

Combining Fractals and Box-Counting Dimension

M. Ndiaye

Mathematics and Computer Sciences Department, Marist College, Poughkeepsie, NY, USA

Email: malick.ndiaye@marist.edu

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Abstract

In this paper, the box-counting dimension is used to derive an explicit formula for the dimension of a fractal constructed using several contractions or by combining fractals. This dimension agrees with the Hausdorff dimension in the particular case when the scales factors considered are all the same. A more general sufficient condition for the box-counting dimension and the Hausdorff dimension to be the same is given. It is also shown that the dimension of the fractal obtained by combining two fractals is the weighted average of the dimensions of the two fractals.

Keywords

Box-Counting Dimension, Combining Fractals

1. Introduction

We can think of fractals as a complex geometric figure that is self similar across different scales. Fractals occupy the space between two discrete dimensions. These are shapes whose dimension is not a whole number unlike any Euclidean space. Since fractals show similar details on different scales, so scaling is at the center of understanding fractals dimension. When one successfully magnifies a section of a fractal image, each magnification reveals repeated fractals shapes that are similar in detail to the images of the main fractal and other magnification. There are various ways to construct fractals mathematically. We have for example a deterministic process like the one used to construct the classic Cantor set [1] [2] by iteratively deleting the middle third section of the unit interval. We can also construct fractals by repeatedly iterating a given map or through the use of IFS (Iterating Function System) [1] [3] or by looking at the basin boundary of infinity of the Mandelbrot set: Julia set [1] [4] [5] [6]. In their article: On combining and

convolving fractals [7], the authors introduced a new process of constructing fractals: The convolution of fractals into one. They then explore the effect of such combinations on the dimensions of the corresponding fractals by using the Moran equations [8] [9]. In general, finding an explicit formula for the dimension in term of elementary functions is quite difficult, and sometime impossible. In this paper, the box-counting dimension [2] [10] is used to derive an alternative explicit formula for the dimension of fractals constructed using several contractions or by combining fractals. The formula for the dimension of fractals constructed using several contractions could be seen as the generalization of the Cantor set dimension in the case we apply a multitude of scales on the unit interval. With this formula, it is much easier to show several properties of the dimension of the fractal obtained by combining two fractals than using the dimension from the Moran equation.

The paper is divided into six sections.

In Section 2, the definition of contractions, iterated functions system (IFS) are introduced. The existence of a limiting set from an IFS is discussed. An example of fractal is also given.

In Section 3, the definition and the properties of the box-counting dimension are introduced and specific examples about the box-counting dimension of the Cantor set and the Sierpinski gasket are given.

In Section 4, the box-counting dimension is applied to fractals that arise from multitude of scaling.

In Section 5, the dimension of fractals obtained by combination of two fractals is explored and it is shown that this dimension is a weighted average.

Section 6 is dedicated to the conclusion.

2. Notion of Contractions, Iterated Function Systems, Fractal Definition

The functions that are used in mathematics to reproduce a contracted self-similar image from a given figure is called contraction. This notion is defined as follow.

Definition 1. A contraction in \mathbb{R}^m with scale factor $r \in (0,1)$ is a map $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ such that for all x and y in \mathbb{R}^m $|f(x) - f(y)| \leq r|x - y|$.

Remark 1. If $|f(x) - f(y)| = r|x - y|$ then f is said to be a similarity with scale factor r .

Matrices provide a very convenient notation for contractions. A contraction can be written as $f(x) = Ax + b$ where $x \in \mathbb{R}^m$ and A is a $m \times m$ matrix whose eigenvalues have magnitudes less than 1 and b is a constant in \mathbb{R}^m .

Definition 2. An iterated function system (IFS) is a collection of contractions $\{f_1, f_2, \dots, f_n\}$ with $n \geq 2$ on a closed subset $\Omega \subset \mathbb{R}^n$. A nonempty compact subset F of Ω is an attractor or fractal of the IFS if $F = \bigcup_{i=1}^n f_i(F)$.

The following theorem shows the existence and the uniqueness of such attractor given any IFS.

Theorem 1. Given any IFS defined by the contractions $C_1, C_2, C_3, \dots, C_m$ on a

closed subset Ω of \mathbb{R}^m and let \mathcal{C} be the set of all nonempty compact subsets of Ω and let f be the map defined as $f(X) = \bigcup_{i=1}^m C_i(X)$ where C_i , then there exists a unique attractor F such that $f(F) = F$. Furthermore, for every set $X \in \mathcal{C}$ such that $f(X) \subset X$, $F = \bigcap_{i=1}^{\infty} f^i(X)$.

Proof. let Ω be a closed set in \mathbb{R}^m . Let $E \subset \Omega$.

The δ -neighborhood of E denoted $N_\delta(E)$ is defined as

$$N_\delta(E) = \{x \in \Omega \mid |x - a| < \delta \text{ for some } a \in E\}.$$

Let \mathcal{C} be the set of compact in Ω .

The Hausdorff metric on \mathcal{C} is defined as:

$$d_H(A, B) = \inf \{ \delta \mid A \subset N_\delta(B), B \subset N_\delta(A) \}.$$

- The set of compact \mathcal{C} in Ω is a complete space using the Hausdorff metric.

Let S_n be a Cauchy sequence in \mathcal{C} that is $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ such that if $n, m \geq N$ then $d_H(S_n, S_m) < \varepsilon$.

Choose $\varepsilon_i = \frac{\varepsilon}{2^i} \exists N_i \in \mathbb{N}$ such that if $n, m \geq N$ then $d_H(S_n, S_m) < \frac{\varepsilon_i}{2^i}$, $i \geq 1$.

If $N_1 < N_2 < N_3 < \dots < N_i < \dots$ then $d_H(S_{N_i}, S_{N_{i+1}}) < \frac{\varepsilon}{2^i}$ since $N_{i+1} > N_i$,

\Rightarrow there exists a sequence $s_{N_i} \in S_{N_i}, i \geq 1$ such that $|s_{N_i} - s_{N_{i+1}}| < \frac{\varepsilon}{2^i}$ and $s_{N_{i+1}} \in S_{N_{i+1}}$.

Let's show that the sequence s_{N_k} is a Cauchy sequence.

Suppose that $m < n$ then

$$\begin{aligned} |s_{N_m} - s_{N_n}| &= |s_{N_m} - s_{N_{m+1}} + s_{N_{m+1}} - s_{N_{m+2}} + \dots + s_{N_{n-1}} - s_{N_n}| \\ &< |s_{N_m} - s_{N_{m+1}}| + |s_{N_{m+1}} - s_{N_{m+2}}| + \dots + |s_{N_{n-1}} - s_{N_n}| \\ &< \frac{\varepsilon}{2^m} + \frac{\varepsilon}{2^{m+1}} + \frac{\varepsilon}{2^{m+2}} + \dots + \frac{\varepsilon}{2^{n-1}} \\ &< \varepsilon \left(\frac{1}{2^m} + \frac{1}{2^{m+1}} + \dots + \frac{1}{2^{n-1}} \right) \\ &< \varepsilon \left(\frac{1}{2} + \frac{1}{2^2} + \dots \right) = \varepsilon \end{aligned}$$

$\Rightarrow s_{N_k}$ is a Cauchy sequence.

