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On Some Topological Properties of Normed Boolean Algebras

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Abstract

This paper concerns the compactness and separability properties of the normed Boolean algebras (N.B.A.) with respect to topology generated by a distance equal to the square root of a measure of symmetric difference between two elements. The motivation arises from studying random elements talking values in N.B.A. Those topological properties are important assumptions that enable us to avoid possible difficulties when generalising concepts of random variable convergence, the definition of conditional law and others. For each N.B.A., there exists a finite measure space $(E,\mathcal{E},\mu)$ such that the N.B.A. is isomorphic to $(\tilde{\mathcal{E}},\tilde{\mu})$ resulting from the factorisation of initial σ-algebra by the ideal of negligible sets. We focus on topological properties $(\tilde{\mathcal{E}},\tilde{\mu})$ in general setting when μ can be an infinite measure. In case when μ is infinite, we also consider properties of $\tilde{\mathcal{E}}_{fin}\subseteq\tilde{\mathcal{E}}$ consisting of classes of measurable sets having finite measure. The compactness and separability of the N.B.A. are characterised using the newly defined terms of approximability and uniform approximability of the corresponding measure space. Finally, conditions on $(E,\mathcal{E},\mu)$ are derived for separability and compactness of $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{E}}_{fin}$.

Keywords

Compact, Locally Compact, Polish Space, Separable

1. Introduction

The motivation for studying the topological properties of normed Boolean algebras arises from probability theory, more precisely from its subfield of stochastic geometry. Nowadays, the mathematical theory of random sets is very popular. The books [1] and [2] provide basic definitions, notions and theoretical results on closed random sets (or compact random sets) as random elements with values in the family of closed subsets of locally compact Hausdorff second countable topological space.

An approach to defining a random set that takes values in a more general family of sets than closed or compact sets is presented in [3]. There, the random set is represented as a random element taking values in a normed Boolean algebra (N.B.A.), *i.e.* a complete Boolean algebra endowed with a strictly positive finite measure, see [4].) These random elements are defined using Borel subsets of N.B.A. generated by a distance on N.B.A. equal to the square root of a measure of symmetric difference between two elements.

If we want to study different types of convergence of these random sets taking values in N.B.A. or generalize some other concepts related to random variables, it is beneficial to ensure that the space of its values is a Polish space or a locally compact, Hausdorff and second countable topological space (LCSH space). This motivated us to study the topological properties of the N.B.A.s with respect to topology generated by the distance equal to the square root of a measure of symmetric difference between two elements.

Let us mention some conveniences we get when working with random elements with a separable metric space of values. In this setting, for every two random elements X and X', a set $\{X = X'\}$ is an event. The distance between two random elements is a random variable, which allows us to introduce convergence in probability (see [5]). In this case, the space of simple random elements is a dense subspace. If the space of values of the random elements is complete and separable (Polish), then the conditional law can be defined and the Doob-Dynkin representation holds (see [6]).

Locally compactness is also a desirable property when considering weak convergence of distributions of random elements (see [7]).

It is worth mentioning that there are some other topologies that can be defined on N.B.A.s that can generate Borel sets (for more details see [4, Chapter 4]). The best known is the o-topology, which in our case coincides with the topology generated by the distance on N.B.A. mentioned above. However, in this paper, we will only focus on the topological properties of N.B.A. when considering the o-topology, since it was the most suitable for defining the notion of random set in [3]. To our knowledge, these topological properties have not yet been studied.

For each complete N.B.A. $(\mathcal{X}, m)$ where m is finite, there exists a finite measure space $(E, \mathcal{E}, \mu)$ such that the N.B.A. $(\mathcal{X}, m)$ and N.B.A. $(\tilde{\mathcal{E}}, \tilde{\mu})$ resulting from the factorisation of initial σ-algebra by the ideal of negligible sets are isomorphic (see [4]). Following this result, we derive that the N.B.A. is homeomorphic to the space of indicator functions $L^p(E, \mathcal{E}, \mu)$.

The measure space analysis approach allows us to apply some well-established properties of topologies on space of measurable functions to the topology on N.B.A.

We generalise this setting allowing μ and corresponding m to obtain infinite values. In this case, the above-mentioned homeomorphism does not hold.

As we mentioned before, if μ is finite, then topological properties of N.B.A. are equivalent to topological properties of a subset of indicators in L^2 space. Following results concerning the separability of L^p spaces are established. If μ is σ-finite and $\mathcal{E}$ is countably generated, then $L^p(E,\mathcal{E},\mu)$ is separable for $1 \le p < +\infty$ (see [8, Proposition 3.4.5.]). Since every metric subspace of separable metric space is separable [9, Theorem VIII, p.~160] if these conditions hold the space of indicators is separable as well.

If measure μ is not finite, then $(\tilde{\mathcal{E}},\tilde{\mu})$ is not homeomorphic to the space of indicator functions in $L^p(E,\mathcal{E},\mu)$. In this case, we also consider $\tilde{\mathcal{E}}_{fin} = \{[A]:\mu(A)<\infty\} \subset \tilde{\mathcal{E}}$, which is homeomorphic to the space of indicator functions in $L^p(E,\mathcal{E},\mu)$. In case $(E,\mathcal{E}) = (\mathbb{R}^d,\mathfrak{B}(\mathbb{R}^d))$ we prove the $\tilde{\mathcal{E}}_{fin}$ and corresponding space of indicators is separable if measure μ is outer regular. Although $\mathfrak{B}(\mathbb{R}^d)$ is countably generated, there are measures on $\mathfrak{B}(\mathbb{R}^d)$ which are outer regular but not σ-finite.

The compactness of subsets of L^p-spaces has already been well studied, and some conditions for the compactness of generally bounded subsets of L^p-spaces can be found in [10] and ([11], Theorems 18, 20, 21 pp.297). Although these conditions can be verified for our case when μ is finite, we introduce conditions that are easier to verify, more intuitive in our setting and can be applied for verifying compactness of $\tilde{\mathcal{E}}$ in case when μ is infinite.

It is well known that a separable space is a space that is "well approximated by a countable subset" and a compact space is a space that is "well approximated by a finite subset". We construct conditions for the corresponding measure space that follow this intuition. We call those conditions approximability and uniform approximability. We prove that if the measure can be well approximated by its values on a countable family or a finite family of measurable sets, then the corresponding N.B.A. is separable or a compact metric space, respectively.

Verifying the conditions of approximability and uniform approximability, we derive conditions and in some cases characterisation for separability and compactness of $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{E}}_{fin}$ based on properties of corresponding measure space $(E,\mathcal{E},\mu)$.

The outline of the paper is as follows.

In the Preliminaries section, we recall basic definitions and results concerning separability and compactness, we also mention some results from the measure theory we use for deriving results. The final subsection is dedicated to the terminology concerning Boolean algebras. The metric spaces $(\tilde{\mathcal{E}},d_\mu)$ and $(\tilde{\mathcal{E}}_{fin},d_\mu)$ are introduced and their completeness is discussed.

In the Main result section, we introduce properties of approximability and uniform approximability of measure with respect to filtration. Separability and compactness are characterised using these terms. Further, we discuss separability and compactness of $(\tilde{\mathcal{E}},d_\mu)$ and $(\tilde{\mathcal{E}}_{fin},d_\mu)$ based on the properties of the corresponding measure space $(E,\mathcal{E},\mu)$.

The paper is concluded by the Discussion section where the obtained results are summarised.

2. Preliminaries

2.1. Topological Properties

Let us first recall definitions and the basic relation of topological properties we study. The definitions and the results we present can be found in [12] and [13].

For some $A \subset X$, let $\mathcal{A} = \{G_i\}$ be a class of subsets of X such that $A \subset \cup_i G_i$. $\mathcal{A}$ is called a **cover** of A, and an **open cover** if each G_i is open. Furthermore, if a finite subclass of A is also a cover of A, *i.e.* if $G_{i_1}, \cdots, G_{i_m} \in A$ such that $A \subset G_{i_1} \cup \cdots \cup G_{i_m}$ then $\mathcal{A}$ contains a **finite subcover**.

Definition 1 A subset A of a topological space X is **compact** if every open cover of A is reducible to a finite cover.

In other words, if A is compact and $A \subset \cup_i G_i$, where the G_i are open sets, then one can select a finite number of the open sets $G_{i_1}, \cdots, G_{i_m}$, so that $A \subset G_{i_1} \cup \cdots \cup G_{i_m}$.

If X is a topological space, a neighbourhood of $x \in X$ is a subset V of X that includes an open set U such that $x \in U$.

Definition 2 A topological space X is **locally compact** if every point in X has a compact neighbourhood.

Definition 3 A subset S of a metric space X is called a **totally bounded** subset of X if, and only if, for each $r \in \mathbb{R}^+$, there is a finite collection of balls of X of radius r that. covers S. A metric space X is said to be totally bounded if, and only if, it is a totally bounded subset of itself.

Theorem 2.1 *A metric space is compact if and only if it is complete and totally bounded.*

Theorem 2.2 *A subspace Y of a complete metric space is complete if and only if Y is closed.*

Theorem 2.3 *Every closed subset of a compact space is compact.*

Definition 4 A topological space X is said to be **separable** if it contains a countable dense subset.

Theorem 2.4 *Every metric subspace of separable metric space is separable.*

2.2. Measure theory

We will need the following definitions and results from measure theory.

Theorem 2.5 ([14]) *Every open subset U of* $\mathbb{R}^d$, $d \geq 1$, *can be written as a countable union of disjoint half-open cubes of form* $A^{(n)}_{i_1,\cdots,i_d} = \prod_{k=1}^{d} \left[\frac{i_k}{2^n}, \frac{i_k+1}{2^n}\right\rangle$, $n \in \mathbb{N}$, $i_1, \cdots, i_d \in \mathbb{Z}$.

