Advances in Pure Mathematics, 2024, 14, 354-366
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

On Two Types of Stability of Solutions to a Pair of Damped Coupled Nonlinear Evolution Equations

Mark Jones

School of Computer Science and Mathematics, Kingston University, London, UK
Email: mark.jones@kingston.ac.uk

How to cite this paper: Jones, M. (2024) On Two Types of Stability of Solutions to a Pair of Damped Coupled Nonlinear Evolution Equations. *Advances in Pure Mathematics*, **14**, 354-366.
https://doi.org/10.4236/apm.2024.145020

Received: December 11, 2023
Accepted: May 20, 2024
Published: May 23, 2024

Abstract

The stability of a set of spatially constant plane wave solutions to a pair of damped coupled nonlinear Schrödinger evolution equations is considered. The equations could model physical phenomena arising in fluid dynamics, fibre optics or electron plasmas. The main result is that any small perturbation to the solution remains small for all time. Here small is interpreted as being both in the supremum sense and the square integrable sense.

Keywords

Nonlinear Schrödinger Equation, Stability, Capillary-Gravity Waves

1. Introduction

We shall be concerned with the following pair of coupled nonlinear partial differential equations:

$$iA_T + rA_{XX} + uA_{YY} + p|A|^2 Ae^{-2\delta T} + q|B|^2 Ae^{-2\delta T} = 0 \quad (Ia)$$

$$iB_T + vB_{XX} + sB_{YY} + w|B|^2 Be^{-2\delta T} + q|A|^2 Be^{-2\delta T} = 0. \quad (Ib)$$

Such systems have been used to model the slow evolution of small amplitude capillary-gravity waves caused by the interaction between the *M*th and *N*th harmonics of the fundamental mode and in the presence of damping. They can be applied both to waves on a free surface of an infinitely deep fluid or to those arising at the interface of two fluids each of infinite vertical extent, for more details and a derivation consult Jones [1] [2] [3]. Other authors who have considered such systems as applied to water waves include Carter [4], Dhar & Das [5], Hammack, Henderson, Segur & Carter [6] [7] [8] and Onorato, Osborne & Serio [9]. However, applications of systems such as (I) are not confined to fluid dynamics, other examples include plasmas [10] [11] and fibre optics [12] [13].

An alternative formulation of the system (I) is

$$i\tilde{A}_T + r\tilde{A}_{XX} + u\tilde{A}_{YY} + p\left|\tilde{A}\right|^2 \tilde{A} + q\left|\tilde{B}\right|^2 \tilde{A} + i\delta\tilde{A} = 0 \qquad (\tilde{I}a)$$

$$i\tilde{B}_T + v\tilde{B}_{XX} + s\tilde{B}_{YY} + w\left|\tilde{B}\right|^2 \tilde{B} + q\left|\tilde{A}\right|^2 \tilde{B} + i\delta\tilde{B} = 0. \qquad (\tilde{I}b)$$

The system ($\tilde{I}$) may be transformed into (I) by means of the substitutions $\tilde{A} = A\mathrm{e}^{-\delta t}$ and $\tilde{B} = B\mathrm{e}^{-\delta t}$. Observe that the two systems coincide when $\delta = 0$. The wave (or interface) profile corresponding to a solution of (I) is given by

$$\begin{aligned} H(x,y,t) &= \varepsilon\left(A\exp\{iN(kx-\omega t)\} + A^*\exp\{-iN(kx-\omega t)\}\right) \\ &\quad + \varepsilon\left(B\exp\{iM(kx-\omega t)\} + B^*\exp\{-iM(kx-\omega t)\}\right) + O\left(\varepsilon^2\right). \end{aligned} \qquad (1.1)$$

Here k is the wavenumber of the fundamental mode; ω is the frequency and ε is a parameter representing the wave steepness that satisfies $0 < \varepsilon \ll 1$. The complex amplitudes A and B are functions of the slowly varying spatial quantities $X = \varepsilon x$, $Y = \varepsilon y$ and the very slowly varying temporal quantity $T = \varepsilon^2 t$. The coefficients r, u, v, s, p, q are real-valued constants, their values in the case of surface waves may be found in [1] and for interfacial waves in [2] [3]. However, our theory will apply to quite arbitrary values of the coefficients. The numbers M and N are positive integers and we shall assume that $M > N$. These equations are not valid when $M = 2N$ or $M = 3N$, for a treatment of these situations see [14] [15] [16] [17] [18]. The damping parameter δ is taken to satisfy $\delta \geq 0$. In the case of non-resonant waves, it represents the effects of dissipation, see [19] [20]. In the resonant case, this could remain true [4] but it might also represent imperfections in the resonance such as the situation when the interacting waves travel at speeds close, rather than precisely equal, to the velocities required for resonance. Other possibilities are that the dispersion relation is satisfied approximately rather than exactly or that a wavemaker operates at a frequency very near but not equal to that which is required for resonance.

