivative of β with respect to α; $\tau=\tau(x)$ is a scalar function and $\theta=\theta_i y^i$ is a 1-form on *M*.

Theorem 3.5. *Let* $F=\mu_1\alpha+\mu_2\beta+\mu_3\frac{\beta^2}{\alpha}$ *the* (α,β)-*metric an n-dimensional manifold M, with* α *is a Riemannian metric*; β *is a* 1-*form. Then F is locally projectively flat if and only if*

$$\begin{aligned}&2\mu_3\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)\left[\frac{\partial\beta}{\partial x^k}-\frac{\beta}{\alpha}\frac{\partial\alpha}{\partial x^k}\right]y^k+\left(\mu_2+2\mu_3\frac{\beta}{\alpha}\right)\left(\frac{\partial b_i}{\partial x^k}-\frac{\partial b_k}{\partial x^i}\right)y^k\\&+\left(\mu_1-\mu_3\frac{\beta^2}{\alpha^2}\right)\left[\frac{\partial}{\partial y^i}\left(\frac{\partial\alpha}{\partial x^k}\right)y^k-\frac{\partial\alpha}{\partial x^i}\right]=0.\end{aligned} \tag{3.10}$$

Proof: We apply lemma 1.1, using

$$\frac{\partial F}{\partial x^j}-\frac{\partial^2F}{\partial x^k\partial y^i}y^k=0.$$

First, we compute

$$\frac{\partial F}{\partial x^k}=\left(\mu_2+2\mu_3\frac{\beta}{\alpha}\right)\frac{\partial\beta}{\partial x^k}+\left(\mu_1-\mu_3\frac{\beta^2}{\alpha^2}\right)\frac{\partial\alpha}{\partial x^k}. \tag{3.11}$$

Then, we obtain

$$\begin{aligned}\frac{\partial}{\partial y^i}\left(\frac{\partial F}{\partial x^k}\right)y^k&=2\mu_3\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)\frac{\partial\beta}{\partial x^k}y^k+\left(\mu_2+2\mu_3\frac{\beta}{\alpha}\right)\frac{\partial b^i}{\partial x^k}y^k\\&-2\mu_3\left(\frac{\beta}{\alpha}\right)\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)\frac{\partial\alpha}{\partial x^k}y^k+\left(\mu_1-\mu_3\frac{\beta^2}{\alpha^2}\right)\frac{\partial}{\partial y^i}\left(\frac{\partial\alpha}{\partial x^k}\right)y^k.\end{aligned} \tag{3.12}$$

From (3.11), replacing *k* and *i* and substituting $\beta=b_k(x)y^k$, we get

$$\frac{\partial F}{\partial x^i}=\left(\mu_2+2\mu_3\frac{\beta}{\alpha}\right)\frac{\partial b_k}{\partial x^i}y^k+\left(\mu_1-\mu_3\frac{\beta^2}{\alpha^2}\right)\frac{\partial\alpha}{\partial x^i}. \tag{3.13}$$

Finally, substituting (3.12) and (3.13) in (2.6), we obtain

$$\begin{aligned}&2\mu_3\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)\frac{\partial\beta}{\partial x^k}y^k+\left(\mu_2+2\mu_3\frac{\beta}{\alpha}\right)\frac{\partial b_i}{\partial x^k}y^k-2\mu_3\left(\frac{\beta}{\alpha}\right)\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)\frac{\partial\alpha}{\partial x^k}y^k\\&+\left(\mu_1-\mu_3\frac{\beta^2}{\alpha^2}\right)\frac{\partial}{\partial y^i}\left(\frac{\partial\alpha}{\partial x^k}\right)y^k-\left(\mu_2+2\mu_3\frac{\beta}{\alpha}\right)\frac{\partial b_k}{\partial x^i}y^k-\left(\mu_1-\mu_3\frac{\beta^2}{\alpha^2}\right)\frac{\partial\alpha}{\partial x^i}=0.\end{aligned} \tag{3.14}$$

