

The Angles and Main Angles of Some Special Graphs

Haicheng Ma, Shang Gao

Department of Mathematics, Qinghai University for Nationalities, Xining, China

Email: qhmymhc@163.com, 1027462041@qq.com

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Abstract

The eigenvalues of the adjacency matrix of a graph are called the eigenvalues of the graph. Let the vector $e_j = (0, \dots, 1, \dots, 0)^T$ and the all -1 vector $j = (1, 1, \dots, 1)^T$, the cosine of the (acute) angle formed by the vector e_j and the eigensubspace is called an angle of the graph. The cosine of the (acute) angle formed by the vector j and the eigensubspace is called a main angle of the graph. The angles and main angles are all important parameters on the graph, and they can be combined with the eigenvalues of the graph to determine the degree sequence of the graph, the number of triangles, quadrilaterals and pentagons on the graph, and the characteristic polynomials of the complement graph, but there is little study on the angles and main angles of the graph. In this paper, we determine the angles and main angles of the complete graph, the cube graph, the Petersen graph, the cycle and the complete bipartite graph.

Keywords

Adjacency Matrix, Eigenvalue, Angle, Main Angle

1. Introduction

This paper considers only finite undirected simple graphs. Let G be a graph with order n , its adjacency matrix is defined as follows: $A(G) = (a_{ij})_{n \times n}$

$$a_{ij} = \begin{cases} 1 & \text{if } i \sim j, \\ 0 & \text{others.} \end{cases}$$

where $i \sim j$ denotes that the vertex i and the vertex j are adjacent. Obviously, $A(G)$ is a real symmetric matrix in which all diagonal elements are 0 and all other elements are 0 or 1, its eigenvalues are all real numbers. The n eigenvalues of $A(G)$ are said to be the eigenvalues of the graph G and to form the spec-

trum of this graph. Let $\mu_1 > \mu_2 > \dots > \mu_m$ be the distinct eigenvalues of the graph G , and their multiplicities are n_1, n_2, \dots, n_m , respectively, then $n_1 + n_2 + \dots + n_m = n$. Throughout this paper, the spectrum of the graph G is often labeled as $Spec G = \{\mu_1^{n_1}, \mu_2^{n_2}, \dots, \mu_m^{n_m}\}$. Let μ be an eigenvalue of the graph G , a subspace of the vector space C^n in the complex field generated by all the vectors x that satisfy $A(G)x = \mu x$ is called the eigensubspace of the eigenvalue μ , and we denote it as $\varepsilon(\mu)$. Its dimension is denoted by $\dim \varepsilon(\mu)$, which is equal to the multiplicity of the eigenvalue μ .

Since the adjacency matrix $A = (a_{ij})$ is a real symmetric matrix, there exists a unitary matrix U such that $\bar{U}^T A U = E$, where $E = \text{diag}(\mu_1, \mu_2, \dots, \mu_n)$ is a diagonal matrix, and the columns of the matrix U are the eigenvectors corresponding to the eigenvalue, and they form a standard orthonormal basis of the vector space C^n , \bar{U}^T denotes the conjugate transpose of the matrix U . If this basis is constructed by stringing together the orthonormal basis of the eigenspaces of A , then $E = \mu_1 E_1 + \mu_2 E_2 + \dots + \mu_m E_m$, where $\mu_1, \mu_2, \dots, \mu_m$ are the distinct eigenvalues of A and each E_i has block diagonal form $\text{diag}(0, \dots, 0, I, 0, \dots, 0), (i = 1, 2, \dots, m)$. Then A has the spectral decomposition [1] [2]

$$A = \mu_1 P_1 + \mu_2 P_2 + \dots + \mu_m P_m, \tag{1}$$

where $P_i = U E_i \bar{U}^T (i = 1, 2, \dots, m)$. For fixed i , if $\varepsilon(\mu_i)$ has $\{x_1, x_1, \dots, x_n\}$ as a unit orthonormal basis, then

$$P_i = x_1 x_1^{-T} + x_2 x_2^{-T} + \dots + x_n x_n^{-T}, \tag{2}$$

and P_i represents the orthonormal projection of C^n on $\varepsilon(\mu_i)$. Moreover, it satisfies $\sum_{i=1}^m P_i = I$, $P_i^2 = P_i = P_i^T$, $P_i P_j = 0 (i \neq j)$.