A spatially constant solution of (I) is given, for non-zero δ, by

$$A_0(T) = \alpha\exp\left\{i\frac{\left(p|\alpha|^2 + q|\beta|^2\right)\left(1-\mathrm{e}^{-2\delta T}\right)}{2\delta}\right\} \qquad (1.2a)$$

$$B_0(T) = \beta\exp\left\{i\frac{\left(w|\beta|^2 + q|\alpha|^2\right)\left(1-\mathrm{e}^{-2\delta T}\right)}{2\delta}\right\} \qquad (1.2b)$$

the corresponding solution to ($\tilde{I}$) is

$$\tilde{A}_0(T) = \alpha\mathrm{e}^{-2\delta T}\exp\left\{i\frac{\left(p|\alpha|^2 + q|\beta|^2\right)\left(1-\mathrm{e}^{-2\delta T}\right)}{2\delta}\right\} \qquad (1.3a)$$

$$\tilde{B}_0(T) = \beta e^{-2\delta T} \exp\left\{ i \frac{\left(w|\beta|^2 + q|\alpha|^2\right)\left(1 - e^{-2\delta T}\right)}{2\delta} \right\}. \tag{1.3b}$$

When $\delta = 0$ the solution is

$$A_0(T) = \alpha \exp\left\{ i\left(p|\alpha|^2 + q|\beta|^2 \right) T \right\} \tag{1.4a}$$

$$B_0(T) = \beta \exp\left\{ i\left(w|\beta|^2 + q|\alpha|^2 \right) T \right\} \tag{1.4b}$$

In these expressions α and β are completely arbitrary constants, they can assume both real and complex values. Observe that both (1.2) and (1.3) tend to (1.4) as $\delta \to 0$. These are significant solutions of the systems ($\mathcal{I}$) and ($\tilde{\mathcal{I}}$) because they represent, via (1.1), a spatially uniform train of resonant plane waves in a deep channel.

The main objective of the paper is to prove that provided $\delta \neq 0$, the solutions (1.2) and (1.3) are stable to small perturbations, in other words, it is shown that any amount of dissipation stabilizes the wavetrain. This extends the results of Segur *et al.* [20] who examined the effects of dissipation on a nearly monochromatic wavetrain. They found that in this, nonresonant case, dissipation stabilizes the instability.

Our main results concern the stability of the solution (1.2), for a more extensive discussion of this concept, see Nemytskii & Stepanov [21]. To fix things, assume that we are seeking solutions of system ($\mathcal{I}$) on the rectangular domain defined by $\mathcal{D} = [0, L_X] \times [0, L_Y]$ and we apply periodic conditions on the boundary of $\mathcal{D}$. Informally, a solution $A_0(T), B_0(T)$ of (1.2) is stable if every solution of (1.2) that is initially close to it at $T = 0$ remains close to it for all $T > 0$. If not, the solution is regarded as unstable. Let us give a more precise definition.

Consider the space of continuous functions on $\mathcal{D}$ equipped with a norm which we shall denote by $\|.\|$. We shall consider two norms, the supremum norm and the $\mathcal{L}^2$, ie square-integrable, norm. Then take the solution (1.2) $A_0(T), B_0(T)$ of ($\mathcal{I}$) and impose a perturbation

$$A(X,Y,T) = A_0(T) + \mathcal{A}(X,Y,T), \quad B(X,Y,T) = B_0(T) + \mathcal{B}(X,Y,T). \tag{1.5}$$

Now substitute in (1.5) into ($\mathcal{I}$) and linearize around $\mathcal{A}(X,Y,T)$ and $\mathcal{B}(X,Y,T)$. Then $A_0(T), B_0(T)$ is a stable solution if for all $\varepsilon > 0$ there exists $\Delta > 0$ such if $\mathcal{A}(X,Y,T)$ that $\mathcal{B}(X,Y,T)$ satisfy

$$\|\mathcal{A}(X,Y,0)\| + \|\mathcal{B}(X,Y,0)\| < \Delta \tag{1.6a}$$

then

$$\|\mathcal{A}(X,Y,T)\| + \|\mathcal{B}(X,Y,T)\| < \varepsilon \tag{1.6b}$$

for all $T > 0$. The main results of this work, which are to be found in section 3, are that the solution (1.2) is stable in respect of both norms. We remark that the type of stability considered here may more properly be referred to as linear stability. However, since it is the only type of stability dealt with in this work our description will suffice. It is hoped to cover the case of nonlinear stability in subsequent work. A detailed analysis of the stability of (1.2) in the absence of dissipation, ie when $\delta = 0$ was carried out by Jones [3]. His results showed that the solutions may be stable or unstable to plane wave perturbations depending on the signs of the coefficients r, u, v, s, p, q and also on the nature of the perturbations themselves ie whether they are longitudinal, transverse or oblique to the carrier wave. However our results in this work show that when $\delta \neq 0$ the solutions are always stable. In other words any amount of dissipation however small, or indeed any other imperfection in the resonance stabilizes the instability.

