Applied Mathematics, 2021, 12, 131-146
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Uniqueness of Positive Radial Solutions for a Class of Semipositone Systems on the Exterior of a Ball

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How to cite this paper: Mohamed, A., Abbakar, K.A., Awad, A., Khalil, O., Acyl, B.M., Youssouf, A.A. and Mousa, M. (2021) Uniqueness of Positive Radial Solutions for a Class of Semipositone Systems on the Exterior of a Ball. *Applied Mathematics*, **12**, 131-146.
https://doi.org/10.4236/am.2021.123009

Received: November 16, 2020
Accepted: March 9, 2021
Published: March 12, 2021

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Abstract

In this paper, we study the positive radial solutions for elliptic systems to the nonlinear BVP: $\begin{cases} -\Delta u = \lambda k_1(|x|) f_1(u,v) & \text{on } \Omega, \\ -\Delta v = \lambda k_2(|x|) f_2(u,v) & \text{on } \Omega, \\ u(x) = v(x) = 0 & \text{on } |x| \to \infty, \\ \frac{\partial u}{\partial \eta} + \tilde{c}_1(u)u = 0 & \text{on } |x| = r_0, \\ \frac{\partial v}{\partial \eta} + \tilde{c}_2(v)v = 0, & \text{on } |x| = r_0. \end{cases}$, where $\Delta u = div(\nabla u)$ and $\Delta v = div(\nabla v)$ are the Laplacian of *u*, λ is a positive parameter, $\Omega = \{x \in \mathbb{R}^n : N > 2, |x| > r_0, r_0 > 0\}$, let $i = [1,2]$ then $K_i : [r_0, \infty) \to (0, \infty)$ is a continuous function such that $\lim_{r\to\infty} k_i(r) = 0$ and $\frac{\partial}{\partial \eta}$ is The external natural derivative, and $\tilde{c}_i : [0,\infty) \to (0,\infty)$ is a continuous function. We discuss existence and multiplicity results for classes of *f* with a) $f_i > 0$, b) $f_i < 0$, and c) $f_i = 0$. We base our presence and multiple outcomes via the Sub-solutions method. We also discuss some unique findings.

Keywords

Elliptic System, Positive Radial Solution, Exterior Domains, Fixed Point Index

1. Introduction

In reaction diffusion processes, steady states define the long term dynamics.

Here we consider a steady state reaction diffusion equation on an exterior domain with a nonlinear boundary condition on the interior boundary. Namely, we study positive radial solutions to:

$$\begin{cases} -\Delta u = \lambda k_1(|x|) f_1(u,v) \quad \text{on } \Omega, \\ -\Delta v = \lambda k_2(|x|) f_2(u,v) \quad \text{on } \Omega, \\ u(x) = v(x) = 0 \quad \text{on } |x| \to \infty, \\ \dfrac{\partial u}{\partial \eta} + \tilde{c}_1(u)u = 0 \quad \text{on } |x| = r_0, \\ \dfrac{\partial v}{\partial \eta} + \tilde{c}_2(v)v = 0, \quad \text{on } |x| = r_0. \end{cases} \tag{1.1}$$

where $\Delta u = div(\nabla u)$ and $\Delta v = div(\nabla v)$ are the Laplacian of u, λ is a positive parameter, $\Omega = \{x \in \mathbb{R}^n : N > 2, |x| > r_0, r_0 > 0\}$, let $i = [1,2]$ then $K_i : [r_0, \infty) \to (0, \infty)$ is a continuous function such that $\lim_{r\to\infty} k_i(r) = 0$ and $\frac{\partial}{\partial \eta}$ is the outward normal derivative, and $\tilde{c}_i : [0,\infty) \to (0,\infty)$ is a is an non decreasing (increasing) function. Here the reaction term $f_i : [0,\infty) \times [0,\infty) \to R$ is a C^1 function. The case when $f_i < 0$ (see [1] [2], that the study of positive solutions to such problems is considerably more challenging than in the case $f_i > 0$ (positone problems). For a rich history on semipositone problems with Dirichlet boundary conditions on bounded domains, (see [3]-[8], and on domains exterior to a ball, see [9] [10] [11]. Such nonlinear boundary conditions occur very naturally in applications see [12] for a detailed description in a model arising in combustion theory. Recently, the existence of a radial positive solution for (1.1) when $\lambda \gg 1$ has been established in [13], via the method of subsuper solutions. Here we discuss the uniqueness of this radial solution when some additional assumptions hold. In [14], the authors study such a uniqueness result for the case of Dirichlet boundary condition on $|x| = r_0$. Our focus in this paper is to consider the uniqueness result for semipositone problem when a class of nonlinear boundary condition is satisfied at $|x| = r_0$. The fact that we have no longer a fixed value of u on $|x| = r_0$ results in quite a challenge in extending the results in [15].

Namely, we need to establish a detailed behavior of u at $|x| = r_0$ to achieve our goal. Instead of working directly with (1), we note that the change of variables $r = |x|$ and $s = \left(\frac{r}{r_0}\right)^{2-N}$ transforms (1) into the following boundary value problem:

$$\begin{cases} -u''(t) = \lambda \tilde{a}_1(t) f_1(u(t), v(t)) \quad t \in [0,1], \\ -v''(t) = \lambda \tilde{a}_2(t) f_2(u(t), v(t)) \quad t \in [0,1], \\ \dfrac{N-2}{r_0} u' + \tilde{c}_1(u(1)) u(1) = 0, \\ \dfrac{N-2}{r_0} v' + \tilde{c}_2(v(1)) v(1) = 0, \\ u(0) = v(0) = 0. \end{cases} \tag{1.2}$$

where $\tilde{a}_i = \frac{r_0^2}{(2-N)^2} t^{\frac{-2(N-1)}{N-2}} k_i\left(r_0 t^{\frac{1}{2-N}}\right)$. We will only assume $k_i \le \frac{1}{r^{N+\mu}}$ for $r \gg 1$ and for some $\mu \in (0, N-2)$. then $\tilde{a}_i \in ((0,1],(0,\infty))$ could be singular at 0.if $\mu \ge N-2$, $\tilde{a}_i$ will be nonsingular at 0 and it will be an easier case to study. Note that $\tilde{a}_i = \inf_{t\in(0,1]} \tilde{a}_i(t) > 0$ and there exists a constant $\tilde{d} > 0$ such that $\tilde{a}_i \le \frac{\tilde{d}}{t^{\alpha}}$ for all $t \in (0,1]$ where $\tilde{\alpha} = \frac{(N-2)-\mu}{N-2}$. Motivated by the above discussion, in this paper, we will study positive solutions in $C^2(0,1) \cap C^1[0,1]$ to the following boundary value problems:

$$\begin{cases} -u''(t) = \lambda a_1(t) f_1(u(t), v(t)) & t \in [0,1], \\ -v''(t) = \lambda a_2(t) f_2(u(t), v(t)) & t \in [0,1], \\ u'(1) + c_1(u(1))u(1) = 0, \\ v'(1) + c_2(v(1))v(1) = 0, \\ u(0) = v(0) = 0. \end{cases} \tag{1.3}$$

where $c_i : [0,\infty) \to (0,\infty)$ is a continuous function and $a_i \in C((0,1],(0,\infty))$ is such that:

(H1) $a_i = \inf_{t\in(0,1]} a_i(t) > 0$;

(H2) there exists a constant $d > 0$ such that $a_i(t) = \frac{d}{t^{\alpha}}$ for all $t \in (0,\varepsilon]$ where $a \in (0,1)$ and $\varepsilon \approx 0$

(H3) a_i is decreasing. We consider various C^1 classes of the reaction term $f_i : [0,\infty) \times [0,\infty) \to R$ satisfying the following:

(F1) $f_i < 0$ and $\lim_{s\to\infty} \frac{f_i(s)}{s} = 0$; $i = 1,2$

(F2) f_i is increasing and $\lim_{s\to\infty} f_i(s) = \infty; i = 1,2$

(F3) f_i is concave on $[0,\infty)$. $i = 1,2$

Theorem 1.1. Assume (H1) - (H3) and (F1) - (F3). Then (1.3) has a unique positive solution for all λ sufficiently large.