The module $\|P_i e_j\|$ of the orthonormal projection of the vector $e_j = (0, \dots, 1, \dots, 0)^T$ into the eigensubspace $\varepsilon(\mu_i)$ is exactly equal to the cosine of the (acute) angle between the vector e_j and the eigensubspace $\varepsilon(\mu_i)$, in [2], which is defined as the angle of the graph, and we denote it as α_{ij} . The matrix $B = (\alpha_{ij})_{m \times n}$ formed by the angles is called the angle matrix of the graph

G . The module $\frac{\|P_i j\|}{\sqrt{n}}$ of the orthonormal projection of the unit vector $\frac{1}{\sqrt{n}} j$

into the eigensubspace $\varepsilon(\mu_i)$ is exactly equal to the cosine of the (acute) angle between the vector j and the eigensubspace $\varepsilon(\mu_i)$, in [1], which is defined as the main angle of the graph, and we denote it as β_i . The vector

$b = (\beta_1, \beta_2, \dots, \beta_m)^T$ formed by the main angles is called the main angle vector of the graph G , where $j = (1, 1, \dots, 1)^T$.

For positive integers m, n , K_n , C_n and $K_{m,n}$ denote the complete graph on n vertices, the n -cycle and the complete bipartite graph on $m+n$ vertices, respectively. Q_8 denotes the cube graph on 8 vertices (as shown in Figure 1), and P denotes the Petersen graph (as shown in Figure 2). There are many studies on the eigenvalues of the graph in [1] [2] [3] [4] [5], since the angles and

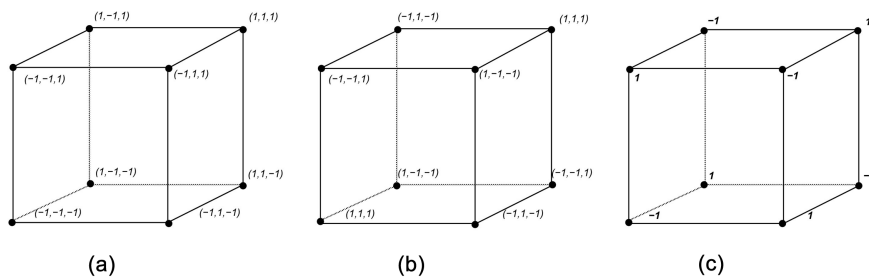


Figure 1. The eigenvectors corresponding to the eigenvalues 1, -1 and -3 of Q_8 . (a) The weight eigenvectors corresponding to 1; (b) The weight eigenvectors corresponding to -1; (c) The weight eigenvectors corresponding to -3.

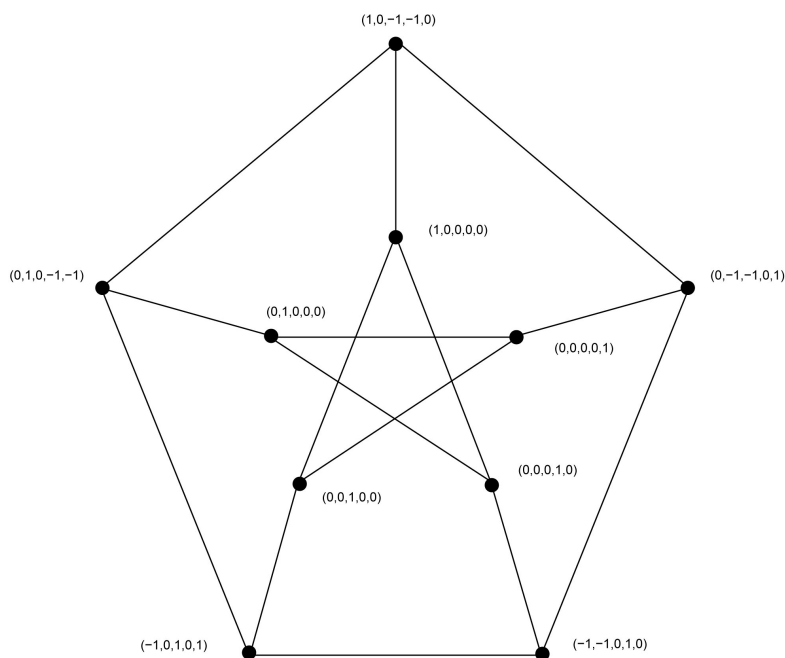


Figure 2. The eigenvectors corresponding to the eigenvalue 1 of P .

main angles are difficult to determine, there is a few study on them. Actually, they are all important parameters on the graph, and they can be combined with the eigenvalues of the graph to determine the degree sequence of the graph, the number of triangles, quadrilaterals and pentagons on the graph, and the characteristic polynomials of the complement graph [1] [2]. In this paper, we determine the angles and main angles of the complete graph, the cube graph, the Petersen graph, the cycle and the complete bipartite graph. Undefined concepts and notations will follow [1].

2. Some Lemmas

Lemma 2.1 [1] Any graph is determined by its eigenvalues and a basis of the corresponding eigenvectors.