Related Work

The system (*1*) may be regarded as a generalisation of the celebrated nonlinear Schrödinger (nlS) equation

$$iA_T + A_{XX} + A_{YY} + |A|^2 A = 0 \tag{1.7}$$

which is known to model several physical situations, among them pulse propagation along optical fibres and waves in a plasma.

It was Zakharov [22] who first showed that the nlS equation may be used to describe the slow evolution of one-dimensional monochromatic deep-water gravity waves. He then proceeded to show that all plane wave solutions are linearly unstable. At around the same time Benjamin & Feir [23], established the important result that a uniform train of gravity waves in deep water is unstable to a perturbation consisting of other waves travelling in the same direction and with almost the same frequency.

Notwithstanding these discoveries, at around the same time, Snodgrass *et al.* [24] (and later Collard, Ardhuin & Chapron [25]) were able to track the propagation of slowly varying trains of surface waves, or ocean swell, across the Pacific for many thousands of kilometers. In addition, Lake *et al.* [26] [27] encountered difficulties in reproducing the Benjamin-Feir instability in the laboratory while Hammack, Henderson & Segur [6] conducted experiments in which they were able to reproduce waves in deep water that showed no evidence of being unstable. This led to the most interesting paper of Segur *et al.* [20] in which dissipative effects were taken into account. The authors considered the equation

$$iA_T + rA_{XX} + uA_{YY} + p|A|^2 A + i\delta A = 0, \tag{1.8}$$

where δ is the dissipation parameter. Their theoretical results showed that any amount of dissipation, however small, has the effect of stabilizing the waves. This paper also contained the results of some experiments backing up these findings. Their results were extended by Canney & Carter [19] who generalised (1.8) by including higher order terms up to and including the fifth derivative (ie ones like A_{XXXYY}). They performed a stability analysis and their conclusions agreed with the earlier ones that dissipation stabilizes the waves. All the above investigations were concerned with nonresonant waves. The first comprehensive study of resonant capillary-gravity waves caused by a general interaction between the Mth and Nth harmonics appears to be that of Chen & Saffman [28] who used weakly nonlinear perturbation theory to present a fairly complete account of the surface waves that may occur. Later Jones used the method of multiple scales to derive the system (I), first to model resonant waves on the surface of deep water [1] and then those which manifest themselves on the interface of two fluids [2]. More recently [3] he conducted a detailed study of the stability of the solutions (1.4) of (I) when $\delta = 0$ ie the undamped case. His findings showed that the stability is dependent on the values of the parameters in the problem, but physically one important parameter is V which is the ratio of the horizontal speed of the two fluids when the system is in its quiescent state. If V is positive he identified a large region where the interfaces are stable but in the cases when V is negative he showed that they are almost always unstable. He also considered the damped version of (I) and proved that any plane wave perturbation to the solution (1.2) is oscillatory. However in the present work this is taken further and it is proved that the solutions (1.2) are stable, in other words irrespective of whether an undamped resonant wavetrain is stable or unstable, any amount of damping (however small) has a stabilizing effect. Systems of equations similar to (I), which is sometimes known as the vector nonlinear Schrödinger equation, have been considered by a number of other researchers. Hammack, Henderson & Segur [6] utilised them to investigate the patterns which occur on the surface of deep water when two spatially uniform carrier waves interact obliquely with each other. Carter [4] considered the linear stability of such waves with and without dissipation. His findings broadly showed that dissipation renders the waves less unstable.

2. Setting the Scene

In this section, we gather together a few preliminary results about the system (I). For a detailed treatment of the nlS equation see [29]. We may first observe that in (I) we have taken the coefficients of $|B|^2 A$ and $|A|^2 B$ to be equal. Our first result shows that it is always possible to do this and hence our analysis is therefore perfectly general. Other works [12] [13] make more restrictive assumptions concerning the coefficients.

Lemma 2.1 *The system*

$$iA_T + rA_{XX} + uA_{YY} + p|A|^2 A + q_1|B|^2 A = 0 \tag{2.1a}$$

$$iB_T + vB_{XX} + sB_{YY} + w|B|^2 B + q_2|A|^2 B = 0 \tag{2.1b}$$

may always be transformed into one in which $q_1 = q_2$

Proof. First note that it is permissible to take both q_1 and q_2 to be positive, for if either of them is not, simply replace the relevant equation with its conjugate. It is then easy to verify that the required transformations are $A = \sqrt{q_1} A_1$ and $B = \sqrt{q_2} B_1$.