In Section two we will establish important a priori estimates. We will first recall some important results from [8] where the authors studied the case of Dirichlet boundary condition, or equivalently (1.3) with the boundary condition $t = 1$ replaced by $u(1) = v(1) = 0$. These results do not depend on the boundary condition at $t = 1$ and hence it is also true for solutions of (1.3). In view of the readers convenience we include the proofs of these results. In Section three, we prove Theorem 1.1.

2. Advance Estimates

Let $F(s) = \int_0^s f_i(t)\mathrm{d}s$. Note that there exist unique positive numbers β, θ such that $f_i(\beta) = 0$ and $F(\theta) = 0$ and $\beta < \theta$

Theorem 2.1. (See [8].) Let u, v are a positive solution of (1.3). Then u and v has only one interior maximum in $(0,1)$, say at t_m, depending on λ, and $u(t_m) > \theta$, $v(t_m) > \theta$.

Proof. Let

$$\begin{cases} E_1(t) = \lambda F(u(t))a_1(t) + \dfrac{|u'(t)|^2}{2}, t \in (0,1), \\ E_2(t) = \lambda F(v(t))a_2(t) + \dfrac{|v'(t)|^2}{2}, t \in (0,1) \end{cases} \tag{1.4}$$

then

$$\begin{cases} E_1'(t) = \lambda F(u(t))a_1'(t), t \in (0,1), \\ E_2'(t) = \lambda F(v(t))a_2'(t), t \in (0,1) \end{cases} \tag{1.5}$$

Note that by (H3), $a_1'(t) < 0$ and $a_2'(t) < 0$ for all $t \in (0,1]$. Hence, $E_1(t)$ and $E_2(t)$ are increases when $u(t) < \theta$, $v(t) < \theta$ and decreases when $u(t) > \theta$, $v(t) > \theta$.

Let $t_m \in (0,1)$ be the first point at which u has a local maximum and assume that $u(t) \le \theta$ and $v(t) \le \theta$ for all $t \in [0,t_m]$. Then $E_1(t)$ and $E_2(t)$ are increases in $[0,t_m]$. Now integrating (1.3) from t to t_m, for $t < t_m$

$$\begin{cases} u'(t) = \int_t^{t_m} \lambda a_1(s) f_1(u(s))\mathrm{d}s \le \lambda f_1(\theta) \int_t^{t_m} \dfrac{c_1}{s^\alpha}\mathrm{d}s \le \lambda \dfrac{c_1 f_1(\theta)}{1-\alpha} \\ v'(t) = \int_t^{t_m} \lambda a_2(s) f_2(v(s))\mathrm{d}s \le \lambda f_2(\theta) \int_t^{t_m} \dfrac{c_2}{s^\alpha}\mathrm{d}s \le \lambda \dfrac{c_2 f_2(\theta)}{1-\alpha} \end{cases} \tag{1.6}$$

where $c_i > d$ are such that $a_i(t) \le \dfrac{c_i}{t^\alpha}$ for all $t \in (0,1)$ using (H2). Integrating again (1.6) from 0 to t, $t \le t_m$

$$\begin{cases} u(t) \le \int_0^t \lambda \dfrac{c_1 f_1(\theta)}{1-\alpha}\mathrm{d}s, \quad C_0 = \dfrac{c_1 f_1(\theta)}{1-\alpha} \\ V(t) \le \int_0^t \lambda \dfrac{c_2 f_2(\theta)}{1-\alpha}\mathrm{d}s, \quad C_0 = \dfrac{c_2 f_2(\theta)}{1-\alpha} \end{cases} \tag{1.7}$$

Since f_i are continuous, there exists $K_0 > 0$ such that $|F(u)| \le K_0 u$ and $|F(v)| \le K_0 v$ for all $(u,v) \in [0,\theta]$. Hence

$$\lim_{t\to 0^+} \lambda |F(u(t))| a_1(t) \le \lim_{t\to 0^+} \lambda K_0 u(t) a_1(t) \le \lim_{t\to 0^+} \lambda^2 K_0 C_0 c_1 d t^{1-\alpha} = 0$$

$$\lim_{t\to 0^+} \lambda |F(v(t))| a_2(t) \le \lim_{t\to 0^+} \lambda K_0 v(t) a_2(t) \le \lim_{t\to 0^+} \lambda^2 K_0 C_0 c_2 d t^{1-\alpha} = 0$$

Hence $\lim_{t\to 0^+} E_i(t) \ge 0$. Since $E_i(t)$ increases on $[0,t_m]$, $E_1(t_m) = \lambda F(u(t_m))a_1(t_m) > 0$ and $E_2(t_m) = \lambda F(v(t_m))a_2(t_m) > 0$ and which is a contra-diction if $u(t_m) \le \theta$ and $v(t_m) \le \theta$. Suppose that u and v has two interior maxima. Then there exists $\tilde{t} \in (\tilde{t},1)$ such that $u'(\tilde{t}) = 0$, $v'(\tilde{t}) = 0$ and $u''(\tilde{t}) \ge 0$, $v''(\tilde{t}) \ge 0$. Since $u''(\tilde{t}) = \lambda a_1(\tilde{t}) f_1(u(\tilde{t})) \ge 0$, and $v''(\tilde{t}) = \lambda a_2(\tilde{t}) f_2(v(\tilde{t})) \ge 0$ we have $f_1(u(\tilde{t})) \le 0$ and $f_2(v(\tilde{t})) \le 0$ which implies that $u(\tilde{t}) \le \beta$ and $u(\tilde{t}) \le \beta$. Thus $E_i(\tilde{t}) < 0$. Let $t_\theta(t_m,\tilde{t})$ such that $u(t_\theta) = \theta$ and $v(t_\theta) = \theta$. Then $E_1(t_\theta) = \dfrac{|u'(t_\theta)|^2}{2} \ge 0$, $E_2(t_\theta) = \dfrac{|v'(t_\theta)|^2}{2} \ge 0$ and E_i increases in $(t_\theta,\tilde{t})$ since $u(t) < \theta$ and since $v(t) < \theta$ in $(t_\theta,\tilde{t})$. Hence $E_i(\tilde{t}) > 0$, which is a contradiction. Therefore, we have only one interior maximum and that maximum value is larger than θ.

