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Global Bounded Solutions for the Keller-Segel Chemotaxis System with Singular Sensitivity

Khalid Ahmed Abbakar[1,2*], Omer Khalil[1], Alhussein Mohamed[1], Bechir Mahamat[1], Abdoulaye Ali[1], Abeer Alhadi[3]

[1]College of Mathematics and Statistics, Northwest Normal University, Lanzhou, China
[2]Department of Mathematics and Physics, Faculty of Education, University of Gadarif, Gadarif, Sudan
[3]College of Physics and Electronic Engineering, Northwest Normal University, Lanzhou, China

Email: *khalidahmed200@hotmail.com

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Abstract

In this paper, we consider the Neumann initial-boundary value problem for the Keller-Segel chemotaxis system with singular sensitivity

$$\begin{cases} u_t = d_1\Delta u - \chi\nabla\cdot\left(\dfrac{u}{v}\nabla v\right), \\ v_t = d_2\Delta v - v + u, \end{cases} \tag{0.1}$$

is considered in a bounded domain with smooth boundary, $\Omega \subset \mathbb{R}^n\ (n \geq 1)$, where $d_1 > 0, d_2 > 0$ with parameter $\chi \in \mathbb{R}$. When $d_1 = d_2 + \chi$, satisfying for all initial data $0 \leq u_0 \in C^0\left(\bar{\Omega}\right)$ and $0 < v_0 \in W^{1,\infty}\left(\Omega\right)$, we prove that the problem possesses a unique global classical solution which is uniformly bounded in $\Omega \times (0,\infty)$.

Keywords

Keller-Segel System, Chemotaxis, Global Bounded Solution, Singular Sensitivity

1. Introduction

The Keller-Segel system is used to model chemotactic movement in biology [1]. The mathematical study of the system has attracted great interest in recent years [2]. In this paper, we consider the Neumann initial-boundary value problem for the chemotaxis system with singular sensitivity

$$\begin{cases} u_t = d_1 \Delta u - \chi \nabla \cdot \left(\dfrac{u}{v} \nabla v \right), & x \in \Omega, t > 0 \\ v_t = d_2 \Delta v - v + u, & x \in \Omega, t > 0 \\ \dfrac{\partial u}{\partial \nu} = 0, \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega \end{cases} \tag{1.2}$$

in a bounded domain $\Omega \subset \mathbb{R}^n (n \geq 1)$ with smooth boundary, where $d_1 > 0$ and $d_2 > 0$ are diffusion coefficients of cell density and chemical stimulus, respectively. The Keller-Segel systems were introduced to describe the aggregation of cellular slime molds, u represents the density of the cells and v represents the concentration of a chemical substance secreted by themselves. The chemical substance is an attractant, they sense a gradient of the chemical substances and move towards higher concentrations. The function χ is called a sensitivity function, and expresses the relation between the chemical concentration and the cells response, the symbol $\dfrac{\partial}{\partial \nu}$ denotes differentiation with respects to the outward normal ν on $\partial\Omega$ and the initial data u_0 and v_0 are sufficiently smooth functions. For system (1.2) with $d_1 = d_2$, the global existence and boundedness of classical solution is proved under the assumption $0 < \chi < \sqrt{\dfrac{2}{n}}$ see [3] [4] [5]. Lankeit [6] extended the range of χ in the two-dimensional case. Also the generalized solutions with large χ are constructed in [3] [7] [8]. More results on the related model with general sensitivity can be found in [9] [10] [11] [12]. In this present paper, we prove the existence of global bounded classical solutions for (1.2) without assumptions on the space dimensions or the smallness assumption on the initial data in the case $d_1 = d_2 + \chi$. Our main result reads as follows.

2. Preliminaries

Lemma 1.2. (Poincaré inequality) [13] Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, then there is exists a constant $C = C(n, p, \Omega)$, such that for all $u \in W^{1,p}(\Omega)$

1) $\|u\|_{W^{1,p}(\Omega)} \leq C\left(\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^q(\Omega)}\right), \ \forall p > 1, \ q > 0.$

2) $\left\| u - \dfrac{1}{|\Omega|}\int_\Omega u(x)\mathrm{d}x \right\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}, \ \forall 1 \leq p \leq +\infty.$

Theorem 1.1. Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary and let the parameters $d_1 > 0, d_2 > 0$ and $\chi \in \mathbb{R}$ satisfy $d_1 = d_2 + \chi$. Then for any nonnegative function $u_0 \in C^0(\bar{\Omega})$ and positive function $\upsilon_0 \in W^{1,\infty}(\Omega)$, the problem (1.2) has a unique global classical solution which is bounded in $\Omega \times (0, \infty)$.

3. Proof of Theorem 1.1

As a preparation to the proof, we first state one result concerning local-in-time classical solution of the problem (1.2), which can be proved by standard contraction mapping arguments and parabolic regularity theory (see ([11], Proposition 2.2) and the references therein).

Lemma 3.1. Suppose that $u_0 \in C^0(\bar{\Omega})$ is a nonnegative function and that $\upsilon_0 \in W^{1,\infty}(\Omega)$ is a positive function in $\bar{\Omega}$. Then there exist the maximal existence time $T_{\max} \le \infty$ and a uniquely determined pair (u, v) of positive functions

$$u \in C^0\left(\bar{\Omega}\times[0,T_{\max})\right)\cap C^{2,1}\left(\bar{\Omega}\times(0,T_{\max})\right),$$

$$v \in C^0\left(\bar{\Omega}\times[0,T_{\max})\right)\cap C^{2,1}\left(\bar{\Omega}\times(0,T_{\max})\right)\cap L^{\infty}_{\log}\left([0,T_{\max});W^{1,\infty}(\Omega)\right)$$

that solves (1.2) classically in $\Omega\times[0,T_{\max})$. In additions, for the second component v of the solution one can find $\eta > 0$ such that

$$\inf_{x\in\Omega} v(x,t) \ge \eta \quad \text{for all } t\in(0,T_{\max})$$

Furthermore, if $T_{\max} < \infty$, Then

$$\|u(\cdot,t)\|_{L^{\infty}(\Omega)} + \|v(\cdot,t)\|_{W^{1,q}(\Omega)} \to \infty \quad \text{as } t \nearrow T_{\max}.$$

The following lemma is a generalization of the maximum principle, which plays a major role in the proof of the main result.