The pair of Equations (I) constitutes a Lagrangian system. The Lagrangian is

$$L=\frac{i}{2}\left(A_T^*A-A^*A_T+B_T^*B-B^*B_T\right)+rA_X^*A_X+vB_X^*B_X+uA_Y^*A_Y$$
$$+sB_Y^*B_Y-\left(\frac{p}{2}|A|^4+q|A|^2|B|^2+\frac{w}{2}|B|^4\right)\mathrm{e}^{-2\delta t}. \tag{2.2}$$

The corresponding Euler-Lagrange equations are

$$\frac{\partial}{\partial T}\left(\frac{\partial L}{\partial A_T^*}\right)+\frac{\partial}{\partial X}\left(\frac{\partial L}{\partial A_X^*}\right)+\frac{\partial}{\partial Y}\left(\frac{\partial L}{\partial A_Y^*}\right)-\frac{\partial L}{\partial A_X^*}=0 \tag{2.3a}$$

and

$$\frac{\partial}{\partial T}\left(\frac{\partial L}{\partial B_T^*}\right)+\frac{\partial}{\partial X}\left(\frac{\partial L}{\partial B_X^*}\right)+\frac{\partial}{\partial Y}\left(\frac{\partial L}{\partial B_Y^*}\right)-\frac{\partial L}{\partial B_X^*}=0. \tag{2.3b}$$

The system admits a number of conservation laws: the first being that the quantities

$$M(T,A)=\iint_{\mathcal{D}}|A|^2\,\mathrm{d}X\mathrm{d}Y,\quad M(T,B)=\iint_{\mathcal{D}}|B|^2\,\mathrm{d}X\mathrm{d}Y \tag{2.4ab}$$

are both constant ie they are independent of T. These results represent the conservation of mass or energy. Furthermore, the following quantities are also constants

$$P(T,A)=\iint_{\mathcal{D}}r\left(A_XA^*-A_X^*A\right)+u\left(A_YA^*-A_Y^*A\right)\mathrm{d}X\mathrm{d}Y \tag{2.5a}$$

and

$$P(T,B)=\iint_{\mathcal{D}}v\left(B_XB^*-B_X^*B\right)+s\left(B_YB^*-B_Y^*B\right)\mathrm{d}X\mathrm{d}Y, \tag{2.5b}$$

and represent the conservation of momentum. There are other conservation laws but we do not pursue this topic here, see [13].
For the alternative system ($\tilde{I}$) the corresponding results are

$$M\left(T,\tilde{A}\right)=M\left(0,\tilde{A}\right)\mathrm{e}^{-2\delta T},\quad M\left(T,\tilde{B}\right)=M\left(0,\tilde{B}\right)\mathrm{e}^{-2\delta T} \tag{2.6}$$

and

$$P\left(T,\tilde{A}\right)=P\left(0,\tilde{A}\right)\mathrm{e}^{-2\delta T},\quad P\left(T,\tilde{B}\right)=P\left(0,\tilde{B}\right)\mathrm{e}^{-2\delta T}. \tag{2.7}$$

The results (2.7) show that $|\tilde{A}|$ and $|\tilde{B}|$ tend to zero in the $\mathcal{L}^2$ sense as $T\to\infty$, however the transformation into system (I) factors out this decay.

3. Stability of the Waves

This section contains the main results of this paper and concerns the stability of the solutions (1.2) to the system (I). The key mathematical tool used is the classical Sturm-Louiville theory of differential equations [30] but we also employ various *ad hoc* techniques to derive our estimates.

3.1. Perturbing the Solutions

We introduce perturbations a,θ,b,ψ to the solutions as follows:

$$A_0(T)=\alpha\exp\left\{i\frac{\left(p|\alpha|^2+q|\beta|^2\right)\left(1-e^{-2\delta T}\right)}{2\delta}\right\}(1+a+i\theta) \tag{3.1a}$$

$$B_0(T)=\beta\exp\left\{i\frac{\left(w|\beta|^2+q|\alpha|^2\right)\left(1-e^{-2\delta T}\right)}{2\delta}\right\}(1+b+i\psi). \tag{3.1b}$$

Then if we substitute (3.1) into (*I*) and assume that the perturbations are sufficiently small that their products may be neglected, the result is

$$\theta_T - ra_{XX} - ua_{YY} - 2p|\alpha|^2 e^{-2\delta T} a - 2q|\beta|^2 e^{-2\delta T} b = 0 \tag{3.2a}$$

$$\psi_T - vb_{XX} - sb_{YY} - 2q|\alpha|^2 e^{-2\delta T} a - 2w|\beta|^2 e^{-2\delta T} b = 0 \tag{3.2b}$$

$$a_T + r\theta_{XX} + u\theta_{YY} = 0 \tag{3.2c}$$

$$b_T + v\psi_{XX} + s\psi_{YY} = 0. \tag{3.2d}$$

Since the coefficients in (3.2) are functions of T but not X and Y it is sensible to seek solutions in the form

$$\begin{pmatrix} a \\ b \\ \theta \\ \psi \end{pmatrix} = \begin{pmatrix} \hat{a} \\ \hat{b} \\ \hat{\theta} \\ \hat{\psi} \end{pmatrix} \exp(imX + ilY), \tag{3.3}$$

where $\hat{a},\hat{b},\hat{\theta},\hat{\psi}$ are functions of T only. It will also prove convenient to introduce the quantities P and Q where $P = rm^2 + ul^2$ and $Q = vm^2 + sl^2$.
Then substituting (3.3) into (3.2) yields

$$\hat{\theta}_T + P\hat{a} - 2p|\alpha|^2 e^{-2\delta T}\hat{a} - 2q|\beta|^2 e^{-2\delta T}\hat{b} = 0, \tag{3.4a}$$

$$\hat{\psi}_T + Q\hat{b} - 2q|\alpha|^2 e^{-2\delta T}\hat{a} - 2w|\beta|^2 e^{-2\delta T}\hat{b} = 0, \tag{3.4b}$$

$$\hat{a}_T - P\hat{\theta} = 0, \tag{3.4c}$$

$$\hat{b}_T - Q\hat{\psi} = 0. \tag{3.4d}$$

We can now prove our first result.