Since s_{N_k} is a sequence of real numbers therefore $\lim_{k \rightarrow \infty} s_{N_k} = L$ exists.

Let S be the set of those limits that is

$$S = \left\{ L \mid L = \lim_{k \rightarrow \infty} s_{N_k}, s_{N_k} \in S_{N_k}, N_1 < N_2 < \dots \right\}$$

Figure 1 shows a subsequence S_{N_k} of sets from S_N . Several subsequences of real numbers a_{N_k}, b_{N_k}, \dots are depicted.

Now, let's show that $\lim_{n \rightarrow \infty} S_n = S$ that is $\forall \varepsilon > 0 \exists K(\varepsilon) \in \mathbb{N}$ such that if $n \geq K(\varepsilon)$ then $d_H(S_n, S) < \varepsilon$ that is $S_n \subset N_\varepsilon(S)$ and $S \subset N_\varepsilon(S_n)$.

First let's start by showing that $S \subset N_\varepsilon(S_n)$.

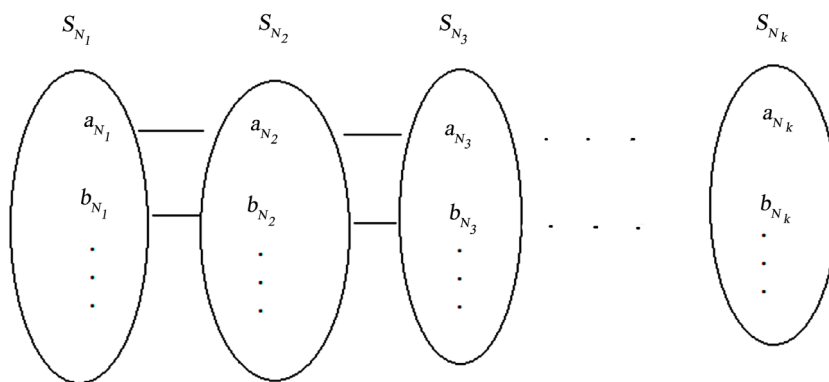


Figure 1. Subsequence S_{N_k} of sets from S_n .

Let $s \in S \Rightarrow \exists s_{N_k} \in S_{N_k}$ such that $\lim_{k \rightarrow \infty} s_{N_k} = s \Rightarrow \forall \varepsilon > 0 \exists P(\varepsilon) \in \mathbb{N}$ such that if $K(\varepsilon) \geq P(\varepsilon)$ then $|s_{N_k} - s| < \frac{\varepsilon}{2}$.

Since S_n is a Cauchy sequence then $\forall \varepsilon > 0 \exists M(\varepsilon) \in \mathbb{N}$ such that if $n, m \geq M(\varepsilon)$ then $d_H(S_n, S_m) < \frac{\varepsilon}{2}$ which means that $S_n \subset N_{\frac{\varepsilon}{2}}(S_m)$ and $S_m \subset N_{\frac{\varepsilon}{2}}(S_n) \quad n \geq M(\varepsilon)$.

Assuming that $N_k \geq M(\varepsilon), n \geq M(\varepsilon)$ then $d_H(S_n, S_{N_k}) < \frac{\varepsilon}{2}$.

\Rightarrow there exists a sequence $s_n \in S_n$ such that $|s_n - s_{N_k}| < \frac{\varepsilon}{2}$.

Now let $L(\varepsilon)$ be the maximum of $P(\varepsilon)$ and $M(\varepsilon)$.

If $n, k > L(\varepsilon)$ then $|s - s_n| \leq |s - s_{N_k}| + |s_{N_k} - s_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$$\Rightarrow s \in N_\varepsilon(A_n) \Rightarrow S \subset N_\varepsilon(S_n)$$

Now, let's show that $S_n \subset N_\varepsilon(S) \quad \forall n \geq K(\varepsilon)$ for some $K(\varepsilon) \in \mathbb{N}$.

Let $b \in S_n$, since S_n is a Cauchy sequence then $\forall \varepsilon_i = \frac{\varepsilon}{2^i} > 0 \exists N_i(\varepsilon) \in \mathbb{N}$

Such that if $n, m \geq N_i(\varepsilon)$ then $d_H(S_n, S_m) < \frac{\varepsilon}{2^i}, i \geq 1$.

If $b \in A_n$ for some $n \geq N_i(\varepsilon)$ then $\exists l \in \mathbb{N}$ such that $b = b_{N_l}$ where $b_{N_l} \in A_{N_l}$.

Since $\lim_{k \rightarrow \infty} b_{N_k} = s$ where $b \in S$, then for $k > l$,

$$\begin{aligned} |b_{N_l} - b_{N_k}| &\leq |b_{N_l} - b_{N_{l+1}}| + |b_{N_{l+1}} - b_{N_{l+2}}| + \dots + |b_{N_{k-1}} - b_{N_k}| \\ &< \frac{\varepsilon}{2^l} + \frac{\varepsilon}{2^{l+1}} + \dots + \frac{\varepsilon}{2^{k-1}} \\ &< \varepsilon \left(\frac{1}{2^l} + \frac{1}{2^{l+1}} + \dots + \frac{1}{2^{k-1}} \right) \\ &< \varepsilon \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots \right) = \varepsilon \end{aligned}$$

$$\Rightarrow \lim_{k \rightarrow \infty} |b_{N_l} - b_{N_k}| \leq \varepsilon \Rightarrow |b_{N_l} - s| \leq \varepsilon \Rightarrow |b - s| \leq \varepsilon \Rightarrow b \in N_\varepsilon(S) \Rightarrow S_n \subset N_\varepsilon(S)$$

$$\Rightarrow \lim_{n \rightarrow \infty} S_n = S.$$

Let's show that the set S is a compact set that is bounded and closed set.