Let G be a graph on n vertices, and its vertices are labeled as $1, 2, \dots, n$. We assign a weight x_i to each vertex on the graph, and we get a vector

$$x = (x_1, x_2, \dots, x_n)^T.$$

Lemma 2.2 [1] The nonzero vector $x = (x_1, x_2, \dots, x_n)^T$ is an eigenvector of the graph G corresponding to the eigenvalue λ if and only if

$$\lambda x_i = \sum_{j \sim i} x_j, (i = 1, 2, \dots, n)$$

holds, where \sum is the sum of the weights x_j of all vertices j adjacent to the vertex i .

Remark Lemma 2.2 shows that the nonzero vector $x = (x_1, x_2, \dots, x_n)^T$ is an eigenvector of the graph G corresponding to eigenvalue λ if and only if the λ times of the weight x_i of the $i (= 1, 2, \dots, n)$ th vertex is exactly equal to the sum of the weight of each vertex adjacent to it.

Lemma 2.3 [1] The angles α_{ij} of a graph satisfy the equalities

$$\sum_{j=1}^n \alpha_{ij}^2 = \dim \mathcal{E}(\mu_i), \sum_{i=1}^m \alpha_{ij}^2 = 1.$$

Lemma 2.4 [1] The main angles β_i of a graph satisfy the equality

$$\sum_{i=1}^m \beta_i^2 = 1.$$

3. The Primary Outcomes

Theorem 3.1 The angle matrix of the complete graph K_n is

$$B(K_n) = \begin{pmatrix} \frac{\sqrt{n}}{n} & \frac{\sqrt{n}}{n} & \dots & \frac{\sqrt{n}}{n} \\ \frac{\sqrt{n-1}}{n} & \frac{\sqrt{n-1}}{n} & \dots & \frac{\sqrt{n-1}}{n} \\ \dots & \dots & \dots & \dots \\ \frac{\sqrt{n-1}}{n} & \frac{\sqrt{n-1}}{n} & \dots & \frac{\sqrt{n-1}}{n} \end{pmatrix},$$

the main angle vector is $b(K_n) = (1, 0)^T$.

Proof. By [5], we know that the eigenvalues of K_n are $n-1$ and -1 , the multiplicities of them are 1 and $n-1$, respectively. So the spectrum of K_n is

$$\text{Spec}K_n = \{(n-1)^1, (-1)^{n-1}\},$$

the eigenvector corresponding to the eigenvalue $n-1$ is the all -1 vector

$j = (1, 1, \dots, 1)^T$, and we unify it into $\frac{1}{\sqrt{n}}j$. So the matrix

$$P_1 = \left(\frac{1}{\sqrt{n}}j \right) \left(\frac{1}{\sqrt{n}}j \right)^T = \frac{1}{n}jj^T, \text{ by the definition of the angle,}$$

$\alpha_{1j} = \|P_1 e_j\| = \left\| \frac{1}{n}jj^T e_j \right\| = \left\| \frac{1}{n}j \right\| = \frac{1}{n}\|j\| = \frac{\sqrt{n}}{n}$. By Lemma 2.3, we can easily get the second row of the angle matrix $B(K_n)$. By the definition of the main angle, and

$\beta_i = \|P_i j\|/\sqrt{n}$, $\beta_1 = \|P_1 j\|/\sqrt{n} = \left\| \frac{1}{n}jj^T j \right\|/\sqrt{n} = \|j\|/\sqrt{n} = \sqrt{n}/\sqrt{n} = 1$. By Lemma 2.4, we can easily get the main angle vector $b(K_n)$. \square

Theorem 3.2 The angle matrix of the cube graph Q_8 is

$$B(Q_8) = \begin{pmatrix} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \dots & \frac{\sqrt{2}}{4} \\ \frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} & \dots & \frac{\sqrt{6}}{4} \\ \frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} & \dots & \frac{\sqrt{6}}{4} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \dots & \frac{\sqrt{2}}{4} \end{pmatrix},$$

the main angle vector is $b(Q_8) = (1, 0, 0, 0)^T$.

Proof. By [4], we know that the eigenvalues of Q_8 are 3, 1, -1 and -3, the multiplicities of them are 1, 3, 3 and 1, respectively. So the spectrum of Q_8 is

$$\text{Spec}Q_8 = \{3^1, 1^3, (-1)^3, (-3)^1\},$$

the eigenvector corresponding to the eigenvalue 3 is the all -1 vector

$j = (1, 1, 1, 1, 1, 1, 1, 1)^T$, and we unify it into $\frac{1}{\sqrt{8}}j$. So the matrix

$$P_1 = \left(\frac{1}{\sqrt{8}}j\right)\left(\frac{1}{\sqrt{8}}j\right)^T = \frac{1}{8}jj^T, \text{ by the definition of the angle,}$$

$\alpha_{1j} = \|P_1 e_j\| = \left\|\frac{1}{8}jj^T e_j\right\| = \left\|\frac{1}{8}j\right\| = \frac{1}{8}\|j\| = \frac{\sqrt{2}}{4}, (j = 1, 2, \dots, 8)$. By the definition of

the main angle, $\beta_1 = \|P_1 j\|/\sqrt{8} = \left\|\frac{1}{8}jj^T j\right\|/\sqrt{8} = \|j\|/\sqrt{8} = \sqrt{8}/\sqrt{8} = 1$. So all angles corresponding to the eigenvalue 3 are $\frac{\sqrt{2}}{4}$, the main angle is 1.