Theorem 2.2. (See [8].) Let u and v be a positive solution of (1.3) and let $t_\beta \in (0, t_m)$ such that $u(t_\beta) = \beta$ and $v(t_\beta) = \beta$. Then $t_\beta \le \bigcirc\left(\lambda^{-\frac{1}{2}}\right)$ as $\lambda \to \infty$.

Proof. Let $t_{\frac{\beta}{2}} \in (0, t_\beta)$ be the point such that $u\left(t_{\frac{\beta}{2}}\right) = \frac{\beta}{2}$ and $v\left(t_{\frac{\beta}{2}}\right) = \frac{\beta}{2}$ from Integrating (1.3) from 0 to t for some $t < t_{\frac{\beta}{2}}$

$$u'(t) = u'(0) - \lambda \int_0^t a_1(s) f_1(u(s)) \mathrm{d}s \ge \lambda \underline{a_1}\left(-f_1\left(\frac{\beta}{2}\right)\right) t$$

$$v'(t) = v'(0) - \lambda \int_0^t a_2(s) f_2(v(s)) \mathrm{d}s \ge \lambda \underline{a_2}\left(-f_2\left(\frac{\beta}{2}\right)\right) t,$$

Integrating the above again from 0 to t_β

$$t_{\frac{\beta}{2}} \le \tilde{c}_1 \lambda^{-\frac{1}{2}},\ t_{\frac{\beta}{2}} \le \tilde{c}_2 \lambda^{-\frac{1}{2}} \quad \text{where} \begin{cases} \tilde{c}_1 = \left(\dfrac{-\beta}{\underline{a_1}\left(-f_1\left(\frac{\beta}{2}\right)\right)^{\frac{1}{2}}}\right) > 0 \\ \tilde{c}_2 = \left(\dfrac{-\beta}{\underline{a_2}\left(-f_2\left(\frac{\beta}{2}\right)\right)^{\frac{1}{2}}}\right) > 0 \end{cases} \tag{1.8}$$

By the mean value theorem, there exists a $\tilde{t} \in \left[0, t_{\frac{\beta}{2}}\right]$ such that $u\left(t_{\frac{\beta}{2}}\right) - u(0) = u'(\tilde{t}) t_{\frac{\beta}{2}}$, $v\left(t_{\frac{\beta}{2}}\right) - v(0) = v'(\tilde{t}) t_{\frac{\beta}{2}}$ and by (2.5) $u'(\tilde{t}) \ge \frac{\beta}{2\tilde{c}_1} \lambda^{\frac{1}{2}}$ and $v'(\tilde{t}) \ge \frac{\beta}{2\tilde{c}_2} \lambda^{\frac{1}{2}}$. Since u' and v' are increases in $[0, t_\beta]$,

$$\begin{cases} u'(t) \ge \dfrac{\beta}{2\tilde{c}_1} \lambda^{\frac{1}{2}}, & t \in \left[t_{\frac{\beta}{2}}, t_\beta\right] \\ v'(t) \ge \dfrac{\beta}{2\tilde{c}_2} \lambda^{\frac{1}{2}}, & t \in \left[t_{\frac{\beta}{2}}, t_\beta\right] \end{cases} \tag{1.9}$$

Integrating (2.6) from $t_{\frac{\beta}{2}}$ to t_β, we have that $\left(t_\beta - t_{\frac{\beta}{2}}\right) \le \tilde{c}_1 \lambda^{-\frac{1}{2}}$ and $\left(t_\beta - t_{\frac{\beta}{2}}\right) \le \tilde{c}_1 \lambda^{-\frac{1}{2}}$. This implies $t_\beta \le \bigcirc\left(\lambda^{-\frac{1}{2}}\right)$ by (2.5). □

Lemma 2.3. Let u and v be a positive solution of (1.3). Then $u(1) \to \infty$ and $v(1) \to \infty$ as $\lambda \to \infty$

Proof. We first claim that $u(1) \ge \frac{\beta + \theta}{2}$ and $v(1) \ge \frac{\beta + \theta}{2}$ for $\lambda \gg 1$. Assume that $u(1)<\frac{\beta+\theta}{2}$ and $v(1)<\frac{\beta+\theta}{2}$ Then there exists a $\tilde{t}_\theta\in(t_m,1)$ such that $u(\tilde{t}_\theta)=\theta$ and $v(\tilde{t}_\theta)=\theta$ where t_m is the point at which u,v achieves are maximum, and $u(t_m)>\theta$, $v(t_m)>\theta$ by Lemma 2.1.

From (2.1) and (2.2), $E_1(\tilde{t}_\theta)=\frac{|u'(\tilde{t}_\theta)|}{2}>0$, $E_2(\tilde{t}_\theta)=\frac{|v'(\tilde{t}_\theta)|}{2}>0$ and $E_i(t)\ge 0$ on $[\tilde{t}_\theta,1]$ since $u(t)\le\theta$ and $v(t)\le\theta$ in $[\tilde{t}_\theta,1]$. Hence we obtain that

$$E_1(1)=\lambda F(u(1))a_1(1)+\frac{|u'(1)|}{2}>0,$$

$$E_2(1)=\lambda F(v(1))a_2(1)+\frac{|v'(1)|}{2}>0$$

and from (1.3), we have

$$\begin{cases} c_1(u(1))u(1)=-u'(1)\ge\sqrt{-2\lambda F(u(1))a_1(1)}, \\ c_2(v(1))v(1)=-v'(1)\ge\sqrt{-2\lambda F(v(1))a_2(1)}. \end{cases} \tag{1.10}$$

This cannot hold unless $u(1)\to 0$ and $v(1)\to 0$ as $\lambda\to\infty$. However, rewriting (1.10) as

$$\begin{cases} c_1(u(1))u(1)^{\frac{1}{2}}\ge\sqrt{-2\lambda\frac{F(u(1))}{u(1)}a_1(1)}, \\ c_2(v(1))v(1)^{\frac{1}{2}}\ge\sqrt{-2\lambda\frac{F(v(1))}{v(1)}a_2(1)}. \end{cases} \tag{1.11}$$

we obtain a contradiction when $\lambda\gg 1$ since $\frac{F(u(1))}{u(1)}\to f_1(0)$, $\frac{F(v(1))}{v(1)}\to f_2(0)$ if $u(1)\to 0$ and $v(1)\to 0$ as $\lambda\to\infty$. Hence,

$$u(1)\ge\frac{\beta+\theta}{2} \text{ and } v(1)\ge\frac{\beta+\theta}{2} \quad \text{for } \lambda\gg 1 \tag{1.12}$$

Next, we claim that $u(t_m)=\|u\|_\infty\to\infty$ and $v(t_m)=\|v\|_\infty\to\infty$ as $\lambda\to\infty$. Let

$h:=u-\beta$ and $w:=v-\beta$ then $h>0$, $w>0$ in $(t_\beta,1]$ and satisfies

$$\begin{cases} -h''=\lambda a_1(t)\frac{f_1(u)}{u-\beta}h & (t_\beta,1) \\ -w''=\lambda a_2(t)\frac{f_2(v)}{v-\beta}w & (t_\beta,1) \\ h(t_\beta)=w(t_\beta)=0, \\ h(1)=u(1)-\beta>0 \\ w(1)=v(1)-\beta>0 \end{cases} \tag{1.13}$$