Definition 5 Let $\mathcal{A}$ be a σ-algebra on $\mathbb{R}^d$ that includes the σ-algebra $\mathfrak{B}(\mathbb{R}^d)$ of Borel sets. A measure μ on $(\mathbb{R}^d, \mathcal{A})$ is **regular** if

(a) (locally finite) each compact subset K of $\mathbb{R}^d$ satisfies $\mu(K) < +\infty$,

(b) (outer regular) Each set A in $\mathcal{A}$ satisfies

$$\mu(A) = \inf\{\mu(U) : U \text{ is open and } A \subseteq U\}, \text{ and}$$

(c) (inner regular) each open subset U of $\mathbb{R}^d$ satisfies

$$\mu(U) = \sup\{\mu(K) : K \text{ is compact and } K \subseteq U\}.$$

Theorem 2.6 ([8]) *Any finite measure on* $\mathbb{R}^d$ *is regular.*

The Lebesgue measure on $\mathbb{R}^d$ is a regular measure (see e.g. [8]). However, not all σ-finite measures on $\mathbb{R}^d$ are regular ([15], Corollary 13.7]. Also, there are some outer regular measures that are not σ-finite. For example, defined $\mu : \mathfrak{B}(\mathbb{R}) \to [0,\infty]$ by $\mu(A) = \begin{cases} \lambda(A), & 0 \notin A \\ \infty, & 0 \in A \end{cases}$. It is easy to see that μ is a measure on $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ that is not σ-finite but is outer regular.

Definition 6 If $(E, \mathcal{E}, \mu)$ is a measure space, a set $A \in \mathcal{E}$ is called an **atom** of μ iff $0 < \mu(A) < \infty$ and for every $C \subset A$ with $C \in \mathcal{E}$, either $\mu(C) = 0$ or $\mu(C) = \mu(A)$.

A measure without any atoms is called **non-atomic.**

A measure space $(E, \mathcal{E}, \mu)$, or the measure μ, is called **purely atomic** if there is a collection $\mathcal{C}$ of atoms of μ such that for each $A \in \mathcal{E}$, $\mu(A)$ is the sum of the numbers $\mu(C)$ for all $C \in \mathcal{C}$ such that $\mu(A \cap C) = \mu(C)$.

Lemma 2.1 ([16]) *An atom of any finite measure* μ *on* $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ *is a singleton* $\{x\}$ *such that* $\mu(\{x\}) > 0$.

Lemma 2.2 ([17]) *Any atom of a Borel measure on a second countable Hausdorff space includes a singleton of positive measure.*

In particular, a Borel measure on a second countable Hausdorff space is nonatomic if and only if every singleton has measure zero.

A measure space $(E, \mathcal{E}, \mu)$ is localizable if there is a collection $\mathcal{A}$ of disjoint measurable sets of finite measure, whose union is all of X, such that for every set $B \subset X$, B is measurable if and only if $B \cap C \in \mathcal{E}$ for all $C \in \mathcal{A}$, and then $\mu(B) = \sum_{C \in \mathcal{A}} \mu(B \cap C)$. Some examples of localisable measures are the σ-finite ones or counting measures on possibly uncountable sets.

Theorem 2.7 ([18]) *Let* $(E, \mathcal{E}, \mu)$ *be a localisable measure space. Then there exist measures* ν *and* ρ *such that* $\mu = \nu + \rho$, ν *are purely atomic and* ρ *non-atomic.*

2.3. Boolean Algebra

In this section, we present the basics concerning Boolean algebras of sets. For more details, see e.g. [4] or [19].

Definition 7 A **Boolean algebra (B.A.)** is a structure $(\mathcal{X}, \cup, \cap, (\cdot)^c, 0, 1)$ with two binary operations $\cup$ and $\cap$, a unary operation $(\cdot)^c$ and two distinguished elements 0 and 1 such that for all A, B and C in $\mathcal{X}$,

$$\begin{aligned} &A \cup (B \cup C) = (A \cup B) \cup C, && A \cap (B \cap C) = (A \cap B) \cap C, \\ &A \cup B = B \cup A, && A \cap B = B \cap A, \\ &A \cup (A \cap B) = A, && A \cap (A \cup B) = A, \\ &A \cap (B \cup C) = (A \cap B) \cup (A \cap C), && A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ &A \cup (A)^c = 1, && A \cap (A)^c = 0. \end{aligned}$$

Definition 8 Let $\mathcal{X}$ be a B.A. The B.A. $\mathcal{X}$ is **normed (N.B.A.)** if there exists a σ-additive strictly positive finite measure μ (*i.e.* $\mu(\mathrm{A})=0$ implies $\mathrm{A}=0$) defined on it. In this case, we use the notation $(\mathcal{X},m)$.

On $\mathcal{X}\times\mathcal{X}$ we can define a relation $\subseteq$ by setting $\mathrm{A}\subseteq\mathrm{B}$ if $\mathrm{A}\cup\mathrm{B}=\mathrm{B}$. It is easy to verify that $\subseteq$ is a partial order relation.

Definition 9 B.A. $\mathcal{X}$ is **complete** if every non-empty subset $\mathcal{C}\subseteq\mathcal{X}$ has its infimum and supremum.

Let $(E,\mathcal{E},\mu)$ be a finite measure space. We can define equivalence relation ~ on $\mathcal{E}\times\mathcal{E}$ by setting $A\sim B$ if and only if $\mu(A\Delta B)=0$, where $A\Delta B=(A\cap B^c)\cup(A^c\cap B)$ is the symmetric difference between the sets A and B (A^c and B^c denote the complements of A and B, respectively).

Let $[A]=\{B\in\mathcal{E}:\mu(A\Delta B)=0\}\in\tilde{\mathcal{E}}$. Then $\tilde{\mathcal{E}}=\{[A]:A\in\mathcal{E}\}$ a quotient space of $\mathcal{E}$ by ~ is a complete N.B.A. endowed with the measure $\tilde{\mu}$ defined by $\tilde{\mu}([A]):=\mu(A)$.

The inverse result also holds. Namely, for each complete N.B.A. $(\mathcal{X},m)$, there exists a measure space $(E,\mathcal{E},\mu)$ such that the N.B.A. $(\mathcal{X},m)$ is isomorphic to $(\tilde{\mathcal{E}},\tilde{\mu})$ (see [4]). Therefore, further on we focus on investigating properties of $(\tilde{\mathcal{E}},\tilde{\mu})$. We generalise the above setting, by letting measure μ be arbitrary, possibly non-finite.

Define $d_\mu:\tilde{\mathcal{E}}\times\tilde{\mathcal{E}}\to[0,\infty]$ by

$$d_\mu([A],[B]):=\left(\tilde{\mu}([A]\Delta[B])\right)^{(1/2)}=\left(\mu(A\Delta B)\right)^{(1/2)}.$$

It is easy to see that d_μ is a metric on $\tilde{\mathcal{E}}$ possibly taking infinite values. We suppose the topology $\tilde{\mathcal{E}}$ is generated by d_μ. We are interested in the topological properties of $(\tilde{\mathcal{E}},d_\mu)$.

Remark 1 Let us mention that there are many topologies introduced in B.A.s. The most popular among them is the order topology. It is known that the topology of the metric space $(\tilde{\mathcal{E}},d_\mu)$ coincides with the ordered topology (see [4]).

Denote $L^2(E,\mathcal{E},\mu)$ a Hilbert space of measurable functions that are square integrable with respect to the measure μ, where functions that agree μ almost everywhere are identified. Let

$\mathcal{I}=\{\mathbb{1}_A,A\in\mathcal{E}\}=\{\mathbb{1}_A:\mu(A)<\infty\}\subset L^2(E,\mathcal{E},\mu)$ where

$$\mathbb{1}_A(x)=\begin{cases}0, & x\notin A,\\ 1, & x\in A,\end{cases}$$

stands for the **indicator** of set A or a **characteristic function** of set A.

If $\mu(E)<\infty$, we can define $\iota:\tilde{\mathcal{E}}\to\mathcal{I}$ by $\iota([A])=\mathbb{1}_A$. Since

$$d_\mu([A],[B])=\left(\int_E|\mathbb{1}_A-\mathbb{1}_B|^2\,\mathrm{d}\mu\right)^{(1/2)},$$

$\mathcal{E}$ and $\mathcal{I}$ are isometric.

Suppose that 1_{A_n} converges to f in $L^2(E,\mathcal{E},\mu)$. Since, $L^2(E,\mathcal{E},\mu)$ is complete, $f\in L^2(E,\mathcal{E},\mu)$. Let us show that f is an indicator function of some measurable set. There exists a subsequence $\left(\mathbb{1}_{A_{n_k}}\right)_k$ such that

$\lim_k \mathbb{1}_{A_{n_k}}(x) = f(x)$, *μ-a.e.* (see e.g. [20, Theorem 16.25]) Also, following

$$f(x) = \liminf_k \mathbb{1}_{A_{n_k}}(x) = \mathbb{1}_{\liminf_k A_{n_k}}(x) \le \mathbb{1}_{\limsup_k A_{n_k}}(x)$$
$$= \limsup \mathbb{1}_{A_{n_k}}(x) = f(x), \ \mu\text{-a.e.},$$

we conclude that $\mathcal{I}$ is closed. Following Theorem 2.2 we can conclude that $\mathcal{I}$ is complete metric subspace of $L^2(E,\mathcal{E},\mu)$. Therefore, we have shown that in case μ is finite $(\tilde{\mathcal{E}}, d_\mu)$ is complete metric space. Let us show that this holds in a general case when μ is not finite.

Theorem 2.8 $(\tilde{\mathcal{E}}, d_\mu)$ *is complete metric space.*

Proof. Suppose that $([A_n])_{n\in\mathbb{N}}$ is a Cauchy sequence in $\tilde{\mathcal{E}}$. Then for fixed, $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for $n,m \ge n_\varepsilon$ $d_\mu([A_n],[A_m]) < \varepsilon$. We define $f_n = \mathbb{1}_{A_n} - \mathbb{1}_{A_{n_\varepsilon}}$, $n \ge n_\varepsilon$. Since $\int |f_n| \mathrm{d}\mu = \mu(A_n \Delta A_{n_\varepsilon}) = d_\mu([A_n]\Delta[A_{n\varepsilon}]) \le \varepsilon$, so $f_n \in L^p(E,\mathcal{E},\mu)$. Also, (f_n) is a Cauchy sequence, and since $L^1(E,\mathcal{E},\mu)$ is complete, there exists $f \in L^1$ such that $\lim_n \int_E |f_n - f|^p \mathrm{d}\mu = 0$. Furthermore, there exists a subsequence $(f_{n_k})_k$ such that $\lim_k f_{nk}(x) = f(x)$, *μ-a.e.* It holds

$$f(x) = \liminf_k \left(\mathbb{1}_{A_{n_k}}(x) - \mathbb{1}_{A_{n_\varepsilon}}\right) = \mathbb{1}_{\liminf_k A_{n_k}}(x) - \mathbb{1}_{A_{n_\varepsilon}}$$
$$\le \mathbb{1}_{\limsup_k A_{n_k}}(x) - \mathbb{1}_{A_{n_\varepsilon}} = \limsup_k \left(\mathbb{1}_{A_{n_k}}(x) - \mathbb{1}_{A_{n_\varepsilon}}\right) = f(x), \ \mu\text{-a.e.},$$

which shows that $\mathbb{1}_{\liminf_k A_{n_k}} = \mathbb{1}_{\limsup_k A_{n_k}}$ *μ-a.e*, or equivalently $[\liminf_k A_{n_k}] = [\limsup_k A_{n_k}]$. Following

$$d_\mu([A_n],[\liminf_k A_{n_k}]) = \int_E \left|\mathbb{1}_{A_n} - \mathbb{1}_{\liminf_k A_{n_k}}\right| \mathrm{d}\mu = \int_E \left|\mathbb{1}_{A_n} - \mathbb{1}_{A_{n_\varepsilon}} + \mathbb{1}_{A_{n_\varepsilon}} - \mathbb{1}_{\liminf_k A_{n_k}}\right| \mathrm{d}\mu,$$
$$= \int_E |f_n - f| \mathrm{d}\mu = 0$$

the sequence $([A_n])_n$ is convergent, so $(\tilde{\mathcal{E}}, d_\mu)$ is complete.