By Lemma 2.1 and Lemma 2.2, we can know that the orthonormal eigenvectors corresponding to the eigenvalue 1 are (as shown in **Figure 1(a)**)

$$\xi_1 = (1, 1, 1, -1, 1, -1, -1, -1)^T,$$

$$\xi_2 = (1, 1, -1, 1, -1, 1, -1, -1)^T,$$

$$\xi_3 = (1, -1, 1, 1, -1, -1, 1, -1)^T,$$

and we unify them into $\frac{1}{\sqrt{8}}\xi_1, \frac{1}{\sqrt{8}}\xi_2, \frac{1}{\sqrt{8}}\xi_3$, then the orthonormal projection matrix

$$\begin{aligned} P_2 &= \left(\frac{1}{\sqrt{8}}\xi_1\right)\left(\frac{1}{\sqrt{8}}\xi_1\right)^T + \left(\frac{1}{\sqrt{8}}\xi_2\right)\left(\frac{1}{\sqrt{8}}\xi_2\right)^T + \left(\frac{1}{\sqrt{8}}\xi_3\right)\left(\frac{1}{\sqrt{8}}\xi_3\right)^T \\ &= \frac{1}{8}(\xi_1\xi_1^T + \xi_2\xi_2^T + \xi_3\xi_3^T), \end{aligned}$$

by the definition of the angle,

$$\alpha_{21} = \|P_2 e_1\| = \left\|\frac{1}{8}(\xi_1\xi_1^T e_1 + \xi_2\xi_2^T e_1 + \xi_3\xi_3^T e_1)\right\| = \frac{1}{8}\|(\xi_1 + \xi_2 + \xi_3)\| = \frac{\sqrt{6}}{4},$$

in the same way,

$$\alpha_{22} = \frac{1}{8}\|(\xi_1 + \xi_2 - \xi_3)\| = \frac{\sqrt{6}}{4}, \quad \alpha_{23} = \frac{1}{8}\|(\xi_1 - \xi_2 + \xi_3)\| = \frac{\sqrt{6}}{4},$$

$$\alpha_{24} = \frac{1}{8} \|(-\xi_1 + \xi_2 + \xi_3)\| = \frac{\sqrt{6}}{4}, \quad \alpha_{25} = \frac{1}{8} \|(\xi_1 - \xi_2 - \xi_3)\| = \frac{\sqrt{6}}{4},$$

$$\alpha_{26} = \frac{1}{8} \|(-\xi_1 + \xi_2 - \xi_3)\| = \frac{\sqrt{6}}{4}, \quad \alpha_{27} = \frac{1}{8} \|(-\xi_1 - \xi_2 + \xi_3)\| = \frac{\sqrt{6}}{4},$$

$$\alpha_{28} = \frac{1}{8} \|(-\xi_1 - \xi_2 - \xi_3)\| = \frac{\sqrt{6}}{4}.$$

Since ξ_1, ξ_2, ξ_3 are orthonormal to the all -1 vector, then

$$\beta_2 = \|P_2 j\|/\sqrt{n} = \|\xi_1 \xi_1^T j + \xi_2 \xi_2^T j + \xi_3 \xi_3^T j\|/\sqrt{n} = 0.$$

So all angles corresponding to the eigenvalue 1 are $\frac{\sqrt{6}}{4}$, the main angle is 0.

By Lemma 2.1 and Lemma 2.2, we can know that the orthonormal eigenvectors corresponding to the eigenvalue -1 are (as shown in **Figure 1(b)**)

$$\eta_1 = (1, -1, -1, 1, 1, -1, -1, 1)^T,$$

$$\eta_2 = (1, -1, 1, -1, -1, 1, -1, 1)^T,$$

$$\eta_3 = (1, 1, -1, -1, -1, -1, 1, 1)^T,$$

and we unify them into $\frac{1}{\sqrt{8}}\eta_1, \frac{1}{\sqrt{8}}\eta_2, \frac{1}{\sqrt{8}}\eta_3$, then the orthonormal projection matrix