Theorem 3.1 *All solutions* $(\hat{a},\hat{b},\hat{\theta},\hat{\psi})$ *of the system (3.4) are oscillatory. For sufficiently large* T *the maximum period of oscillation is bounded above by* $\pi/\left|rm^2+ul^2\right|$.

Proof. First differentiate (3.4c) wrt T and substitute into (3.4a) thus obtaining the equation

$$\hat{a}_{TT} + P\left\{P - 2p|\alpha|^2 e^{-2\delta T}\right\}\hat{a} - 2qP|\beta|^2 e^{-2\delta T}\hat{b} = 0. \tag{3.5}$$

Now observe that there is a positive constant l such that for all large enough T, the coefficient of $\hat{a}$ in (3.5) is greater than l^2. In fact, we may take l to be anything less than P^2. Manifestly it can be seen that (3.5) is equivalent to

$$\begin{aligned}&\hat{a}_{TT}\sin(lT) + l^2\sin(lT)\hat{a} + \left[P\left\{P - 2p|\alpha|^2 e^{-2}\right\} - l^2\right]\sin(lT)\hat{a}\\ &-|\beta|^2 e^{-2\delta T}\sin(lT)\hat{b} = 0.\end{aligned} \tag{3.6}$$

Now let n be an even positive integer and suppose that $\hat{a}(T)$ is positive between $T = n\pi/l$ and $T = (n+1)\pi/l$, other cases are similar. Now integrate (3.6) between $n\pi/l$ and $(n+1)\pi/l$. Using integration by parts on the first two terms yields $l\left(a(n\pi/l) + a((n+1)\pi/l)\right)$ which is positive by our assumption. The integrand of the remaining terms involving $\hat{a}$ is positive by the choice of l and the fact that n is even while the coefficient of $\hat{a}$ can be rendered arbitrarily small by choosing T sufficiently large. Hence the integral cannot be zero which proves the first sentence of the theorem.

The second sentence follows from the fact that l^2 can be chosen to be arbitrarily close to P^2 which is equal to $\left(rm^2 + ul^2\right)^2$.

Remark 3.2 Similar results to Thm 3.1 can be found in [3] [20]. Equation (3.6) is of Sturm-Louiville type, for a general treatment of such equations see [30].

We are now able to proceed to our main results on stability. Recall that if we have any solution to (I), the spatial variables X and Y belong to the rectangular domain $\mathcal{D} = [0, L_X] \times [0, L_Y]$ and we impose periodic boundary conditions on $\mathcal{D}$.

3.2. Supremum Type Stability

Since, as far as is known, the set (3.4) has no explicit solutions available, we estimate them by the following means. First consider a truncated version of (3.4ac) ie those parts with constant coefficients, that is

$$\hat{\theta}_T + P\hat{a} = 0, \quad \hat{a}_T - P\hat{\theta} = 0. \tag{3.7ab}$$

It is easy to see that these have solutions

$$\hat{a} = \exp(\pm iPT), \quad \hat{\theta} = \pm i \exp(\pm iPT). \tag{3.8ab}$$

This suggests introducing a new set of variables $a_1, b_1, \theta_1, \psi_1$ via the transformations

$$\begin{aligned}\hat{a} &= a_1 \exp(iPT) + \theta_1 \exp(-iPT),\ \hat{\theta} = ia_1 \exp(iPT) - i\theta_1 \exp(-iPT),\\ \hat{b} &= b_1 \exp(iQT) + \psi_1 \exp(-iQT),\ \hat{\psi} = ib_1 \exp(iQT) - i\psi_1 \exp(-iQT).\end{aligned} \tag{3.9}$$

Then rewriting (3.4) in terms of these new variables leads us to

$$\begin{aligned}ia_{1T} \exp(iPT) = p|\alpha|^2 &\left[a_1 \exp(iPT) + \theta_1 \exp(-iPT)\right] \mathrm{e}^{-2\delta T}\\ &+ q|\beta|^2 b_1 \left[\exp(iQT) + \psi_1 \exp(-iQT)\right] \mathrm{e}^{-2\delta T}\end{aligned} \tag{3.10}$$

from which we may deduce the following inequality

$$|a_{1T}| \le p|\alpha|^2 \left(|a_1| + |\theta_1|\right) \mathrm{e}^{-2\delta T} + q|\beta|^2 \left(|b_1| + |\psi_1|\right) \mathrm{e}^{-2\delta T} \tag{3.11}$$

which clearly implies the weaker inequality

$$|a_{1T}| \le \left\{\left(|p| + |q|\right)|\alpha|^2 + \left(|q| + |w|\right)|\beta|^2\right\}\left(|a_1| + |\theta_1| + |b_1| + |\psi_1|\right) \mathrm{e}^{-2\delta T}. \tag{3.12}$$