Let $\psi=:\sin\left(\frac{\pi(t-t_\beta)}{t-t_\beta}\right)$. Then ψ satisfies:

$$\begin{cases} -\psi'' = \dfrac{\pi^2}{\left(t-t_\beta\right)^2}\psi, & \left(t_\beta,1\right) \\ \psi\left(t_\beta\right) = \psi(1) = 0 \end{cases} \tag{1.14}$$

Multiplying (1.13) by ψ and (1.14) by h,and w integrating both from t_β to 1 and subtracting, we have

$$\int_{t_\beta}^{1}\left(h''\psi-\psi''h\right)\mathrm{d}t = \int_{t_\beta}^{1}\left(\frac{\pi^2}{\left(1-t_\beta\right)^2}-\lambda\frac{f_1(u)}{u-\beta}a_1(t)\right)h\psi\,\mathrm{d}t,$$

$$\int_{t_\beta}^{1}\left(w''\psi-\psi''w\right)\mathrm{d}t = \int_{t_\beta}^{1}\left(\frac{\pi^2}{\left(1-t_\beta\right)^2}-\lambda\frac{f_2(v)}{v-\beta}a_2(t)\right)w\psi\,\mathrm{d}t$$

Since $\int_{t_\beta}^{1}\left(h''\psi-\psi''h\right)\mathrm{d}t = -\psi'(1)h(1)(>0)$ and $\int_{t_\beta}^{1}\left(w''\psi-\psi''w\right)\mathrm{d}t = -\psi'(1)w(1)(>0)$ by integration by parts, we can see that

$$\frac{\pi^2}{\left(1-t_\beta\right)^2} > \lambda\frac{f_1(u)}{u-\beta}a_1(t) \text{ and } \frac{\pi^2}{\left(1-t_\beta\right)^2} > \lambda\frac{f_2(v)}{v-\beta}a_2(t) \text{ for some } t\in\left(t_\beta,1\right) \tag{1.15}$$

Note that $\inf_{(0,1]} a_1(t) > 0$, $\inf_{(0,1]} a_2(t) > 0$ and from Lemma 2.2 we can assume $\left(1-t_\beta\right) > \frac{1}{2}$. Thus (2.11) is only true if $\frac{f_1(u)}{u-\beta}\to 0$ and $\frac{f_2(v)}{v-\beta}\to 0$ when $\lambda \gg 1$. By (F1) (F2), we conclude that $u(t_m) = \|u\|_\infty$ and $v(t_m) = \|v\|_\infty$ as $\lambda\to\infty$. Notice that since $u''<0$ and $v''<0$ in $\left(t_\beta,1\right]$, we have

$$u(t) \ge \frac{u(t_m)-\beta}{t_m-t_\beta}\left(t-t_\beta\right)+\beta,\quad t\in\left[t_\beta,t_m\right]$$

$$v(t) \ge \frac{v(t_m)-\beta}{t_m-t_\beta}\left(t-t_\beta\right)+\beta,\quad t\in\left[t_\beta,t_m\right]$$

$$u(t) \ge \frac{u(t_m)-\dfrac{\beta+\theta}{2}}{1-t_m}(1-t)+\frac{\beta+\theta}{2},\quad t\in\left[t_m,1\right]$$

$$v(t) \ge \frac{v(t_m)-\dfrac{\beta+\theta}{2}}{1-t_m}(1-t)+\frac{\beta+\theta}{2},\quad t\in\left[t_m,1\right]$$

Since $u(t_m)\to\infty$, $v(t_m)\to\infty$ and $t_\beta\to 0$ as $\lambda\to\infty$, it is all true that

$$v(t) \ge \frac{\beta+\theta}{2} \text{ and } u(t) \ge \frac{\beta+\theta}{2}, \text{ in } \left[\frac{1}{4},1\right] \quad \text{for } \lambda \gg 1 \tag{1.16}$$

Now we show that $u(1)\to\infty$ and $v(1)\to\infty$ as $\lambda\to\infty$. Since u,v is a solution of (1.3), u and v can be written as: (see Appendix 8.2 in [5] for details)

$$u(t) = \lambda\int_0^1 G(t,s)a_1(s)f_1(u(s))\mathrm{d}s - c_1(u(1))u(1)t \tag{1.17}$$

$$v(t) = \lambda\int_0^1 G(t,s)a_2(s)f_2(v(s))\mathrm{d}s - c_2(v(1))v(1)t \tag{1.18}$$

where

$$G(t,s)=\begin{cases} s, & 0\le s\le t\le 1 \\ t, & 0\le t\le s\le 1 \end{cases}$$

Let $t=1$. Then from (1.17) and (1.18), we have

$$\begin{aligned}&\left[1+c_1\left(u(1)\right)\right]u(1)\\ &=\lambda\left[\int_0^{t_\beta}G(1,s)a_1(s)f_1\left(u(s)\right)\mathrm{d}s+\int_{t_\beta}^1 G(1,s)a_1(s)f_1\left(u(s)\right)\mathrm{d}s\right]\end{aligned}\tag{1.19}$$

$$\begin{aligned}&\left[1+c_2\left(v(1)\right)\right]v(1)\\ &=\lambda\left[\int_0^{t_\beta}G(1,s)a_2(s)f_2\left(v(s)\right)\mathrm{d}s+\int_{t_\beta}^1 G(1,s)a_2(s)f_2\left(v(s)\right)\mathrm{d}s\right]\end{aligned}\tag{1.20}$$

Then using the fact $G(1,s)=s$ and $t_\beta\to\infty$ as $\lambda\to\infty$, for λ large we obtain

$$\left[1+c_1\left(u(1)\right)\right]u(1)=\lambda\left(\int_0^{t_\beta}sa_1(s)f_1\left(u(s)\right)\mathrm{d}s+\int_{t_\beta}^1 sa_1(s)f_1\left(u(s)\right)\mathrm{d}s\right)$$

$$\begin{aligned}\left[1+c_2\left(v(1)\right)\right]v(1)&=\lambda\left(\int_0^{t_\beta}sa_2(s)f_2\left(v(s)\right)\mathrm{d}s+\int_{t_\beta}^1 sa_2(s)f_2\left(v(s)\right)\mathrm{d}s\right)\\ &\ge\lambda\left(\int_0^{t_\beta}sa_1(s)f_1\left(u(s)\right)\mathrm{d}s+\int_{\frac{1}{4}}^1 sa_1(s)f_1\left(u(s)\right)\mathrm{d}s\right)\\ &\ge\lambda\left(\int_0^{t_\beta}sa_2(s)f_2\left(v(s)\right)\mathrm{d}s+\int_{\frac{1}{4}}^1 sa_2(s)f_2\left(v(s)\right)\mathrm{d}s\right)\\ &\ge\frac{\lambda}{2}f_1\left(\frac{\beta+\theta}{2}\right)\int_{\frac{1}{4}}^1 sa_1(s)\mathrm{d}s\\ &\ge\frac{\lambda}{2}f_2\left(\frac{\beta+\theta}{2}\right)\int_{\frac{1}{4}}^1 sa_2(s)\mathrm{d}s\end{aligned}$$

where the last inequality is obtained by (1.16). Hence, we have

$$\left[1+c_1\left(u(1)\right)\right]u(1)\ge\lambda K \text{ and } \left[1+c_2\left(v(1)\right)\right]v(1)\ge\lambda K,\tag{1.21}$$

where $K=\frac{1}{2}f_1\left(\frac{\beta+\theta}{2}\right)\int_{\frac{1}{4}}^1 sa_1(s)\mathrm{d}s>0$ and $K=\frac{1}{2}f_2\left(\frac{\beta+\theta}{2}\right)\int_{\frac{1}{4}}^1 sa_2(s)\mathrm{d}s>0$.