$$P_3 = \left(\frac{1}{\sqrt{8}}\eta_1\right)\left(\frac{1}{\sqrt{8}}\eta_1\right)^T + \left(\frac{1}{\sqrt{8}}\eta_2\right)\left(\frac{1}{\sqrt{8}}\eta_2\right)^T + \left(\frac{1}{\sqrt{8}}\eta_3\right)\left(\frac{1}{\sqrt{8}}\eta_3\right)^T$$

$$= \frac{1}{8}(\eta_1\eta_1^T + \eta_2\eta_2^T + \eta_3\eta_3^T),$$

then

$$\alpha_{31} = \alpha_{38} = \frac{1}{8} \|(\eta_1 + \eta_2 + \eta_3)\| = \frac{\sqrt{6}}{4}, \quad \alpha_{32} = \alpha_{37} = \frac{1}{8} \|(-\eta_1 - \eta_2 + \eta_3)\| = \frac{\sqrt{6}}{4},$$

$$\alpha_{33} = \alpha_{36} = \frac{1}{8} \|(-\eta_1 + \eta_2 - \eta_3)\| = \frac{\sqrt{6}}{4}, \quad \alpha_{34} = \alpha_{35} = \frac{1}{8} \|(\eta_1 - \eta_2 - \eta_3)\| = \frac{\sqrt{6}}{4}.$$

Since η_1, η_2, η_3 are orthonormal to the all -1 vector, then

$$\beta_3 = \|P_3 j\|/\sqrt{n} = \|\eta_1\eta_1^T j + \eta_2\eta_2^T j + \eta_3\eta_3^T j\|/\sqrt{n} = 0.$$

So all angles corresponding to the eigenvalue -1 are $\frac{\sqrt{6}}{4}$, the main angle is 0.

By Lemma 2.1 and Lemma 2.2, we can know that the orthonormal eigenvector corresponding to the eigenvalue -3 is $\zeta = (1, -1, -1, -1, 1, 1, 1, -1)^T$ (as shown in **Figure 1(c)**), and we unify them into $\frac{1}{\sqrt{8}}\zeta$. So the orthonormal projection

matrix $P_4 = \left(\frac{1}{\sqrt{8}}\zeta\right)\left(\frac{1}{\sqrt{8}}\zeta\right)^T = \frac{1}{8}\zeta\zeta^T$, then

$$\alpha_{41} = \alpha_{45} = \alpha_{46} = \alpha_{47} = \frac{1}{8} \|\zeta\| = \frac{\sqrt{2}}{4},$$

$$\alpha_{42} = \alpha_{43} = \alpha_{44} = \alpha_{48} = \frac{1}{8} \|\zeta\| = \frac{\sqrt{2}}{4}.$$

$$\beta_4 = \|P_4 j\|/\sqrt{8} = \left\| \frac{1}{8} \zeta \zeta^T j \right\|/\sqrt{8} = 0.$$

So all angles corresponding to the eigenvalue -3 are $\frac{\sqrt{2}}{4}$, the main angle is 0 .
 \square

Theorem 3.3 The angle matrix of the Petersen graph P is

$$B(P) = \begin{pmatrix} \frac{\sqrt{10}}{10} & \frac{\sqrt{10}}{10} & \dots & \frac{\sqrt{10}}{10} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \dots & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{10}}{5} & \frac{\sqrt{10}}{5} & \dots & \frac{\sqrt{10}}{5} \end{pmatrix},$$

the main angle vector is $b(P) = (1, 0, 0)^T$.

Proof. By [4], we know that the eigenvalues of P are $3, 1$ and -2 , the multiplicities of them are $1, 5$ and 4 , respectively. So the spectrum of P is

$$Spec P = \{3^1, 1^5, (-2)^4\},$$

the eigenvector corresponding to the eigenvalue 3 is the all -1 vector

$j = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^T$, and we unify it into $\frac{1}{\sqrt{10}} j$. So the matrix

$$P_1 = \left(\frac{1}{\sqrt{10}} j \right) \left(\frac{1}{\sqrt{10}} j \right)^T = \frac{1}{10} j j^T, \text{ then}$$

$$\alpha_{1j} = \|P_1 e_j\| = \frac{1}{10} \|j\| = \frac{\sqrt{10}}{10}, (j = 1, 2, \dots, 10).$$

$$\beta_1 = \|P_1 j\|/\sqrt{10} = \left\| \frac{1}{10} j j^T j \right\|/\sqrt{10} = \|j\|/\sqrt{10} = \sqrt{10}/\sqrt{10} = 1.$$

So all angles corresponding to the eigenvalue 3 are $\frac{\sqrt{10}}{10}$, the main angle is 1 .