Let's start showing that S is closed. let x_n be a sequence in S such that

$$\lim_{n \rightarrow \infty} x_n = x.$$

Let's show that $x \in S$.

$$\lim_{n \rightarrow \infty} x_n = x \Leftrightarrow \forall \varepsilon > 0 \exists N(\varepsilon) \in \mathbb{N} \text{ such that if } n \geq N(\varepsilon) \text{ then } |x_n - x| < \frac{\varepsilon}{2}.$$

$$\text{Since } x_n \in S \Rightarrow \exists \text{ a sequence } (x_n)_{N_k} \in S_{N_k} \text{ such that } \lim_{k \rightarrow \infty} (x_n)_{N_k} = x_n.$$

$$\Rightarrow \forall \varepsilon > 0 \exists M(\varepsilon) \in \mathbb{N} \text{ such that if } k \geq M(\varepsilon) \text{ then } \left| (x_n)_{N_k} - x_n \right| < \frac{\varepsilon}{2}.$$

Assuming that $n \geq N(\varepsilon)$ and $k \geq M(\varepsilon)$ then

$$\left| (x_n)_{N_k} - x \right| \leq \left| (x_n)_{N_k} - x_n \right| + |x_n - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$$\rightarrow \lim_{n \rightarrow \infty} (x_n)_{N_k} = x \Rightarrow x \in S \text{ therefore } S \text{ is closed.}$$

Let's show that the set S is a bounded set.

$$\text{Since } \lim_{n \rightarrow \infty} S_n = S \Rightarrow \text{for } \varepsilon = 1 \exists N(\varepsilon) \in \mathbb{N} \text{ such that } d_H(S_n, S) < \varepsilon \rightarrow S \subset N_1(S_n) \text{ for } n \geq N(\varepsilon).$$

$$\text{Since } S_n \text{ is bounded then } A_n \subset \bar{B}_r(x) \text{ for some } x \in S_n \text{ and some } r > 0, \text{ where } \bar{B}_r(x) = \{y \in \mathbb{R} / |y - x| \leq r\} \text{ then } N_1(A_n) \subset \bar{B}_{r+1}(x) \Rightarrow S \subset \bar{B}_{r+1}(x).$$

Therefore S is bounded.

- The function f defined as $f(X) = \bigcup_{1 \leq i \leq m} C_i(X)$ is a contraction using the Hausdorff metric.

So, we need to show that $d_H(f(X), f(Y)) < r d_H(X, Y)$ where $r < 1$.

$$d_H(f(X), f(Y)) = d_H\left(\bigcup_{1 \leq i \leq m} C_i(X), \bigcup_{1 \leq i \leq m} C_i(Y)\right).$$

Let's show that if $\max_{1 \leq i \leq m} d_H(C_i(X), C_i(Y)) < l$ then

$$d_H\left(\bigcup_{1 \leq i \leq m} C_i(X), \bigcup_{1 \leq i \leq m} C_i(Y)\right) < l.$$

Suppose that $\max_{1 \leq i \leq m} d_H(C_i(X), C_i(Y)) < l$ then $d_H(C_i(X), C_i(Y)) < l$ for all $i \Rightarrow C_i(X) \subset N_l(C_i(Y))$ and $C_i(Y) \subset N_l(C_i(X))$ for all i .

$$C_i(X) \subset N_l(C_i(Y)) \Rightarrow \bigcup_{1 \leq i \leq m} C_i(X) \subset \bigcup_{1 \leq i \leq m} N_l(C_i(Y)).$$

Let's show that $\bigcup_{1 \leq i \leq m} N_l(C_i(Y)) = N_l\left(\bigcup_{1 \leq i \leq m} C_i(Y)\right)$.

Let $x \in \bigcup_{1 \leq i \leq m} N_l(C_i(Y)) \Leftrightarrow \exists i$ such that $x \in N_l(C_i(Y)) \Leftrightarrow |x - y| < l$ for some $y \in C_i(Y)$.

Then for some

$$\begin{aligned} y \in \bigcup_{1 \leq i \leq m} C_i(Y) &\Leftrightarrow x \in N_l\left(\bigcup_{1 \leq i \leq m} C_i(Y)\right) \\ &\Rightarrow \bigcup_{1 \leq i \leq m} N_l(C_i(Y)) = N_l\left(\bigcup_{1 \leq i \leq m} C_i(Y)\right) \\ &\Rightarrow \bigcup_{1 \leq i \leq m} C_i(X) \subset N_l\left(\bigcup_{1 \leq i \leq m} C_i(Y)\right). \end{aligned}$$

Same thing, we can show that

$$\bigcup_{1 \leq i \leq m} C_i(Y) \subset N_i \left(\bigcup_{1 \leq i \leq m} C_i(X) \right).$$

Therefore

$$\begin{aligned} d_H \left(\bigcup_{1 \leq i \leq m} C_i(X), \bigcup_{1 \leq i \leq m} C_i(Y) \right) &< l \\ \Rightarrow d_H \left(\bigcup_{1 \leq i \leq m} C_i(X), \bigcup_{1 \leq i \leq m} C_i(Y) \right) &\leq \max_{1 \leq i \leq m} d_H(C_i(X), C_i(Y)) \end{aligned}$$

Since C_i is a contraction then $d_H(C_i(x), C_i(Y)) < r_i d_H(X, Y)$, $0 < r_i < 1$.

$$\Rightarrow d_H \left(\bigcup_{1 \leq i \leq m} C_i(X), \bigcup_{1 \leq i \leq m} C_i(Y) \right) \leq \frac{c}{1} d_H(X, Y)$$

where $c = \max_{1 \leq i \leq m} r_i$ so $0 < c < 1$.

Therefore f is a contraction. Since \mathcal{C} is a complete space and f is a contraction, then according to the Banach's Contraction Mapping Theorem, we conclude that f has a fix point, that is there exists F such $f(F) = F$ therefore $F = \bigcup_{i=1}^m C_i(F)$ has a solution.

We can also prove that $F = \bigcap_{i=0}^{\infty} f^i(X)$ where $X \in \mathcal{C}$.

$\lim_{k \rightarrow \infty} f^k(X) = f \left(\lim_{k \rightarrow \infty} f^{k-1}(X) \right)$ since f is continuous because f is a contraction.

$$\Rightarrow \lim_{k \rightarrow \infty} f^k(X) = F.$$

Since $f(X) \subset X$ because f is a contraction, therefore $\{f^k(X), k \geq 0\}$ are nested sets $\Rightarrow \lim_{k \rightarrow \infty} f^k(X) = \bigcap_{k \geq 0} f^k(X) \Rightarrow \bigcap_{k \geq 0} f^k(X) = F$. □

Figure 2 below shows the successive iterations of f defined as $f(X) = \bigcup_{i=1}^k C_i(X)$ starting with the initial compact set F_0 .