Lemma 3.2. Suppose that $\Omega\subset\mathbb{R}^n\,(n\ge1)$ is a bounded domain with smooth boundary, $d > 0$ is a positive constant and is a positive continuous function satisfying $\int_0^{\infty} a(t)\mathrm{d}t < \infty$. Let $z\in C^0\left(\bar{\Omega}\times[0,\infty)\right)\cap C^{2,1}\left(\bar{\Omega}\times(0,\infty)\right)$, $z\ge0$ in $\bar{\Omega}\times[0,\infty)$. If

$$\begin{cases} z_t \le d\Delta z + a(t)z, & x\in\Omega, t>0 \\ \dfrac{\partial z}{\partial \nu} = 0, & x\in\partial\Omega, t>0 \\ z(x,0) = z_0(x), & x\in\Omega \end{cases} \tag{1.3}$$

then z is bounded in $\Omega\times(0,\infty)$.

Proof. Set

$$y(t) := \max_{x\in\bar{\Omega}} z_0(x)\cdot \mathrm{e}^{\int_0^t a(s)\mathrm{d}s} \quad \text{for all } t\ge0.$$

By simple calculations we can show that y is the solution of

$$\begin{cases} y' = a(t)y(t), & t>0, \\ y(0) := \max_{x\in\bar{\Omega}} z_0(x), \end{cases} \tag{1.4}$$

and it is bounded in $(0,\infty)$ by our supposition. Therefore, by the comparison principle, we see that z is bounded in $\Omega\times(0,\infty)$.

We are now in the position to prove global boundedness of solutions for (1.2).

4. Proof of the Main Result

Motivated by [14], let us introduce the function $w = \dfrac{u}{v}$. by using this assumption $d_1 = \chi + d_2$, we shall transform the system (1.2) into

$$\begin{cases} w_1 = d_1\Delta w + \dfrac{2d_1-\chi}{v}\Delta v\cdot\Delta w + (1-w)w, & (x,t)\in\Omega\times(0,T_{\max}), \\ v_t = d_2\Delta v - v + vw, & (x,t)\in\Omega\times(0,T_{\max}), \\ \dfrac{\partial w}{\partial v} = 0, \dfrac{\partial v}{\partial v} = 0, & (x,t)\in\partial\Omega\times(0,T_{\max}), \\ w(x,0) = \dfrac{u_0(x)}{v_0(x)}, v(x,0) = v_0(x), & x\in\Omega \end{cases} \tag{1.5}$$

and then, by the comparison principle we will obtain

$$w(x,t) \le \frac{y_0 e^t}{y_0 e^t - y_0 + 1}, \ (x,t)\in\Omega\times(0,T_{\max})$$

where $y_0 := \max_{x\in\bar{\Omega}} \dfrac{u_0(x)}{v_0(x)}$. Hence, the second equation in (1.5) implies that

$$v_t \le d_2\Delta v + \frac{y_0 - 1}{y_0 e^t - y_0 + 1}v, \ (x,t)\in\Omega\times(0,T_{\max}) \tag{1.6}$$

If $y_0 \le 1$, we deduce that

$$v(x,t) \le \max_{x\in\bar{\Omega}} v_0 x, \ (x,t)\in\Omega\times(0,T_{\max})$$

by using the maximum principle. For $y_0 > 1$, Let $\bar{v} = e^{-(y_0-1)t}v$. Through direct computation we establish that

$$\bar{v}_t \le d_2\Delta\bar{v} - \frac{y_0(y_0-1)(e^t-1)}{y_0 e^t - y_0 + 1}\bar{v}, \ \ (x,t)\in\Omega\times(0,T_{\max}).$$

We shall also use the maximum principle for the second time, it follows that

$$\bar{v}(x,t) \le \max_{x\in\bar{\Omega}} v_0 x, \ \ (x,t)\in\Omega\times(0,T_{\max}),$$

which implies that

$$v(x,t) \le e^{(y_0-1)} \max_{x\in\bar{\Omega}} v_0 x, \ \ (x,t)\in\Omega\times(0,T_{\max}).$$

Along with this, the Lemma 3.1 guarantees that $v(x,t)$ is global in time. Then the integral

$$\int_0^\infty \frac{y_0 - 1}{y_0 e^t - y_0 + 1}dt < \infty,$$

we apply the Lemma 3.2 to (1.6), it follows that v is bounded in $\Omega\times(0,\infty)$, and hence $u = vw$ is bounded in $\Omega\times(0,\infty)$ with smooth boundary, $\Omega\subset\mathbb{R}^n(n\ge 1)$, Thus we complete the proof.

5. Conclusion and Remarks

In the paper, we presented that the Neumann initial-boundary value problem for the chemotaxis system with singular sensitivity in problem (0.1) is bounded in $\Omega\times(0,\infty)$ with smooth boundary, $\Omega\subset\mathbb{R}^n(n\ge 1)$. Then we established that the problem (1.2) has a unique global classical solution which is bounded in $\Omega\times(0,\infty)$. And we showed that $\Omega\subset\mathbb{R}^n(n\ge 1)$ is a bounded domain with smooth boundary, $d > 0$ is a positive constant and a is a positive continuous function.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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