By Lemma 2.1 and Lemma 2.2, we can know that the eigenvectors corresponding to the eigenvalue 1 are (as shown in **Figure 2**)

$$\delta_1 = (1, 0, -1, -1, 0, 1, 0, 0, 0, 0)^T, \delta_2 = (0, 1, 0, -1, -1, 0, 1, 0, 0, 0)^T,$$

$$\delta_3 = (-1, 0, 1, 0, -1, 0, 0, 1, 0, 0)^T, \delta_4 = (-1, -1, 0, 1, 0, 0, 0, 0, 1, 0)^T,$$

$$\delta_5 = (0, -1, -1, 0, 1, 0, 0, 0, 0, 1)^T,$$

and we unify and orthogonality them into

$$\gamma_1 = \left(\frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{2}, 0, 0, 0, 0 \right)^T,$$

$$\gamma_2 = \left(-\frac{1}{2\sqrt{15}}, \frac{2}{\sqrt{15}}, \frac{1}{2\sqrt{15}}, -\frac{3}{2\sqrt{15}}, -\frac{2}{\sqrt{15}}, -\frac{1}{2\sqrt{15}}, \frac{2}{\sqrt{15}}, 0, 0, 0 \right)^T,$$

$$\gamma_3 = \left(-\frac{1}{\sqrt{15}}, \frac{1}{\sqrt{15}}, \frac{1}{\sqrt{15}}, -\frac{1}{2\sqrt{15}}, -\frac{3}{2\sqrt{15}}, \frac{3}{2\sqrt{15}}, -\frac{1}{\sqrt{15}}, \frac{5}{2\sqrt{15}}, 0, 0 \right)^T,$$

$$\gamma_4 = \left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, \frac{2}{3}, 0\right)^T,$$

$$\gamma_5 = \left(-\frac{1}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T.$$

Then the orthonormal projection matrix $P_2 = \gamma_1\gamma_1^T + \gamma_2\gamma_2^T + \gamma_3\gamma_3^T + \gamma_4\gamma_4^T + \gamma_5\gamma_5^T$,

$$\begin{aligned} \alpha_{21} &= \|P_2 e_1\| = \|\gamma_1\gamma_1^T e_1 + \gamma_2\gamma_2^T e_1 + \gamma_3\gamma_3^T e_1 + \gamma_4\gamma_4^T e_1 + \gamma_5\gamma_5^T e_1\| \\ &= \left\| \frac{1}{2}\gamma_1 - \frac{1}{2\sqrt{15}}\gamma_2 - \frac{1}{\sqrt{15}}\gamma_3 - \frac{1}{3}\gamma_4 - \frac{1}{3\sqrt{2}}\gamma_5 \right\| = \frac{\sqrt{2}}{2}, \end{aligned}$$

in the same way, we have

$$\alpha_{2j} = \frac{\sqrt{2}}{2}, (j = 2, 3, \dots, 10).$$

Since $\gamma_i (i = 1, 2, 3, 4, 5)$ are orthonormal to the all -1 vector j , then

$$\beta_2 = \|P_2 j\| / \sqrt{10} = \|\gamma_1\gamma_1^T j + \gamma_2\gamma_2^T j + \gamma_3\gamma_3^T j + \gamma_4\gamma_4^T j + \gamma_5\gamma_5^T j\| / \sqrt{10} = 0.$$

So all angles corresponding to the eigenvalue 1 are $\frac{\sqrt{2}}{2}$, the main angle is 0.

By Lemma 2.1 and Lemma 2.2, we can easily know that all angles corresponding to the eigenvalue -2 are $\frac{\sqrt{10}}{5}$, the main angle is 0. \square

Theorem 3.4 The angle matrix of the n -cycle C_n is

$$B(C_n) = \begin{pmatrix} \frac{\sqrt{n}}{n} & \frac{\sqrt{n}}{n} & \dots & \frac{\sqrt{n}}{n} \\ \frac{\sqrt{n}}{n} & \frac{\sqrt{n}}{n} & \dots & \frac{\sqrt{n}}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sqrt{n}}{n} & \frac{\sqrt{n}}{n} & \dots & \frac{\sqrt{n}}{n} \end{pmatrix},$$

the main angle vector is $b(C_n) = (1, 0, \dots, 0)^T$.