The advantage of inequality (3.12) is that by similar reasoning that it can be shown to apply to the other quantities, ie (3.12) remains true if the right side is unchanged and the left side is replaced by any one of $b_{1T}, \theta_{1T}, \psi_{1T}$. Now let

$$\Gamma=\left(|p|+|q|\right)|\alpha|^{2}+\left(|q|+|w|\right)|\beta|^{2}. \tag{3.13}$$

From (3.12) it then follows by standard iterative methods, [30, Ch 3] see also [20], that

$$|a_1|\le\sqrt{|a_1^2(0)|+|b_1^2(0)|+|\theta_1^2(0)|+|\psi_1^2(0)|}\exp\left(2\Gamma\left(1-e^{-2\delta T}\right)/\delta\right) \tag{3.14}$$

so that

$$|a_1|\le\sqrt{|a_1^2(0)|+|b_1^2(0)|+|\theta_1^2(0)|+|\psi_1^2(0)|}\exp\left(2\Gamma/\delta\right) \tag{3.15}$$

providing us with a bound independent of *T*. Now let $\|.\|$ denote the supremum norm on $\mathcal{D}$. Transforming back to our original variables by means of (3.9) and (3.3) we then obtain the inequality

$$\|a\|\le\frac{1}{\sqrt{2}}\sqrt{|a^2(0)|+|b^2(0)|+|\theta^2(0)|+|\psi^2(0)|}\exp\left(2\Gamma/\delta\right) \tag{3.16}$$

It then follows from (3.14) that $\|a\|<\varepsilon$ and the same inequality applies to $\|b\|,\|\theta\|,\|\psi\|$. Supremum stability now follows.

Remark 3.3 Similar results for the single nlS equation may be found in [20].

Remark 3.4 Our result may be modified to apply to the case when the functions are defined on the whole of the X,Y plane if we impose suitable boundary conditions at infinity.

3.3. $\mathcal{L}^2$ Type Stability

The calculation proceeds the same way until we reach the system (3.2). This time we multiply (3.2a) by θ, (3.2c) by *a* and add to obtain

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial T}\left(a^2+\theta^2\right)+r\frac{\partial}{\partial X}\left(a\theta_X-\theta a_X\right)+u\frac{\partial}{\partial Y}\left(a\theta_Y-\theta a_Y\right)\\&-p|\alpha|^2a\theta e^{-2\delta T}-2q|\beta|^2b\theta e^{-2\delta T}=0.\end{aligned} \tag{3.17}$$

Then integrating over $\mathcal{D}$ and using the Cauchy-Schwartz inequality there results that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dT}\iint\left(a^2+\theta^2\right)dA\\&\le|p||\alpha|^2e^{-2\delta T}\iint\left(a^2+\theta^2\right)dA+|q||\beta|^2e^{-2\delta T}\iint\left(b^2+\theta^2\right)dA.\end{aligned} \tag{3.18a}$$

The analogous equation resulting from (3.2bd) is

$$\begin{aligned}&\frac{1}{2}\frac{d}{dT}\iint\left(b^2+\psi^2\right)dA\\&\le|w||\beta|^2e^{-2\delta T}\iint\left(b^2+\psi^2\right)dA+|q||\alpha|^2e^{-2\delta T}\iint\left(a^2+\psi^2\right)dA.\end{aligned} \tag{3.18b}$$

Putting these together yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dT}\iint\left(a^2+\theta^2+b^2+\psi^2\right)dA\\&\le\left\{p|\alpha|^2+w|\beta|^2+q\left(|\alpha|^2+|\beta|^2\right)\right\}e^{-2\delta T}\iint\left(a^2+\theta^2+b^2+\psi^2\right)dA.\end{aligned} \tag{3.19}$$

Now let

$$\sigma(T)=\iint\left(a^2+\theta^2+b^2+\psi^2\right)\mathrm{d}A.$$

Then integrating (3.19) gives

$$\sigma(T)\le\sigma(0)\exp\left\{\left(p|\alpha|^2+w|\beta|^2+q\left(|\alpha|^2+|\beta|^2\right)\right)\left(\frac{1-\mathrm{e}^{-2\delta T}}{\delta}\right)\right\} \quad (3.20)$$

implying

$$\sigma(T)\le\sigma(0)\exp\left\{\frac{p|\alpha|^2+w|\beta|^2+q\left(|\alpha|^2+|\beta|^2\right)}{\delta}\right\}. \quad (3.21)$$

Stability now readily follows. Given $\varepsilon>0$ choose

$$\Delta=\varepsilon\exp\left\{-\frac{p|\alpha|^2+w|\beta|^2+q\left(|\alpha|^2+|\beta|^2\right)}{\delta}\right\}. \quad (3.22)$$

Then if $\sigma(0)<\Delta$ we have $\sigma(T)<\varepsilon$ for all T and the result is shown.