Remark 2 In this case $\mu(E) = \infty$, one can also consider $\tilde{\mathcal{E}}_{fin} = \{[A] : A \in \mathcal{E}, \mu(A) < \infty\}$. It is easy to see that $(\tilde{\mathcal{E}}_{fin}, d_\mu)$ is isometric to $\mathcal{I} \subset L^2(E,\mathcal{E},\mu)$, so it is a complete metric subset of $(\tilde{\mathcal{E}}, d_\mu)$. However, $\tilde{\mathcal{E}}_{fin}$ is not a B.A. since it is not e.g. closed under complements.

3. Main Result

Before we show the main result, in order to get intuition, we first start with a motivating example.

Suppose that $K = [0,1] \times [0,1]$ (an observation window) and consider $(E,\mathcal{E},\mu) = \left(K, \mathfrak{B}(K), \lambda|_{\mathfrak{B}(K)}\right)$, where $\mathfrak{B}(K)$ is Borel σ-algebra on K and $\lambda|_{\mathfrak{B}(K)}$ Lebesgue measure.

If we consider a ball in $(\tilde{\mathcal{E}}, d_\mu) = \left(\widetilde{\mathfrak{B}(K)}, d_\lambda\right)$ of radius $\varepsilon > 0$ it holds:

$$B([A],\varepsilon) = \left\{[B] \in \widetilde{\mathfrak{B}(K)} : d_\lambda([A],[B]) < \varepsilon\right\}$$
$$= \left\{[B] \in \widetilde{\mathfrak{B}(K)}, \lambda(A\Delta B) < \varepsilon^2\right\}.$$

For each $n \in \mathbb{N}$, we can partition K into 2^{2n} smaller squares $A_{(i,j)}^{(n)} = \left[(i-1)/2^n, i/2^n\right] \times \left[(j-1)/2^n, j/2^n\right]$, $i, j = 1, \cdots, 2^n$. Intuitively, we pixelise the unit square by a $2^n \times 2^n$ net.

We show that for each ε we can pixelise the unit square fine enough so that the error of approximation of the set B would be less than ε. Denote by $I_n = \{1, 2, \cdots, 2^n\} \times \{1, 2, \cdots, 2^n\}$.

Lemma 3.1 *For an arbitrary* $B \in \mathfrak{B}(K)$ *and an arbitrary* $\varepsilon > 0$, *there exists* $n \in \mathbb{N}$ *and* $I \subset I_n$ *such that*

$$\lambda\left(\bigcup_{(i,j)\in I} A_{(i,j)}^{(n)} \Delta B\right) < \varepsilon.$$

This result follows directly from Lemma 3.2 which we prove later in the paper.

Note that the family $\left\{\left[A_{(i,j)}^{(n)}\right] : i, j, n \in \mathbb{N}, i, j \le 2^n\right\}$ is countable dense subset of $\left(\widetilde{\mathfrak{B}(K)}, d_\lambda\right)$, so $\left(\widetilde{\mathfrak{B}(K)}, d_\lambda\right)$ is separable.

Following Lemma 3.1, for arbitrary $\varepsilon > 0$ the collection of balls

$$\mathcal{C}_\varepsilon = \left\{B([A], \varepsilon) : \exists n \in \mathbb{N} \text{ such that } A = \bigcup_{(i,j)\in I} A_{(i,j)}^{(n)} \text{ for some } I \subset I_n\right\} \tag{3.1}$$

is an infinite (countable) open cover $\left(\widetilde{\mathfrak{B}(K)}, d_\lambda\right)$. (Since for every $\varepsilon > 0$ and arbitrary $B \in \mathfrak{B}(K)$ there exists A in form $A = \bigcup_{(i,j)\in I} A_{(i,j)}^{(n)}$ for some $I \subset I_n$ such that $[B] \in B([A], \varepsilon)$)

Suppose that $\left(\widetilde{\mathfrak{B}(K)}, d_\lambda\right)$ is compact, therefore the open cover (3.1) should have a finite subcover. It means that there exists $m \in \mathbb{N}$ such that the collection of open balls

$$\mathcal{C}_{\varepsilon, fin}^m = \left\{B([A], \varepsilon) : A = \bigcup_{(i,j)\in I} A_{(i,j)}^{(m)} \text{ for some } I \subset I_m\right\}$$

covers $\left(\widetilde{\mathfrak{B}(K)}, d_\lambda\right)$.

However, if we take $\varepsilon = \frac{1}{\sqrt{2}}$ and define a set

$$T_m = \bigcup_{i,j=1}^{m} T_{(i,j)}, \tag{3.2}$$

where

$$T_{(i,j)} = \left\{(x, y) \in K : x \in \left[(i-1)/2^m, i/2^m\right], y \in \left[(j-1)/2^m, j/2^m\right], y \le x - i/2^m + j/2^m\right\}$$

it is easy to see (Left plot in **Figure 1** provides a visualisation of the set T_2) that $\lambda(A \Delta T_m) = 1/2$, for each A such that $A = \bigcup_{(i,j)\in I} A_{(i,j)}^{(n)}$ for some $I \subset I_n$, $n \le m$. Therefore $[T_m]$ is not contained in any ball in $\mathcal{C}_{\frac{1}{\sqrt{2}}, fin}^m$ so $\mathcal{C}_{\frac{1}{\sqrt{2}}, fin}^m$ cannot be a

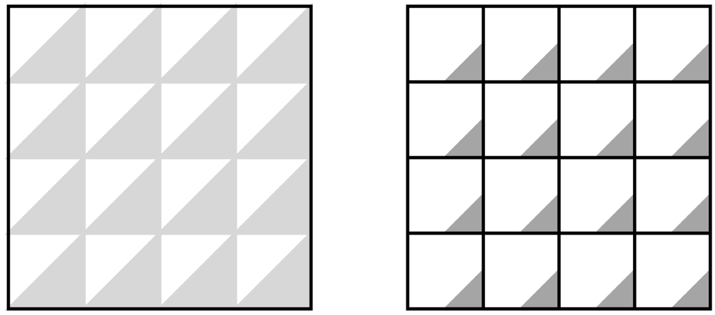

Figure 1. Left plot: Visualisation of set $T_2^{\frac{1}{2}}$ (coloured in grey) defined by (3.2). Right plot: Visualisation of the set T_2^{ε} for $\varepsilon = \frac{1}{8}$ (coloured in dark grey) defined by (3.3).

cover of $\left(\widetilde{\mathfrak{B}(K)}, d_\lambda\right)$. So countable open cover $\mathcal{C}_{\frac{1}{\sqrt{2}}}$ of $\left(\widetilde{\mathfrak{B}(K)}, d_\lambda\right)$ has no finite subcover. We can conclude that $\left(\widetilde{\mathfrak{B}(K)}, d_\lambda\right)$ is not compact.

In order to show that $\left(\widetilde{\mathfrak{B}(K)}, d_\lambda\right)$ is not locally compact, we will prove that closed ball with the centre in $[\varnothing]$ and radius $\varepsilon < \frac{1}{\sqrt{2}}$ denoted by $\bar{B}([A], \varepsilon) = \left\{[B] \in \widetilde{\mathfrak{B}(K)} : \lambda(A \Delta B) \le \varepsilon^2\right\}$ is not compact.

Since $\mathcal{C}_\varepsilon$ covers $\widetilde{\mathfrak{B}(K)}$ it also covers $\bar{B}([\varnothing], \varepsilon)$ We will show that open cover $\mathcal{C}_\varepsilon$ cannot be reduced to a finite subcover. For that purpose, for $0 < \varepsilon^2 \le \frac{1}{2}$ we define a set

$$T_m^{(\varepsilon)} = \bigcup_{(i,j) \in I_m} T_{(i,j)}^{(\varepsilon)}, \tag{3.3}$$

where

$T_{(i,j)}^{(\varepsilon)} = \left\{(x,y) \in K : x \in \left[(i-1)/2^m + \left(1-\varepsilon^2/2\right)/2^m, i/2^m\right], y \in \left[(j-1)/2^m, j/2^m\right], y \le x - i/2^m + j/2^m\right\}$ of 2^{2m} disjoint triangles whose union has Lebesgue measure equal to ε^2, so $[T_m] \in \bar{B}([\varnothing], \varepsilon)$ for each $m \in \mathbb{N}$. The right plot in Figure 1 provides a visualisation of the set T_2^{ε} for $\varepsilon = \frac{1}{8}$. For each A such that $A = \bigcup_{(i,j) \in I} A_{(i,j)}^{(m)}$ for some $I \subset I_m$, it holds

$$\begin{aligned} \mu\left(A \Delta T_m^{(\varepsilon)}\right) &= \mu\left(A \setminus T_m^{(\varepsilon)}\right) + \mu\left(T_m^{(\varepsilon)} \setminus A\right) \\ &= \sum_{(i,j) \in I} \mu\left(A_{(i,j)} \setminus T_m^{(\varepsilon)}\right) + \sum_{(i,j) \in I_m \setminus I} \mu\left(T_m^{(\varepsilon)} \setminus A_{(i,j)}\right) \\ &= \left(1 - \frac{\varepsilon^2}{2^{2m}}\right)|I| + \frac{\varepsilon^2}{2^{2m}}\left(|I_m| - |I|\right) \\ &\ge \varepsilon^2, \end{aligned} \tag{3.4}$$

where the last inequality follows from the fact that from $0 < \varepsilon^2 \le \frac{1}{2}$ follows $1-\varepsilon^2 > \varepsilon^2$. So $\left[T_m^{(\varepsilon)}\right] \in \bar{B}([\varnothing],\varepsilon)$ but it is not in any ball in $\mathcal{C}_{\varepsilon,fin}^m$. We conclude $\mathcal{C}_{\varepsilon,fin}^m$ cannot be a cover of $\bar{B}([\varnothing],\varepsilon)$. Since the countable open cover $\mathcal{C}_\varepsilon$ of $\bar{B}([\varnothing],\varepsilon)$ has no finite subcover, $\bar{B}([\varnothing],\varepsilon)$ is not compact. We conclude that $[\varnothing]$ does not have a compact neighbourhood, so $\widetilde{\mathfrak{B}(K)}$ is not locally compact.