Proof. By [1], we know that the eigenvalues of C_n are

$2 \cos\left(\frac{2\pi j}{n}\right) (j = 0, 1, 2, \dots, n-1)$, the multiplicities of them are all 1. So the spectrum of C_n is

$$Spec C_n = \left\{ 2 \cos\left(\frac{2\pi j}{n}\right) (j = 0, 1, 2, \dots, n-1) \right\},$$

when $j = 0$, the eigenvector corresponding to the eigenvalue 2 is the all -1 vector $j = (1, 1, \dots, 1)^T$, and we unify it into $\frac{1}{\sqrt{n}} j$. So the orthonormal projection matrix

$$P_1 = \left(\frac{1}{\sqrt{n}} j\right) \left(\frac{1}{\sqrt{n}} j\right)^T = \frac{1}{n} j j^T, \text{ then}$$

$$\alpha_{1j} = \|P_1 e_j\| = \frac{1}{n} \|j\| = \frac{\sqrt{n}}{n}, (j = 1, 2, \dots, n),$$

$$\beta_1 = \|P_1 j\|/\sqrt{n} = \left\| \frac{1}{n} j j^T j \right\|/\sqrt{n} = \|j\|/\sqrt{n} = \sqrt{n}/\sqrt{n} = 1.$$

So all angles corresponding to the eigenvalue 2 are $\frac{\sqrt{n}}{n}$, the main angle is 1.

By Lemma 2.1 and Lemma 2.2, we can know that the eigenvectors corresponding to the eigenvalue $2 \cos\left(\frac{2\pi j}{n}\right)$ ($j = 1, 2, \dots, n-1$) are

$$\begin{aligned} \theta_1 &= (1, \omega, \omega^2, \dots, \omega^{n-1})^T, \\ \theta_2 &= (1, \omega^2, \omega^4, \dots, \omega^{2(n-1)})^T, \\ &\vdots \\ \theta_{n-1} &= (1, \omega^{n-1}, \omega^{2(n-1)}, \dots, \omega^{(n-1)(n-1)})^T, \end{aligned}$$

where $\|\theta_j\| = \sqrt{1 + \omega\bar{\omega} + \dots + \omega^{n-1}\bar{\omega}^{n-1}} = \sqrt{n}$ ($j = 1, 2, \dots, n-1$), then the orthonormal projection matrix $P_2 = \frac{1}{n} \theta_1 \bar{\theta}_1^T, P_3 = \frac{1}{n} \theta_2 \bar{\theta}_2^T, \dots, P_n = \frac{1}{n} \theta_{n-1} \bar{\theta}_{n-1}^T$, for example, when $j = 1$, we have the orthonormal projection matrix P_2 corresponding to the eigenvalue $2 \cos\left(\frac{2\pi}{n}\right)$,

$$\begin{aligned} \alpha_{21} &= \|P_2 e_1\| = \left\| \frac{1}{n} \theta_1 \bar{\theta}_1^T e_1 \right\| = \frac{1}{n} \|\theta_1\| = \frac{\sqrt{n}}{n}, \\ \alpha_{22} &= \|P_2 e_2\| = \left\| \frac{1}{n} \theta_1 \bar{\theta}_1^T e_2 \right\| = \frac{1}{n} \|\omega \theta_1\| = \frac{\sqrt{n}}{n}, \\ &\vdots \\ \alpha_{2n} &= \|P_2 e_n\| = \left\| \frac{1}{n} \theta_1 \bar{\theta}_1^T e_n \right\| = \frac{1}{n} \|\omega^{n-1} \theta_1\| = \frac{\sqrt{n}}{n}. \end{aligned}$$

Since the eigenvectors corresponding to distinct eigenvalues are orthonormal, then

$$\beta_2 = \|P_2 j\|/\sqrt{n} = \left\| \frac{1}{n} \theta_1 \bar{\theta}_1^T j \right\|/\sqrt{n} = 0.$$

So all angles corresponding to the eigenvalue $2 \cos\left(\frac{2\pi}{n}\right)$ are $\frac{\sqrt{n}}{n}$, the main angle is 0.

In the same way, all angles corresponding to the eigenvalue

$$2 \cos\left(\frac{2\pi j}{n}\right) (j = 1, 2, \dots, n-1) \text{ are } \frac{\sqrt{n}}{n}, \text{ the main angle is 0. } \square$$

Theorem 3.5 The angle matrix of the complete bipartite graph $K_{m,n}$ is

$$B(K_{m,n}) = \begin{pmatrix} \frac{1}{\sqrt{2m}} & \cdots & \frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2n}} & \cdots & \frac{1}{\sqrt{2n}} \\ \sqrt{\frac{m-1}{m}} & \cdots & \sqrt{\frac{m-1}{m}} & \sqrt{\frac{n-1}{n}} & \cdots & \sqrt{\frac{n-1}{n}} \\ \frac{1}{\sqrt{2m}} & \cdots & \frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2n}} & \cdots & \frac{1}{\sqrt{2n}} \end{pmatrix},$$

the main angle vector is $b(K_{m,n}) = \left(\frac{\sqrt{m} + \sqrt{n}}{\sqrt{2(m+n)}}, 0, \frac{|\sqrt{m} - \sqrt{n}|}{\sqrt{2(m+n)}} \right)^T$.