By induction, we obtain

$$f^n(F_0) = \bigcup_{i=1, j=1, \dots, l=1}^k C_i(C_j(\dots C_l(F_0)))$$

So we can define the fractal F_∞ associated to the function f and F_0 as the subset of F_0 defined by

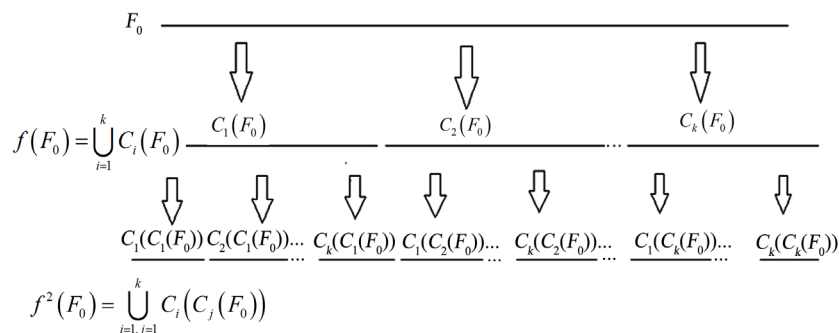


Figure 2. Successive iteration of f starting with the initial compact F_0 .

$$F_\infty = \bigcap_{n=0}^\infty f^n(F_0) = F_0 \cap f(F_0) \cap f(f(F_0)) \cap f(f(f(F_0))) \cap \dots$$

The next example is an example of fractals that can be obtained using IFS.

Example 1. (Cantor Set)

Starting with $K_0 = [0,1]$, we obtain the subsequent subsets by acting two contractions with the same scale factor of $\frac{1}{3}$: $f_1(x) = \frac{1}{3}x$ and $f_2(x) = 1 - \frac{1}{3}x$ on $K_0 = [0,1]$, then we get K_1 . After repeating this process recursively in the contracted intervals, we obtain the Cantor set K_∞ as shown **Figure 3** below.

And then we can see that: $K_\infty = \bigcup_{i=1}^2 f_i(K_\infty)$.

3. The Box-Counting Dimension

We know that from our Euclidean understanding of dimensions, a point has zero dimension, a line segment has one dimension, a square has two dimensions, and a cube has three dimensions. We also know that this understanding comes from our designation of the number of parameters we use to locate points and measure distance within given space.

Now, we use Hausdorff dimension [11] when we deal with a non-Euclidean space for instance fractals. The Hausdorff dimension is more sophisticated.

Fortunately, when the fractal arises from contractions, we have the following theorem that gives a way to estimate the Hausdorff dimension easily.

Theorem 2. Let $\{C_1, C_2, \dots, C_n\}$ be an iterated function system (IFS) with $n \geq 2$. Let r_1, r_2, \dots, r_n be respectively the scales of the contractions.

If F_∞ is the attractor of the IFS, then the Hausdorff dimension d of F_∞ satisfies the equation:

$$\sum_{i=1}^n r_i^d = 1$$

the equation: $\sum_{i=1}^n r_i^d = 1$ is known as the Moran Equation.

With the Moran equation, we can approximate the dimension of a fractal obtained using multiple contractions.

Another way of computing the dimension of a fractal is through the box counting method I describe below.

Let S be a bounded d -dimensional set. Imagine covering S by d -dimensional cubes of side length ε .

Let $N(\varepsilon)$ be the number of cubes of side length ε needed to cover S .

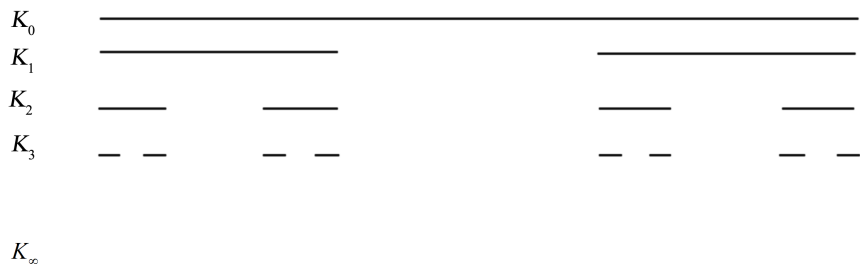


Figure 3. Cantor set.

If $d = 1$. So we can consider line segments of length ε . $N(\varepsilon) \approx \frac{C}{\varepsilon}$ where C is a constant. The exponent 1 of ε represents the dimension of the line segment.

If $d = 2$, we can consider squares of side ε then $N(\varepsilon) \approx \frac{C}{\varepsilon^2}$. The exponent 2 of ε represents the dimension of the square.

If $d = 3$, then we can consider cubes of side ε and then $N(\varepsilon) \approx \frac{C}{\varepsilon^3}$. The exponent 3 of ε represents the dimension of the cube.

We conclude that for a bounded d -dimensional set, $N(\varepsilon) \approx \frac{C}{\varepsilon^d}$. To find the general formula for the dimension of a given bounded shape whether it is an Euclidean shape or a fractal shape we can solve the equation:

$$N(\varepsilon) \approx \frac{C}{\varepsilon^d}$$

$$\ln(N(\varepsilon)) \approx \ln\left(\frac{C}{\varepsilon^d}\right)$$

$$d \approx \frac{\ln(N(\varepsilon)) - \ln(C)}{\ln\left(\frac{1}{\varepsilon}\right) - \ln\left(\frac{1}{\varepsilon}\right)}$$

When ε is very small that is when ε approaches 0 then

$$\lim_{\varepsilon \rightarrow 0^+} \left(\frac{\ln(N(\varepsilon)) - \ln(C)}{\ln\left(\frac{1}{\varepsilon}\right) - \ln\left(\frac{1}{\varepsilon}\right)} \right) = \lim_{\varepsilon \rightarrow 0^+} \frac{\ln(N(\varepsilon))}{\ln\left(\frac{1}{\varepsilon}\right)}$$

which leads to the definition of box-counting dimension.

Definition 3. The box counting dimension denoted $\text{boxdim}(S)$ of a bounded set S is given by:

$$\text{boxdim}(S) = \lim_{\varepsilon \rightarrow 0^+} \frac{\ln(N(\varepsilon))}{\ln\left(\frac{1}{\varepsilon}\right)} \tag{1}$$

We can easily show that the box-counting dimension agrees with the standard notion of dimension see [1]. Now let's provide some simplifications to this definition.

Proposition 1. Let S be a bounded set of \mathbb{R}^m .