In order to generalise these ideas on arbitrary measure space $(E,\mathcal{E},\mu)$ we introduce the following definitions.

Definition 10 Let $(E,\mathcal{E},\mu)$ be a measure space and $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ a filtration, *i.e.* $\mathcal{F}_n \subseteq \mathcal{E}$ is σ-algebra such that $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$. Let $\mathcal{E}'$ be an arbitrary subset of $\mathcal{E}$.

Measure μ is **approximable** on $\mathcal{E}'$ with respect to $\mathcal{F}$ if for each $A \in \mathcal{E}'$ and each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ and $A' \in \mathcal{F}_{n_\varepsilon}$ such that

$$\mu(A\Delta A') < \varepsilon.$$

Measure μ is **uniformly approximable** on $\mathcal{E}'$ with respect to $\mathcal{F}$ if for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for every $A \in \mathcal{E}'$ there exists $A' \in \mathcal{F}_{n_\varepsilon}$ so that

$$\mu(A\Delta A') < \varepsilon.$$

For arbitrary $\mathcal{E}' \subseteq \mathcal{E}$ denote by $\widetilde{\mathcal{E}'} = \{[A] : A \in \mathcal{E}'\}$.

Theorem 3.1 *$\widetilde{\mathcal{E}'}$ is separable in $(\tilde{\mathcal{E}}, d_\mu)$ if and only if there exists $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ a filtration for which $|\mathcal{F}_n|$ is finite for each $n \in \mathbb{N}$ such that μ is approximable on $\mathcal{E}'$ with respect to $\mathcal{F}$.*

Proof. Suppose that μ is approximable on $\mathcal{E}'$ wih respect to $\mathcal{F}$. Denote by $\widetilde{\mathcal{E}'}_c = \{[B] : B \in \cup_{n\in\mathbb{N}} \mathcal{F}_n\}$ Since $\cup_{n\in\mathbb{N}} \mathcal{F}_n$ is countable, $\widetilde{\mathcal{E}'}_c$ is also countable. Let us show that $\widetilde{\mathcal{E}'}_c$ is dense in $\widetilde{\mathcal{E}'}$. For arbitrary $\varepsilon > 0$ and arbitrary $A \in \mathcal{E}'$ there exists $A' \in \cup_{n\in\mathbb{N}} \mathcal{F}_n$ such that $\mu(A\Delta A') < \varepsilon^2$, so $[A] \in B([A'],\varepsilon)$, and therefore $\widetilde{\mathcal{E}'}_c$ is countable dense subset of $\widetilde{\mathcal{E}'}$ and $\widetilde{\mathcal{E}'}$ is separable.

If $\widetilde{\mathcal{E}'}$ is separable, then there exists a countable dense subset of $\widetilde{\mathcal{E}'}$, denote it by $\widetilde{\mathcal{E}'}_c$. W can represent $\widetilde{\mathcal{E}'}_c$ as $\{[B] : B \in \mathcal{B}\}$ for some countable $\mathcal{B} = \{B_n : n \in \mathbb{N}\} \subset \mathcal{E}'$. Take $\mathcal{F}_n = \sigma(B_1 \cdots B_n)$. Then for each $\varepsilon > 0$ and each, $A \in \mathcal{E}'$ there exists $B_n \in \mathcal{B}$ such that $[A] \in B([B_n],\varepsilon)$ so that $\mu(A\Delta B_n) < \varepsilon^2$. Therefore, μ is aproximable on $\mathcal{E}'$ with respect to $(F_n)_{n\in\mathbb{N}}$. □

Theorem 3.2 *$\widetilde{\mathcal{E}'}$ is totally bounded in $(\tilde{\mathcal{E}}, d_\mu)$ if and only if there exist exists a filtration $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ such that μ is uniformly approximable on $\mathcal{E}'$ with respect to $\mathcal{F}$ and $|\mathcal{F}_n|$ is finite for each $n \in \mathbb{N}$.*

Proof. Suppose that $\widetilde{\mathcal{E}'}$ is totally bounded. Then for each $\varepsilon > 0$ there exists a finite family of sets $\mathcal{A}_\varepsilon = \left\{A_1^{(\varepsilon)}, \cdots, A_{m_\varepsilon}^{(\varepsilon)}\right\}$ such that $\widetilde{\mathcal{E}'} \subseteq \cup_{i=1}^m B\left(\left[A_i^{(\varepsilon)}\right],\varepsilon\right)$. For arbitrary $B \in \mathcal{E}'$, $[B] \in \cup_{i=1}^n B\left(\left[A_i^{(\varepsilon)}\right],\varepsilon\right)$, so there exists $A_i^{(\varepsilon)}$ such that $\mu\left(A_i^{(\varepsilon)}\Delta B\right) < \varepsilon^2$. If we consider $\varepsilon_n = \frac{1}{n}$ and take $\mathcal{F}_n = \sigma\left(\cup_{i=1}^n \mathcal{A}_{\frac{1}{n}}\right)$ we get that

μ is uniformly approximable on $\mathcal{E}'$ with respect to $(\mathcal{F}_n)$. □

Conversely, if there exists $(\mathcal{F}_n)$ where $|\mathcal{F}_n|$ is finite for each $n \in \mathbb{N}$ and μ is uniformly approximable on $\mathcal{E}'$ with respect to $(\mathcal{F}_n)$. For each $\varepsilon > 0$ there exists n such that for each $B \in \mathcal{E}'$ there exists $A_n \in \mathcal{F}_n$, $\mu(A_n \Delta B) < \varepsilon^2$. So $[B] \in B([A_n], \varepsilon)$ and therefore $\widetilde{\mathcal{E}'} \subseteq \cup_{A \in \mathcal{F}_n} B([A], \varepsilon)$, which shows that $\widetilde{\mathcal{E}'}$ is totally bounded.

Corollary 3.2.1.

(a) $(\tilde{\mathcal{E}}, d_\mu)$ is separable if and only if there exists $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{N}}$ a filtration for which $|\mathcal{F}_n|$ is finite for each $n \in \mathbb{N}$ such that μ is approximable on $\mathcal{E}$ with respect to $\mathcal{F}$.

(b) $(\tilde{\mathcal{E}}, d_\mu)$ is compact if and only if there exists $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{N}}$ a filtration for which $|\mathcal{F}_n|$ is finite for each $n \in \mathbb{N}$ such that μ is approximable on $\mathcal{E}$ with respect to $\mathcal{F}$.

Proof. The (a) part follows directly from Theorem 3.1. The (b) part follows from Theorem 2.1, Theorem 2.8 and Theorem 3.2. □

Intuitively speaking, we can imagine a finite filtration $(\mathcal{F}_n)$ as a way to pixelise E that in each step (as n grows) we get a finer "grid". Following Theorem 3.1 and Theorem 3.2, $(\tilde{\mathcal{E}}, d_\mu)$ is separable if for each measurable set we can find a level of pixelization such that the error is smaller than arbitrary $\varepsilon > 0$ and $(\tilde{\mathcal{E}}, d_\mu)$ is compact if for each $\varepsilon > 0$ we can find a level of pixelisation such that all measurable sets are well approximated on this level, *i.e.* the error of pixelisation is smaller than ε for each measurable set.

Let us now classify measures μ on $(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d))$ based on topological properties of corresponding $(\widetilde{\mathfrak{B}(\mathbb{R}^d)}, d_\mu)$.

Further on, denote by $A^{(n)}_{(i_1,\cdots,i_d)} = \prod_{j=1}^{d} \left[(i_j - 1)/2^n, i_j/2^n \right\rangle$, $i_1,\cdots,i_d \in \mathbb{Z}$, a d-dimensional half-open interval in $\mathbb{R}^d$. We prove that an arbitrary Borel set in $\mathbb{R}^d$ can be approximated by the finite union of disjoint half-open d-intervals in a sense that the measure of symmetric difference between the Borel set and the union is arbitrary small.

Lemma 3.2 *If μ is a outer regular measure on $\mathbb{R}^d$ then for an arbitrary $B \in \mathfrak{B}(\mathbb{R}^d)$ such that $\mu(B) < \infty$ and an arbitrary $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ and finite $I \subseteq \{-n_0 2^{n_0} + 1, \cdots, 0, \cdots, n_0 2^{n_0}\}^d$ such that*

$$\mu\left(\bigcup_{(i_1,\cdots,i_d) \in I} A^{(n_0)}_{(i_1,\cdots,i_d)} \Delta B \right) < \varepsilon.$$

Proof. Let us take an arbitrary $B \in \mathfrak{B}(\mathbb{R}^d)$, $\mu(B) < \infty$ and an arbitrary $\varepsilon > 0$.

Space $\mathbb{R}^d$ can be represented as a decreasing union of the half-open d-intervals $[-n, n\rangle^d$, $n \in \mathbb{N}$. It holds that

$$\mu(B) = \mu(B \cap \mathbb{R}^d) = \mu\left(B \cap \cup_{n \in \mathbb{N}} [-n, n\rangle^d\right)$$
$$= \mu\left(\cup_{n \in \mathbb{N}} B \cap [-n, n\rangle^d\right) = \lim_{n \to \infty} \mu\left(B \cap [-n, n\rangle^d\right).$$

The last equality follows from continuity of the measure μ form below with respect to the increasing sequence $\left\{B\cap\left[-n,n\right\rangle^d\right\}_n$.