Thus

$$2\mu_3\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)\left[\frac{\partial\beta}{\partial x^k}-\frac{\beta}{\alpha}\frac{\partial\alpha}{\partial x^k}\right]y^k+\left(\mu_2+2\mu_3\frac{\beta}{\alpha}\right)\left(\frac{\partial b_i}{\partial x^k}-\frac{\partial b_k}{\partial x^i}\right)y^k$$
$$+\left(\mu_1-\mu_3\frac{\beta^2}{\alpha^2}\right)\left[\frac{\partial}{\partial y^i}\left(\frac{\partial\alpha}{\partial x^k}\right)y^k-\frac{\partial\alpha}{\partial x^i}\right]=0$$

This completes the proof of necessity. The converse part follow easily.

Theorem 3.6. *Let* $F=\mu_1\alpha+\mu_2\beta+\mu_3\frac{\beta^2}{\alpha}$ *the* (α,β) *-metric given by* (1.1), *be locally projectively flat. Assume that* α *is locally projectively flat. Then*

$$\mu_3\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)(P-Q)=\frac{1}{2}\left(\frac{\mu_2}{2\beta}+\frac{\mu_3}{\alpha}\right)\left[\frac{\partial b_i}{\partial x^k}-\frac{\partial b_k}{\partial x^i}\right]y^k. \tag{3.15}$$

where $P=\frac{1}{2\alpha}\frac{\partial\alpha}{\partial x^k}y^k;Q=\frac{1}{2\beta}\frac{\partial\beta}{\partial x^k}y^k$

Since α is locally projectively flat and from (2.6), we get

$$\frac{\partial}{\partial y^i}\left(\frac{\partial\alpha}{\partial x^k}\right)y^k-\frac{\partial\alpha}{\partial x^i}=0. \tag{3.16}$$

From (3.10) and (3.16), we get

$$2\mu_3\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)\left[\frac{\partial\beta}{\partial x^k}-\frac{\beta}{\alpha}\frac{\partial\alpha}{\partial x^k}\right]y^k=-\left(\mu_2+2\mu_3\frac{\beta}{\alpha}\right)\left(\frac{\partial b_i}{\partial x^k}-\frac{\partial b_k}{\partial x^i}\right)y^k \tag{3.17}$$

Use definitions of P and Q and dividing with 2β in (3.17), we get

$$\mu_3\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)(P-Q)=\frac{1}{2}\left(\frac{\mu_2}{2\beta}+\frac{\mu_3}{\alpha}\right)\left[\frac{\partial b_i}{\partial x^k}-\frac{\partial b_k}{\partial x^i}\right]y^k.$$

Hence the proof.

From [13], we have the following:

Definition 3.3. *We say that an* (α,β) *-metric* $F=\alpha\phi\left(\frac{\beta}{\alpha}\right)$ *on a manifold M, satisfy the sign property, if the function*

$$A_\phi(s)=\phi'(-s)\phi(s)+\phi(-s)\phi'(s)$$

has a fix sign on a symmetric interval $(-b_0,b_0)$. Here, with s is denoted $s=\frac{\beta}{\alpha}$.

Let us consider the metric (1.1), $F=\mu_1\alpha+\mu_2\beta+\mu_3\frac{\beta^2}{\alpha}$, with $\phi(s)=\mu_1+\mu_2 s+\mu_3 s^2$.

In this case, we have:

$$A_\phi(s)=\phi'(-s)\phi(s)+\phi(-s)\phi'(s)=2\mu_1\mu_2-2\mu_2\mu_3 s^2.$$

We conclude that, for $s\in\left(-\sqrt{a},\sqrt{a}\right)$, $A_\phi(s)$ has a fix sign.

Thus metric (1.1) satisfy the sign property.

4. Conclusion

In this paper, we have obtained some important results concerning the projective change and locally projective flatness of the generalized (α,β)-metric $F=\mu_1\alpha+\mu_2\beta+\mu_3\frac{\beta^2}{\alpha}$ (μ_1, μ_2 and $\mu_3\neq 0$ are constants). Further, we have shown that the generalized (α,β)-metric satisfy the sign property.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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