For all $\varepsilon \in (0,1)$, $\exists b_n$ such that $b_{n+1} < \varepsilon < b_n$ and $\lim_{n \rightarrow \infty} b_n = 0$ and

$$\lim_{n \rightarrow \infty} \frac{\ln(b_{n+1})}{\ln(b_n)} = 1.$$

Proof. For any positive real number a , there exists n such that $n \leq a \leq n+1$ in

particular when $a = \frac{\ln\left(\frac{1}{\varepsilon}\right)}{\ln 2} > 0$. Therefore we conclude that

$$\ln 2^{-(n+1)} \leq \ln(\varepsilon) \leq \ln 2^{-n}$$

$$2^{-(n+1)} \leq \varepsilon \leq 2^{-n}$$

So we can choose $b_n = 2^{-n}$ then $\lim_{n \rightarrow \infty} b_n = 0$ and $\lim_{n \rightarrow \infty} \frac{\ln(b_{n+1})}{\ln(b_n)} = 1$

□

Theorem 3. Assuming that $b_1 > b_2 > b_3 > \dots > b_n > \dots$ and $\lim_{n \rightarrow \infty} b_n = 0$ and $\lim_{n \rightarrow \infty} \frac{\ln(b_{n+1})}{\ln(b_n)} = 1$. If $\lim_{n \rightarrow \infty} \frac{\ln(N(b_n))}{\ln\left(\frac{1}{b_n}\right)} = d$ then $\lim_{\varepsilon \rightarrow 0} \frac{\ln(N(\varepsilon))}{\ln\left(\frac{1}{\varepsilon}\right)} = d$

Proof. We start the proof with a bounded set S in \mathbb{R}^2 . Let $N(\varepsilon)$ be the number of boxes of side length ε needed to cover S .

Let S_1 be any subset of S being covered by a box of side length b_{n+1} .

We then can cover the subset S_1 by almost 4 boxes of side length ε . Therefore $N(\varepsilon) \leq 4N(b_{n+1})$. Since $\varepsilon \leq b_n$ then $N(b_n) \leq 4N(\varepsilon)$. This implies that

$$\frac{N(b_n)}{2^2} \leq N(\varepsilon) \leq 2^2 N(b_{n+1})$$

And thus in \mathbb{R}^m we obtain

$$\frac{N(b_n)}{2^m} \leq N(\varepsilon) \leq 2^m N(b_{n+1})$$

Since \ln is an increasing function then

$$-m \ln(2) + \ln(N(b_n)) \leq \ln(N(\varepsilon)) \leq m \ln(2) + \ln(N(b_{n+1}))$$

Now using the fact that

$$b_{n+1} \leq \varepsilon \leq b_n$$

we obtain the following inequality

$$\frac{\ln\left(\frac{1}{b_n}\right) - m \ln(2) + \ln(N(b_n))}{\ln\left(\frac{1}{b_{n+1}}\right)} \leq \frac{\ln(N(\varepsilon))}{\ln\left(\frac{1}{\varepsilon}\right)} \leq \frac{\ln\left(\frac{1}{b_{n+1}}\right) + m \ln(2) + \ln(N(b_{n+1}))}{\ln\left(\frac{1}{b_n}\right)}$$

Now using the squeeze theorem, we see that if $\lim_{n \rightarrow \infty} \frac{\ln(N(b_n))}{\ln\left(\frac{1}{b_n}\right)} = d$ then

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\ln(N(\varepsilon))}{\ln\left(\frac{1}{\varepsilon}\right)} = d \text{ therefore the boxdim} = d. \quad \square$$

There are other simplifications we can use to get the dimension of the fractals. We can replace $N(\varepsilon)$ by $N_0(\varepsilon)$ which is the smallest number of boxes needed to cover the set. Also, instead of using squares, we can use disks of radius ε or triangles of side length ε .

Example 2. Let's find the box counting dimension of the Cantor set. From the figure in example 1, the length of each interval of K_1 equals $\frac{1}{3}$ so $b_1 = \frac{1}{3}$.

The number of boxes needed to cover these intervals is 2 so $N(b_1) = 2$.

Step 2, the length of each interval of K_2 equals $\frac{1}{3^2}$ so $b_2 = \frac{1}{3^2}$. The number of boxes needed to cover these intervals is 2^2 so $N(b_2) = 2^2$.

Continuing until the n th step, we see that $b_n = \frac{1}{3^n}$ and $N(b_n) = 2^n$

$$\text{boxdim}(K) = \lim_{n \rightarrow \infty} \frac{\ln(2^n)}{\ln(3^n)} = \frac{\ln(2)}{\ln(3)}$$

4. Box-Counting Dimension for Fractals That Arise from Multitude of Scalings

4.1. The Case of a One Dimensional Set

Theorem 4. Let K_0 be a bounded set of dimension 1.

Let $C_i, 1 \leq i \leq k$ be contractions on K_0 with corresponding scale factors

$$r_i = \frac{a_i}{b_i}, 1 \leq i \leq k \text{ where } a_i \text{ and } b_i \text{ are both integers.}$$

Let r be the largest rational number such that $\frac{r_i}{r}$ is an integer for all $1 \leq i \leq k$

$$\text{that is } r = \frac{\text{gcd}(a_i)}{\text{lcm}(b_i)}, 1 \leq i \leq k.$$

If K_∞ is the fractal that results from K_0 using the contractions $C_1, C_2, C_3, \dots, C_k$ then

$$\text{boxdim}(K_\infty) = -\frac{\ln\left(\frac{r_1}{r} + \frac{r_2}{r} + \frac{r_3}{r} + \dots + \frac{r_k}{r}\right)}{\ln(r)} \tag{2}$$

Proof. Consider a set of dimension one for instance the segment $K_0 = [0,1]$. Let K_1 be the set obtained by acting k contractions with scales factors r_1, r_2, \dots, r_k on the segment $K_0 = [0,1]$. For the next step, let K_2 be the set obtained by acting the n contractions on each interval of the set K_1 . Repeating this process, we obtain an attractor denoted K_∞ as seen in **Figure 4** below. K_∞ is a fractal.

Let's find the dimension of the fractal K_∞ . Let r be the largest rational number such that $\frac{r_i}{r}$ is an integer for all $1 \leq i \leq k$ that is $r = \frac{\text{gcd}(a_i)}{\text{lcm}(b_i)}$.

Define $b_1 = r$ and let $N(b_1)$ be the number of intervals of length b_1 needed to cover K_1 so

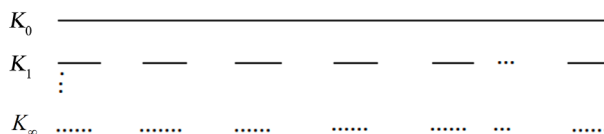


Figure 4. Fractal from one dimensional that arises from a multitude of scaling.

$$N(b_1) = \frac{r_1}{r} + \frac{r_2}{r} + \frac{r_3}{r} + \dots + \frac{r_k}{r}$$