Now, from (1.21), clearly $u(1)\to\infty$, $v(1)\to\infty$ as $\lambda\to\infty$. □

Lemma 2.4. Let u and v be a positive solution of (1.3). Then there exists $[\alpha,\mu]\subset\left[\frac{1}{4},1\right]$, $\alpha\ne\mu$, both independent of λ, such that $\inf_{[\alpha,\mu]}u(t)\to\infty$ and $\inf_{[\alpha,\mu]}v(t)\to\infty$ as $\lambda\to\infty$

Proof. As $\lambda\to\infty$, t_m may converge to 1 or to any other point in $(0,1)$. First we consider the case $t_m\nrightarrow 1$ as $\lambda\to\infty$. Since $u(1)<u(t_m)$ and $v(1)<v(t_m)$ clearly there exists $\alpha<1$ such that $\inf_{[\alpha,1]}u(t)\to\infty$ and $\inf_{[\alpha,1]}v(t)\to\infty$ as $\lambda\to\infty$ by Lemma 2.3.

Now, let us consider the case when $t_m\to 1$ as $\lambda\to\infty$. By differentiating (1.17) and (1.18) (or integrating (1.3)), we obtain

$$u'(t)=\lambda\int_t^1 a_1(s)f_1\left(u(s)\right)\mathrm{d}s-c_1\left(u(1)\right)u(1),\quad t\in[0,1],$$

$$v'(t)=\lambda\int_t^1 a_2(s)f_2\left(v(s)\right)\mathrm{d}s-c_2\left(v(1)\right)v(1),\quad t\in[0,1]$$

which gives us that

$$u'(t)=\lambda\int_t^1 a_1(s)f_1(u(s))\mathrm{d}s-c_1(u(1))u(1)=0, \tag{1.22}$$

$$v'(t)=\lambda\int_{t_m}^1 a_2(s)f_2(v(s))\mathrm{d}s-c_2(v(1))v(1)=0 \tag{1.23}$$

Now we rewrite (1.17) and (1.18) by using (1.22), (1.23) as:

$$u(t)=\lambda\int_0^1 G(t,s)a_1(s)f_1(u(s))\mathrm{d}s-\lambda\left(\int_{t_m}^1 a_1(s)f_1(u(s))\mathrm{d}s\right)t,$$

$$\begin{aligned}
v(t)&=\lambda\int_0^1 G(t,s)a_2(s)f_2(v(s))\mathrm{d}s-\lambda\left(\int_{t_m}^1 a_2(s)f_2(v(s))\mathrm{d}s\right)t\\
&=\lambda\int_0^{t_\beta} G(s,t)a_1(s)f_1(u(s))\mathrm{d}s+\lambda\int_{t_\beta}^{t_m} G(t,s)a_1(s)f_1(u(s))\mathrm{d}s\\
&\quad+\lambda\int_{t_m}^1\left[G(t,s)-t\right]a_1(s)f_1(u(s))\mathrm{d}s.\\
&=\lambda\int_0^{t_\beta} G(s,t)a_2(s)f_2(v(s))\mathrm{d}s+\lambda\int_{t_\beta}^{t_m} G(t,s)a_2(s)f_2(v(s))\mathrm{d}s\\
&\quad+\lambda\int_{t_m}^1\left[G(t,s)-t\right]a_2(s)f_2(v(s))\mathrm{d}s.
\end{aligned}$$

Note that if $t\in[0,t_m]$, then

$$\int_{t_m}^1\left[G(t,s)-t\right]a_1(s)f_1(u(s))\mathrm{d}s=0$$

and

$$\int_{t_m}^1\left[G(t,s)-t\right]a_2(s)f_2(v(s))\mathrm{d}s=0$$

since $t\le t_m\le s\le 1$ implies $G(t,s)=t$. Now $t_\beta\to 0$ and $t_m\to 1$ as $\lambda\to\infty$. Hence, for $t\in\left[\frac{1}{4},\frac{3}{4}\right]$ and λ large we obtain

$$u(t)=\lambda\left(\int_0^{t_\beta} G(t,s)a_1(s)f_1(u(s))\mathrm{d}s+\int_{t_\beta}^{t_m} G(t,s)a_1(s)f_1(u(s))\mathrm{d}s\right),$$

$$\begin{aligned}
v(t)&=\lambda\left(\int_0^{t_\beta} G(t,s)a_2(s)f_2(v(s))\mathrm{d}s+\int_{t_\beta}^{t_m} G(t,s)a_2(s)f_2(v(s))\mathrm{d}s\right)\\
&\ge\lambda\left(\int_0^{t_\beta} G(t,s)a_1(s)f_1(u(s))\mathrm{d}s+\int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)a_1(s)f_1(u(s))\mathrm{d}s\right),\\
&\ge\lambda\left(\int_0^{t_\beta} G(t,s)a_2(s)f_2(v(s))\mathrm{d}s+\int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)a_2(s)f_2(v(s))\mathrm{d}s\right)\\
&\ge\lambda\frac{a_1}{2}f_1\left(\frac{\beta+\theta}{2}\right)\int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)\mathrm{d}s,\\
&\ge\lambda\frac{a_2}{2}f_2\left(\frac{\beta+\theta}{2}\right)\int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)\mathrm{d}s.
\end{aligned}$$

Thus,

$$u(t)\ge\lambda\frac{a_1}{2}f_1\left(\frac{\beta+\theta}{2}\right)\inf_{\left[\frac{1}{4},\frac{3}{4}\right]}\int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)\mathrm{d}s$$

and

$$v(t)\ge\lambda\frac{a_2}{2}f_2\left(\frac{\beta+\theta}{2}\right)\inf_{\left[\frac{1}{4},\frac{3}{4}\right]}\int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)\mathrm{d}s$$

on $\left[\frac{1}{4},\frac{3}{4}\right]$, which means that $u(t)\to\infty$ and $v(t)\to\infty$ for all $t\in\left[\frac{1}{4},\frac{3}{4}\right]$ as $\lambda\to\infty$.

Lemma 2.5. Let u and v be a positive solution of (1.3). Then there exists $\tilde{\lambda}$ such that if $\lambda>\tilde{\lambda}$, then

$$u(t)\ge\lambda Cd(t,\partial\Omega)\ \text{ and }\ v(t)\ge\lambda Cd(t,\partial\Omega) \tag{1.24}$$

for some positive constant C, independent of λ. Here $\Omega=(0,1)$.