Proof. By [2], we know that the eigenvalues of $K_{m,n}$ are \sqrt{mn} , 0 and $-\sqrt{mn}$, the multiplicities of them are 1, $m+n-2$ and 1, respectively. So the spectrum of $K_{m,n}$ is

$$\text{Spec}K_{m,n} = \left\{ (\sqrt{mn})^1, 0^{(m+n-2)}, (-\sqrt{mn})^1 \right\}.$$

The vertices of complete bipartite graph $K_{m,n}$ can be divided into two color classes X and Y , $|X|=m$, $|Y|=n$. If we assign a weight $\frac{1}{\sqrt{2m}}$ to each vertex

of X and a weight $\frac{1}{\sqrt{2n}}$ to each vertex of Y , by Lemma 2.2, we can know that

the vector $\rho_1 = \left(\frac{1}{\sqrt{2m}}, \dots, \frac{1}{\sqrt{2m}}, \frac{1}{\sqrt{2n}}, \dots, \frac{1}{\sqrt{2n}} \right)^T$ is a unit eigenvector corresponding to the eigenvalue \sqrt{mn} , so the orthonormal projection matrix corresponding to the eigenvalue \sqrt{mn} is $P_1 = \rho_1 \rho_1^T$, then

$$\alpha_{1i} = \|P_1 e_i\| = \|\rho_1 \rho_1^T e_i\| = \left\| \frac{1}{\sqrt{2m}} \rho_1 \right\| = \frac{1}{\sqrt{2m}}, (i = 1, 2, \dots, m),$$

$$\alpha_{1i} = \|P_1 e_i\| = \|\rho_1 \rho_1^T e_i\| = \left\| \frac{1}{\sqrt{2n}} \rho_1 \right\| = \frac{1}{\sqrt{2n}}, (i = m+1, m+2, \dots, m+n),$$

$$\beta_1 = \|P_1 j\|/\sqrt{m+n} = \|\rho_1 \rho_1^T j\|/\sqrt{m+n} = \frac{\sqrt{m} + \sqrt{n}}{\sqrt{2(m+n)}}.$$

So all angles corresponding to the eigenvalue \sqrt{mn} are $m \frac{1}{\sqrt{2m}}$ s and n

$\frac{1}{\sqrt{2n}}$ s, the main angle is $\frac{\sqrt{m} + \sqrt{n}}{\sqrt{2(m+n)}}$.

If we assign a weight $\frac{1}{\sqrt{2m}}$ to each vertex of X and a weight $-\frac{1}{\sqrt{2n}}$ to each vertex of Y , by Lemma 2.2, we can know that the vector

$\rho_2 = \left(\frac{1}{\sqrt{2m}}, \dots, \frac{1}{\sqrt{2m}}, -\frac{1}{\sqrt{2n}}, \dots, -\frac{1}{\sqrt{2n}} \right)^T$ is a unit eigenvector corresponding

to the eigenvalue $-\sqrt{mn}$, so the orthonormal projection matrix corresponding to the eigenvalue $-\sqrt{mn}$ is $P_2 = \rho_2 \rho_2^T$, then

$$\alpha_{2i} = \|P_2 e_i\| = \|\rho_2 \rho_2^T e_i\| = \left\| \frac{1}{\sqrt{2m}} \rho_2 \right\| = \frac{1}{\sqrt{2m}}, (i = 1, 2, \dots, m),$$

$$\alpha_{2i} = \|P_2 e_i\| = \|\rho_2 \rho_2^T e_i\| = \left\| -\frac{1}{\sqrt{2n}} \rho_2 \right\| = \frac{1}{\sqrt{2n}}, (i = m+1, m+2, \dots, m+n),$$

$$\beta_2 = \|P_2 j\|/\sqrt{m+n} = \|\rho_2 \rho_2^T j\|/\sqrt{m+n} = \frac{|\sqrt{m} - \sqrt{n}|}{\sqrt{2(m+n)}}.$$

So all angles corresponding to the eigenvalue $-\sqrt{mn}$ are $m \frac{1}{\sqrt{2m}}$ s and $n \frac{1}{\sqrt{2n}}$ s, the main angle is $\frac{|\sqrt{m}-\sqrt{n}|}{\sqrt{2(m+n)}}$.

By Lemma 2.3, all angles corresponding to the eigenvalue 0 are $m \sqrt{\frac{m-1}{m}}$ s and $n \sqrt{\frac{n-1}{n}}$ s. By Lemma 2.4, the main angle corresponding to the eigenvalue 0 is $\beta_3 = 0$. \square

4. Conclusion

In this paper, the angles and main angles of the complete graph, the cube graph, the Petersen graph, the cycle and the complete bipartite graph are determined.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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