For the second step, define $b_2 = r^2$ and let $N(b_2)$ be the number of intervals of length b_2 needed to cover K_2 . So

$$\begin{aligned} N(b_2) &= \left(\frac{r_1^2}{r^2} + \frac{r_1 r_2}{r^2} + \frac{r_1 r_3}{r^2} + \dots + \frac{r_1 r_k}{r^2} \right) + \left(\frac{r_1^2 r_2}{r^2} + \frac{r_2^2}{r^2} + \frac{r_2 r_3}{r^2} + \frac{r_2 r_4}{r^2} + \dots + \frac{r_2 r_k}{r^2} \right) \\ &\quad + \left(\frac{r_1 r_3}{r^2} + \frac{r_2 r_3}{r^2} + \frac{r_3^2}{r^2} + \frac{r_3 r_4}{r^2} + \dots + \frac{r_3 r_k}{r^2} \right) + \dots \\ &\quad + \left(\frac{r_1 r_k}{r^2} + \frac{r_2 r_k}{r^2} + \frac{r_3 r_k}{r^2} + \frac{r_4 r_k}{r^2} + \dots + \frac{r_k^2}{r^2} \right) \\ &= \frac{r_1}{r} \left(\frac{r_1}{r} + \frac{r_2}{r} + \frac{r_3}{r} + \dots + \frac{r_k}{r} \right) + \frac{r_2}{r} \left(\frac{r_1}{r} + \frac{r_2}{r} + \frac{r_3}{r} + \dots + \frac{r_k}{r} \right) \\ &\quad + \frac{r_3}{r} \left(\frac{r_1}{r} + \frac{r_2}{r} + \frac{r_3}{r} + \dots + \frac{r_k}{r} \right) + \dots + \frac{r_k}{r} \left(\frac{r_1}{r} + \frac{r_2}{r} + \frac{r_3}{r} + \dots + \frac{r_k}{r} \right) \\ N(b_2) &= \left(\frac{r_1}{r} + \frac{r_2}{r} + \frac{r_3}{r} + \dots + \frac{r_k}{r} \right)^2 \end{aligned}$$

For the third step,

$$b_3 = r^3 \text{ and } N(b_3) = \left(\frac{r_1}{r} + \frac{r_2}{r} + \frac{r_3}{r} + \dots + \frac{r_k}{r} \right)^3$$

By induction,

$$b_n = r^n \text{ and } N(b_n) = \left(\frac{r_1}{r} + \frac{r_2}{r} + \frac{r_3}{r} + \dots + \frac{r_k}{r} \right)^n$$

Now, we can evaluate the boxdim of K_∞ :

$$\begin{aligned} \text{boxdim}(K_\infty) &= \lim_{n \rightarrow \infty} \frac{\ln(N(b_n))}{\ln\left(\frac{1}{b_n}\right)} \\ \text{boxdim}(K_\infty) &= - \frac{\ln\left(\frac{r_1}{r} + \frac{r_2}{r} + \frac{r_3}{r} + \dots + \frac{r_k}{r}\right)}{\ln(r)} \end{aligned}$$

Remark 2. This formula could be also seen as the generalization of the Cantor set dimension in the case we apply a multitude of scales on the unit interval.

Example 3. If $r_1 = \frac{2}{5}, r_2 = \frac{1}{3}, r_3 = \frac{1}{6}$ then $r = \frac{\text{GCD}(2,1,1)}{\text{LCM}(5,3,6)} = \frac{1}{30}$ therefore

$$\text{boxdim}(K_\infty) = - \frac{\ln(12+10+5)}{\ln\left(\frac{1}{30}\right)} = \frac{\ln(27)}{\ln(30)} \approx 0.97 .$$

Remark 3. Using the Moran equation $\left(\frac{2}{5}\right)^d + \left(\frac{1}{3}\right)^d + \left(\frac{1}{6}\right)^d = 1$, we would find the Hausdorff dimension $d \approx 0.92$ which is very close to the boxplot dimension.

4.2. The Case of a Bounded Set of Any Arbitrary Dimension

Theorem 5. Let K_0 be a bounded set of dimension m .

Let $C_1, C_2, C_3, \dots, C_k$ be contractions on K_0 with corresponding scale factors $r_i = \frac{a_i}{b_i}, 1 \leq i \leq k$ where a_i and b_i are both integers.

Let r be the largest rational number such that $\frac{r_i}{r}$ is an integer for all $1 \leq i \leq k$ that is $r = \frac{\text{gcd}(a_i)}{\text{lcm}(b_i)}, 1 \leq i \leq k$.

If K_∞ is the fractal that results from K_0 using the contractions $C_1, C_2, C_3, \dots, C_k$ then

$$\text{boxdim}(K_\infty) = -\frac{\ln\left(\left(\frac{r_1}{r}\right)^m + \left(\frac{r_2}{r}\right)^m + \left(\frac{r_3}{r}\right)^m + \dots + \left(\frac{r_k}{r}\right)^m\right)}{\ln(r)} \tag{3}$$

Proof. Starting with a bounded set K_0 in \mathbb{R}^2 .

Let K_1 be the set obtained after acting a set of contractions C_1, C_2, \dots, C_k with scales factors $r_1, r_2, r_3, \dots, r_k$ on the set K_0 . Repeating the same process indefinitely, we obtain the limiting set denoted K_∞ as shown in **Figure 5** (using $r_1 = \frac{1}{3}, r_2 = \frac{1}{4}, r_3 = \frac{1}{5}, r_4 = \frac{1}{6}, r_5 = \frac{1}{7}$).

K_∞ is a fractal.

To find the dimension of K_∞ , we consider B the smallest box of side length l to cover K_0 . Let r be the largest rational number such that $\frac{r_i}{r}$ is an integer for all $1 \leq i \leq k$.

So for the first step of the fractal construction, if $b_1 = rl$, then the number of boxes needed to cover K_1 is given by

$$N(b_1) = \frac{(r_1 l)^2}{(rl)^2} + \frac{(r_2 l)^2}{(rl)^2} + \frac{(r_3 l)^2}{(rl)^2} + \frac{(r_4 l)^2}{(rl)^2} + \dots + \frac{(r_k l)^2}{(rl)^2}$$

$$N(b_1) = \frac{(r_1)^2}{r^2} + \frac{(r_2)^2}{r^2} + \frac{(r_3)^2}{r^2} + \frac{(r_4)^2}{r^2} + \dots + \frac{(r_k)^2}{r^2}$$

For the second step of the fractal construction, if $b_2 = r^2 l$, then

$$N(b_2) = \left(\frac{(r_1^2 l)^2}{(r^2 l)^2} + \frac{(r_1 r_2 l)^2}{(r^2 l)^2} + \frac{(r_1 r_3 l)^2}{(r^2 l)^2} + \frac{(r_1 r_4 l)^2}{(r^2 l)^2} + \dots + \frac{(r_1 r_k l)^2}{(r^2 l)^2} \right)$$

$$+ \left(\frac{(r_1 r_2 l)^2}{(r^2 l)^2} + \frac{(r_2^2 l)^2}{(r^2 l)^2} + \frac{(r_2 r_3 l)^2}{(r^2 l)^2} + \dots + \frac{(r_2 r_k l)^2}{(r^2 l)^2} \right) + \dots$$

$$+ \left(\frac{(r_1 r_k l)^2}{(r^2 l)^2} + \frac{(r_2 r_k l)^2}{(r^2 l)^2} + \frac{(r_3 r_k l)^2}{(r^2 l)^2} + \dots + \frac{(r_k l)^2}{(r^2 l)^2} \right)$$