Proof. Let ϕ_i be the unique solution of the problems

$$\begin{cases}-\phi_i''=\omega_{\alpha,\mu}a_i(t), & (t_\beta,1)\\ \psi_i(t_\beta)=\psi_i(1)=0, & \text{and } i=1,2\end{cases} \tag{1.25}$$

where ω is the characteristic function. By the Hopf maximum principle there exists $\tilde{c}_i>0$ such that $\phi_i(t)>\tilde{c}_ie_i(t)$ for all $t\in[0,1]$, where e_i are a solution of

$$\begin{cases}-e_i''=\omega_{\alpha,\mu}a_i(t), & (t_\beta,1)\\ e_i(0)=e_i(1)=0, & \text{and } i=1,2\end{cases} \tag{1.26}$$

Let $H>0$ be such that $D:=\tilde{c}_i f_i(H)+f_i(0)>0$, and this is possible by (F2). Let u_1,v_1 and u_2,v_2 satisfy

$$-u_1''=\lambda f_1(H)\omega[\alpha,\mu]a_1(t),\ t\in(0,1),\ u_1(0)=0=u_1(1)$$
$$-v_1''=\lambda f_2(H)\omega[\alpha,\mu]a_2(t),\ t\in(0,1),\ v_1(0)=0=v_1(1)$$

and

$$-u_2''=\lambda f_1(0)\omega[\alpha,\mu]a_1(t),\ t\in(0,1),\ u_2(0)=0=u_2(1)$$
$$-v_2''=\lambda f_2(0)\omega[\alpha,\mu]a_2(t),\ t\in(0,1),\ v_2(0)=0=v_2(1)$$

Now by Lemma 2.4, there exists $\tilde{\lambda}>0$ such that if $\lambda>\tilde{\lambda}$, then

$$u(t)\ge H\ \text{ and }\ v(t)\ge H\ \text{ on } [\alpha,\mu]. \tag{1.27}$$

Hence, by (1.27), for $\lambda\gg1$ we have that for $t\in(0,1)$

$$-u''=\lambda f_1(u)a_1(t)\ge\lambda f_1(u)a_1(t)\omega[0,t_B]+\lambda f_1(u)a_1(t)\omega[\alpha,\mu],$$
$$-v''=\lambda f_2(v)a_2(t)\ge\lambda f_2(v)a_2(t)\omega[0,t_B]+\lambda f_2(v)a_2(t)\omega[\alpha,\mu]$$
$$-v''\ge\lambda f_2(0)a_2(t)+\lambda f_2(H)a_2(t)\omega[\alpha,\mu]$$
$$-u''\ge\lambda f_1(0)a_1(t)+\lambda f_1(H)a_1(t)\omega[\alpha,\mu]$$
$$=-(u_1-u_2)''(t)$$
$$=-(v_1-v_2)''(t)$$

$u(0)-(u_1-u_2)(0)=0$, $v(0)-(v_1-v_2)(0)=0$ and $u(1)-(u_1-u_2)(1)=u(1)>0$, $v(1)-(v_1-v_2)(1)=v(1)>0$. By the maximum principle, $u(t)=u_1(t)-u_2(t)=\lambda f_1(H)\phi_1(t)+\lambda f_1(0)e_1(t)$ and $v(t)=v_1(t)-v_2(t)=\lambda f_2(H)\phi_2(t)+\lambda f_2(0)e_2(t)$ in $[0,1]$. Hence

$$u(t)\ge f_1(H)\tilde{c}_1e_1(t)+\lambda f_1(0)e_1(t)=\lambda De_1(t)$$

and

$$v(t) \geq f_2(H)\tilde{c}_2 e_2(t) + \lambda f_2(0) e_2(t) = \lambda D e_2(t)$$

for all $t \in [0,1]$.

Note that there exists $L > 0$ such that $e_1(t) \geq Ld(t, \partial\Omega)$ and $e_2(t) \geq Ld(t, \partial\Omega)$ for all $t \in [0,1]$. Hence, for λ large $u(t) \geq \lambda Cd(t, \partial\Omega)$ and $v(t) \geq \lambda Cd(t, \partial\Omega)$ for all $t \in [0,1]$, where $C := DL > 0$.

Lemma 2.6. Let u and v be a positive solution of (1.3). Then there exists H_λ such that $\|u\|_\infty \leq H_\lambda$ and $\|v\|_\infty \leq H_\lambda$.

Proof.

Let $B = \int_0^1 a_i(s)\,\mathrm{d}s$. Then $B < \infty$ since $a_i(t) \leq c_i t_\alpha$ for all $t \in (0,1)$ for some $c_i > 0$. Now for each given $\lambda > 0$, there exists $W_\lambda > 0$ such that if $W > W_\lambda$, then $\frac{f_i(W)}{W} \leq \frac{1}{2\lambda B}$ due to the hypothesis (F1). Also since $f_i \in C^1([0,\infty), R)$, there exists $K_\lambda > 0$ such that $f_i(W) \leq K_\lambda$ on $[0, W_\lambda]$. Hence,

$$f_i(W) \leq \frac{W}{2\lambda B} + K_\lambda, \quad W \in [0,\infty). \tag{1.28}$$

Now by Lemma 2.1 and (1.28), we have

$$\|u\|_\infty = u(t_m) = \lambda \int_0^1 G(t_m, s) a_1(s) f_1(u(s))\,\mathrm{d}s - c_1(u(1))u(1)t_m$$

$$\begin{aligned}
\|v\|_\infty = v(t_m) &= \lambda \int_0^1 G(t_m, s) a_2(s) f_2(v(s))\,\mathrm{d}s - c_2(u(1))u(1)t_m \\
&\leq \lambda \int_0^1 G(t_m, s) a_1(s) f_1(u(s))\,\mathrm{d}s \\
&\leq \lambda \int_0^1 G(t_m, s) a_2(s) f_2(v(s))\,\mathrm{d}s \\
&\leq \lambda \int_0^1 G(t_m, s) a_1(s) \left[\frac{u(t_m)}{2\lambda B} + K_\lambda\right] \mathrm{d}s \\
&\leq \lambda \int_0^1 G(t_m, s) a_2(s) \left[\frac{v(t_m)}{2\lambda B} + K_\lambda\right] \mathrm{d}s \\
&\leq \lambda \int_0^1 a_1(s) \left[\frac{1}{2\lambda B} u(t_m) + K_\lambda\right] \mathrm{d}s \\
&\leq \lambda \int_0^1 a_2(s) \left[\frac{1}{2\lambda B} v(t_m) + K_\lambda\right] \mathrm{d}s \quad \left(\text{since } G(t,s) \leq 1 \text{ in } [0,1]\times[0,1]\right) \\
&= \frac{1}{2} u(t_m) + \lambda B K_\lambda \\
&= \frac{1}{2} v(t_m) + \lambda B K_\lambda
\end{aligned}$$

Hence, for each $\lambda > 0$, $\|u\|_\infty \leq H_\lambda$ and $\|v\|_\infty \leq H_\lambda$, where $H_\lambda = 2\lambda B K_\lambda$.