Since $\mu(B)<\infty$ for given ε there exists $n_1\in\mathbb{N}$ such that for each $n\geq n_1$

$$\mu\left(B\Delta\left[-n,n\right\rangle^d\right)=\mu(B)-\mu\left(B\cap\left[-n,n\right\rangle^d\right)\leq\frac{\varepsilon}{3}. \quad (3.5)$$

Since μ is outer regular, for $B'=B\cap\left[-n_1,n_1\right\rangle$ and arbitrary $\varepsilon>0$ there exist an open set O such that $B\subseteq O$ and

$$\mu(B'\Delta O)=\mu(O\setminus B')<\frac{\varepsilon}{3}.$$

Following Theorem 2.5, O can be represented as a countable union of almost disjoint half-open cubes $(A_k)_{k\in\mathbb{N}}$. Since

$$\mu(O)=\sum_{k\in\mathbb{N}}\mu(A_k)<\infty,$$

for chosen ε there exists $N\in\mathbb{N}$ such that

$$\mu\left(\cup_{k=1}^N A_k\right)=\sum_{k=1}^N\mu(A_k)\geq\mu(O)-\frac{\varepsilon}{3},$$

so since $\cup_{k=1}^N A_k\subseteq O$

$$\mu\left(\cup_{k=1}^N A_k\Delta O\right)\leq\frac{\varepsilon}{3}.$$

Set $n_2=\max\left\{n\in\mathbb{N}:A_k=A^{(n)}_{i_1,\cdots,i_d},k=1,\cdots,N,(i_1,\cdots,i_d)\in\mathbb{Z}^d\right\}$. Note that if $m_1<m_2$, each $A^{(m_1)}_{(j_1,\cdots,j_d)}$ can be represented as finite union of disjointed d-intervals $A^{(m_2)}_{(j'_1,\cdots,j'_d)}$.

We take $n_0=\max\{n_1,n_2\}$ and define
$I=\left\{(i_1,\cdots,i_d)\in\left\{-n_0 2^{n_0},\cdots,n_0 2^{n_0}\right\}^d:A^{(n_0)}_{(i_1,\cdots,i_d)}\subset A_k,k=1,\cdots,N\right\}$.

Note that $\cup_{k=1}^N=\cup_{(i_1,\cdots,i_d)\in I}A^{(n_0)}_{(i_1,\cdots,i_d)}$

Therefore,

$$\mu\left(\cup_{(i_1,\cdots,i_d)\in I}A^{(n_0)}_{(i_1,\cdots,i_d)}\Delta B\right)\leq\mu\left(\cup_{k=1}^N A_k\Delta O\right)+\mu(O\Delta B')+\mu(B\Delta B)<\varepsilon.$$

Theorem 3.3 *Let* $(E,\mathcal{E},\mu)=\left(\mathbb{R}^d,\mathfrak{B}\left(\mathbb{R}^d\right),\mu\right)$ *where* μ *is an outer regular measure. Then* $\left(\tilde{\mathcal{E}}_{fin},d_\mu\right)$ *is separable.* □

Proof. For each, $n\in\mathbb{N}$ the family
$\mathcal{A}_n=\left\{A^{(n)}_{i_1,\cdots,i_d}:(i_1,\cdots,i_d)\in\left\{-n2^n+1,n2^n\right\}\right\}\cup\mathbb{R}^d\setminus\left[-n,n\right\rangle^d$ is finite (see Figure 2 for visualization). Denote by

$$\mathcal{F}_n=\left\{A\in\mathcal{B}\left(\mathbb{R}^d\right):A\text{ can be written as a union of sets from }\mathcal{A}_n\right\}\cup\{\varnothing\}.$$

Note that since $\mathcal{A}_n$ forms a finite partition of $\mathbb{R}^d$, $\mathcal{F}_n=\sigma(\mathcal{A}_n)$. It holds that $|\mathcal{F}_n|<\infty$. Also note that $\mathcal{F}_n\subset\mathcal{F}_{n+1}$. Let $\mathcal{F}=\cup_{n\in\mathbb{N}}\mathcal{F}_n$, and note that $\mathcal{F}$ is countable. Note that from Lemma 3.2 it follows that the μ is approximable on $\mathcal{E}_{fin}$ with respect to $\mathcal{F}_n$), and from Theorem 3.1 it follows that $\left(\tilde{\mathcal{E}}_{fin},d_\mu\right)$ is separable. □

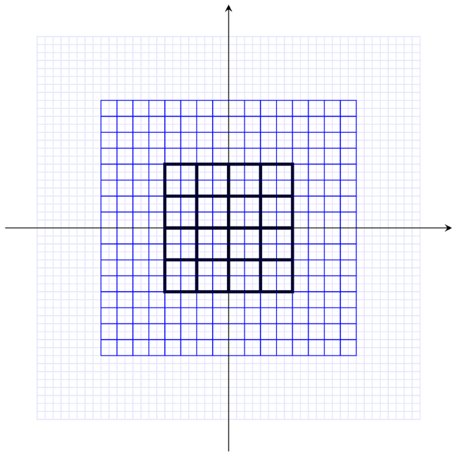

Figure 2. Visualisation of sets in family $\left\{A^{(n)}_{i_1,\cdots,i_d} : (i_1,\cdots,i_d) \in \{-n2^n+1, n2^n\}\right\}$ for $n=1$ (black), $n=2$ (blue) and $n=3$ (light blue).

Corollary 3.3.1 *For any outer regular Borel measure μ on $\left(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d)\right)$, $\left(\widetilde{\mathfrak{B}(\mathbb{R}^d)}_{fin}, d_\mu\right)$ is a Polish space (i.e. complete separable metric space).*

Remark 3 *Since $\left(\widetilde{\mathfrak{B}(\mathbb{R}^d)}_{fin}, d_\mu\right)$ is isometric to set of indicators in $L^2\left(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d), \mu\right)$, we can also conclude that in case of outer regular measure μ set of indicators in $L^2\left(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d), \mu\right)$ is a Polish space.*

Corollary 3.3.2 *For any finite Borel measure μ on $\left(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d)\right)$, $\left(\widetilde{\mathfrak{B}(\mathbb{R}^d)}, d_\mu\right)$ is a Polish space (i.e. complete separable metric space).*

Note that Corollary 3.3.2 could be proven using separability of set indicators in $L^2\left(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d), \mu\right)$.

As it has been already mentioned in Introduction, if $\mathcal{E}$ is countably generated and μ is a σ-finite measure then $L^2(E, \mathcal{E}, \mu)$ and set of indicators in $L^2(E, \mathcal{E}, \mu)$ are separable. For a finite μ, since $\left(\tilde{\mathcal{E}}_{fin}, d_\mu\right) = \left(\tilde{\mathcal{E}}, d_\mu\right)$ and $\left(\tilde{\mathcal{E}}_{fin}, d_\mu\right)$ is homeomorfic to set of indicators, we can conclude $\left(\tilde{\mathcal{E}}, d_\mu\right)$ is separable. We provide an alternative proof of this fact using the notion of approximability.

Theorem 3.4 *Suppose that there exists $\mathcal{C}$ a countable family of subsets of E such that $\mathcal{E} = \sigma(\mathcal{C})$ and $(E, \mathcal{E}, \mu)$ is a finite measure space. Then $\left(\tilde{\mathcal{E}}, d_\mu\right)$ is separable.*

Proof. Without loss of generality, we can suppose that the family $\mathcal{C}$ is a family of disjointed sets that cover E and $\mathcal{C} = \{C_n : n \in \mathbb{N}\}$. We define $\mathcal{F}_n = \sigma(C_1, \cdots, C_n)$. For an arbitrary $A \in \mathcal{E}$ there exists $\{C_k : k \in I\}$ such that $C_k \in \mathcal{C}$ and I is at most countable and $A = \cup_{k \in I} C_k$. It holds $\mu(A) = \mu(\cup_{k \in I} C_k) = \sum_{k \in I} \mu(C_k) < \infty$. If I is finite, we can take $n_\varepsilon = \max I$ and for $A' = A = \cup_{k \in I} C_k \in \mathcal{F}_{n_\varepsilon}$ it holds $\mu(A \Delta A') = 0 < \varepsilon$. If I is countable, we can suppose $I = \{k_n : n \in \mathbb{N}\}$. Since $\mu(A) = \sum_{n \in \mathbb{N}} \mu\left(C_{k_n}\right) < \infty$, for $\varepsilon > 0$ there exists

$n' \in \mathbb{N}$ such that $\sum_{n=n'+1}^{\infty} \mu\left(C_{k_n}\right) < \varepsilon$. If we take $n_\varepsilon = k_{n'}$ and $A' = \cup_{n=1}^{n'} C_{k_n} \in \mathcal{F}_{n_\varepsilon}$, it holds $\mu(A \Delta A') = \sum_{n=n'+1}^{\infty} \mu\left(C_{k_n}\right) < \varepsilon$. So, μ is approximable on $\mathcal{E}$ with respect to finite $(\mathcal{F}_n)$ and therefore $\left(\tilde{\mathcal{E}}, d_\mu\right)$ is separable. □

Theorem 3.5 *Suppose* $\mu = \mu_1 + \mu_2$ *and suppose that* μ_2 *is not (uniformly) approximable on* $\mathcal{E}'$ *with respect to a finite filtration, then* μ *is not (uniformly) approximable on* $\mathcal{E}'$ *with respect to any finite filtration.*

Proof. We prove the result for uniformity approximability since the proof in a case of approximability is similar.

Since μ_2 is not uniformly approximable on $\mathcal{E}'$ with respect to any finite filtration, for arbitrary (F_n) and $\varepsilon > 0$ there exists $B \in \mathcal{E}'$ such that $\mu_2(A \Delta B) \geq \varepsilon$ for all $A \in \cup_{n \in \mathbb{N}} \mathcal{F}_n$. But then

$$\mu(A \Delta B) = \mu_1(A \Delta B) + \mu_2(A \Delta B) \geq \varepsilon,$$

from which follows that μ is not uniformly approximable approximable on $\mathcal{E}'$ with respect to finite filtration. □

Localizable measures can be decomposed into a non-atomic part and a purely atomic part (Theorem 2.7). Following Theorem 3.5, measure μ is (uniformly) approximable if its non-atomic and purely atomic parts are (uniformly) approximable. In other words, $\left(\tilde{\mathcal{E}}, d_\mu\right)$ is separable (compact) if non-atomic and purely atomic part of μ are (uniformly) approximable. Therefore, we focus on separability and compactness properties of $\left(\tilde{\mathcal{E}}, d_\mu\right)$, first in a case when μ is non-atomic and then in a case of purely atomic μ.