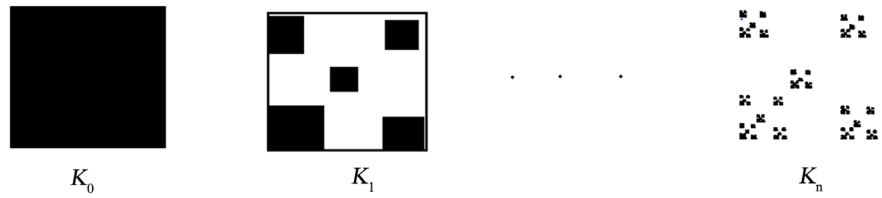


Figure 5. Fractal from two dimension that arises from multiple scaling.

$$N(b_2) = \frac{r_1^2}{r^2} \left(\frac{r_1^2}{r^2} + \frac{r_2^2}{r^2} + \frac{r_3^2}{r^2} + \dots + \frac{r_k^2}{r^2} \right) + \frac{r_2^2}{r^2} \left(\frac{r_1^2}{r^2} + \frac{r_2^2}{r^2} + \frac{r_3^2}{r^2} + \dots + \frac{r_k^2}{r^2} \right) + \dots + \frac{r_k^2}{r^2} \left(\frac{r_1^2}{r^2} + \frac{r_2^2}{r^2} + \frac{r_3^2}{r^2} + \dots + \frac{r_k^2}{r^2} \right)$$

Therefore

$$N(b_2) = \left(\frac{r_1^2}{r^2} + \frac{r_2^2}{r^2} + \frac{r_3^2}{r^2} + \dots + \frac{r_k^2}{r^2} \right)^2$$

By induction

$$b_n = r^n l, \quad N(b_n) = \left(\frac{r_1^2}{r^2} + \frac{r_2^2}{r^2} + \frac{r_3^2}{r^2} + \dots + \frac{r_k^2}{r^2} \right)^n$$

The boxdim of F_∞ is given by

$$\text{boxdim}(K_\infty) = - \frac{\ln \left(\frac{r_1^2}{r^2} + \frac{r_2^2}{r^2} + \frac{r_3^2}{r^2} + \dots + \frac{r_k^2}{r^2} \right)}{\ln(r)}$$

We conclude that in general that if K_0 is a bounded subset of \mathbb{R}^m then the dimension of the fractal K_∞ is given by

$$\text{boxdim}(K_\infty) = - \frac{\ln \left(\frac{r_1^m}{r^m} + \frac{r_2^m}{r^m} + \frac{r_3^m}{r^m} + \dots + \frac{r_k^m}{r^m} \right)}{\ln(r)}$$

Corollary 1. If $r_1 = r_2 = \dots = r_k$ then the box counting dimension and the Hausdorff dimension of F_∞ are both the same.

Proof. If $r_1 = r_2 = \dots = r_k$ then The boxdim of F_∞ is $-\frac{\ln(k)}{\ln(r_1)}$.

The Hausdorff dimension d is such that $\sum_{i=1}^k r_i^d = 1$. Since $r_1 = r_2 = \dots = r_k$ then $d = -\frac{\ln(k)}{\ln(r_1)}$

Corollary 2. If $\sum_{m=1}^k \frac{1}{\left(\sum_{i=1}^k \frac{r_i}{r} \right)^{\ln(r_m)}} = 1$ where r be the largest rational number such that $\frac{r_i}{r}$ is an integer for all $1 \leq i \leq k$, then the boxdim and the Hausdorff dimension of K_∞ agree.

The proof is straightforward.

Example 4. Consider the special case with 2 scale factors r_1 and r_2 and $r_2 = r_1^2$. The sum in the corollary 2 becomes

$$\frac{1}{1 + \frac{r_2}{r_1}} + \frac{1}{\left(1 + \frac{r_2}{r_1}\right)^{\frac{\ln(r_2)}{\ln(r_1)}}} = 1$$

Since $r_2 = r_1^2$ then $r_1 = \frac{-1 + \sqrt{5}}{2}$ and therefore $\frac{r_1}{r_2} = \frac{1}{r_1} = \frac{1 + \sqrt{5}}{2}$ which is the golden ratio.

5. Convoluting Fractals and Dimension

Definition 4. (Convolution) Let G_∞ and H_∞ be two fractals that represent the limiting sets of a given bounded set S_0 of \mathbb{R}^2 with scale factors $r_1, r_2, r_3, \dots, r_p$ and $s_1, s_2, s_3, \dots, s_q$ respectively. If $g(X) = \bigcup_{1 \leq i \leq p} r_i(X)$ and $h(X) = \bigcup_{1 \leq i \leq q} s_i(X)$ where $X \in \mathcal{C}$, then the convolution of G_∞ and H_∞ denoted $G_\infty \star H_\infty$ is given by

$$G_\infty \star H_\infty = \bigcap_{i=0}^{\infty} (h \circ g)^i(S_0) = S_0 \cap (h(g(S_0))) \cap (h(g(h(g(S_0)))) \cap \dots$$

In short, the fractal $G_\infty \star H_\infty$ is constructed by acting alternatively the functions g on S_0 and h on $g(S_0)$ Ad infinitum.

Example 5. We suppose that g is the function whose action consists of constructing a $\frac{1}{3}$ -copy of the circle whose interior is white and a $\frac{1}{2}$ -copy of the same circle. The action of the second function consists of constructing two $\frac{1}{2}$ -copies of the circle and two $\frac{1}{4}$ -copies of the same circle. The first stage and second stage of the convoluted fractal are obtained as shown in **Figure 6** below.

Next, we will explore the dimension of the fractal $G_\infty \star H_\infty$ and see how it is related to the dimensions of G_∞ and H_∞ . We will show in the next theorem that it is simply a weighted average of the two fractal dimensions.