4. Conclusions

We have conducted an investigation into the solutions of the coupled nonlinear Schrödinger equations which correspond to a monochromatic wavetrain on deep water consisting of capillary-gravity waves arising from the resonant or near-resonant interaction of the Mth and Nth harmonics of the fundamental mode. Our equations each include a term that could represent the effects of dissipation or other imperfections. Such effects are significant because all physical systems exhibit dissipation to a larger or smaller extent. Our results prove that any amount of dissipation (however small) has the effect of stabilising the waves in the presence of small perturbations known as sidebands. In the absence of dissipation, it was shown [3] that the waves may or may not be stable depending on the precise modes involved in the interactions, ie the values of M and N, and also whether the direction of the perturbations is normal, transverse or oblique to the main wavetrain. This is in contrast to the nonresonant case when the classical Benjamin-Feir theory predicts instability in the absence of dissipation [23] while the later results of Segur *et al.* [20] showed that such waves are rendered stable by dissipative effects. Physically this shows that the presence of dissipation causes energy to be transferred from the carrier wave to the sidebands. Then provided the initial amplitudes of all the sidebands are sufficiently small to begin with, as in (3.3), they fail to grow very large and the nonlinear interactions between them do not become of sufficient magnitude to become significant enough to cause instability. This is the physical significance of our results on $\mathcal{L}^2$ stability. As well as describing wavetrains, the equations considered in this work may also be applied to the evolution of ocean swells [31] and to account for the propagation of coherent waves across the Pacific. Another application of our results could be to predict the onset of rogue waves [9]. Our discussions and physical interpretation have been very largely focussed on the topic of water waves but as prevoiously remarked they may also be of interest to researchers who are concerned with nonlinear fibre optics which may also be described by coupled nonlinear Schrödinger equations, a particularly relevant case is the cross coupling instability of plane waves.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Jones, M.C.W. (1993) On Coupled Differential Equations Which Model the Evolution of Interacting Capillary-Gravity Wave Modes and Related Questions of Stability. *IMA Journal of Applied Mathematics*, **50**, 13-28. https://doi.org/10.1093/imamat/50.1.13

[2] Jones, M.C.W. (1997) Nonlinear Ripples of Kelvin-Helmholtz Type Which Rise from an Interfacial Mode Interaction. *Journal of Fluid Mechanics*, **341**, 295-315. https://doi.org/10.1017/S0022112097005624

[3] Jones, M.C.W. (2018) Resonant Interfacial Capillary-Gravity Waves in the Presence of Damping Effects. *European Journal of Mechanics-B/Fluids*, **68**, 20-29. https://doi.org/10.1016/j.euromechflu.2017.10.012

[4] Carter, J.D. (2012) Plane Wave Solutions of a Dissipative Generalization of the Vector Nonlinear Schrödinger Equation. *Mathematics and Computers in Simulation*, **82**, 1038-1046. https://doi.org/10.1016/j.matcom.2010.07.032

[5] Dhar, A.K. and Das, K.P. (1991) Fourth-Order Nonlinear Evolution Equations for Two Stokes Wave Trains in Deep Water. *Physics of Fluids A*: *Fluid Dynamics*, **3**, 3021-3026. https://doi.org/10.1063/1.858209

[6] Hammack, J.L., Henderson, D.M. and Segur, H. (2005) Progressive Waves with Persistent Two-Dimensional Surface Patterns in Deep Water. *Journal of Fluid Mechanics*, **532**, 1-52. https://doi.org/10.1017/S0022112005003733

[7] Henderson, D.M. and Segur, H. (2007) Stable Deep-Water Waves Propagating in One and Two Dimensions. *Proceedings in Applied Mathematics and Mechanics*, **7**, 1100401-1100402. https://doi.org/10.1002/pamm.200700393

[8] Henderson, D.M., Segur, H. and Carter, J.D. (2010) Experimental Evidence of Stable Wave Patterns of Deep Water. *Journal of Fluid Mechanics*, **658**, 247-278. https://doi.org/10.1017/S0022112010001643

[9] Onorato, M., Osborne, A.R. and Serio, M. (2006) Modulational Instability in Crossing Sea States: A Possible Mechanism for the Formation of Freak Waves. *Physical Review Letters*, **96**, Article 014503. https://doi.org/10.1103/PhysRevLett.96.014503

[10] Hasegawa, A. (1971) Theory and Computer Experiment on Self-Trapping Instability Plasma Cyclotron Waves. *The Physics of Fluids*, **15**, 870-881. https://doi.org/10.1063/1.1693996