Theorem 3.6 *If* μ *is non-atomic measure and* $\mu(E) = \infty$, *then* $\left(\tilde{\mathcal{E}}, \mu\right)$ *is not separable.*

Proof. Suppose that $\left(\tilde{\mathcal{E}}, \mu\right)$ is separable. Then there exists a filtration $(\mathcal{F}_n)$, $|\mathcal{F}_n| < \infty$ on $(E, \mathcal{E}, \mu)$ such that μ is approximable on $\mathcal{E}$ with respect to $(\mathcal{F}_n)$. We can suppose that $\mathcal{F}_n = \sigma\left(A_1^{(n)}, \cdots, A_{m_n}^{(n)}\right)$, where $A_i^{(n)} \in \mathcal{E}$ are disjoint and $\bigcup_{i=1}^{m_n} A_i^{(n)} = E$. Since $\infty = \mu(E) = \sum_{i=1}^{m_n} \mu\left(A_i^{(n)}\right)$, for each n we can find i_n, $1 \leq i_n \leq m_n$ such that $\mu\left(A_{i_n}^{(n)}\right) = \infty$ and since $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ we can choose $(i_n)_n$ in a way that $A_{i_{n+1}}^{(n+1)} \subseteq A_{i_n}^{(n)}$, $n \in \mathbb{N}$. Since μ is non-atomic, we can construct inductively a sequence of measurable sets $B_n \subset A_{i_n}^{(n)}$ in a following way: $\mu(B_n) = 2^{-n}$ and $B_{n+1} \subset A_{i_{n+1}}^{(n+1)} \setminus B_n$. Note that B_n are disjointed. Let $B = \cup_{n=1}^{\infty} B_n$. For each $n \in \mathbb{N}$, and an arbitrary $A' \in \mathcal{F}_n$,

$$\mu(B \Delta A') = \begin{cases} \mu(B) + \mu(A') \geq 1 & A' \cap A_{i_n}^{(n)} = \varnothing, \\ \mu\left(A_{i_n}^{(n)} \setminus B\right) = \infty, & A_{i_n}^{(n)} \subseteq A'. \end{cases}$$

So, $\mu(B \Delta A') \geq 1$ for each $n \in \mathbb{N}$ which contradicts the assumption of approximability. □

Theorem 3.7 *If measure* μ *on* $(E, \mathcal{E})$ *is non-atomic than* $\left(\tilde{\mathcal{E}}, d_\mu\right)$ *is not* *compact or locally compact.*

Proof. Suppose that μ is non-atomic, then for each $B \in \mathcal{E}$ such that $\mu(B) > 0$, there exists $A \in \mathcal{E}$ such that $A \subseteq B$ and $0 < \mu(A) < \mu(B)$. So for each $B \in \mathcal{E}$ and for each $r \in \mathbb{R}$, $0 \le r < \mu(B)$ there exists a measurable set $A \subset B$ such that $\mu(A) = r$.

Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be an arbitrary filtration on $\mathcal{E}$, such that $\mathcal{F}_n$ is finite for each $n \in \mathbb{N}$. In this case, we can assume that $\mathcal{F}_n = \sigma\left\{A_i^{(n)}, i \in I_n\right\}$ where $A_i^{(n)}$ are disjoint, $\cup_{i\in I_n} A_i^{(n)} = \mathbb{R}^d$ and I_n is a finite set of indices.

Suppose first that μ is finite. Let $0 < \varepsilon < \dfrac{\mu(E)}{2}$ and $\alpha = \dfrac{\varepsilon}{\mu(E)}$. Note that $0 < \alpha < \dfrac{1}{2}$. For each $A_i^{(n)}$ we can find Borel set $C_i^{(n)}$ such that $\mu\left(C_i^{(n)}\right) = \alpha\mu\left(A_i^{(n)}\right)$. If we define set $C_n = \cup_{i\in\mathbb{I}_n} C_i^{(n)}$, calculation similar to (3.4) yields $\mu(C_n) = \varepsilon$ $\mu(C_n \Delta A) \ge \varepsilon$ for each $A \in \mathcal{F}_n$. This shows that μ is not uniformly approximable (F_n), so $\left(\tilde{\mathcal{E}}, d_\mu\right)$ is not compact.

Let $\mu(E) = \infty$. If $\mu\left(A_i^{(n)}\right) = \infty$ for all $i \in I_n$, we can take $C_n = \varnothing$. Then $\mu(C_n) = 0 \le \varepsilon$ and $\mu(C_n \Delta A) = \infty \ge \varepsilon$.

Otherwise, let $I_n' = \left\{i \in I_n : \mu\left(A_i^{(n)}\right) < \infty\right\}$. Let $0 < \varepsilon < \dfrac{\mu\left(\cup_{i\in I_n'} A_i^{(n)}\right)}{2}$ and $\alpha = \dfrac{\varepsilon}{\mu\left(\cup_{i\in I_n'} A_i^{(n)}\right)}$. For each $A_i^{(n)}, i \in I_n'$ we take measurable set $C_i^{(n)}$ such that $\mu\left(C_i^{(n)}\right) = \alpha\mu\left(A_i^{(n)}\right)$. If we define set $C_n = \cup_{i\in\mathbb{I}_n'} C_i^{(n)}$, again we have $\mu(C_n) = \varepsilon$ and $\mu(C_n \Delta A) \ge \varepsilon$.

Let $\mathcal{E}_\varepsilon' = \{A \in \mathcal{E} : \mu(A) \le \varepsilon\}$ Previous discussion show that μ is not uniformly approximable on $\mathcal{E}_\varepsilon' \subset \mathcal{E}$ with respect to any filtration containing finite σ-algebras. We conclude that $\left(\tilde{\mathcal{E}}, d_\mu\right)$ is not compact, and each closed ball $\bar{B}\left([\varnothing], \sqrt{\varepsilon}\right) = \{[B] : B \in \mathcal{E}_\varepsilon'\}$ is not totally bounded, but it is also not compact since $\bar{B}\left([\varnothing], \sqrt{\varepsilon}\right)$ is closed. So, $[\varnothing]$ has no compact neighbourhood. We conclude that $\left(\tilde{\mathcal{E}}, d_\mu\right)$ is not locally compact. □

From Theorems 3.6 and 3.7 we see that in the case of non-atomic measure μ, $\left(\tilde{\mathcal{E}}, d_\mu\right)$ is not separable if μ is an infinite measure and also not compact when μ is an arbitrary measure.

Further on, let μ be a purely atomic measure on $(E, \mathcal{E})$. Suppose that atoms of the μ are singletons. If E is second countable Hausdorff and $\mathcal{E}$ is a Borel σ-algebra on E, following Lemma 2.2, every measure μ on $(E, \mathcal{E})$ satisfies the condition.

We define set $E_{fin} = \{x \in E : 0 < \mu(\{x\}) < \infty\}$ of all atoms with a finite measure and set $E_\infty = \{x \in E : \mu(\{x\}) = \infty\}$ of all atoms with an infinite measure. To prove that corresponding $\left(\tilde{\mathcal{E}}_{fin}, d_\mu\right)$ is separable if and only if E_{fin} is countable we need the following result.

Lemma 3.3 *Let* $(E,\mathcal{E})$ *be a measurable space. Let* $(\mathcal{F}_n)$ *be a filtration on* $\mathcal{E}$ *such that* $\mathcal{F}_n$ *can be represented as* $\sigma\left(A_1^{(n)},\cdots,A_{m_n}^{(n)}\right)$ *where* $m_n \in \mathbb{N}$ *and* $A_1^{(n)},\cdots,A_{m_n}^{(n)}$ *is a finite partition of E. If* E_{fin} *is a uncountable subset of E, there exists a decreasing sequence* $\left(A_{j_n}^{(n)}\right)_n$, $j_n \in \{1,\cdots,m_n\}$ *such that* $A_{j_n}^{(n)} \cap E_{fin}$ *is infinite for each* $n \in \mathbb{N}$ *and* $\bigcap_{n\in\mathbb{N}} A_{j_n}^{(n)} \cap E_{fin} \neq \varnothing$.

Proof. It holds $E = \bigcap_{n\in\mathbb{N}} E = \bigcap_{n\in\mathbb{N}} \bigcup_{j_n=1}^{m_n} A_{j_n}^{(n)} = \bigcup_{(j_n)_n \in \prod_{n=1}^{\infty}\{1,\cdots,m_n\}} \bigcap_{n\in N} A_{j_n}^{(n)}$. It follows that $E_{fin} = E \cap E_{fin} = \bigcup_{(j_n)_n \in \prod_{n=1}^{\infty}\{1,\cdots,m_n\}} \bigcap_{n\in\mathbb{N}} A_{j_n}^{(n)} \cap E_{fin}$. Note that partition $A_1^{(n+1)},\cdots,A_{m_{n+1}}^{(n+1)}$ refines partition $A_1^{(n)},\cdots,A_{m_n}^{(n)}$, so if $\bigcap_{n\in N} A_{j_n}^{(n)} \neq \varnothing$ then $A_{j_{n+1}}^{(n+1)} \subseteq A_{j_n}^{(n)}$.

Suppose conversely, that for each $\left(A_{j_n}^{(n)}\right)_n$, $j_n \in \{1,\cdots,m_n\}$ such that $A_{j_n}^{(n)} \cap E_{fin}$ is infinite for each $n \in \mathbb{N}$, $\bigcap_{n\in\mathbb{N}} A_{j_n}^{(n)} \cap E_{fin} = \varnothing$. Then $E_{fin} = \bigcup_{(j_n)_n \in J'} \bigcap_{n\in\mathbb{N}} A_{j_n}^{(n)} \cap E_{fin}$, where

$J' = \left\{ j = (j_n)_n \in \prod_{n=1}^{\infty}\{1,\cdots,m_n\} : \text{there exists } m_j \in \mathbb{N} \text{ such that } \left|A_{j_k}^{(k)} \cap E_{fin}\right| < \infty, k \geq m_j \right\}$.