3. Proof of Theorem 1.1

We first claim that (1.3) has a maximal positive solutions $\tilde{u}, \tilde{v}$ for $\lambda \gg 1$. Let φ_i be the solutions of the problems

$$\begin{cases} -\varphi_1'' = a_i(t), & t \in (0,1) \\ \varphi_i'(1) = c_i(\varphi_i(1))\varphi_i(1) = 0, \\ \varphi_i(0) = 0 \quad \text{and} \quad i = 1,2 \end{cases} \tag{1.29}$$

Note that (1.29) has the unique solution since e_i, φ_0 are sub solutions and super solutions of (1.29), respectively, where e_i is defined in (1.26) and φ_0 is the solutions of the linear boundary condition problems

$$\begin{cases} -\varphi_0'' = a_i(t), \quad t \in (0,1) \\ \varphi_0'(1) = c_i(\varphi_0(1))\varphi_0(1) = 0, \\ \varphi_0(0) = 0 \quad \text{and} \quad i = 1,2 \end{cases} \tag{1.30}$$

Since f_i satisfies (F1), given $\lambda > 0$, we can choose $Z_\lambda \ge 1$ such that $Z_\lambda > \lambda f_i\left(Z_\lambda \|\varphi_i\|_\infty\right)$ and $Z_\lambda > \lambda f_i(H_\lambda)$ where H_λ is as in Lemma 2.6. Then, $Z_\lambda \varphi_i$ are a super solutions of (1.3) since

$$\begin{cases} -(Z_\lambda \varphi_i)'' = Z_\lambda a_i(t) \ge \lambda a_i f_i\left(Z_\lambda \|\varphi_i\|_\lambda\right) \ge \lambda a_i f_i(Z_\lambda \varphi_i), \quad t \in (0,1) \\ (Z_\lambda \varphi_i'(1)) + c_i\left(Z_\lambda(\varphi_i(1))\right) Z_\lambda \varphi_i(1) = Z_\lambda\left(\varphi_i'(1) + c_i\left(Z_\lambda(\varphi_i(1))\right)\varphi_i(1)\right) \\ \ge Z_\lambda\left(\varphi_i'(1) + c_i(\varphi_i(1))\varphi_i(1)\right) = 0, \\ Z_\lambda \varphi_i(0) = 0 \quad \text{and} \quad i = 1,2 \end{cases}$$

Next, we show that this super solution $Z_\lambda \varphi_i$ is larger than any positive solutions of (1.3). Let θ_i be any positive solutions of (1.3). From Lemma 2.6, we have

$$\begin{aligned} -(Z_\lambda \varphi_i - \theta_i)'' &= Z_\lambda a_i(t) - \lambda a_i f_i(\theta_i) = a_i\left[Z_\lambda - \lambda f_i(\theta_i)\right] \\ &\ge a_i\left[Z_\lambda - \lambda f_i(H_\lambda)\right] > 0, \quad t \in (0,1) \end{aligned}$$

by the choice of Z_λ. Note that $(Z_\lambda \varphi_i - \theta_i)(0) = 0$. Now we show that $(Z_\lambda \varphi_i - \theta_i)(1) \ge 0$. Indeed, since $Z_\lambda \varphi_i'(1) + c_i\left(Z_\lambda(\varphi_i(1))\right) Z_\lambda \varphi_i(1) \ge 0 = \theta_i'(1) + c_i(\theta_i(1))\theta_i(1)$, we have

$$Z_\lambda \varphi_i'(1) - \theta_i'(1) + c_i\left(Z_\lambda(\varphi_i(1))\right) Z_\lambda \varphi_i(1) - c_i(\theta_i(1))\theta_i(1) \ge 0 \tag{1.31}$$

If we assume that $Z_\lambda \varphi_i(1) - \theta_i(1) < 0$, then $c_i\left(Z_\lambda(\varphi_i(1))\right) Z_\lambda \varphi_i(1) - c_i(\theta_i(1))\theta_i(1) < 0$ since c_i increases. Hence from (3.3) we obtain $Z_\lambda \varphi_i'(1) - \theta_i'(1) > 0$. However, $-(Z_\lambda \varphi_i - \theta_i)'' > 0$ in $(0,1)$, $(Z_\lambda \varphi_i - \theta_i)(0) = 0$ and $(Z_\lambda \varphi_i - \theta_i)(1) < 0$ implies that $(Z_\lambda \varphi_i - \theta_i)'(1) > 0$, which is a contradiction. Hence $Z_\lambda \varphi_i(1) - \theta_i(1) \ge 0$. Therefore, by the maximum principle, $Z_\lambda \varphi_i \ge \varphi_i$ in $[0,1]$. Therefore, (1.3) has a maximal positive solutions $\tilde{u}, \tilde{v}$. Now, let u and v be any other positive solutions of (1.3). To establish our theorem, we will show that $u \equiv \tilde{u}$ and $v \equiv \tilde{v}$ for $\lambda \gg 1$. Since u, v and $\tilde{u}, \tilde{v}$ are solutions of (1.3), we obtain

$$\begin{cases} -(\tilde{u} - u)''(t) = \lambda a_1(t)\left(f_1(\tilde{u}(t)) - f_1(u(t))\right), \quad t \in (0,1) \\ -(\tilde{v} - v)''(t) = \lambda a_2(t)\left(f_2(\tilde{v}(t)) - f_2(v(t))\right), \quad t \in (0,1) \\ (\tilde{u} - u)(0) = (\tilde{v} - v)(0) = 0, \\ (\tilde{u} - u)'(1) + c_1(\tilde{u}(1))\tilde{u}(1) - c_1(u(1))u(1) = 0, \\ (\tilde{v} - v)'(1) + c_2(\tilde{v}(1))\tilde{v}(1) - c_2(v(1))v(1) = 0 \end{cases}$$

By the mean value theorem, there exists ξ such that $u \le \xi \le \tilde{u}$ and $v \le \xi \le \tilde{v}$ quadin $[0,1]$ and

$$\begin{cases} -(\tilde{u}-u)''(t) = \lambda a_1(t) f_1'(\xi)(\tilde{u}(t)-u(t)), & t \in (0,1) \\ -(\tilde{v}-v)''(t) = \lambda a_2(t) f_2'(\xi)(\tilde{v}(t)-v(t)), & t \in (0,1) \\ (\tilde{u}-u)(0) = (\tilde{v}-v)(0) = 0, \\ (\tilde{u}-u)'(1) + c_1(\tilde{u}(1))\tilde{u}(1) - c_1(u(1))u(1) = 0, \\ (\tilde{v}-v)'(1) + c_2(\tilde{v}(1))\tilde{v}(1) - c_2(v(1))v(1) = 0 \end{cases} \tag{1.32}$$