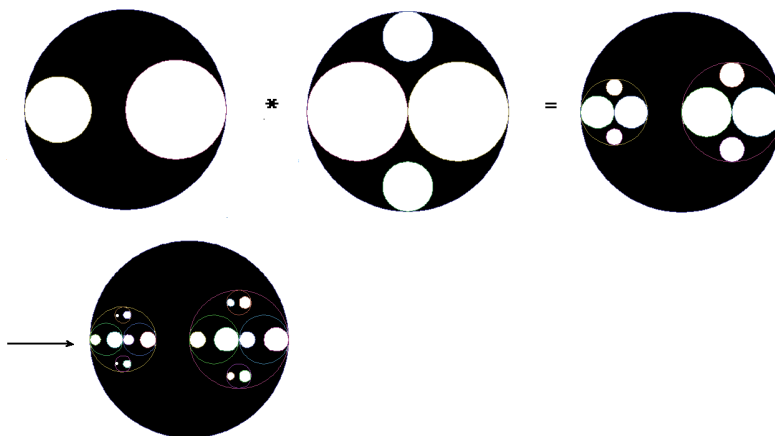


Figure 6. First two stages of the convoluted fractal.

Theorem 6. If d_{G_∞} , d_{H_∞} and $d_{G_\infty * H_\infty}$ represent the box-counting dimension of G_∞ , H_∞ and $G_\infty * H_\infty$ respectively with r be the largest rational number such that $\frac{r_i}{r}$ is an integer for all $1 \leq i \leq k$ and s be the largest rational number such that $\frac{s_i}{s}$ is an integer for all $1 \leq i \leq k$, then $d_{G_\infty * H_\infty}$ is given by the weighted average of d_{G_∞} and d_{H_∞} that is

$$d_{G_\infty * H_\infty} = \frac{d_{G_\infty}}{1 + \frac{\ln(s)}{\ln(r)}} + \frac{d_{H_\infty}}{1 + \frac{\ln(r)}{\ln(s)}} \tag{4}$$

and if $d_{G_\infty} \leq d_{H_\infty}$ then $d_{G_\infty} \leq d_{G_\infty * H_\infty} \leq d_{H_\infty}$.

Proof. It is easy to see that if C and T are both contractions defined respectively on the sets S_0 and $C(S_0)$ with scale factors respectively r and s then the composite $T \circ C$ is also a contraction with scale factor rs . Let

$$g(X) = \bigcup_{j=1}^q T_j(X) \text{ and } h(X) = \bigcup_{i=1}^p C_i(X).$$

Since $g \circ h(S_0) = \bigcup_{i=1}^p \bigcup_{j=1}^q T_j(C_i(S_0))$ then $g \circ h$ is a function with pq associated contractions $T_j \circ C_i, 1 \leq i \leq p, 1 \leq j \leq q$ whose scale factors are respectively $r_i s_j, 1 \leq i \leq p, 1 \leq j \leq q$. then according to the theorem 4.

$$d_{G_\infty * H_\infty} = \frac{\ln \left(\left(\frac{r_1 s_1}{rs} \right)^m + \left(\frac{r_1 s_2}{rs} \right)^m + \left(\frac{r_1 s_3}{rs} \right)^m + \dots + \left(\frac{r_1 s_q}{rs} \right)^m + \left(\frac{r_2 s_1}{rs} \right)^m + \left(\frac{r_2 s_2}{rs} \right)^m + \dots + \left(\frac{r_2 s_q}{rs} \right)^m + \dots \right)}{\ln(rs)} + \frac{\ln \left(\left(\frac{r_p s_1}{rs} \right)^m + \left(\frac{r_p s_2}{rs} \right)^m + \dots + \left(\frac{r_p s_q}{rs} \right)^m \right)}{\ln(rs)}$$

then

$$d_{G_\infty * H_\infty} = \frac{\ln \left(\left(\frac{r_1}{r} \right)^m + \left(\frac{r_2}{r} \right)^m + \left(\frac{r_3}{r} \right)^m + \dots + \left(\frac{r_p}{r} \right)^m \right) + \ln \left(\left(\frac{s_1}{s} \right)^m + \left(\frac{s_2}{s} \right)^m + \left(\frac{s_3}{s} \right)^m + \dots + \left(\frac{s_q}{s} \right)^m \right)}{\ln(s) + \ln(r)}$$

$$d_{G_\infty * H_\infty} = \frac{d_{G_\infty}}{1 + \frac{\ln(s)}{\ln(r)}} + \frac{d_{H_\infty}}{1 + \frac{\ln(r)}{\ln(s)}}$$

Since $d_{G_\infty} \leq d_{H_\infty}$ then

$$\frac{d_{G_\infty}}{1 + \frac{\ln(s)}{\ln(r)}} + \frac{d_{G_\infty}}{1 + \frac{\ln(r)}{\ln(s)}} \leq d_{G_\infty * H_\infty} \leq \frac{d_{H_\infty}}{1 + \frac{\ln(s)}{\ln(r)}} + \frac{d_{H_\infty}}{1 + \frac{\ln(r)}{\ln(s)}}$$

Therefore $d_{G_\infty} \leq d_{G_\infty * H_\infty} \leq d_{H_\infty}$.

Example 6. From the previous picture,

$r_1 = \frac{1}{3}, r_2 = \frac{1}{2}, s_1 = \frac{1}{4}, s_2 = \frac{1}{4}, s_3 = \frac{1}{2}, s_4 = \frac{1}{2}$ then $r = \frac{1}{6}$ and $s = \frac{1}{4}$ therefore ac-

According to Equation (4) $d_{G_\infty} = \frac{\ln(13)}{\ln 6} \approx 1.43$, $d_{H_\infty} = \frac{\ln(10)}{\ln 4} \approx 1.66$, using Equation (6), we calculate the box-counting dimension of the convoluted fractal and find: $d_{G_\infty * H_\infty} = 1.53$.

All the corollaries from [7] follow easily.

Corollary 3. If $d_{G_\infty} = d_{H_\infty}$ then $d_{G_\infty} = d_{G_\infty * H_\infty} = d_{H_\infty}$

Corollary 4. $d_{G_\infty * H_\infty} = d_{H_\infty * G_\infty}$

Corollary 5. If $r_1 = r_2 = r_3 = \dots = r_p = r$ and $s_1 = s_2 = s_3 = \dots = s_q = s$ then

$$d_{G_\infty * H_\infty} = -\frac{\ln(pq)}{\ln(rs)}$$

6. Conclusion

In this paper, the box-counting dimension is used to compute an explicit formula for the dimension of fractals that arise from multiple contractions. This formula allows a very convenient way to compute the dimension of fractals obtained by using the convolution of two fractals. The dimension of the fractal obtained also turns to be explicit and very easy to manipulate in order to prove the properties of the dimension of the convoluted fractal. I believe that much more properties related to the dimension of a fractal can be proved easily using the box-counting dimension. Also, in this paper, two fractals are combined using the composition of functions since the composite of contractions is always a contraction. The multiplication however is not necessary a contraction. It could be interesting in the case we deal with a contraction to study the properties of the dimension of the fractal obtained by combining two fractals using the multiplication instead of the composition. The box-counting dimension might be an appropriate tool to derive any property of the dimension of the fractal obtained.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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