If we denote by $J_{fin} = \left\{ (j_1,\cdots,j_M) : j \in J', M \geq m_j \right\}$, in this case it holds

$$E_{fin} \subseteq \bigcup_{j\in J'} \bigcap_{n=1}^{m_j} A_{j_n}^{(n)} \cap E_{fin} \subseteq \bigcup_{(j_1,\cdots,j_M)\in J_{fin}} \bigcap_{n=1}^{M} A_{j_n}^{(n)} \cap E_{fin}.$$

Since J_{fin} is at most countable (as a subset of all finite sequences $(j_1,\cdots,j_M)$, $M \in \mathbb{N}$, $j_n \in 1,\cdots,m_n$) and $\left|\bigcap_{n=1}^{M} A_{j_n}^{(n)} \cap E_{fin}\right| < \infty$, it follows that E_{fin} is at most countable, which is contradicts the assumption that E_{fin} is uncountable. □

Theorem 3.8 *Let* μ *be a purely atomic measure on* $(E,\mathcal{E})$ *where all the atoms are singletons.*

(a) $\left(\tilde{\mathcal{E}},d_\mu\right)$ is separable if and only if $\sum_{x\in E_{fin}} \mu(\{x\}) < \infty$ and E_∞ is finite.

(b) $\left(\tilde{\mathcal{E}}_{fin},d_\mu\right)$ is separable if and only if E_{fin} is countable.

Proof. Suppose E_{fin} is countable, so it can be written in a form $E_{fin} = \{x_n \in E, n \in \mathbb{N}\}$. We first prove that $\left(\tilde{\mathcal{E}}_{fin},d_\mu\right)$ is separable. Denote by $\mathcal{E}_{fin} = \{B \in \mathcal{E} : \mu(B) < \infty\}$. We set $\mathcal{F}_n = \sigma(\{x_1\},\cdots,\{x_n\})$, $n \in \mathbb{N}$.

For $B \in \mathcal{E}_{fin}$ it holds

$\mu(B) = \mu(B \cap E_{fin}) = \sum_{n=1}^{\infty} \mu(B \cap \{x_n\}) = \sum_{i\in I_B} \mu(\{x_i\}) < \infty$, where $I_B = \{n \in \mathbb{N} : x_n \in B\}$. For arbitrary $\varepsilon > 0$ and arbitrary $B \in \mathcal{E}_{fin}$ there exists finite $I_\varepsilon \subset I_B$ such that $\sum_{i\in I_B \setminus I_\varepsilon} \mu(\{x_i\}) = \mu\left(B \Delta \bigcup_{i\in I_\varepsilon} \{x_i\}\right) < \varepsilon$. If we choose $n_\varepsilon = \max I_\varepsilon$, $\bigcup_{i\in I_\varepsilon} \{x_i\} \in \mathcal{F}_{n_\varepsilon}$. We conclude that μ is approximable on $\mathcal{E}_{fin}$ with respect to $(\mathcal{F}_n)$, so $(\tilde{\mathcal{E}}_{fin}, d_\mu)$ is separable.

Further on, suppose $\sum_{x \in E_{fin}} \mu(\{x\}) < \infty$ and E_∞ is finite, *i.e.* $E_\infty = \{y_1, \cdots, y_m\}$ for fixed $m \in N$ and let $\mathcal{F}_n^\infty = \sigma(\{y_1\}, \cdots, \{y_m\}, \{x_1\}, \cdots, \{x_n\})$, $n \in \mathbb{N}$. For $B \in \mathcal{E}$ such that $\mu(B) = \infty$ it holds $B \cap E_\infty \neq \emptyset$ and $\mu(B \setminus E_\infty) < \infty$. For $B' = B \setminus E_\infty \in \mathcal{E}_{fin}$ we have already proven that for $\varepsilon > 0$ there exists n and $A' \in \mathcal{F}_n$ such that $\mu(B' \Delta A') < \varepsilon$. If we take $A'' = (B \cap E_\infty) \cup A' \in \mathcal{F}_n^\infty$ it holds $\mu(B \Delta A'') = \mu(B' \Delta A') < \varepsilon$, so μ is approximable on $\mathcal{E}$ with respect to $(\mathcal{F}_n^\infty)$ and we conclude $(\tilde{\mathcal{E}}, d_\mu)$ is separable.

To prove the separability of $(\tilde{\mathcal{E}}_{fin}, d_\mu)$ implies E_{fin} is countable, we suppose conversely that $(\tilde{\mathcal{E}}_{fin}, d_\mu)$ is separable and E_{fin} is uncountable. Since $(\tilde{\mathcal{E}}_{fin}, d_\mu)$ is separable, there exists filtration $(\mathcal{F}_n)$ such that $\mathcal{F}_n$ is finite and μ is approximable on $\mathcal{E}$ with respect to $(\mathcal{F}_n)$. Each $\mathcal{F}_n$ can be represented as $\sigma(A_1^{(n)}, \cdots, A_{m_n}^{(n)})$ where $m_n \in \mathbb{N}$ and $A_1^{(n)}, \cdots, A_{m_n}^{(n)}$ is a finite partition of *E*.

Since $E_{fin} \subseteq E$ is uncountable, following Lemma 3.3 there exists $(j_n) \in \prod_{n=1}^\infty \{1, \cdots, m_n\}$ such that $E_{fin} \cap \left(\bigcap_{n \in N} A_{j_n}^{(n)} \right)$ is uncountable. In other words, there exists a decreasing sequence $\left(A_{j_n}^{(n)}\right)_n$ such that $E_{fin} \cap A_{j_n}^{(n)}$ is infinite for each $n \in \mathbb{N}$ and the intersection $\cap_{n \in \mathbb{N}} A_{j_n}^{(n)} \cap E_{fin}$ is non-empty. We take $x \in \cap_{n \in \mathbb{N}} A_{j_n}^{(n)} \cap E_{fin}$, set $B = \{x\}$ and take ε such that $0 < \varepsilon \le \mu(\{x\})$. For arbitrary $n \in \mathbb{N}$, if $A \in \mathcal{F}_n$ such $A_{j_n}^{(n)} \subseteq A$ then $\mu(B \Delta A) \ge \mu\left(A_{j_n}^{(n)} \setminus B\right) \ge \sum_{y \in E_{fin} \cap A_{j_n}^{(n)}} \mu(\{y\}) = \infty$, since the sum is the uncountable sum of non-negative numbers. If $A_{j_n}^{(n)} \cap A = \emptyset$ then $\mu(B \Delta A) \ge \mu(\{x\}) = \varepsilon$. So, for each $n \in \mathbb{N}$ and each $A \in \mathcal{F}_n$ it holds

$$\mu(B \Delta A) \ge \varepsilon.$$

This is a contradiction to the fact that μ is approximable on $\mathcal{E}_{fin}$ with respect to $(\mathcal{F}_n)$ and therefore $(\tilde{\mathcal{E}}_{fin}, d_\mu)$ is not separable.

Let us now prove that if $(\tilde{\mathcal{E}}, d_\mu)$ is separable then E_{fin} is countable and E_∞ is finite. We suppose, conversely, $(\tilde{\mathcal{E}}, d_\mu)$ is separable and E_{fin} is uncountable or E_∞ infinite. If E_{fin} is uncountable then $(\tilde{\mathcal{E}}_{fin}, d_\mu)$ is not separable. Since $(\tilde{\mathcal{E}}_{fin}, d_\mu)$ is a subspace of $(\tilde{\mathcal{E}}, d_\mu)$, using Theorem [9, Theorem VIII, p.~160] we can conclude $(\tilde{\mathcal{E}}, d_\mu)$ is not separable.

If E_∞ is infinite, its partitive set is also infinite. So, for an arbitrary filtration $(\mathcal{F}_n)$ $\mathcal{E}$ with $|\mathcal{F}_n| < \infty$ we can find $B \subset E_\infty$ such that $B \notin \mathcal{F}_n$, $n \in \mathbb{N}$. It holds $\mu(B \Delta A') = \infty$ for all $A' \in \mathcal{F}_n$, $n \in \mathbb{N}$. So, μ is not approximable on $\mathcal{E}$ with respect to any $(\mathcal{F}_n)$ where $|\mathcal{F}_n| < \infty$ and therefore $(\tilde{\mathcal{E}}, d_\mu)$ is not separable. □

Theorem 3.9 *Let* μ *be purely atomic measure on* $(E, \mathcal{E})$ *where all the atoms are singletons.*

(a) $(\tilde{\mathcal{E}}, d_\mu)$ is compact if and only if $\sum_{x \in E_{fin}} \mu(\{x\})$ is finite and E_∞ is finite.

(b) $(\tilde{\mathcal{E}}_{fin}, d_\mu)$ is compact if and only if $\sum_{x \in E_{fin}} \mu(\{x\})$ is finite.

Proof. First, let us prove that $\sum_{x \in E_{fin}} \mu(\{x\}) < \infty$ implies that $(\tilde{\mathcal{E}}_{fin}, d_\mu)$ is compact. Since $\sum_{x \in E_{fin}} \mu(\{x\})$ implies that E_{fin} is countable (see e.g. ([21] Theorem 3.12.6., p. 131)). We can assume that $E_{fin} = \{x_n, n \in \mathbb{N}\}$. We denote $\mathcal{E}_{fin} = \{B \in \mathcal{E} : \mu(B) < \infty\}$. It holds

$$\sum_{n=1}^{\infty} \mu(\{x_n\}) < \infty.$$

So, for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\sum_{n=n_\varepsilon}^{\infty} \mu(\{x_n\}) < \varepsilon$. Define $\mathcal{F}_n = \sigma(\{x_1\}, \cdots, \{x_n\})$. Let us show that μ is uniformly approximable on $\mathcal{E}_{fin}$ with respect to $\mathcal{F}_n$. If we take arbitrary $B \in \mathcal{E}_{fin}$ and set $A = B \cap \{x_1, \cdots, x_{n_\varepsilon}\} \in \mathcal{F}_{n_\varepsilon}$ it holds that

$$\mu(A \Delta B) \le \sum_{n=n_\varepsilon}^{\infty} \mu(\{x_n\}) < \varepsilon,$$

so following Theorem 3.2, $\tilde{\mathcal{E}}_{fin} = \{[B] : B \in \mathcal{E}_{fin}\}$ is compact.

Suppose $E_\infty = \{y_1, \cdots, y_m\}$, for fixed $m \in \mathbb{N}$. To prove $(\tilde{\mathcal{E}}, d_\mu)$ is compact, we set $\mathcal{F}_n^\infty = \sigma(\{y_1\}, \cdots, \{y_m\}, \{x_1\}, \cdots, \{x_n\})$, $m \in \mathbb{N}$. For $B \in \mathcal{E}$ such that $\mu(B) = \infty$ it holds $B \cap E_\infty \neq \varnothing$ and $\mu(B \setminus E_\infty) < \infty$. We have shown that $\varepsilon > 0$ there exists n such that for every $B' \in \mathcal{E}_{fin}$ there exists $A' \in \mathcal{F}_n$ such that $\mu(B' \Delta A') < \varepsilon$. If we take $A'' = (B \cap E_\infty) \cup A' \in \mathcal{F}_n^\infty$ it holds $\mu(B \Delta A'') = \mu(B' \Delta A') < \varepsilon$, so μ is uniformly approximable on $\mathcal{E}$ with respect to $(\mathcal{F}_n^\infty)$ and we conclude $(\tilde{\mathcal{E}}, d_\mu)$ is compact.