By multiplying (1.3) by $(\tilde{u}-u),(\tilde{v}-v)$ and (1.32) by u,v and integrating, we first obtain, using integration by parts,

$$\begin{aligned} \int_0^1 \left[(\tilde{u}-u)'' u - (\tilde{u}-u)u''\right] dt &= \tilde{u}'(1)u(1) - u'(1)\tilde{u}(1) \\ &= u(1)\left[-c_1(\tilde{u}(1))\tilde{u}(1)\right] + \tilde{u}(1)\left[c_1(u(1))u(1)\right] \\ &= u(1)\tilde{u}(1)\left[c_1(u(1)) - c_1(\tilde{u}(1))\right] \\ &\le 0 \end{aligned}$$

$$\begin{aligned} \int_0^1 \left[(\tilde{v}-v)'' v - (\tilde{v}-v)v''\right] dt &= \tilde{v}'(1)v(1) - v'(1)\tilde{v}(1) \\ &= v(1)\left[-c_2(\tilde{v}(1))\tilde{v}(1)\right] + \tilde{v}(1)\left[c_2(v(1))v(1)\right] \\ &= v(1)\tilde{v}(1)\left[c_2(v(1)) - c_2(\tilde{v}(1))\right] \\ &\le 0 \end{aligned}$$

since c_i are increasing. Using that f_i is concave, we also have

$$\begin{aligned} \int_0^1 \left[(\tilde{u}-u)'' u - (\tilde{u}-u)u''\right] dt &= \lambda \int_0^1 a_1(t)\left[f_1(u) - f_1'(\xi)u\right](\tilde{u}-u)\,dt \\ &\ge \lambda \int_0^1 a_1(t)\left[f_1(u) - f_1'(u)u\right](\tilde{u}-u)\,dt \end{aligned} \tag{1.33}$$

$$\begin{aligned} \int_0^1 \left[(\tilde{v}-v)'' v - (\tilde{v}-v)v''\right] dt &= \lambda \int_0^1 a_2(t)\left[f_2(v) - f_2'(\xi)v\right](\tilde{v}-v)\,dt \\ &\ge \lambda \int_0^1 a_2(t)\left[f_2(v) - f_2'(v)v\right](\tilde{v}-v)\,dt \end{aligned} \tag{1.34}$$

From (F1), there exist $r_i > 0$, $b_i > 0$ such that $f_i(s) - f'(s)s \ge b_i$ whenever $s \ge r_i$. From (1.20), for $\lambda \gg 1$, $u(t) \ge r_i$ and $v(t) \ge r_i$ if $d(t,\partial\Omega) \ge \frac{r_i}{\lambda C}$. Let $\Omega_+ = \left[\frac{r_i}{\lambda C}, 1-\frac{r_i}{\lambda C}\right]$ and $\Omega_- = \left(0,\frac{r_i}{\lambda C}\right) \cup \left(1-\frac{r_i}{\lambda C},1\right)$. Then from (1.33), we have

$$0 \ge \lambda \int_{\Omega_+} a_1(t) b_1(\tilde{u}-u)\,dt + \lambda \int_{\Omega_-} a_1(t) f_1(0)(\tilde{u}-u)\,dt$$

$$0 \ge \lambda \int_{\Omega_+} a_2(t) b_2(\tilde{v}-v)\,dt + \lambda \int_{\Omega_-} a_2(t) f_2(0)(\tilde{v}-v)\,dt. \tag{1.35}$$

since when f_i is concave $f_i(W) - W f_i'(W) \ge f_i(0)$ for all $W \ge 0$.

Next let m and h satisfy

$$-m''(t) = \omega_{\Omega_+} a_1(t), \quad t \in (0,1), \quad m(0) = m(1) = 0,$$

$$m''(t) = \omega_{\Omega_+} a_2(t), \quad t \in (0,1), \quad m(0) = m(1) = 0$$

and

$$-h''(t) = \omega_{\Omega_-} a_1(t), \quad t \in (0,1), \quad h(0) = h(1) = 0,$$

$$-h''(t) = \omega_{\Omega_-} a_2(t), \quad t \in (0,1), \quad h(0) = h(1) = 0$$

respectively. Now multiplying (1.32) by $b_i m + f_i(0) h$ and integrating, we obtain

$$I := \int_0^1 -(\tilde{u} - u)'' \left[b_1 m + f_1(0) h \right] \mathrm{d}t$$

$$\begin{aligned} J &:= \int_0^1 -(\tilde{v} - v)'' \left[b_2 m + f_2(0) h \right] \mathrm{d}t \\ &= \left(\tilde{u}(1) - u(1)\right)\left[b_1 m'(1) + f_1(0) h'(1) \right] + \int_{\Omega_+} a_1(t) b_1 (\tilde{u} - u) \mathrm{d}t \\ &\quad + \int_{\Omega_-} a_1(t) f_1(0)(\tilde{u} - u) \mathrm{d}t \\ &= \left(\tilde{v}(1) - v(1)\right)\left[b_2 m'(1) + f_2(0) h'(1) \right] + \int_{\Omega_+} a_2(t) b_2 (\tilde{v} - v) \mathrm{d}t \\ &\quad + \int_{\Omega_-} a_2(t) f_2(0)(\tilde{v} - v) \mathrm{d}t \end{aligned}$$

$$\begin{cases} = I_1 + I_2 \\ = J_1 + J_2 \end{cases} \tag{1.36}$$

Note that as $\lambda \to \infty$, $m \to e_i$ and $h \to 0$ in $C^1[0,1]$. Hence, for λ large, we obtain $b_i m + f_i(0) h > 0$, in $(0,1)$ and $i = 1, 2$

$$b_i m + f_i(0) h > 0 \quad \text{in } (0,1) \tag{1.37}$$

and

$$b_i m' + f_i(0) h' < 0 \tag{1.38}$$

Hence for $\lambda \gg 1$, (1.37) implies $I_1 \le 0$, $J_1 \le 0$ and combining with (1.34) (which implies $I_2 \le 0$ and $J_2 \le 0$) we have $I \le 0$ and $J \le 0$. However, by (1.32), we also have

$$\begin{aligned} I &:= \int_0^1 -(\tilde{u} - u)'' \left[b_1 m + f_1(0) h \right] \mathrm{d}t \\ &\quad + \lambda \int_0^1 a_1(t) f_1'(\xi) \left(\tilde{u}(t) - u(t)\right)\left[b_1 m + f_1(0) h \right] \mathrm{d}t \end{aligned}$$

$$\begin{aligned} J &:= \int_0^1 -(\tilde{v} - v)'' \left[b_2 m + f_2(0) h \right] \mathrm{d}t \\ &\quad + \lambda \int_0^1 a_2(t) f_2'(\xi) \left(\tilde{v}(t) - v(t)\right)\left[b_2 m + f_2(0) h \right] \mathrm{d}t \end{aligned}$$

Now for $\lambda \gg 1$, using (1.36), $a_i > 0$, and $f_i' \ge 0$ we get $I \ge 0$ and $J \ge 0$. Hence, we conclude that $I \equiv 0$ and $J \equiv 0$ for $\lambda \gg 1$, which implies that $\tilde{v} \equiv v$ and $\tilde{u} \equiv u$ in $[0,1]$. This proves that (1.3) has a unique positive solution for all λ large.

4. Conclusion

In the paper, were studied the positive radial solutions for elliptic systems to the nonlinear Boundary Value problems. And then, we presented that by the Theorem 1.1, and Theorem 2.2, we can obtain a solution of the problem (1.3). Moreover, for all $\lambda \gg 1$, then (1.3) has a unique positive solution.

Acknowledgements

The authors would like to thank the anonymous referees for their helpful comments.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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