[11] Pecseli, H.L. (1985) Solitons and Weakly Nonlinear Waves in Plasmas. *IEEE Transactions on Plasma Science*, **13**, 53-86. https://doi.org/10.1109/TPS.1985.4316366

[12] Forest, M.G., McLaughlin, D.W., Muraki, D.J. and Wright, O.C. (2000) Nonfocusing Instabilities in Coupled Integrable Nonlinear Schrödinger pdes. *Journal of Nonlinear Science*, **10**, 291-331. https://doi.org/10.1007/s003329910012

[13] Wadati, M., Isuka, T. and Hisakado, M. (1992) A Coupled Nonlinear Schrödinger Equation and Optical Solitons. *Journal of the Physical Society of Japan*, **61**, 2241-2245. https://doi.org/10.1143/JPSJ.61.2241

[14] Jones, M.C.W. (1992) On the Stability of a Third Harmonic Resonant Wavetrain. *Stability and Applied Analysis of Continuous Media*, **2**, 323-338.

[15] Jones, M.C.W. (1992) Soliton Solutions to Coupled Nonlinear Evolution Equations Modelling a Third Harmonic Resonance in the Theory of Capillary-Gravity Waves. *Applied Mathematical Modelling*, **40**, 2134-2142. https://doi.org/10.1016/j.apm.2015.09.017

[16] McGoldrick, L.F. (1970) On Wilton's Ripples: A Special Case of Resonant Interactions. *Journal of Fluid Mechanics*, **42**, 193-200. https://doi.org/10.1017/S0022112070001179

[17] Nayfeh, A.H. (1971) Third-Harmonic Resonance in the Interaction of Capillary and Gravity Waves. *Journal of Fluid Mechanics*, **48**,385-395. https://doi.org/10.1017/S0022112071001630

[18] Verheest, F. (1981) Third Harmonic and Subharmonic Generation in General Media. *Wave Motion*, **3**, 231-236. https://doi.org/10.1016/0165-2125(81)90016-0

[19] Canney, N. and Carter, J.D. (2007) Stability of Plane Waves on Deep Water with Dissipation. *Mathematics and Computers in Simulation*, **74**, 159-167. https://doi.org/10.1016/j.matcom.2006.10.010

[20] Segur, H., Henderson, D.M., Carter, J.D., Hammack, J.H., Li, C.-M., Pheiff, D. and Socha, K. (2005) Stabilizing the Benjamin-Feir Instability. *Journal of Fluid Mechanics*, **539**, 229-271. https://doi.org/10.1017/S002211200500563X

[21] Nemytskii, V.V. (1960) Qualitative Theory of Differential Equations. Princeton University Press, Princeton. https://doi.org/10.1515/9781400875955

[22] Zakharov, V.E. (1968) Stability of Periodic Waves of Finite Amplitude on the Surface of a Deep Fluid. *Journal of Applied Mechanics and Technical Physics*, **9**, 190-194. https://doi.org/10.1007/BF00913182

[23] Benjamin, T.B. and Feir, J.E. (1967) The Disintegration of Waves on Deep Water. Part 1. Theory. *Journal of Fluid Mechanics*, **27**, 417-430. https://doi.org/10.1017/S002211206700045X

[24] Snodgrass, F.E., Groves, G.W., Hasselmann, K., Miller, G.R., Munk, W.H. and Powers, W.H. (1966) Propagation of Ocean Swell across the Pacific. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, **259**, 431-497. https://doi.org/10.1098/rsta.1966.0022

[25] Collard, F., Ardhuin, F. and Chapron, B. (2009) Monitoring and Analysis of Ocean Swell Fields from Space: New Methods for Routine Observations. *Journal of Geophysical Research: Oceans*, **114**, Article C07023. https://doi.org/10.1029/2008JC005215

[26] Lake, B.M. and Yuen, H.C. (1977) A Note on Some Water-Wave Experiments and the Comparison of Data with Theory. *Journal of Fluid Mechanics*, **83**, 75-81. https://doi.org/10.1017/S0022112077001049

[27] Lake, B.M., Yuen, H.C., Rungaldier, C. and Ferguson, W.E. (1977) Nonlinear Deep-Water Waves Theory and Experiment. Part 2. Evolution of a Continuous Wave Train. *Journal of Fluid Mechanics*, **83**, 49-74. https://doi.org/10.1017/S0022112077001037

[28] Chen, B. and Saffman, P.G. (1979) Steady Gravity-Capillary Waves on Deep Water—Weakly Nonlinear Waves. *Studies in Applied Mathematics*, **68**, 183-210.

https://doi.org/10.1002/sapm1979603183

[29] Sulem, C. and Sulem, P.L. (1999) The Nonlinear Schrödinger Equation. Springer, New York.

[30] Ince, E.L. (1956) Ordinary Differential Equations. Dover, New York.

[31] Henderson, D.M. and Segur, H. (2012) The Benjamin-Feir Instability and the Propagation of Swelle across the Pacific. *Mathematics and Computers in Simulation*, **82**, 1172-1184. https://doi.org/10.1016/j.matcom.2010.06.015