Let us prove that if $(\tilde{\mathcal{E}}_{fin}, d_\mu)$ is compact then $\sum_{x \in E_{fin}} \mu(\{x\}) < \infty$. Suppose conversely, that $(\tilde{\mathcal{E}}_{fin}, d_\mu)$ is compact and $\sum_{x \in E_{fin}} \mu(\{x\}) = \infty$. If E_{fin} is uncountable, then following Theorem 3.8 $(\tilde{\mathcal{E}}_{fin}, d_\mu)$ is not separable, so it cannot be compact. Suppose now that $(\tilde{\mathcal{E}}_{fin}, d_\mu)$ is compact and $E_{fin} = \{x_n, n \in \mathbb{N}\}$ is countable and $\sum_{x \in E_{fin}} \mu(\{x\}) = \sum_{n \in \mathbb{N}} \mu(\{x_n\}) = \infty$. Since $(\tilde{\mathcal{E}}_{fin}, d_\mu)$ is compact there exists a filtration $(\mathcal{F}_n)$ such that $\mathcal{F}_n$ is finite for each $n \in \mathbb{N}$ and μ is uniformly approximable on $\mathcal{E}_{fin} = \{B \in \mathcal{E} : \mu(B) < \infty\}$ with respect to $(\mathcal{F}_n)$. For each, $\mathcal{F}_n$ it holds that $\mathcal{F}_n = \sigma(A_1^{(n)}, \cdots, A_{m_n}^{(n)})$ where $m_n \in \mathbb{N}$ and $A_1^{(n)}, \cdots, A_{m_n}^{(n)}$ is a finite partition of E. We take an arbitrary $\varepsilon > 0$. For each n there exists $A_{j_n}^{(n)}$ such that $A_{j_n}^{(n)} \cap E_{fin}$ is infinite, $A_{j_n}^{(n)} \cap E_{fin} = \{x'_n : n \in \mathbb{N}\}$ and $\sum_{n \in \mathbb{N}} \mu(\{x'_n\}) = \infty$. We an find $m \in \mathbb{N}$ such that $\sum_{n=1}^{m} \mu(\{x'_n\}) \ge \varepsilon$. If we take

$A=\{x_1',\cdots,x_m'\}\subseteq A_{j_n}^{(n)}\cap E_{fin}$, then it is easy to see that $\mu(A\Delta A')\geq\varepsilon$ for each $A'\in\mathcal{F}_n$ since if $A'\cap A_{j_n}^{(n)}=\varnothing$ $\mu(A\Delta A')\geq\mu(A)\geq\varepsilon$ and if $A_{j_n}^{(n)}\subseteq A'$, $\mu(A\Delta A')\geq\mu\left(A_{j_n}^{(n)}\setminus A\right)=\sum_{n=m+1}^{\infty}\mu(\{x_n'\})=\infty\geq\varepsilon$. So, $(\tilde{\mathcal{E}}_{fin},d_\mu)$ cannot be uniformly approximable on $\mathcal{E}_{fin}$ with respect to $(\mathcal{F}_n)$ and $(\tilde{\mathcal{E}}_{fin},d_\mu)$ is not compact.

Let us now prove that if $(\tilde{\mathcal{E}},d_\mu)$ is compact then $\sum_{x\in E_{fin}}\mu(\{x\})<\infty$ and E_∞ is finite. We suppose, conversely, $(\tilde{\mathcal{E}},d_\mu)$ is compact and $\sum_{x\in E_{fin}}\mu(\{x\})=\infty$ or E_∞ is infinite. If $\sum_{x\in E_{fin}}\mu(\{x\})=\infty$ then $(\tilde{\mathcal{E}}_{fin},d_\mu)$ is not compact. Since $(\tilde{\mathcal{E}}_{fin},d_\mu)$ is a closed subspace of $(\tilde{\mathcal{E}},d_\mu)$, using Theorem [9, Theorem VIII, p.~160] we can conclude $(\tilde{\mathcal{E}},d_\mu)$ is not compact.

If E_∞ is infinite, $(\tilde{\mathcal{E}},d_\mu)$ is not separable, so it cannot be compact. □

We conclude the main part of the paper with a few examples.

Example 1 Let μ be a counting measure on $(\mathbb{N},2^{\mathbb{N}})$, *i.e.* $\mu(A)=|A|$ where $|A|$ stands for the cardinal number of A if A is finite and ∞ otherwise. Since E_{fin} is countable, $(\tilde{\mathcal{E}}_{fin},d_\mu)$ is separable, but it is not compact since $\sum_{x\in E_{fin}}\mu(\{x\})=\infty$. Also $(\tilde{\mathcal{E}},d_\mu)$ is not separable. For arbitrary finite B and arbitrary $0<\varepsilon<1$, $B([B],\varepsilon)=\overline{B}([B],\varepsilon)=\{[B]\}$. So, the closed ball with a radius of less than 1 around each element $(\tilde{\mathcal{E}}_{fin},d_\mu)$ and $(\tilde{\mathcal{E}},d_\mu)$ is compact since it is finite. Therefore, they are both locally compact.

Example 2 Let μ be a counting measure on $\left(\mathbb{R}^d,2^{\mathbb{R}^d}\right)$, $\mu(A)=|A|$ where $|A|$ stands for the cardinal number of A if A is finite and ∞ otherwise. Since E_{fin} is uncountable, the corresponding $(\tilde{\mathcal{E}}_{fin},d_\mu)$ is not separable and not compact. However, $(\tilde{\mathcal{E}},d_\mu)$ and $(\tilde{\mathcal{E}}_{fin},d_\mu)$ are locally compact. Similarly to the previous example, for arbitrary finite B and arbitrary $0<\varepsilon<1$, $B([B],\varepsilon)=\overline{B}([B],\varepsilon)=\{[B]\}$. So, the closed ball with a radius less than 1 around each element of $(\tilde{\mathcal{E}},d_\mu)$ is compact since it is finite.

Example 3 Let μ be a counting measure and λ a Lebesgue measure on $\left(\mathbb{R}^d,\mathfrak{B}(\mathbb{R}^d)\right)$. Let $\nu(A)=\mu(A\cap B(0,1))+\lambda(A)$, $A\in\mathfrak{B}(\mathbb{R}^d)$, where $B(0,1)$ stands for a unit ball with centre at the origin. It is easy to see $(\tilde{\mathcal{E}}_{fin},d_\mu)$ and $(\tilde{\mathcal{E}},d_\mu)$ are not separable not compact (since for ν, E_{fin} is uncountable) and not locally compact.

4. Discussion

Using newly defined terms of approximability and uniformly approximability, the conditions for separability and compactness of $(\tilde{\mathcal{E}},d_\mu)$ and $(\tilde{\mathcal{E}}_{fin},d_\mu)$ can be summarised in Table 1. Table 2 provides the topological properties of $(\tilde{\mathcal{E}},d_\mu)$ and $(\tilde{\mathcal{E}}_{fin},d_\mu)$ based on finiteness and atomicity properties of the corresponding measure space $(E,\mathcal{E},\mu)$.

Table 1. Condition on $(E,\mathcal{E},\mu)$ for separability and compactness of $(\tilde{\mathcal{E}},d_\mu)$ and $(\tilde{\mathcal{E}}_{fin},d_\mu)$.

	$(\tilde{\mathcal{E}},d_\mu)$	$(\tilde{\mathcal{E}}_{fin},d_\mu)$
Separable	for $\mathcal{E}$ countably generated if and only if μ is finite (Theorem 3.4 and Theorem 3.7).	• for $\tilde{\mathcal{E}}_{fin}=\widetilde{\mathfrak{B}(\mathbb{R}^d)}_{fin}$, if μ is outer regular (Theorem 3.3), • for μ purely atomic measure where all the atoms are singletons, if and only if the set of atoms with finite measure E_{fin} is countable (Theorem 3.8).
Compact	if and only if μ is purely atomic with all atoms singletons and $\sum_{x\in E_{fin}}\mu(\{x\})<\infty$ and and E_∞ is finite (Theorem 3.7 and Theorem 3.8 (a)).	for μ is purely atomic measure where all atoms are singletons if and only if $\sum_{x\in E_{fin}}\mu(\{x\})<\infty$ (Theorem 3.9).

Table 2. Separability and compactness of $(\tilde{\mathcal{E}},d_\mu)$ and $(\tilde{\mathcal{E}}_{fin},d_\mu)$ depending on whether μ is finite or infinite, purely atomic or non-atomic.

	μ non-atomic	μ purely atomic
μ finite	$(\tilde{\mathcal{E}},\tilde{\mu})=(\tilde{\mathcal{E}}_{fin},\tilde{\mu})$ is: • not compact (Theorem 3.7), • is separable if $\mathcal{E}$ is countably generated (Theorem 3.4).	$(\tilde{\mathcal{E}},\tilde{\mu})=(\tilde{\mathcal{E}}_{fin},\tilde{\mu})$ is: • separable and compact (Theorem 3.8 and Theorem 3.9).
μ infinite	$(\tilde{\mathcal{E}},\tilde{\mu})$ is: • not separable (Theorem 3.6), • not compact (Theorem 3.7). $(\tilde{\mathcal{E}}_{fin},\tilde{\mu})$ is: • is separable if μ is outer regular and $(E,\mathcal{E})=(\mathbb{R}^d,\mathfrak{B}(\mathbb{R}^d))$.	$(\tilde{\mathcal{E}},\tilde{\mu})$ is: • separable and compact if and only if $\sum_{x\in E_{fin}}\mu(\{x\})$ is finite and E_∞ is finite ((Theorem 3.8 (a) and Theorem 3.9 (a)). $(\tilde{\mathcal{E}}_{fin},\tilde{\mu})$ is: • separable if and only if E_{fin} is countable (Theorem 3.8 (b)), • compact if and only if $\sum_{x\in E_{fin}}\mu(\{x\})$ is finite (Theorem 3.9 (b)).

Future research will involve using the results from this paper to generalise the notions of different types of convergence of random sets as random elements taking values in N.B.A. and exploring their properties.

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Conflicts of Interest

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